06/17/2016

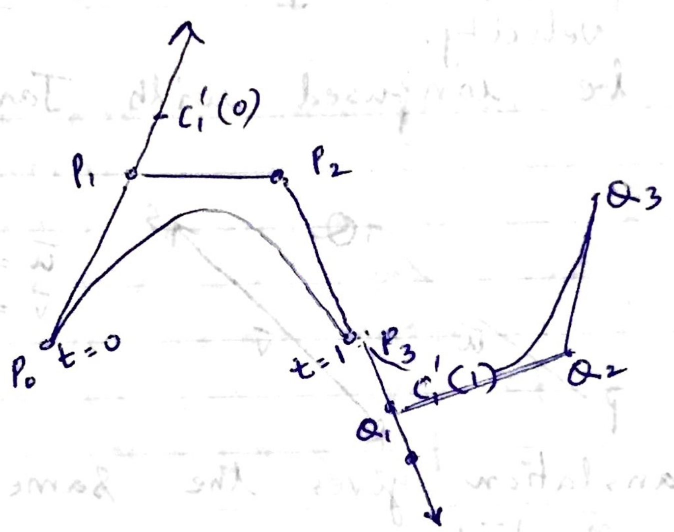
Computer Graphics

$C_1(t)$ is the position of the curve at time t, derivative $C_1'(t)$ of equation (1) is the velocity:

$$C_1'(t) = -3(t-1)^2 P_0 + 3(3t-1)(t-1)P_1 - 3t(3t-2)P_2 + 3t^2 P_3.$$

So, $C_1'(0) = 3(P_1 - P_0)$

$C_1'(1) = 3(P_3 - P_2)$



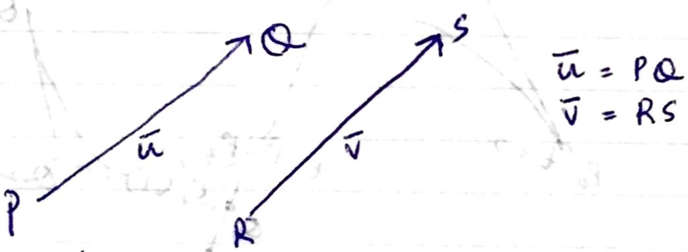
When two Bezier curves a ($P_0 \sim P_3$) & b ($Q_0 \sim Q_3$) are combined, then make the connecting point smooth, we make sure $C_{1a}'(1) = C_{1b}'(0)$, i.e. the final velocity of curve a equals the initial velocity of curve b. This is guaranteed if $P_3 (= Q_0)$ is the midpoint of line P_2Q_1.

Chapter 2

Applied Geometry - Math Basics.

2.1. Vectors

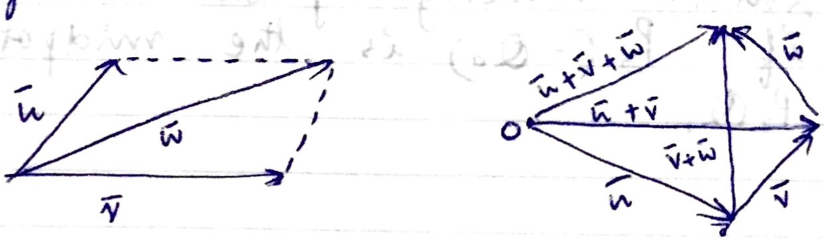
Vector = length + direction of a line segment
Useful representation of real world measurements
such as velocity.
Not to be confused with Java vector.



A translation gives the same vector.
Length of $\bar{u} = |\bar{u}|$,
$-u$ has the same length but opposite direction.

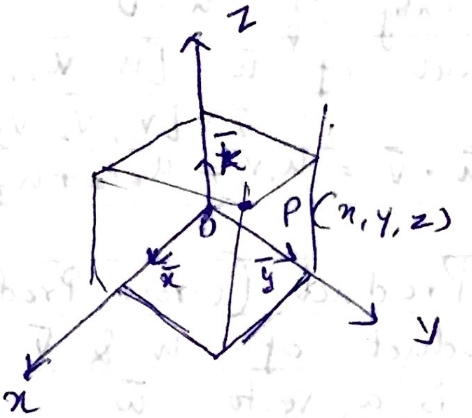
Vector addition

$\bar{w} = \bar{u} + \bar{v}$ is the diagonal of the parallelogram formed by $\bar{u}$ & $\bar{v}$



C being a real number, the length of vector $C\bar{u}$ is $|C||\bar{u}|$. A vector of unit length is called a unit vector.

