AN MATHEMATICAL INSTRUCTION OF THE GRP SOLVER FOR TWO-DIMENSIONAL EULER EQUATIONS

1. Mathematical set-up of the governing equations and the GRP

We consider generalized Riemann problem (GRP) solver for solving two-dimension Euler equations

$$\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}+\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y}=0, \tag{1.1}$$

where

$$\begin{aligned}
&\mathbf{u}=(\rho,\rho u,\rho v,\rho E)^\top,\\
&\mathbf{f}(\mathbf{u})=(\rho u,\rho u^2+p,\rho uv,(\rho E+p)u)^\top,\\
&\mathbf{g}(\mathbf{u})=(\rho v,\rho uv,\rho v^2+p,(\rho E+p)v)^\top,\\
&E=\frac{u^2+v^2}{2}+e,\ \ e=e(\rho,S).
\end{aligned} \tag{1.2}$$

The dependence of the internal energy e on ρ and p is determined by the equation of state (EOS). This version of solver is constructed by incorporating linearized dealing of the tangential derivatives into the nonlinear GRP solver for one-dimensional Euler equations constructed in [1]. So this solver can be regarded as a one-half dimensional GRP solver.

The GRP considered here is the initial value problem of (1.1) equipped with piece-wise polynomial initial data

$$\mathbf{u}(x,y,t=0)=\begin{cases}\mathbf{u}_-(x,y), & x<0,\\ \mathbf{u}_+(x,y), & x>0,\end{cases} \tag{1.3}$$

Denote the limiting values at the origin as

$$\begin{aligned}
&\mathbf{u}_L=\lim_{x\to 0-}\mathbf{u}_-(x,0),\quad (\mathbf{u}_x)_L=\lim_{x\to 0-}\frac{\partial \mathbf{u}_-}{\partial x}(x,0),\quad (\mathbf{u}_y)_L=\lim_{x\to 0-}\frac{\partial \mathbf{u}_-}{\partial y}(x,0),\\
&\mathbf{u}_R=\lim_{x\to 0+}\mathbf{u}_+(x,0),\quad (\mathbf{u}_x)_R=\lim_{x\to 0+}\frac{\partial \mathbf{u}_+}{\partial x}(x,0),\quad (\mathbf{u}_y)_R=\lim_{x\to 0+}\frac{\partial \mathbf{u}_+}{\partial y}(x,0).
\end{aligned} \tag{1.4}$$

The first step is to solve the associated Riemann problem

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t}+\frac{\partial \mathbf{f}(\mathbf{u})}{\partial x}=0,\quad t>0,\\
&\mathbf{u}(x,y,t=0)=\begin{cases}\mathbf{u}_L, & x<0,\\ \mathbf{u}_R, & x>0,\end{cases}
\end{aligned} \tag{1.5}$$

to get the Riemann solution $R^A(\lambda; u_L, u_R)$. Next, solve the GPR in the normal direction of the inital discontinuity

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = -\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y}, \quad t > 0, \\
&\mathbf{u}(x, y, t = 0) = \begin{cases} \mathbf{u}_-(x, y), & x < 0, \\ \mathbf{u}_+(x, y), & x > 0, \end{cases}
\end{aligned} \tag{1.6}
$$

The transversal term $\frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}}$ is regarded as a source term and can be linearly approximated in the next section.

2. Linearized calculation of tangential derivatives

By using the Gibbs relation, governing PDEs for primative variables are derived from the conservative Euler equations as

$$
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{v}}{\partial x} + \mathbf{B}\frac{\partial \mathbf{v}}{\partial y} = 0, \tag{2.1}
$$

where

$$
\mathbf{v} = \begin{bmatrix} \rho \\ u \\ v \\ S \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u & \rho & 0 & 0 \\ \frac{c^2}{\rho} & u & 0 & \frac{1}{\rho}\frac{\partial p}{\partial S} \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ \frac{c^2}{\rho} & 0 & v & \frac{1}{\rho}\frac{\partial p}{\partial S} \\ 0 & 0 & 0 & v \end{bmatrix}, \tag{2.2}
$$

and the sound speed is defined as $c^2 = \partial p / \partial \rho$. Furthermore, the coefficient matrix A has the characteristic decomposition $A = L \Lambda R$ where

$$
R = \begin{bmatrix} -\frac{\rho}{2c} & 0 & -\frac{1}{c^2}\frac{\partial p}{\partial S} & \frac{\rho}{2c} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} -\frac{c}{\rho} & 1 & 0 & -\frac{1}{\rho c}\frac{\partial p}{\partial S} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{c}{\rho} & 1 & 0 & \frac{1}{\rho c}\frac{\partial p}{\partial S} \end{bmatrix}. \tag{2.3}
$$

Linearize (2.1) by evalueating $\mathbf{A}$ and $\mathbf{B}$ at the background state $\mathbf{v}^*$ derived from the Riemann solution $\mathbf{u}^* = R^A(0, \mathbf{u}_L, \mathbf{u}_R)$. Denote the linearized coefficient matrix as $\mathbf{A}^*$ and $\mathbf{B}^*$ and the linearized governing equation is

$$
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A}^*\frac{\partial \mathbf{v}}{\partial x} + \mathbf{B}^*\frac{\partial \mathbf{v}}{\partial y} = 0. \tag{2.4}
$$

The Riemann problem

$$
\begin{aligned}
&\frac{\partial \mathbf{w}}{\partial t} + A^* \frac{\partial \mathbf{w}}{\partial x} = 0, \quad t > 0, \\
&\mathbf{w}(x, y, t = 0) = \begin{cases} (\mathbf{u}_y)_L, & x < 0, \\ (\mathbf{u}_y)_R, & x > 0, \end{cases}
\end{aligned} \tag{2.5}
$$

with respect to $\mathbf{w} = \dfrac{\partial \mathbf{v}}{\partial y}$ is obtained by differentiating (2.4) togenther with the initial data (1.3) and ommiting second-order y-derivatives. The Riemann problem (2.5) is easy to solve by the approach of the characteristic decomposition.

