A MATHEMATICAL INSTRUCTION OF THE GRP SOLVER FOR TWO-DIMENSIONAL EULER EQUATIONS

1. Mathematical set-up of the governing equations and the GRP

We consider generalized Riemann problem (GRP) solver for solving two-dimension Euler equations

$$\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}+\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y}=0, \tag{1.1}$$

where

$$\begin{aligned}
&\mathbf{u}=(\rho,\rho u,\rho v,\rho E)^\top,\\
&\mathbf{f}(\mathbf{u})=(\rho u,\rho u^2+p,\rho uv,(\rho E+p)u)^\top,\\
&\mathbf{g}(\mathbf{u})=(\rho v,\rho uv,\rho v^2+p,(\rho E+p)v)^\top,\\
&E=\frac{u^2+v^2}{2}+e,\ \ e=e(\rho,S).
\end{aligned} \tag{1.2}$$

The dependence of the internal energy e on ρ and p is determined by the equation of state (EOS). This version of solver is constructed by incorporating linearized dealing of the tangential derivatives into the nonlinear GRP solver for one-dimensional Euler equations constructed in [1]. So this solver can be regarded as a *one-and-a-half dimensional* GRP solver.

The GRP considered here is the initial value problem of (1.1) equipped with piece-wise polynomial initial data

$$\mathbf{u}(x,y,t=0)=\begin{cases}\mathbf{u}_-(x,y), & x<0,\\ \mathbf{u}_+(x,y), & x>0.\end{cases} \tag{1.3}$$

Denote the limiting values at the origin as

$$\begin{aligned}
&\mathbf{u}_L=\lim_{x\to 0-}\mathbf{u}_-(x,0),\quad (\mathbf{u}_x)_L=\lim_{x\to 0-}\frac{\partial \mathbf{u}_-}{\partial x}(x,0),\quad (\mathbf{u}_y)_L=\lim_{x\to 0-}\frac{\partial \mathbf{u}_-}{\partial y}(x,0),\\
&\mathbf{u}_R=\lim_{x\to 0+}\mathbf{u}_+(x,0),\quad (\mathbf{u}_x)_R=\lim_{x\to 0+}\frac{\partial \mathbf{u}_+}{\partial x}(x,0),\quad (\mathbf{u}_y)_R=\lim_{x\to 0+}\frac{\partial \mathbf{u}_+}{\partial y}(x,0).
\end{aligned} \tag{1.4}$$

The first step is to solve the associated Riemann problem

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=0,\quad t>0,\\
&\mathbf{u}(x,y,t=0)=\begin{cases}\mathbf{u}_L, & x<0,\\ \mathbf{u}_R, & x>0,\end{cases}
\end{aligned} \tag{1.5}$$

to get the Riemann solution $R^A(\lambda; u_L, u_R)$. Next, solve the GPR in the normal direction of the inital discontinuity

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = -\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y}, \quad t > 0, \\ &\mathbf{u}(x, y = 0, t = 0) = \begin{cases} \mathbf{u}_-(x, y), & x < 0, \\ \mathbf{u}_+(x, y), & x > 0. \end{cases} \end{aligned} \tag{1.6}$$

where the transversal term $\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y}$ is regarded as a source. To achieve that, we firstly solve the homogeneous one-dimensional GRP

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0, \quad t > 0, \\ &\mathbf{u}(x, y, t = 0) = \begin{cases} \mathbf{u}_-(x, y = 0), & x < 0, \\ \mathbf{u}_+(x, y = 0), & x > 0, \end{cases} \end{aligned} \tag{1.7}$$

to obtain $\widetilde{(\partial \mathbf{u}/\partial t)}^*$. Refer to [1] for details. Next we get the intermediate value of the transversal term $(\partial \mathbf{g}/\partial y)^*$ which can be linearly approximated in the next section. At last, the desired time derivative is given by

$$\left(\frac{\partial \mathbf{u}}{\partial t}\right)^* = \left(\frac{\widetilde{\partial \mathbf{u}}}{\partial t}\right)^* - \left(\frac{\partial \mathbf{g}}{\partial y}\right)^*. \tag{1.8}$$