Right-handed co-ordinate system for 3D



$\bar{i}, \bar{j}, \bar{k}$ are 3 orthogonal unit vectors in a 3D system.

Linear combination of $\bar{i}, \bar{j}, \bar{k}$ ($\bar{v}$):

$$\bar{v} = x\bar{i} + y\bar{j} + z\bar{k} = OP$$

x, y, z are co-ordinates of P and are called elements or components of $\bar{v}$:

$$\bar{v} = [x, y, z]$$

2.2 Dot product (inner Product)

Dot product of $\bar{u}$ and $\bar{v}$ i.e $\bar{u} \cdot \bar{v}$ is a real number:

$$\bar{u} \cdot \bar{v} = \begin{cases} |\bar{u}||\bar{v}| \cos\theta & \text{if } \bar{u} \neq 0 \ \& \ \bar{v} \neq 0,\ \theta \text{ is } \angle^{le} \text{ b/w } \bar{u} \ \& \ \bar{v} \\ 0 & \text{if } \bar{u} = 0,\ \bar{v} = 0,\ \text{or } \theta = 90^\circ \end{cases}$$

So for unit vectors $\bar{i}, \bar{j}, \bar{k}$

$$\bar{i} \cdot \bar{i} = \bar{j} \cdot \bar{j} = \bar{k} \cdot \bar{k} = 1$$

$$\bar{i} \cdot \bar{j} = \bar{i} \cdot \bar{k} = \bar{j} \cdot \bar{k} = \bar{k} \cdot \bar{j} = \bar{j} \cdot \bar{i} = \bar{k} \cdot \bar{i} = 0$$

$(\because \cos 90^\circ = 0)$

Since $\bar{u} \cdot \bar{u} = |\bar{u}|^2$

$$|\bar{u}| = \sqrt{|\bar{u} \cdot \bar{u}|} \qquad (\because \cos 0^\circ = 1)$$

Properties of dot product:

$$c(k\, \bar{u} \cdot \bar{v}) = ck(\bar{u} \cdot \bar{v})$$

$$(c\bar{u} + k\bar{v}) \cdot \bar{w} = c\bar{u} \cdot \bar{w} + k\bar{v} \cdot \bar{w}$$

$\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$

$\bar{u} \cdot \bar{u} = 0$ only if $\bar{u} = 0$

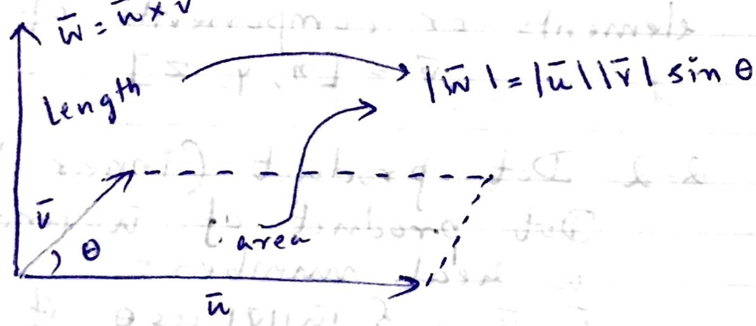
The dot product of $\bar{u} = [u_1, u_2, u_3]$ &
$\bar{v} = [v_1, v_2, v_3]$ is

$$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

2.3 Vector Product (Cross Product)

vector product of $\bar{u}$ & $\bar{v}$ i.e
$\bar{u} \times \bar{v}$ is a vector $\bar{w}$
$\bar{w} = 0$ if $\bar{u} = c\bar{v}$, otherwise :
$|\bar{w}| = |\bar{u}||\bar{v}| \sin\theta$ (θ is the angle b/w $\bar{u}$ & $\bar{v}$)

This is infact the area size of the parallelogram formed by $\bar{u}$ & $\bar{v}$.



