

# A MATHEMATICAL INSTRUCTION OF THE GRP SOLVER FOR TWO-DIMENSIONAL EULER EQUATIONS

## 1. MATHEMATICAL SET-UP OF THE GOVERNING EQUATIONS AND THE GRP

We consider generalized Riemann problem (GRP) solver for solving two-dimension Euler equations

$$(1.1) \quad \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} = 0,$$

where

$$(1.2) \quad \begin{aligned} \mathbf{u} &= (\rho, \rho u, \rho v, \rho E)^\top, \\ \mathbf{f}(\mathbf{u}) &= (\rho u, \rho u^2 + p, \rho uv, (\rho E + p)u)^\top, \\ \mathbf{g}(\mathbf{u}) &= (\rho v, \rho uv, \rho v^2 + p, (\rho E + p)v)^\top, \\ E &= \frac{u^2 + v^2}{2} + e, \quad e = e(\rho, S). \end{aligned}$$

The dependence of the internal energy  $e$  on  $\rho$  and  $p$  is determined by the equation of state (EOS). This version of solver is constructed by incorporating linearized dealing of the tangential derivatives into the nonlinear GRP solver for one-dimensional Euler equations constructed in [1]. So this solver can be regarded as a *one-and-a-half dimensional* GRP solver.

The GRP considered here is the initial value problem of (1.1) equipped with piece-wise polynomial initial data

$$(1.3) \quad \mathbf{u}(x, y, t = 0) = \begin{cases} \mathbf{u}_-(x, y), & x < 0, \\ \mathbf{u}_+(x, y), & x > 0. \end{cases}$$

Denote the limiting values at the origin as

$$(1.4) \quad \begin{aligned} \mathbf{u}_L &= \lim_{x \rightarrow 0^-} \mathbf{u}_-(x, 0), \quad (\mathbf{u}_x)_L = \lim_{x \rightarrow 0^-} \frac{\partial \mathbf{u}_-}{\partial x}(x, 0), \quad (\mathbf{u}_y)_L = \lim_{x \rightarrow 0^-} \frac{\partial \mathbf{u}_-}{\partial y}(x, 0), \\ \mathbf{u}_R &= \lim_{x \rightarrow 0^+} \mathbf{u}_+(x, 0), \quad (\mathbf{u}_x)_R = \lim_{x \rightarrow 0^+} \frac{\partial \mathbf{u}_+}{\partial x}(x, 0), \quad (\mathbf{u}_y)_R = \lim_{x \rightarrow 0^+} \frac{\partial \mathbf{u}_+}{\partial y}(x, 0). \end{aligned}$$

The first step is to solve the associated Riemann problem

$$(1.5) \quad \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} &= 0, \quad t > 0, \\ \mathbf{u}(x, y, t = 0) &= \begin{cases} \mathbf{u}_L, & x < 0, \\ \mathbf{u}_R, & x > 0, \end{cases} \end{aligned}$$

to get the Riemann solution  $R^A(\lambda; u_L, u_R)$ . Next, solve the GPR in the normal direction of the initial discontinuity

$$(1.6) \quad \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} &= -\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y}, \quad t > 0, \\ \mathbf{u}(x, y=0, t=0) &= \begin{cases} \mathbf{u}_-(x, y), & x < 0, \\ \mathbf{u}_+(x, y), & x > 0, \end{cases} \end{aligned}$$

where the transversal term  $\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y}$  is regarded as a source. To achieve that, we firstly solve the homogeneous one-dimensional GRP

$$(1.7) \quad \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} &= 0, \quad t > 0, \\ \mathbf{u}(x, y, t=0) &= \begin{cases} \mathbf{u}_-(x, y=0), & x < 0, \\ \mathbf{u}_+(x, y=0), & x > 0, \end{cases} \end{aligned}$$

to obtain  $(\widetilde{\partial \mathbf{u} / \partial t})^*$ . Refer to [1] for details. Next we get the intermediate value of the transversal term  $(\partial \mathbf{g} / \partial y)^*$  which can be linearly approximated in the next section. At last, the desired time derivative is given by

$$(1.8) \quad \left( \frac{\partial \mathbf{u}}{\partial t} \right)^* = \left( \widetilde{\frac{\partial \mathbf{u}}{\partial t}} \right)^* - \left( \frac{\partial \mathbf{g}}{\partial y} \right)^*.$$