2. Linearized calculation of tangential derivatives

By using the Gibbs relation, governing PDEs for primative variables are derived from the conservative Euler equations as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{v}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{v}}{\partial y} = 0, \tag{2.1}$$

where

$$\mathbf{v} = \begin{bmatrix} \rho \\ u \\ v \\ S \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u & \rho & 0 & 0 \\ \frac{c^2}{\rho} & u & 0 & \frac{1}{\rho}\frac{\partial p}{\partial S} \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ \frac{c^2}{\rho} & 0 & v & \frac{1}{\rho}\frac{\partial p}{\partial S} \\ 0 & 0 & 0 & v \end{bmatrix}, \tag{2.2}$$

and the sound speed is defined as $c^2 = \partial p/\partial \rho$. Furthermore, the coefficient matrix $\mathbf{A}$ has the characteristic decomposition $\mathbf{A} = L\Lambda R$ where

$$R = \begin{bmatrix} -\dfrac{\rho}{2c} & 0 & -\dfrac{1}{c^2}\dfrac{\partial p}{\partial S} & \dfrac{\rho}{2c} \\ \dfrac{1}{2} & 0 & 0 & \dfrac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} -\dfrac{c}{\rho} & 1 & 0 & -\dfrac{1}{\rho c}\dfrac{\partial p}{\partial S} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{c}{\rho} & 1 & 0 & \dfrac{1}{\rho c}\dfrac{\partial p}{\partial S} \end{bmatrix}. \tag{2.3}$$

Linearize (2.1) by evalueating $\mathbf{A}$ and $\mathbf{B}$ at the background state $\mathbf{v}^*$ derived from the Riemann solution $\mathbf{u}^* = R^A(0, \mathbf{u}_L, \mathbf{u}_R)$. Denote the linearized coefficient matrix as $\mathbf{A}^*$ and $\mathbf{B}^*$ and the linearized governing equation is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A}^* \frac{\partial \mathbf{v}}{\partial x} + \mathbf{B}^* \frac{\partial \mathbf{v}}{\partial y} = 0. \tag{2.4}$$

The Riemann problem

$$\begin{aligned} &\frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}^* \frac{\partial \mathbf{w}}{\partial x} + \mathbf{B}^* \frac{\partial \mathbf{w}}{\partial y} = 0, \quad t > 0, \\ &\mathbf{w}(x, y, t = 0) = \begin{cases} (\mathbf{v}_y)_L, & x < 0, \\ (\mathbf{v}_y)_R, & x > 0, \end{cases} \end{aligned} \tag{2.5}$$

with respect to $\mathbf{w} = \frac{\partial \mathbf{v}}{\partial y}$ is obtained by differentiating (2.4) togenther with the initial data (1.3). The Riemann problem (2.5) is easy to solve by the approach of the characteristic decomposition. Note that the transversal term $\mathbf{B}^* \frac{\partial \mathbf{w}}{\partial y}$ has no effect on the Riemann solution and can be dropped.

Assume that $u^* - c^* < 0 < u^*$, we have

$$\mathbf{w}^* = R^* I^+ L^* (\mathbf{v}_y)_L + R^* I^- L^* (\mathbf{v}_y)_R, \tag{2.6}$$

where R^* and L^* are matrices in (2.3) evaluated at $\mathbf{v}^*$ and

$$I^+ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I^- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Explicit calculations of (2.6) gives us

$$\begin{aligned} &\left(\frac{\partial \rho}{\partial y}\right)^* = -\frac{\rho^*}{2c^*}\Big(\Phi_R - \Phi_L\Big) - \frac{1}{c^{*2}}\left(\frac{\partial p}{\partial S}\right)^* \left(\frac{\partial S}{\partial y}\right)_L, \quad \left(\frac{\partial u}{\partial y}\right)^* = \frac{1}{2}\Big(\Phi_R + \Phi_L\Big), \\ &\left(\frac{\partial v}{\partial y}\right)^* = \left(\frac{\partial v}{\partial y}\right)_L, \quad \left(\frac{\partial S}{\partial y}\right)^* = \left(\frac{\partial S}{\partial y}\right)_L, \end{aligned} \tag{2.7}$$

where

$$\Phi_R = \Big(\frac{\partial u}{\partial y}\Big)_R - \frac{1}{\rho^* c^*}\Big(\frac{\partial p}{\partial y}\Big)_R, \quad \Phi_L = \Big(\frac{\partial u}{\partial y}\Big)_L + \frac{1}{\rho^* c^*}\Big(\frac{\partial p}{\partial y}\Big)_L.$$

At last, we have

$$\Big(\frac{\partial p}{\partial y}\Big)^* = c^{*2}\Big(\frac{\partial \rho}{\partial y}\Big)^* + \Big(\frac{\partial p}{\partial S}\Big)^*\Big(\frac{\partial S}{\partial y}\Big)^* = -\frac{\rho^* c^*}{2}\Big(\Phi_R - \Phi_L\Big). \tag{2.8}$$