Direction : right-handed screw rule.

Properties : $(k\bar{u}) \times \bar{v} = k(\bar{u} \times \bar{v})$ ----- (1)

$\bar{u} \times (\bar{v} + \bar{w}) = \bar{u} \times \bar{v} + \bar{u} \times \bar{w}$ ---- (2)

$\bar{u} \times \bar{v} = -\bar{v} \times \bar{u}$ ----- (3)

$$\left.\begin{aligned}
&\bar{i} \times \bar{i} = \bar{j} \times \bar{j} = \bar{k} \times \bar{k} = 0 \\
&\bar{i} \times \bar{j} = \bar{k} \\
&\bar{j} \times \bar{i} = -\bar{k} \\
&\bar{j} \times \bar{k} = \bar{i} \\
&\bar{k} \times \bar{j} = -\bar{i} \\
&\bar{k} \times \bar{i} = \bar{j} \ ; \ \bar{i} \times \bar{k} = -\bar{j}
\end{aligned}\right\} \text{-----} (4)$$

So, we have

$$\bar{u} \times \bar{v} = (\bar{u}_1\bar{i} + u_2\bar{j} + u_3\bar{k}) \times (\bar{v}_1\bar{i} + v_2\bar{j} + \bar{v}_3\bar{k})$$

$$= u_1v_1(\bar{i}\times\bar{i}) + u_2v_1(\bar{j}\times\bar{i}) + u_3v_1(\bar{k}\times\bar{i})$$
$$+ u_1v_2(\bar{i}\times\bar{j}) + u_2v_2(\bar{j}\times\bar{j}) + u_3v_2(\bar{k}\times\bar{j})$$
$$+ u_1v_3(\bar{i}\times\bar{k}) + u_2v_3(\bar{j}\times\bar{k}) + u_3v_3(\bar{k}\times\bar{k})$$

$$= (u_2v_3 - u_3v_2)\bar{i} + (u_3v_1 - u_1v_3)\bar{j} + (u_1v_2 - u_2v_1)\bar{k}$$

using ② & ④.

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\bar{i} + \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}\bar{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\bar{k}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

2.4 Determinants

To solve 2 linear equations $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$

We multiply the first equation by b_2, second $b_3 - b_1$ and then add them up to cancel y, we get:

$$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2$$

We can similarly obtain:

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1$$

If $(a_1b_2 - a_2b_1) \neq 0$, we get

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

We can write: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

For a 3 order determinant :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The sign of a_{ij} is determined by $(-1)^{i+j}$

Properties of Determinants :-

① $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

② $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

③ $$\begin{vmatrix} ca_1 & b_1 \\ ca_2 & b_2 \end{vmatrix} = c \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

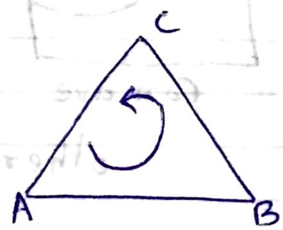
④ $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 + ka_1 & b_3 + kb_1 & c_3 + kc_1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

a row (column) is adding to a constant multiply of another row (column)

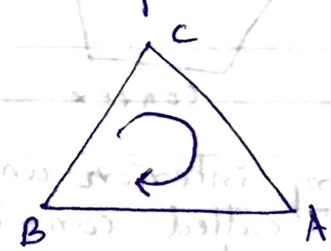
⑤ $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 3a_1-2a_2 & 3b_1-2b_2 & 3c_1-2c_2 \end{vmatrix} = 0$

a row (column) is a linear combination of another row (column)

2.5 Orientation of 3 points



Orientation >0



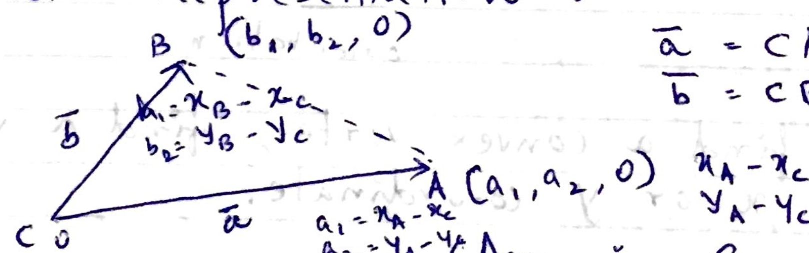
Orientation <0

When A, B & C are on a straight line, orientation $=0$.