## 2. LINEARIZED CALCULATION OF TANGENTIAL DERIVATIVES

By using the Gibbs relation, governing PDEs for primitive variables are derived from the conservative Euler equations as

$$(2.1) \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{v}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{v}}{\partial y} = 0,$$

where

$$(2.2) \quad \mathbf{v} = \begin{bmatrix} \rho \\ u \\ v \\ S \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u & \rho & 0 & 0 \\ \frac{c^2}{\rho} & u & 0 & \frac{1}{\rho} \frac{\partial p}{\partial S} \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ \frac{c^2}{\rho} & 0 & v & \frac{1}{\rho} \frac{\partial p}{\partial S} \\ 0 & 0 & 0 & v \end{bmatrix},$$

and the sound speed is defined as  $c^2 = \partial p / \partial \rho$ . Furthermore, the coefficient matrix  $\mathbf{A}$  has the characteristic decomposition  $\mathbf{A} = L\Lambda R$  where

$$(2.3) \quad R = \begin{bmatrix} -\frac{\rho}{2c} & 0 & -\frac{1}{c^2} \frac{\partial p}{\partial S} & \frac{\rho}{2c} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} -\frac{c}{\rho} & 1 & 0 & -\frac{1}{\rho c} \frac{\partial p}{\partial S} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{c}{\rho} & 1 & 0 & \frac{1}{\rho c} \frac{\partial p}{\partial S} \end{bmatrix}.$$

Linearize (2.1) by evaluating  $\mathbf{A}$  and  $\mathbf{B}$  at the background state  $\mathbf{v}^*$  derived from the Riemann solution  $\mathbf{u}^* = R^A(0, \mathbf{u}_L, \mathbf{u}_R)$ . Denote the linearized coefficient matrices as  $\mathbf{A}^*$  and  $\mathbf{B}^*$ . The linearized governing equation is

$$(2.4) \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{A}^* \frac{\partial \mathbf{v}}{\partial x} + \mathbf{B}^* \frac{\partial \mathbf{v}}{\partial y} = 0.$$

The Riemann problem

$$(2.5) \quad \begin{aligned} \frac{\partial \mathbf{w}}{\partial t} + \mathbf{A}^* \frac{\partial \mathbf{w}}{\partial x} + \mathbf{B}^* \frac{\partial \mathbf{w}}{\partial y} &= 0, \quad t > 0, \\ \mathbf{w}(x, y, t = 0) &= \begin{cases} (\mathbf{v}_y)_L, & x < 0, \\ (\mathbf{v}_y)_R, & x > 0, \end{cases} \end{aligned}$$

with respect to  $\mathbf{w} = \frac{\partial \mathbf{v}}{\partial y}$  is obtained by differentiating (2.4) together with the initial data (1.3). The Riemann problem (2.5) is easy to solve by the approach of the characteristic decomposition. Note that the transversal term  $\mathbf{B}^* \frac{\partial \mathbf{w}}{\partial y}$  has no effect on the Riemann solution and can be dropped.

Assume that  $u^* - c^* < 0 < u^*$ , we have

$$(2.6) \quad \mathbf{w}^* = R^* I^+ L^* (\mathbf{v}_y)_L + R^* I^- L^* (\mathbf{v}_y)_R,$$

where  $R^*$  and  $L^*$  are matrices in (2.3) evaluated at  $\mathbf{v}^*$  and

$$I^+ = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad I^- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Explicit calculations of (2.6) gives us

$$(2.7) \quad \begin{aligned} \left( \frac{\partial \rho}{\partial y} \right)^* &= -\frac{\rho^*}{2c^*} (\Phi_R - \Phi_L) - \frac{1}{c^{*2}} \left( \frac{\partial p}{\partial S} \right)^* \left( \frac{\partial S}{\partial y} \right)_L, \quad \left( \frac{\partial u}{\partial y} \right)^* = \frac{1}{2} (\Phi_R + \Phi_L), \\ \left( \frac{\partial v}{\partial y} \right)^* &= \left( \frac{\partial v}{\partial y} \right)_L, \quad \left( \frac{\partial S}{\partial y} \right)^* = \left( \frac{\partial S}{\partial y} \right)_L, \end{aligned}$$

where

$$\Phi_R = \left(\frac{\partial u}{\partial y}\right)_R - \frac{1}{\rho^* c^*} \left(\frac{\partial p}{\partial y}\right)_R, \quad \Phi_L = \left(\frac{\partial u}{\partial y}\right)_L + \frac{1}{\rho^* c^*} \left(\frac{\partial p}{\partial y}\right)_L.$$

At last, we have

$$(2.8) \quad \left(\frac{\partial p}{\partial y}\right)^* = c^{*2} \left(\frac{\partial \rho}{\partial y}\right)^* + \left(\frac{\partial p}{\partial S}\right)^* \left(\frac{\partial S}{\partial y}\right)^* = -\frac{\rho^* c^*}{2} (\Phi_R - \Phi_L).$$

