DOE 4

# Randomized Complete Block Design

## 1 Model and parameter estimation

Suppose that there are $r$ different treatments that are to be compared and each subject is tested or treated randomly in each of $r$ treatments. There is one observation per treatment for each subject and the order in which the treatments are randomly determined within each subject which is called *block*. Especially, if every treatment appears in each of the blocks, the blocks are called *complete* block. This design is called a randomized complete block design (RCBD). On the other hand, the one-way ANOVA in DOE-3 is a completely randomized design (CRD) since the observations are taken completely randomly throughout the design (not restricted within each block). In general, the purpose of using a block lies in removing variation, not in finding block differences, while that of using a treatment is finding treatment effect (differences). The idea of blocking originated in agriculture, where farmers or experimenters took advantage of similar growing conditions to observe and control experimental variances.

### 1.1 Point estimation

Let $\mu_i$ be the population mean (effect) of $i$th treatment and $\beta_j$ be the mean (effect) of $j$the subject. We assume that there is one observation for each treatment-block combination.

Then we can consider the statistical model for observations

$$Y_{ij} = \mu_i + \beta_j + \epsilon_{ij}, \tag{1}$$

where $i = 1, 2, \ldots, r$, $j = 1, 2, \ldots, b$, and $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$.

Don't be confused with $r$ and $b$. In this handout, $r$ is the number of treatments and $b$ is the number of blocks.

Note that the treatments are always a fixed factor while, by definition, blocks are always a random factor. However, whether we assume blocks to be fixed or random, statistical inferences are unchanged. For more details, see §11.3 of Casella and Berger (1990). Thus, many textbooks including Montgomery (2017) treat blocks as a fixed factor.

We are interested in testing the effect of the treatment. Thus, we can consider the following hypothesis testing

$$\boxed{\begin{aligned} &H_0 : \mu_1 = \mu_2 = \cdots = \mu_r \\ &H_1 : \mu_i \neq \mu_j \quad \text{for at least one pair } (i, j). \end{aligned}} \tag{2}$$

However, the model $Y_{ij} = \mu_i + \beta_j + \epsilon_{ij}$ in (1) is overparameterized so that it is not identifiable. For example, the model $Y_{ij} = \mu_i + \beta_j + \epsilon_{ij}$ is equivalent to $Y_{ij} = \mu_i^* + \beta_j^* + \epsilon_{ij}$ where $\mu_i^* = \mu_i - \gamma$ and $\beta_j^* = \beta_j + \gamma$. Thus, for any value of $\gamma$, both $\mu_i + \beta_j$ and $\mu_i^* + \beta_j^*$ give the same value. An alternative version of (1) is

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}, \tag{3}$$

where $\sum_{i=1}^{r} \tau_i = 0$ and $\sum_{j=1}^{b} \beta_j = 0$. Notice that there are $b+r$ free parameters in (1) while there are $b + r - 1$ free parameters in (3) due to two restrictions on $\tau_i$ and $\beta_j$. Note that the hypotheses in (2) are equivalent to

$$\boxed{\begin{aligned} &H_0 : \tau_1 = \tau_2 = \cdots = \tau_r = 0 \\ &H_1 : \tau_i \neq 0 \quad \text{for at least one } i, \end{aligned}} \tag{4}$$

which are more simple than (2). The hypotheses above imply that

$$\begin{aligned} &H_0 : Y_{ij} = \mu + \beta_j + \epsilon_{ij} && \text{(reduced)} \\ &H_1 : Y_{ij} = \mu + \boxed{\tau_i} + \beta_j + \epsilon_{ij}. && \text{(full)} \end{aligned}$$

The parameters, $\mu$, $\tau_i$ and $\beta_j$ can be estimated by using the least squares method. We define $Q_2$ by

$$Q_2 = \sum_{i=1}^{r}\sum_{j=1}^{b}\epsilon_{ij}^2 = \sum_{i=1}^{r}\sum_{j=1}^{b}\big(Y_{ij} - \mu - \tau_i - \beta_j\big)^2. \tag{5}$$