3D vector representation :

$\bar{a} = CA$

$\bar{b} = CB$



$x_A - x_c$

$y_A - y_c$

(Assuming C at the origin)

So, $\bar{a} = a_1\bar{i} + a_2\bar{j} + 0\bar{k}$

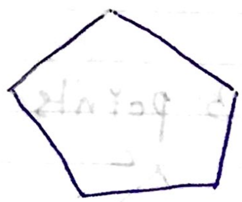
$\bar{b} = b_1\bar{i} + b_2\bar{j} + 0\bar{k}$

$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \bar{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

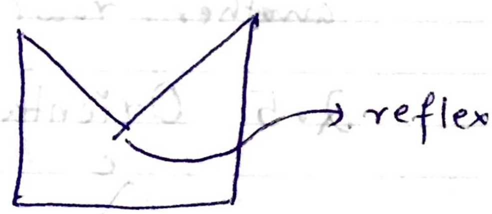
$= (a_1 b_2 - a_2 b_1)\bar{k}$ — ①

$$a_1 b_2 - a_2 b_1 \begin{cases} > 0 : \text{thumb pointing to us, CCW} \\ = 0 : \text{on the same line} \\ < 0 : \text{clockwise (CC)} \end{cases}$$

2.6 Polygon Shapes:



Convex



Concave

Every interior angle $< 180^\circ$ called convex vertex

otherwise.

If we know a convex vertex of a polygon, the triangle constructed by the convex vertex and its two neighboring vertices can be used to determine the order of vertex sequence orientation

To find a convex vertex, find a vertex with least x or y co-ordinate.

A useful method in class Tools2D:

```
Class Tools2D {
    static float area2(Point2D A, point2D B,
                       Point2D C)
    {
        return (Ax-Cx)*(By-Cy) - (Ay-Cy)*
               (Bx-Cx);
    } // return Area(ABC)*2, >0 if CCW,
                              <0 if CW
}
```

$$\underbrace{(A_x - C_x)}_{a_1} * \underbrace{(B_y - C_y)}_{b_2} - \underbrace{(A_y - C_y)}_{a_2} * \underbrace{(B_x - C_x)}_{b_1}$$

Because:

$$2 * \text{Area}(ABC) = |a \times b| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$= (x_A - x_C)(y_B - y_C) - (y_A - y_C)(x_B - x_C)$$

$$= (x_A y_B - y_A x_B) + (x_B y_C - y_B x_C) + (x_C y_A - y_C x_A)$$

Could be generalized to obtain a polygon area:-

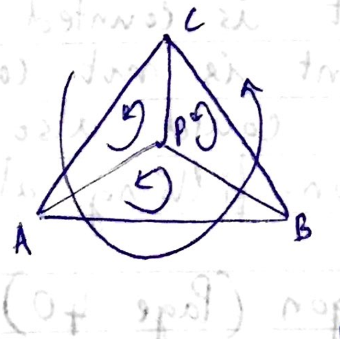
$$2 * \text{Area}(P_0 \cdots P_{n-1}) = \sum_{i=0}^{n-1} (x_i y_{i+1} - y_i x_{i+1})$$

(P_0 & P_n are the same vertex)