In computations, the second term on the right hand side of  $\left(\frac{\partial \rho}{\partial y}\right)^*$  can be approximated by

$$\begin{aligned} \frac{1}{c^{*2}} \left(\frac{\partial p}{\partial S}\right)^* \left(\frac{\partial S}{\partial y}\right)_L &\approx \frac{1}{c^{*2}} \left(\frac{\partial p}{\partial S}\right)_L \left(\frac{\partial S}{\partial y}\right)_L \\ &\approx \frac{1}{c^{*2}} \left[ \left(\frac{\partial p}{\partial y}\right)_L - c_L^2 \left(\frac{\partial \rho}{\partial y}\right)_L \right] \\ &\approx \frac{1}{c^{*2}} \left(\frac{\partial p}{\partial y}\right)_L - \left(\frac{\partial \rho}{\partial y}\right)_L. \end{aligned}$$

Therefore, we have

$$(2.9) \quad \left(\frac{\partial \rho}{\partial y}\right)^* = \left(\frac{\partial \rho}{\partial y}\right)_L + \frac{1}{c^{*2}} \left[ \left(\frac{\partial p}{\partial y}\right)^* - \left(\frac{\partial p}{\partial y}\right)_L \right].$$

The transversal term is

$$(2.10) \quad \left(\frac{\partial \mathbf{g}}{\partial y}\right)^* = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)^* \left(\frac{\partial \mathbf{v}}{\partial y}\right)^*.$$

### 3. THREE-DIMENSIONAL EXTENSION

This *one-and-a-half dimensional* GRP solver can be easily extended to solver GRPs of three-dimensional Euler equations

$$(3.1) \quad \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} + \frac{\partial \mathbf{h}(\mathbf{u})}{\partial z} = 0,$$

where

$$\begin{aligned} \mathbf{u} &= (\rho, \rho u, \rho v, \rho w, \rho E)^\top, \\ \mathbf{f}(\mathbf{u}) &= (\rho u, \rho u^2 + p, \rho uv, \rho uw, (\rho E + p)u)^\top, \\ \mathbf{g}(\mathbf{u}) &= (\rho v, \rho uv, \rho v^2 + p, \rho vw, (\rho E + p)v)^\top, \\ \mathbf{h}(\mathbf{u}) &= (\rho w, \rho uw, \rho vw, \rho w^2 + p, (\rho E + p)w)^\top, \\ E &= \frac{u^2 + v^2 + w^2}{2} + e, \quad e = e(\rho, S). \end{aligned} \quad (3.2)$$

The initial value problem to be solved is

$$(3.3) \quad \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} &= -\frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} - \frac{\partial \mathbf{h}(\mathbf{u})}{\partial z}, \quad t > 0, \\ \mathbf{u}(x, y, z, t = 0) &= \begin{cases} \mathbf{u}_-(x, y, z), & x < 0, \\ \mathbf{u}_+(x, y, z), & x > 0. \end{cases} \end{aligned}$$

By dropping the source, we first solve the homogeneous one-dimensional GRP

$$(3.4) \quad \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} &= 0, \quad t > 0, \\ \mathbf{u}(x, y = 0, z = 0, t = 0) &= \begin{cases} \mathbf{u}_-(x, y = 0, z = 0), & x < 0, \\ \mathbf{u}_+(x, y = 0, z = 0), & x > 0, \end{cases} \end{aligned}$$

to obtain  $(\widetilde{\partial \mathbf{u} / \partial t})^*$ .

The second step is to get  $(\partial \mathbf{g} / \partial y)^*$ . To achieve that, write the quasi linear form of (3.1) and (3.2) as

$$(3.5) \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{v}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{v}}{\partial y} + \mathbf{C} \frac{\partial \mathbf{v}}{\partial z} = 0,$$

where

$$(3.6) \quad \mathbf{v} = \begin{bmatrix} \rho \\ u \\ v \\ w \\ S \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} u & \rho & 0 & 0 & 0 \\ \frac{c^2}{\rho} & u & 0 & 0 & \frac{1}{\rho} \frac{\partial p}{\partial S} \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{bmatrix}.$$