Assume that $u^* - c^* < 0 < u^*$, we have

$$
\mathbf{w}^* = R^* I^+ L^* (\mathbf{u}_y)_L + R^* I^- L^* (\mathbf{u}_y)_R, \tag{2.6}
$$

where R^* and L^* are matrices in (2.3) evaluated at $\mathbf{v}^*$ and

$$
I^+ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I^- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
$$

Explicit calculations of (2.6) gives us

$$
\begin{aligned}
&\left(\frac{\partial \rho}{\partial y}\right)^* = -\frac{\rho^*}{2c^*}\Big(\Phi_R - \Phi_L\Big) - \frac{1}{c^{*2}}\left(\frac{\partial p}{\partial S}\right)^*\left(\frac{\partial S}{\partial y}\right)_L, \quad \left(\frac{\partial u}{\partial y}\right)^* = \frac{1}{2}\Big(\Phi_R + \Phi_L\Big), \\
&\left(\frac{\partial v}{\partial y}\right)^* = \left(\frac{\partial v}{\partial y}\right)_L, \quad \left(\frac{\partial S}{\partial y}\right)^* = \left(\frac{\partial S}{\partial y}\right)_L,
\end{aligned} \tag{2.7}
$$

where

$$
\Phi_R = \left(\frac{\partial u}{\partial y}\right)_R - \frac{1}{\rho^* c^*}\left(\frac{\partial p}{\partial y}\right)_R, \quad \Phi_L = \left(\frac{\partial u}{\partial y}\right)_L - \frac{1}{\rho^* c^*}\left(\frac{\partial p}{\partial y}\right)_L.
$$

At last, we have

$$
\left(\frac{\partial p}{\partial y}\right)^* = c^{*2}\left(\frac{\partial \rho}{\partial y}\right)^* + \left(\frac{\partial p}{\partial S}\right)^*\left(\frac{\partial S}{\partial y}\right)^* = -\frac{\rho^* c^*}{2}\Big(\Phi_R - \Phi_L\Big). \tag{2.8}
$$

In computations, the second term on the right hand side of $\left(\dfrac{\partial \rho}{\partial y}\right)^*$ can be approximated by

$$
\begin{aligned}
\frac{1}{c^{*2}}\left(\frac{\partial p}{\partial S}\right)^*\left(\frac{\partial S}{\partial y}\right)_L &\approx \frac{1}{c^{*2}}\left(\frac{\partial p}{\partial S}\right)_L\left(\frac{\partial S}{\partial y}\right)_L \\
&\approx \frac{1}{c^{*2}}\left[\left(\frac{\partial p}{\partial y}\right)_L - {c_L}^2\left(\frac{\partial \rho}{\partial y}\right)_L\right] \\
&\approx \frac{1}{c^{*2}}\left(\frac{\partial p}{\partial y}\right)_L - \left(\frac{\partial \rho}{\partial y}\right)_L.
\end{aligned}
$$

Therefore, we have

$$
\left(\frac{\partial \rho}{\partial y}\right)^* = \left(\frac{\partial \rho}{\partial y}\right)_L + \frac{1}{c^{*2}}\left[\left(\frac{\partial p}{\partial y}\right)^* - \left(\frac{\partial p}{\partial y}\right)_L\right]. \tag{2.9}
$$

The transversal terms added on temporal derivatives $[(\partial\rho/\partial t)^*, (\partial u/\partial t)^*, (\partial v/\partial t)^*, (\partial p/\partial t)^*]$ are

$$
-\begin{bmatrix} v^* & 0 & \rho^* & 0 \\ 0 & v^* & 0 & 0 \\ \dfrac{c^{*2}}{\rho^*} & 0 & v^* & \dfrac{1}{\rho^*} \\ 0 & 0 & \rho^* c^{*2} & v^* \end{bmatrix} \begin{bmatrix} \left(\dfrac{\partial\rho}{\partial y}\right)^* \\ \left(\dfrac{\partial u}{\partial y}\right)^* \\ \left(\dfrac{\partial v}{\partial y}\right)^* \\ \left(\dfrac{\partial p}{\partial y}\right)^* \end{bmatrix}, \tag{2.10}
$$

where the coefficient matrix is the Jacobian matrix of $\mathbf{g}$ with respect to $(\rho, u, v, p)^\top$

References

[1] M. Ben-Artzi, J. Li, and G. Warnecke, *A direct Eulerian GRP scheme for compressible fluid flows*, J. Comput. Phys., 218 (2006), pp. 19–43.