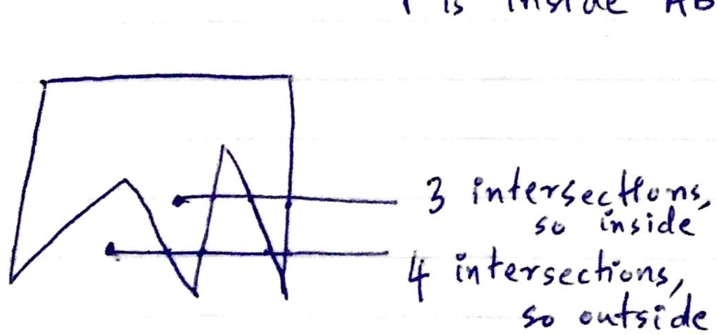
2.7 Point-in-Polygon Test

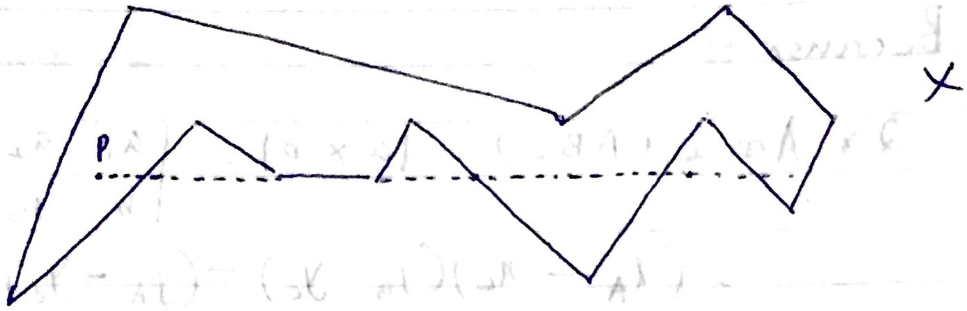
(Real world application in UI design: Mouse click on an icon)

A point P is inside a triangle ABC if the orientation of ABP, BCP and CAP are the same as that of ABC

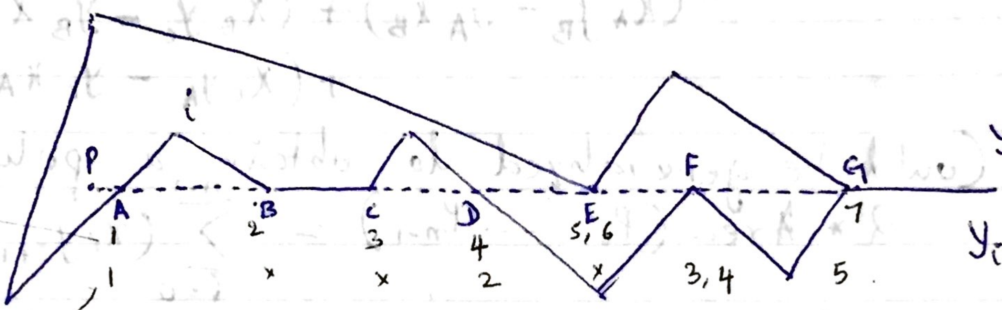


If orientation of ABC is CCW, then if Tools2Darea2(A, B, P) >= 0
Tools2Darea2(B, C, P) >= 0
Tools2Darea2(C, A, P) >= 0
P is inside ABC, or on edge of it





Count vertices for edges above the line



$y_i < y_p \leq y_{i+1}$

$y_{i+1} < y_p \leq y_i$

Count vertices for edges below the line.

Special Cases :

An edge $(i, i+1)$ is counted if

$$y_i \leq y_p < y_{i+1} \quad \text{or}$$

$$y_{i+1} \leq y_p < y_i$$

So, lower endpoint is counted once for each edge, upper endpoint is not counted.

This method, could also be used to implement Polygon-filling algorithm.

Method insidePolygon (Page 40).

2.8 Point and line Relationships.

① Testing whether a point P is on a line defined by A & B.

if (Math.abs (Tools 2D. area2(A, B, P)) < eps)

eps - a small positive value for tolerance.

② Testing whether a point P is on a line segment AB : (inclusive of A & B):

For the mid term.

$$\text{if } \left\{ \begin{array}{l} x_A \neq x_B \text{ AND } x_P \text{ inbetween } x_A \ \& \ x_B \\ x_A = x_B \text{ AND } y_P \text{ inbetween } y_A \ \& \ y_B \end{array} \right\}$$

AND {Tools 2D. area2 (A, B, P) < eps}.