In computations, the second term on the right hand side of $\left(\frac{\partial \rho}{\partial y}\right)^*$ can be approximated by

$$\begin{aligned}\frac{1}{c^{*2}}\Big(\frac{\partial p}{\partial S}\Big)^*\Big(\frac{\partial S}{\partial y}\Big)_L &\approx \frac{1}{c^{*2}}\Big(\frac{\partial p}{\partial S}\Big)_L\Big(\frac{\partial S}{\partial y}\Big)_L \\ &\approx \frac{1}{c^{*2}}\left[\Big(\frac{\partial p}{\partial y}\Big)_L - {c_L}^2\Big(\frac{\partial \rho}{\partial y}\Big)_L\right] \\ &\approx \frac{1}{c^{*2}}\Big(\frac{\partial p}{\partial y}\Big)_L - \Big(\frac{\partial \rho}{\partial y}\Big)_L.\end{aligned}$$

Therefore, we have

$$\Big(\frac{\partial \rho}{\partial y}\Big)^* = \Big(\frac{\partial \rho}{\partial y}\Big)_L + \frac{1}{c^{*2}}\left[\Big(\frac{\partial p}{\partial y}\Big)^* - \Big(\frac{\partial p}{\partial y}\Big)_L\right]. \tag{2.9}$$

The transversal term is

$$\Big(\frac{\partial \mathbf{g}}{\partial y}\Big)^* = \Big(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\Big)^*\Big(\frac{\partial \mathbf{v}}{\partial y}\Big)^*. \tag{2.10}$$

3. Three-dimensional extension

This *one-and-a-half dimensional* GRP solver can be easily extended to solver GRPs of three-dimensional Euler equations

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} + \frac{\partial \mathbf{h}(\mathbf{u})}{\partial z} = 0, \tag{3.1}$$

where

$$\begin{aligned}&\mathbf{u} = (\rho, \rho u, \rho v, \rho w, \rho E)^\top, \\ &\mathbf{f}(\mathbf{u}) = (\rho u, \rho u^2 + p, \rho uv, \rho uw, (\rho E + p)u)^\top, \\ &\mathbf{g}(\mathbf{u}) = (\rho v, \rho uv, \rho v^2 + p, \rho vw, (\rho E + p)v)^\top, \\ &\mathbf{h}(\mathbf{u}) = (\rho w, \rho uw, , \rho vw, \rho w^2 + p, (\rho E + p)w)^\top, \\ &E = \frac{u^2 + v^2 + w^2}{2} + e, \ \ e = e(\rho, S).\end{aligned} \tag{3.2}$$

The initial value problem to be solved is

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=-\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y}-\frac{\partial \mathbf{h}(\mathbf{u})}{\partial z}, \quad t>0, \\
&\mathbf{u}(x,y,z,t=0)=\begin{cases}\mathbf{u}_-(x,y,z), & x<0, \\ \mathbf{u}_+(x,y,z), & x>0.\end{cases}
\end{aligned} \tag{3.3}
$$

By dropping the source, we first solve the homogeneous one-dimensional GRP

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=0, \quad t>0, \\
&\mathbf{u}(x,y=0,z=0,t=0)=\begin{cases}\mathbf{u}_-(x,y=0,z=0), & x<0, \\ \mathbf{u}_+(x,y=0,z=0), & x>0,\end{cases}
\end{aligned} \tag{3.4}
$$

to obtain $\widetilde{(\partial \mathbf{u}/\partial t)}^*$.

The second step is to get $(\partial \mathbf{g}/\partial y)^*$. To achieve that, write the quasi linear form of (3.1) and (3.2) as

$$
\frac{\partial \mathbf{v}}{\partial t}+\mathbf{A}\frac{\partial \mathbf{v}}{\partial x}+\mathbf{B}\frac{\partial \mathbf{v}}{\partial y}+\mathbf{C}\frac{\partial \mathbf{v}}{\partial z}=0, \tag{3.5}
$$

where

$$
\mathbf{v}=\begin{bmatrix}\rho \\ u \\ v \\ w \\ S\end{bmatrix}, \quad \mathbf{A}=\begin{bmatrix} u & \rho & 0 & 0 & 0 \\ \frac{c^2}{\rho} & u & 0 & 0 & \frac{1}{\rho}\frac{\partial p}{\partial S} \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{bmatrix}. \tag{3.6}
$$