Differentiating (5) by $\mu$ and setting zero, we have

$$\frac{\partial Q_2}{\partial \mu} = -2\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij} - \mu - \tau_i - \beta_j) = 0,$$

which results in

$$\sum_{i=1}^{r}\sum_{j=1}^{b} Y_{ij} - N\mu - b\sum_{i=1}^{r}\tau_i - r\sum_{j=1}^{b}\beta_j = 0,$$

where $N = rb$. Since $\sum_{i=1}^{r}\tau_i = 0$ and $\sum_{j=1}^{b}\beta_j = 0$, we have

$$\sum_{i=1}^{r}\sum_{j=1}^{b} Y_{ij} - N\mu = Y_{\bullet\bullet} - N\mu = 0,$$

where $Y_{\bullet\bullet} = \sum_{i=1}^{r}\sum_{j=1}^{b} Y_{ij}$. Then we have

$$\hat{\mu} = \overline{Y}_{\bullet\bullet}, \tag{6}$$

where $\overline{Y}_{\bullet\bullet} = Y_{\bullet\bullet}/N = Y_{\bullet\bullet}/(rb)$.

Differentiating (5) by $\tau_i$ and $\beta_j$ and setting zeros, we have

$$\frac{\partial Q_2}{\partial \tau_i} = -2\sum_{j=1}^{b}(Y_{ij} - \mu - \tau_i - \beta_j) = -2\Big[\sum_{j=1}^{b} Y_{ij} - b\mu - b\tau_i\Big] = 0$$

and

$$\frac{\partial Q_2}{\partial \beta_j} = -2\sum_{i=1}^{r}(Y_{ij} - \mu - \tau_i - \beta_j) = -2\Big[\sum_{i=1}^{r} Y_{ij} - r\mu - r\beta_j\Big] = 0,$$

which result in

$$\tau_i = \frac{1}{b}\sum_{j=1}^{b} Y_{ij} - \mu \quad\text{and}\quad \beta_j = \frac{1}{r}\sum_{i=1}^{r} Y_{ij} - \mu. \tag{7}$$

Substituting (6) into (7), we have

$$\hat{\tau}_i = \frac{1}{b}\sum_{j=1}^{b} Y_{ij} - \hat{\mu} = \frac{1}{b}\,Y_{i\bullet} - \overline{Y}_{\bullet\bullet} = \overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet} \tag{8}$$

and

$$\hat{\beta}_j = \frac{1}{r}\sum_{i=1}^{r} Y_{ij} - \hat{\mu} = \frac{1}{r} Y_{\bullet j} - \overline{Y}_{\bullet\bullet} = \overline{Y}_{\bullet j} - \overline{Y}_{\bullet\bullet}, \tag{9}$$

where $Y_{i\bullet} = \sum_{j=1}^{b} Y_{ij}$, $\overline{Y}_{i\bullet} = Y_{i\bullet}/b$, $Y_{\bullet j} = \sum_{i=1}^{r} Y_{ij}$ and $\overline{Y}_{\bullet j} = Y_{\bullet j}/r$. It is easily seen that

$$\sum_{i=1}^{r} \hat{\tau}_i = 0 \quad \text{and} \quad \sum_{j=1}^{b} \hat{\beta}_j = 0.$$

We also denote the residuals by $\hat{\epsilon}_{ij} = Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j$ Then it is immediate from (6), (8) and (9) that we have

$$\hat{\epsilon}_{ij} = Y_{ij} - \overline{Y}_{i\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet}.$$

## 1.2 Interval estimation

Recall that $Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$ from (3) where $\epsilon_{ij} \overset{iid}{\sim} N(0, \sigma^2)$, $i = 1, 2, \ldots, r$, and $j = 1, 2, \ldots, b$. Then we have $Y_{i\bullet} \sim N(b\mu + b\tau_i + \sum_{j=1}^{b} \beta_j, b\sigma^2)$. Since $\sum_{j=1}^{b} \beta_j = 0$, we have $Y_{i\bullet} \sim N(b\mu + b\tau_i, b\sigma^2)$ and $\overline{Y}_{i\bullet} \sim N(\mu + \tau_i, \sigma^2/b)$, where $i = 1, 2, \ldots, r$.