Definitions of  $\mathbf{B}$  and  $\mathbf{C}$  are omitted since they do not matter. Again, the sound speed is defined as  $c^2 = \partial p / \partial \rho$ . The coefficient matrix  $\mathbf{A}$  has the characteristic decomposition  $\mathbf{A} = \mathbf{L} \mathbf{\Lambda} \mathbf{R}$  where

$$(3.7) \quad R = \begin{bmatrix} -\frac{\rho}{2c} & 0 & 0 & -\frac{1}{c^2} \frac{\partial p}{\partial S} & \frac{\rho}{2c} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} -\frac{c}{\rho} & 1 & 0 & 0 & -\frac{1}{\rho c} \frac{\partial p}{\partial S} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{c}{\rho} & 1 & 0 & 0 & \frac{1}{\rho c} \frac{\partial p}{\partial S} \end{bmatrix}.$$

Fix the three coefficient matrices in (3.5) by the Riemann solution  $\mathbf{u}^*$  and differentiate the linearized PDE together with the initial data in (3.3). By denoting  $\mathbf{w} = \frac{\partial \mathbf{v}}{\partial y}$ , we get the

following Riemann problem

$$(3.8) \quad \begin{aligned} \frac{\partial \mathbf{w}^{(y)}}{\partial t} + \mathbf{A}^* \frac{\partial \mathbf{w}^{(y)}}{\partial x} + \mathbf{B}^* \frac{\partial \mathbf{w}^{(y)}}{\partial y} + \mathbf{C}^* \frac{\partial \mathbf{w}^{(y)}}{\partial z} &= 0, \quad t > 0, \\ \mathbf{w}^{(y)}(x, y, z, t = 0) &= \begin{cases} (\mathbf{v}_y)_L, & x < 0, \\ (\mathbf{v}_y)_R, & x > 0. \end{cases} \end{aligned}$$

Once again, transversal terms  $\mathbf{B}^* \frac{\partial \mathbf{w}}{\partial y}$  and  $\mathbf{C}^* \frac{\partial \mathbf{w}}{\partial z}$  can be dropped.

Assume that  $u^* - c^* < 0 < u^*$ , we have

$$(3.9) \quad (\mathbf{w}^{(y)})^* = R^* I^+ L^* (\mathbf{v}_y)_L + R^* I^- L^* (\mathbf{v}_y)_R,$$

where  $R^*$  and  $L^*$  are matrices in (3.7) evaluated at  $\mathbf{v}^*$  and

$$I^+ = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad I^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Explicit calculations of (3.9) gives us

$$(3.10) \quad \begin{aligned} \left(\frac{\partial \rho}{\partial y}\right)^* &= -\frac{\rho^*}{2c^*} \left(\Phi_R^{(y)} - \Phi_L^{(y)}\right) - \frac{1}{c^{*2}} \left(\frac{\partial p}{\partial S}\right)^* \left(\frac{\partial S}{\partial y}\right)_L, \quad \left(\frac{\partial u}{\partial y}\right)^* = \frac{1}{2} \left(\Phi_R^{(y)} + \Phi_L^{(y)}\right), \\ \left(\frac{\partial v}{\partial y}\right)^* &= \left(\frac{\partial v}{\partial y}\right)_L, \quad \left(\frac{\partial w}{\partial y}\right)^* = \left(\frac{\partial w}{\partial y}\right)_L, \quad \left(\frac{\partial S}{\partial y}\right)^* = \left(\frac{\partial S}{\partial y}\right)_L, \end{aligned}$$

where

$$\Phi_R^{(y)} = \left(\frac{\partial u}{\partial y}\right)_R - \frac{1}{\rho^* c^*} \left(\frac{\partial p}{\partial y}\right)_R, \quad \Phi_L^{(y)} = \left(\frac{\partial u}{\partial y}\right)_L + \frac{1}{\rho^* c^*} \left(\frac{\partial p}{\partial y}\right)_L.$$

By the EOS  $p = p(\rho, S)$ , we have

$$(3.11) \quad \left(\frac{\partial p}{\partial y}\right)^* = c^{*2} \left(\frac{\partial \rho}{\partial y}\right)^* + \left(\frac{\partial p}{\partial S}\right)^* \left(\frac{\partial S}{\partial y}\right)^* = -\frac{\rho^* c^*}{2} \left(\Phi_R^{(y)} - \Phi_L^{(y)}\right).$$

In computations, the second term on the right hand side of  $\left(\frac{\partial \rho}{\partial y}\right)^*$  can be approximated by

$$\begin{aligned} \frac{1}{c^{*2}} \left(\frac{\partial p}{\partial S}\right)^* \left(\frac{\partial S}{\partial y}\right)_L &\approx \frac{1}{c^{*2}} \left(\frac{\partial p}{\partial S}\right)_L \left(\frac{\partial S}{\partial y}\right)_L \\ &\approx \frac{1}{c^{*2}} \left[ \left(\frac{\partial p}{\partial y}\right)_L - c_L^2 \left(\frac{\partial \rho}{\partial y}\right)_L \right] \\ &\approx \frac{1}{c^{*2}} \left(\frac{\partial p}{\partial y}\right)_L - \left(\frac{\partial \rho}{\partial y}\right)_L. \end{aligned}$$