Definitions of $\mathbf{B}$ and $\mathbf{C}$ are ommitted since they do not matter. Again, the sound speed is defined as $c^2=\partial p/\partial\rho$. The coefficient matrix $\mathbf{A}$ has the characteristic decomposition $\mathbf{A}=L\Lambda R$ where

$$
R=\begin{bmatrix} -\frac{\rho}{2c} & 0 & 0 & -\frac{1}{c^2}\frac{\partial p}{\partial S} & \frac{\rho}{2c} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad L=\begin{bmatrix} -\frac{c}{\rho} & 1 & 0 & 0 & -\frac{1}{\rho c}\frac{\partial p}{\partial S} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{c}{\rho} & 1 & 0 & 0 & \frac{1}{\rho c}\frac{\partial p}{\partial S} \end{bmatrix}. \tag{3.7}
$$

Fix the three coefficient matrices in (3.5) by the Riemann solution $\mathbf{u}^*$ and differentiate the linearized PDE together with the initial data in (3.3). By denoting $\mathbf{w}=\frac{\partial \mathbf{v}}{\partial y}$, we get the following Riemann problem

$$
\begin{aligned}
&\frac{\partial \mathbf{w}^{(y)}}{\partial t} + \mathbf{A}^* \frac{\partial \mathbf{w}^{(y)}}{\partial x} + \mathbf{B}^* \frac{\partial \mathbf{w}^{(y)}}{\partial y} + \mathbf{C}^* \frac{\partial \mathbf{w}^{(y)}}{\partial z} = 0, \quad t > 0, \\
&\mathbf{w}^{(y)}(x, y, z, t = 0) = \begin{cases} (\mathbf{v}_y)_L, & x < 0, \\ (\mathbf{v}_y)_R, & x > 0. \end{cases}
\end{aligned} \tag{3.8}
$$

Once again, transversal terms $\mathbf{B}^* \frac{\partial \mathbf{w}}{\partial y}$ and $\mathbf{C}^* \frac{\partial \mathbf{w}}{\partial z}$ can be dropped.

Assume that $u^* - c^* < 0 < u^*$, we have

$$
(\mathbf{w}^{(y)})^* = R^* I^+ L^* (\mathbf{v}_y)_L + R^* I^- L^* (\mathbf{v}_y)_R, \tag{3.9}
$$

where R^* and L^* are matrices in (3.7) evaluated at $\mathbf{v}^*$ and

$$
I^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad I^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Explicit calculations of (3.9) gives us

$$
\begin{aligned}
&\left(\frac{\partial \rho}{\partial y}\right)^* = -\frac{\rho^*}{2c^*}\left(\Phi_R^{(y)} - \Phi_L^{(y)}\right) - \frac{1}{c^{*2}}\left(\frac{\partial p}{\partial S}\right)^*\left(\frac{\partial S}{\partial y}\right)_L, \quad \left(\frac{\partial u}{\partial y}\right)^* = \frac{1}{2}\left(\Phi_R^{(y)} + \Phi_L^{(y)}\right), \\
&\left(\frac{\partial v}{\partial y}\right)^* = \left(\frac{\partial v}{\partial y}\right)_L, \quad \left(\frac{\partial w}{\partial y}\right)^* = \left(\frac{\partial w}{\partial y}\right)_L, \quad \left(\frac{\partial S}{\partial y}\right)^* = \left(\frac{\partial S}{\partial y}\right)_L,
\end{aligned} \tag{3.10}
$$

where

$$
\Phi_R^{(y)} = \left(\frac{\partial u}{\partial y}\right)_R - \frac{1}{\rho^* c^*}\left(\frac{\partial p}{\partial y}\right)_R, \quad \Phi_L^{(y)} = \left(\frac{\partial u}{\partial y}\right)_L + \frac{1}{\rho^* c^*}\left(\frac{\partial p}{\partial y}\right)_L.
$$

By the EOS $p = p(\rho, S)$, we have

$$
\left(\frac{\partial p}{\partial y}\right)^* = c^{*2}\left(\frac{\partial \rho}{\partial y}\right)^* + \left(\frac{\partial p}{\partial S}\right)^*\left(\frac{\partial S}{\partial y}\right)^* = -\frac{\rho^* c^*}{2}\left(\Phi_R^{(y)} - \Phi_L^{(y)}\right). \tag{3.11}
$$