We let $\mu_i = \mu + \tau_i$. Then we have $\overline{Y}_{i\bullet} \sim N(\mu_i, \sigma^2/b)$ and thus

$$\frac{\overline{Y}_{i\bullet} - \mu_i}{\sqrt{\sigma^2/b}} \sim N(0, 1).$$

As we discussed in DOE-3, we need to estimate $\sigma^2$ since it is a nuisance parameter. We have

$$\text{SSE} = \sum_{i=1}^{r}\sum_{j=1}^{b} \hat{\epsilon}_{ij}^2 = \sum_{i=1}^{r}\sum_{j=1}^{b} (Y_{ij} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j)^2 = \sum_{i=1}^{r}\sum_{j=1}^{b} (Y_{ij} - \overline{Y}_{i\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet})^2. \tag{10}$$

We showed that there are $b + r - 1$ free parameters in (3). Since there are $rb$ squared terms in the summations in (10) with $b + r - 1$ estimates, the degrees of freedom are given by

$$\text{df} = rb - (b + r - 1) = (r - 1)(b - 1).$$

As will be shown in details, the distribution of $\text{SSE}/\sigma^2$ is given by

$$\frac{1}{\sigma^2}\text{SSE} = \frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b} (Y_{ij} - \overline{Y}_{i\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet})^2 \sim \chi^2\big((r-1)(b-1)\big).$$

Analogous to the approach in DOE-3, we have

$$\frac{\overline{Y}_{i\bullet} - \mu_i}{\sqrt{\text{MSE}/b}} \sim t\big(\text{df} = (r-1)(b-1)\big),$$

where

$$\text{MSE} = \frac{\text{SSE}}{(r-1)(b-1)}.$$

Then the endpoints for the interval estimation of $\mu_i$ with $100(1-\alpha)\%$ confidence level are given by

$$\boxed{\overline{Y}_{i\bullet} \ \pm \ t\big(1-\tfrac{\alpha}{2};(r-1)(b-1)\big)\sqrt{\frac{\text{MSE}}{b}},}$$

where $t(\gamma;\nu)$ is the *lower* $\gamma$th quantile of the $t$ distribution with $\nu$ degrees of freedom.

We can also obtain the $100(1-\alpha)\%$ confidence interval of $\mu_\ell - \mu_m$ as follows. Since $\overline{Y}_{i\bullet} \sim N(\mu_i, \sigma^2/b)$, we have

$$\overline{Y}_{\ell\bullet} - \overline{Y}_{m\bullet} \sim N\left(\mu_\ell - \mu_m, \sigma^2\Big(\frac{1}{b}+\frac{1}{b}\Big)\right)$$

and

$$\frac{(\overline{Y}_{\ell\bullet} - \overline{Y}_{m\bullet}) - (\mu_\ell - \mu_m)}{\sqrt{\dfrac{2\sigma^2}{b}}} \sim N(0,1).$$

Thus, we have

$$\frac{(\overline{Y}_{\ell\bullet} - \overline{Y}_{m\bullet}) - (\mu_\ell - \mu_m)}{\sqrt{\dfrac{2\text{MSE}}{b}}} \sim t\big(\text{df} = (r-1)(b-1)\big).$$

Then the endpoints for the $100(1-\alpha)\%$ confidence interval of $\mu_\ell - \mu_m$ are given by

$$\boxed{\overline{Y}_{\ell\bullet} - \overline{Y}_{m\bullet} \ \pm \ t\big(1-\tfrac{\alpha}{2};(r-1)(b-1)\big)\sqrt{\frac{2\text{MSE}}{b}}.}$$

We obtained the $100(1-\alpha)\%$ confidence interval of the treatment means. Similarly, we can also obtain the $100(1-\alpha)\%$ confidence interval of the *block means* although this interval is of no interest in practice. We let $\eta_j = \mu + \beta_j$. Then we have $Y_{\bullet j} \sim N(\sum_{i=1}^r \tau_i + r\eta_j, r\sigma^2)$. Since $\sum_{i=1}^r \tau_i = 0$, we have $\overline{Y}_{\bullet j} \sim N(\eta_j, \sigma^2/r)$, which results in

$$\frac{\overline{Y}_{\bullet j} - \eta_j}{\sqrt{\sigma^2/r}} \sim N(0,1).$$

Then we have

$$\frac{\overline{Y}_{\bullet j} - \eta_j}{\sqrt{\text{MSE}/r}} \sim t\big(\text{df} = (r-1)(b-1)\big).$$

The endpoints for the interval estimation of $\eta_j$ with $100(1-\alpha)\%$ confidence level are given by

$$\boxed{\overline{Y}_{\bullet j} \;\pm\; t\big(1-\tfrac{\alpha}{2};(r-1)(b-1)\big)\sqrt{\frac{\text{MSE}}{r}}.}$$