Therefore, we have

$$(3.12) \quad \left(\frac{\partial \rho}{\partial y}\right)^* = \left(\frac{\partial \rho}{\partial y}\right)_L + \frac{1}{c^{*2}} \left[ \left(\frac{\partial p}{\partial y}\right)^* - \left(\frac{\partial p}{\partial y}\right)_L \right].$$

The transversal term is

$$(3.13) \quad \left(\frac{\partial \mathbf{g}}{\partial y}\right)^* = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{v}}\right)^* \left(\frac{\partial \mathbf{v}}{\partial y}\right)^*.$$

Similarly, in order to obtain  $(\partial \mathbf{h}/\partial z)^*$ , solve the Riemann problem

$$(3.14) \quad \begin{aligned} \frac{\partial \mathbf{w}^{(z)}}{\partial t} + \mathbf{A}^* \frac{\partial \mathbf{w}^{(z)}}{\partial z} &= 0, \quad t > 0, \\ \mathbf{w}^{(z)}(x, y, z, t = 0) &= \begin{cases} (\mathbf{v}_z)_L, & x < 0, \\ (\mathbf{v}_z)_R, & x > 0, \end{cases} \end{aligned}$$

where  $\mathbf{w}^{(z)} = \partial \mathbf{v}/\partial z$ . The Riemann solution is

$$(3.15) \quad (\mathbf{w}^{(z)})^* = R^* I^+ L^* (\mathbf{v}_z)_L + R^* I^- L^* (\mathbf{v}_z)_R,$$

which leads to

$$(3.16) \quad \begin{aligned} \left(\frac{\partial \rho}{\partial z}\right)^* &= -\frac{\rho^*}{2c^*} \left(\Phi_R^{(z)} - \Phi_L^{(z)}\right) - \frac{1}{c^{*2}} \left(\frac{\partial p}{\partial S}\right)^* \left(\frac{\partial S}{\partial z}\right)_L, \quad \left(\frac{\partial u}{\partial z}\right)^* = \frac{1}{2} \left(\Phi_R^{(z)} + \Phi_L^{(z)}\right), \\ \left(\frac{\partial v}{\partial z}\right)^* &= \left(\frac{\partial v}{\partial z}\right)_L, \quad \left(\frac{\partial w}{\partial z}\right)^* = \left(\frac{\partial w}{\partial z}\right)_L, \quad \left(\frac{\partial S}{\partial z}\right)^* = \left(\frac{\partial S}{\partial z}\right)_L, \\ \left(\frac{\partial p}{\partial z}\right)^* &= c^{*2} \left(\frac{\partial \rho}{\partial z}\right)^* + \left(\frac{\partial p}{\partial S}\right)^* \left(\frac{\partial S}{\partial z}\right)^* = -\frac{\rho^* c^*}{2} \left(\Phi_R^{(z)} - \Phi_L^{(z)}\right), \end{aligned}$$

where

$$\Phi_R^{(z)} = \left(\frac{\partial u}{\partial z}\right)_R - \frac{1}{\rho^* c^*} \left(\frac{\partial p}{\partial z}\right)_R, \quad \Phi_L^{(z)} = \left(\frac{\partial u}{\partial z}\right)_L + \frac{1}{\rho^* c^*} \left(\frac{\partial p}{\partial z}\right)_L.$$

The transversal term is

$$(3.17) \quad \left(\frac{\partial \mathbf{h}}{\partial z}\right)^* = \left(\frac{\partial \mathbf{h}}{\partial \mathbf{v}}\right)^* \left(\frac{\partial \mathbf{v}}{\partial z}\right)^*.$$

At last,

$$\left(\frac{\partial \mathbf{u}}{\partial t}\right)^* = \left(\frac{\partial \widetilde{\mathbf{u}}}{\partial t}\right)^* - \left(\frac{\partial \mathbf{g}}{\partial y}\right)^* - \left(\frac{\partial \mathbf{h}}{\partial z}\right)^*.$$

## REFERENCES

- [1] M. BEN-ARTZI, J. LI, AND G. WARNECKE, *A direct Eulerian GRP scheme for compressible fluid flows*, J. Comput. Phys., 218 (2006), pp. 19–43.