In computations, the second term on the right hand side of $\left(\frac{\partial \rho}{\partial y}\right)^*$ can be approximated by

$$
\begin{aligned}
\frac{1}{c^{*2}}\left(\frac{\partial p}{\partial S}\right)^*\left(\frac{\partial S}{\partial y}\right)_L &\approx \frac{1}{c^{*2}}\left(\frac{\partial p}{\partial S}\right)_L\left(\frac{\partial S}{\partial y}\right)_L \\
&\approx \frac{1}{c^{*2}}\left[\left(\frac{\partial p}{\partial y}\right)_L - {c_L}^2\left(\frac{\partial \rho}{\partial y}\right)_L\right] \\
&\approx \frac{1}{c^{*2}}\left(\frac{\partial p}{\partial y}\right)_L - \left(\frac{\partial \rho}{\partial y}\right)_L.
\end{aligned}
$$

Therefore, we have

$$
\left(\frac{\partial \rho}{\partial y}\right)^* = \left(\frac{\partial \rho}{\partial y}\right)_L + \frac{1}{c^{*2}}\left[\left(\frac{\partial p}{\partial y}\right)^* - \left(\frac{\partial p}{\partial y}\right)_L\right]. \tag{3.12}
$$

The transversal term is

$$\left(\frac{\partial \mathbf{g}}{\partial y}\right)^* = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)^* \left(\frac{\partial \mathbf{v}}{\partial y}\right)^*. \tag{3.13}$$

Similarly, in order to obtain $(\partial \mathbf{h}/\partial z)^*$, solve the Riemann problem

$$\begin{aligned} &\frac{\partial \mathbf{w}^{(z)}}{\partial t} + \mathbf{A}^* \frac{\partial \mathbf{w}^{(z)}}{\partial z} = 0, \quad t > 0, \\ &\mathbf{w}^{(z)}(x, y, z, t = 0) = \begin{cases} (\mathbf{v}_z)_L, & x < 0, \\ (\mathbf{v}_z)_R, & x > 0, \end{cases} \end{aligned} \tag{3.14}$$

where $\mathbf{w}^{(z)} = \partial \mathbf{v}/\partial z$. The Riemann solution is

$$(\mathbf{w}^{(z)})^* = R^* I^+ L^* (\mathbf{v}_z)_L + R^* I^- L^* (\mathbf{v}_z)_R, \tag{3.15}$$

which leads to

$$\begin{aligned} &\left(\frac{\partial \rho}{\partial z}\right)^* = -\frac{\rho^*}{2c^*}\left(\Phi_R^{(z)} - \Phi_L^{(z)}\right) - \frac{1}{c^{*2}}\left(\frac{\partial p}{\partial S}\right)^*\left(\frac{\partial S}{\partial z}\right)_L, \quad \left(\frac{\partial u}{\partial z}\right)^* = \frac{1}{2}\left(\Phi_R^{(z)} + \Phi_L^{(z)}\right), \\ &\left(\frac{\partial v}{\partial z}\right)^* = \left(\frac{\partial v}{\partial z}\right)_L, \quad \left(\frac{\partial w}{\partial z}\right)^* = \left(\frac{\partial w}{\partial z}\right)_L, \quad \left(\frac{\partial S}{\partial z}\right)^* = \left(\frac{\partial S}{\partial z}\right)_L, \\ &\left(\frac{\partial p}{\partial z}\right)^* = c^{*2}\left(\frac{\partial \rho}{\partial z}\right)^* + \left(\frac{\partial p}{\partial S}\right)^*\left(\frac{\partial S}{\partial z}\right)^* = -\frac{\rho^* c^*}{2}\left(\Phi_R^{(z)} - \Phi_L^{(z)}\right), \end{aligned} \tag{3.16}$$

where

$$\Phi_R^{(z)} = \left(\frac{\partial u}{\partial z}\right)_R - \frac{1}{\rho^* c^*}\left(\frac{\partial p}{\partial z}\right)_R, \quad \Phi_L^{(z)} = \left(\frac{\partial u}{\partial z}\right)_L + \frac{1}{\rho^* c^*}\left(\frac{\partial p}{\partial z}\right)_L.$$

The transversal term is

$$\left(\frac{\partial \mathbf{h}}{\partial z}\right)^* = \left(\frac{\partial \mathbf{h}}{\partial \mathbf{v}}\right)^* \left(\frac{\partial \mathbf{v}}{\partial z}\right)^*. \tag{3.17}$$

References

[1] M. Ben-Artzi, J. Li, and G. Warnecke, *A direct Eulerian GRP scheme for compressible fluid flows*, J. Comput. Phys., 218 (2006), pp. 19–43.