We can also obtain the $100(1-\alpha)\%$ confidence interval of $\eta_\ell - \eta_m$ as follows. Since $\overline{Y}_{\bullet j} \sim N(\eta_j, \sigma^2/r)$, we have

$$\overline{Y}_{\bullet \ell} - \overline{Y}_{\bullet m} \sim N\left(\eta_\ell - \eta_m, \sigma^2\Big(\frac{1}{r}+\frac{1}{r}\Big)\right)$$

and

$$\frac{(\overline{Y}_{\bullet \ell} - \overline{Y}_{\bullet m}) - (\eta_\ell - \eta_m)}{\sqrt{\dfrac{2\sigma^2}{r}}} \sim N(0,1).$$

Thus, we have

$$\frac{(\overline{Y}_{\bullet \ell} - \overline{Y}_{\bullet m}) - (\eta_\ell - \eta_m)}{\sqrt{\dfrac{2\text{MSE}}{r}}} \sim t\big(\text{df} = (r-1)(b-1)\big).$$

Then the endpoints for the $100(1-\alpha)\%$ confidence interval of $\eta_\ell - \eta_m$ are given by

$$\boxed{\overline{Y}_{\bullet \ell} - \overline{Y}_{\bullet m} \;\pm\; t\big(1-\tfrac{\alpha}{2};(r-1)(b-1)\big)\sqrt{\frac{2\text{MSE}}{r}}.}$$

## 2 Analysis of variance (ANOVA)

Let $\hat{Y}_{ij}$ denote the fitted values of $Y_{ij}$. Then we have $\hat{Y}_{ij} = \hat{\mu}+\hat{\tau}_i+\hat{\beta}_j$ and $\hat{\epsilon}_{ij} = Y_{ij}-\hat{Y}_{ij} = Y_{ij}-\hat{\mu}-\hat{\tau}_i-\hat{\beta}_j$. It is easily seen that $Y_{ij}-\hat{\mu} = \hat{\tau}_i+\hat{\beta}_j+\hat{\epsilon}_{ij}$, $\sum_i\sum_j \hat{\tau}_i\hat{\beta}_j = 0$, $\sum_i\sum_j \hat{\tau}_i\hat{\epsilon}_{ij} = 0$, and $\sum_i\sum_j \hat{\beta}_j\hat{\epsilon}_{ij} = 0$. Using these, we have

$$\begin{aligned}
\underbrace{\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\hat{\mu})^2}_{\text{SSTo}} &= \sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\tau}_i+\hat{\beta}_j+\hat{\epsilon}_{ij})^2 \\
&= \underbrace{\sum_{i=1}^{r}\sum_{j=1}^{b}\hat{\tau}_i^2}_{\text{SStr}} + \underbrace{\sum_{i=1}^{r}\sum_{j=1}^{b}\hat{\beta}_j^2}_{\text{SSblock}} + \underbrace{\sum_{i=1}^{r}\sum_{j=1}^{b}\hat{\epsilon}_{ij}^2}_{\text{SSE}},
\end{aligned}$$

that is,

$$\underbrace{\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\overline{Y}_{\bullet\bullet})^2}_{\text{SSTo}}$$
$$=\sum_{i=1}^{r}\sum_{j=1}^{b}(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet})^2+\sum_{i=1}^{r}\sum_{j=1}^{b}(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet})^2+\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\overline{Y}_{i\bullet}-\overline{Y}_{\bullet j}+\overline{Y}_{\bullet\bullet})^2$$
$$=\underbrace{b\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet})^2}_{\text{SStr}}+\underbrace{r\sum_{j=1}^{b}(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet})^2}_{\text{SSblock}}+\underbrace{\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\overline{Y}_{i\bullet}-\overline{Y}_{\bullet j}+\overline{Y}_{\bullet\bullet})^2}_{\text{SSE}}. \tag{11}$$

Since $Y_{ij}=\mu+\tau_i+\beta_j+\epsilon_{ij}$ and $\epsilon_{ij}\overset{iid}{\sim}N(0,\sigma^2)$, we have

$$Y_{ij}\sim N(\mu+\tau_i+\beta_j,\sigma^2) \tag{12}$$
$$\overline{Y}_{i\bullet}\sim N\big(\mu+\tau_i,\sigma^2/b\big) \tag{13}$$
$$\overline{Y}_{\bullet j}\sim N\big(\mu+\beta_j,\sigma^2/r\big) \tag{14}$$
$$\overline{Y}_{\bullet\bullet}\sim N\big(\mu,\sigma^2/(rb)\big), \tag{15}$$

under $H_0$ or $H_1$.

Since $rb(\overline{Y}_{\bullet\bullet}-\mu)^2/\sigma^2\sim\chi^2(1)$ from (15) and $Y_{ij}\sim N(\mu,\sigma^2)$ under $\tau_i=0$ and $\beta_j=0$ from (12), we have

$$\underbrace{\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\mu)^2}_{\chi^2(rb)\text{ under }\tau_i=0\text{ and }\beta_j=0}=\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\overline{Y}_{\bullet\bullet})^2+\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\overline{Y}_{\bullet\bullet}-\mu)^2$$
$$=\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\overline{Y}_{\bullet\bullet})^2+\underbrace{\frac{rb}{\sigma^2}(\overline{Y}_{\bullet\bullet}-\mu)^2}_{\chi^2(1)}.$$

Thus, using Lemma 3 in DOE-3, we have

$$\frac{1}{\sigma^2}\text{SSTo}=\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\overline{Y}_{\bullet\bullet})^2\sim\chi^2(rb-1)\text{ under }\tau_i=0\text{ and }\beta_j=0. \tag{16}$$

It is immediate from $\overline{Y}_{i\bullet}\sim N\big(\mu,\sigma^2/b\big)$ under $\tau_i=0$ from (13) and

$$\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\mu)^2=\sum_{i=1}^{r}\big\{(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet})+(\overline{Y}_{\bullet\bullet}-\mu)\big\}^2=\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet})^2+r(\overline{Y}_{\bullet\bullet}-\mu)^2$$

that we have

$$\underbrace{\frac{b}{\sigma^2}\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\mu)^2}_{\chi^2(r)\ \text{ under }\tau_i=0} = \frac{b}{\sigma^2}\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet})^2 + \underbrace{\frac{rb}{\sigma^2}(\overline{Y}_{\bullet\bullet}-\mu)^2}_{\chi^2(1)}.$$

Thus, using Lemma 3 in DOE-3, we have

$$\frac{1}{\sigma^2}\text{SStr} = \frac{b}{\sigma^2}\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\overline{Y}_{\bullet\bullet})^2 \sim \chi^2(r-1) \quad \text{under } \tau_i = 0. \tag{17}$$

Similarly, we have $\overline{Y}_{\bullet j} \sim N\big(\mu, \sigma^2/r\big)$ under $\beta_j = 0$ from (14) and

$$\sum_{j=1}^{b}(\overline{Y}_{\bullet j}-\mu)^2 = \sum_{j=1}^{b}\left\{(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet}) + (\overline{Y}_{\bullet\bullet}-\mu)\right\}^2 = \sum_{j=1}^{b}(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet})^2 + b(\overline{Y}_{\bullet\bullet}-\mu)^2$$

that we have

$$\underbrace{\frac{r}{\sigma^2}\sum_{j=1}^{b}(\overline{Y}_{\bullet j}-\mu)^2}_{\chi^2(b)\ \text{ under }\beta_j=0} = \frac{r}{\sigma^2}\sum_{j=1}^{b}(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet})^2 + \underbrace{\frac{rb}{\sigma^2}(\overline{Y}_{\bullet\bullet}-\mu)^2}_{\chi^2(1)}.$$

Thus, using Lemma 3 in DOE-3, we have

$$\frac{1}{\sigma^2}\text{SSblock} = \frac{r}{\sigma^2}\sum_{j=1}^{b}(\overline{Y}_{\bullet j}-\overline{Y}_{\bullet\bullet})^2 \sim \chi^2(b-1) \quad \text{under } \beta_j = 0. \tag{18}$$

Using (16), (17) and (18) along with (11), we have

$$\underbrace{\frac{1}{\sigma^2}\text{SSTo}}_{\chi^2(rb-1)\text{ under }\tau_i=0\text{ and }\beta_j=0} = \underbrace{\frac{1}{\sigma^2}\text{SStr}}_{\chi^2(r-1)\text{ under }\tau_i=0} + \underbrace{\frac{1}{\sigma^2}\text{SSblock}}_{\chi^2(b-1)\text{ under }\beta_j=0} + \frac{1}{\sigma^2}\text{SSE}.$$

Thus, we have

$$\frac{1}{\sigma^2}\text{SSE} \sim \chi^2(\text{df}) \quad \text{under } \tau_i = 0 \text{ and } \beta_j = 0,$$

where $\text{df} = (rb-1)-(r-1)-(b-1) = (r-1)(b-1)$.

Actually, $\text{SSE}/\sigma^2$ has a chi-square distribution with $(r-1)(b-1)$ degrees of freedom under $H_0$ or $H_1$. We show this as follows. Since $\epsilon_{ij} = Y_{ij}-\mu-\tau_i-\beta_j \overset{iid}{\sim} N(0,\sigma^2)$, we have

$$\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\mu-\tau_i-\beta_j)^2 \sim \chi^2(rb),$$

under $H_0$ or $H_1$. Considering $(\hat{\epsilon}_{ij}) \perp (\epsilon_{ij}-\hat{\epsilon}_{ij})$, we have

$$\sum_{i=1}^{r}\sum_{j=1}^{b}\epsilon_{ij}^2 = \sum_{i=1}^{r}\sum_{j=1}^{b}\hat{\epsilon}_{ij}^2 + \sum_{i=1}^{r}\sum_{j=1}^{b}(\epsilon_{ij}-\hat{\epsilon}_{ij})^2,$$

that is

$$\begin{aligned}
&\underbrace{\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\mu-\tau_i-\beta_j)^2}_{\chi^2(rb)\text{ under } H_0 \text{ or } H_1} \\
&\quad = \frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\hat{\mu}-\hat{\tau}_i-\hat{\beta}_j)^2 + \frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\mu}-\mu+\hat{\tau}_i-\tau_i+\hat{\beta}_j-\beta_j)^2 \\
&\quad = \frac{1}{\sigma^2}\text{SSE} + \frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\mu}-\mu+\hat{\tau}_i-\tau_i+\hat{\beta}_j-\beta_j)^2. \qquad (19)
\end{aligned}$$

The last term in the above can be decomposed as

$$\begin{aligned}
&\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\mu}-\mu+\hat{\tau}_i-\tau_i+\hat{\beta}_j-\beta_j)^2 \\
&\quad = \frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\mu}-\mu)^2 + \frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\tau}_i-\tau_i)^2 + \frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\beta}_j-\beta_j)^2 \\
&\quad = \frac{rb}{\sigma^2}(\hat{\mu}-\mu)^2 + \frac{b}{\sigma^2}\sum_{i=1}^{r}(\hat{\tau}_i-\tau_i)^2 + \frac{r}{\sigma^2}\sum_{j=1}^{b}(\hat{\beta}_j-\beta_j)^2. \qquad (20)
\end{aligned}$$

Since $\hat{\mu}-\mu = \overline{Y}_{\bullet\bullet}-\mu$, we have $\hat{\mu}-\mu \sim N\big(0,\sigma^2/(rb)\big)$ from (15) so that

$$\frac{rb}{\sigma^2}(\hat{\mu}-\mu)^2 \sim \chi^2(1) \quad \text{under } H_0 \text{ or } H_1. \qquad (21)$$

Since $\hat{\tau}_i-\tau_i = \overline{Y}_{i\bullet}-\tau_i-\overline{Y}_{\bullet\bullet}$, we have

$$\begin{aligned}
\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\tau_i-\mu)^2 &= \sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\tau_i-\overline{Y}_{\bullet\bullet}+\overline{Y}_{\bullet\bullet}-\mu)^2 = \sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\tau_i-\overline{Y}_{\bullet\bullet})^2 + \sum_{i=1}^{r}(\overline{Y}_{\bullet\bullet}-\mu)^2 \\
&= \sum_{i=1}^{r}(\hat{\tau}_i-\tau_i)^2 + r(\overline{Y}_{\bullet\bullet}-\mu)^2,
\end{aligned}$$

which results in

$$\frac{b}{\sigma^2}\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\tau_i-\mu)^2 = \frac{b}{\sigma^2}\sum_{i=1}^{r}(\hat{\tau}_i-\tau_i)^2 + \frac{rb}{\sigma^2}(\overline{Y}_{\bullet\bullet}-\mu)^2.$$

We have $\overline{Y}_{i\bullet}-\tau_i-\mu \sim N(0,\sigma^2/b)$ from (13), which results in $\frac{b}{\sigma^2}\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\tau_i-\mu)^2 \sim \chi^2(r)$ under $H_0$ or $H_1$. Also, we have $\overline{Y}_{\bullet\bullet}-\mu \sim N\big(0,\sigma^2/(rb)\big)$ from (15) again, which results in $\frac{rb}{\sigma^2}(\overline{Y}_{\bullet\bullet}-\mu)^2 \sim \chi^2(1)$ under $H_0$ or $H_1$. Using these, we have

$$\underbrace{\frac{b}{\sigma^2}\sum_{i=1}^{r}(\overline{Y}_{i\bullet}-\tau_i-\mu)^2}_{\chi^2(r)} = \frac{b}{\sigma^2}\sum_{i=1}^{r}(\hat{\tau}_i-\tau_i)^2 + \underbrace{\frac{rb}{\sigma^2}(\overline{Y}_{\bullet\bullet}-\mu)^2}_{\chi^2(1)}.$$

Thus, we have

$$\frac{b}{\sigma^2}\sum_{i=1}^{r}(\hat{\tau}_i-\tau_i)^2 \sim \chi^2(r-1) \quad \text{under } H_0 \text{ or } H_1. \qquad (22)$$

Similarly, we have

$$\frac{r}{\sigma^2}\sum_{j=1}^{b}(\hat{\beta}_j-\beta_j)^2 \sim \chi^2(b-1) \quad \text{under } H_0 \text{ or } H_1. \tag{23}$$

It is immediate from (21), (22) and (23) along with (20) that

$$\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\mu}-\mu+\hat{\tau}_i-\tau_i+\hat{\beta}_j-\beta_j)^2 = \underbrace{\frac{rb}{\sigma^2}(\hat{\mu}-\mu)^2}_{\chi^2(1)} + \underbrace{\frac{b}{\sigma^2}\sum_{i=1}^{r}(\hat{\tau}_i-\tau_i)^2}_{\chi^2(r-1)} + \underbrace{\frac{r}{\sigma^2}\sum_{j=1}^{b}(\hat{\beta}_j-\beta_j)^2}_{\chi^2(b-1)}$$

under $H_0$ or $H_1$ again. Thus, we have

$$\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\mu}-\mu+\hat{\tau}_i-\tau_i+\hat{\beta}_j-\beta_j)^2 \sim \chi^2(r+b-1). \tag{24}$$

Considering (19) and (24), we have

$$\underbrace{\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(Y_{ij}-\mu-\tau_i-\beta_j)^2}_{\chi^2(rb) \text{ under } H_0 \text{ or } H_1} = \frac{1}{\sigma^2}\mathrm{SSE} + \underbrace{\frac{1}{\sigma^2}\sum_{i=1}^{r}\sum_{j=1}^{b}(\hat{\mu}-\mu+\hat{\tau}_i-\tau_i+\hat{\beta}_j-\beta_j)^2}_{\chi^2(r+b-1) \text{ under } H_0 \text{ or } H_1},$$

which results in

$$\frac{1}{\sigma^2}\mathrm{SSE} \sim \chi^2\big((r-1)(b-1)\big) \quad \text{under } H_0 \text{ or } H_1.$$

# 3 Paired sample $t$-test as RCB design

The test statistic for the paired $t$-test is

$$T = \frac{\overline{D}}{S_D/\sqrt{b}} \sim t(b-1),$$

equivalently

$$F = \frac{b\overline{D}^2}{\sum_{j=1}^{b}(D_j-\overline{D})^2/(b-1)} \sim F(r-1, b-1),$$

where $r=2$. We will show that

$$b\cdot\overline{D}^2 = \frac{1}{b}\Big[\sum_{j=1}^{b}D_j\Big]^2 = 2\cdot\mathrm{SStr} \quad\text{and}\quad (b-1)S_D^2 = \sum_{j=1}^{b}\big(D_j-\overline{D}\big)^2 = 2\cdot\mathrm{SSE}.$$

which implies

$$F = \frac{\mathrm{SStr}/(r-1)}{\mathrm{SSE}/(b-1)} \sim F(r-1, b-1),$$

where $r=2$ again.

First, it is immediate from $\sum_{j=1}^{b} D_j = \sum_{j=1}^{b}(Y_{1j} - Y_{2j}) = b(\overline{Y}_{1\bullet} - \overline{Y}_{2\bullet})$ that we have

$$b \cdot \overline{D}^2 = \frac{1}{b}\Big[\sum_{j=1}^{b} D_j\Big]^2 = b(\overline{Y}_{1\bullet} - \overline{Y}_{2\bullet})^2. \tag{25}$$

On the other hand, we have

$$\mathrm{SStr} = \sum_{j=1}^{b}\sum_{i=1}^{2} \hat{\tau}_i^2 = b\sum_{i=1}^{2}(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet})^2,$$

where $\overline{Y}_{\bullet\bullet} = (\overline{Y}_{1\bullet} + \overline{Y}_{2\bullet})/2$. Thus, we have

$$\begin{aligned}\mathrm{SStr} &= b\sum_{i=1}^{2}\Big[\overline{Y}_{i\bullet} - \frac{\overline{Y}_{1\bullet} + \overline{Y}_{2\bullet}}{2}\Big]^2 = b\Big[\frac{\overline{Y}_{1\bullet} - \overline{Y}_{2\bullet}}{2}\Big]^2 + b\Big[\frac{\overline{Y}_{2\bullet} - \overline{Y}_{1\bullet}}{2}\Big]^2 \\ &= \frac{b}{2}(\overline{Y}_{1\bullet} - \overline{Y}_{2\bullet})^2. \end{aligned} \tag{26}$$

It is easily seen from (25) and (26) that

$$b \cdot \overline{D}^2 = \frac{1}{b}\Big[\sum_{j=1}^{b} D_j\Big]^2 = 2 \cdot \mathrm{SStr}.$$

Next, we will show that $(b-1)S_D^2 = \sum_{j=1}^{b}(D_j - \overline{D})^2 = 2 \cdot \mathrm{SSE}$. We have

$$\begin{aligned}\mathrm{SSE} &= \sum_{i=1}^{2}\sum_{j=1}^{b}\big(Y_{ij} - \overline{Y}_{i\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet}\big)^2 \\ &= \sum_{j=1}^{b}\big(Y_{1j} - \overline{Y}_{1\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet}\big)^2 + \sum_{j=1}^{b}\big(Y_{2j} - \overline{Y}_{2\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet}\big)^2 \end{aligned} \tag{27}$$

It is immediate upon using $\overline{Y}_{\bullet\bullet} = \frac{1}{2}(\overline{Y}_{1\bullet} + \overline{Y}_{2\bullet})$ and $\overline{Y}_{\bullet j} = \frac{1}{2}(Y_{1j} + Y_{2j})$ that we have

$$Y_{1j} - \overline{Y}_{1\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet} = \frac{1}{2}\Big[(Y_{1j} - Y_{2j}) - (\overline{Y}_{1\bullet} - \overline{Y}_{2\bullet})\Big] \tag{28}$$

and

$$Y_{2j} - \overline{Y}_{2\bullet} - \overline{Y}_{\bullet j} + \overline{Y}_{\bullet\bullet} = -\frac{1}{2}\Big[(Y_{1j} - Y_{2j}) - (\overline{Y}_{1\bullet} - \overline{Y}_{2\bullet})\Big] \tag{29}$$

Substituting (28) and (29) into (27), we have

$$\mathrm{SSE} = \frac{1}{2}\sum_{j=1}^{b}\Big[(Y_{1j} - Y_{2j}) - (\overline{Y}_{1\bullet} - \overline{Y}_{2\bullet})\Big]^2 = \frac{1}{2}\sum_{j=1}^{b}\big(D_j - \overline{D}\big)^2,$$

which results in

$$(b-1)S_D^2 = \sum_{j=1}^{b}\big(D_j - \overline{D}\big)^2 = 2 \cdot \mathrm{SSE}.$$

# References

Casella, G. and Berger, R. L. (1990). *Statistical Inference*. Wadsworth, Belmont, CA.

Montgomery, D. C. (2017). *Design and Analysis of Experiments*. John Wiley & Sons, 9th edition.