Probability Distributions

1 **Binomial** (n,p), $\quad (x=0,1,\ldots,n)$ R:$x$binom

$f(x)=\binom{n}{x}p^x(1-p)^{n-x}$
$M(t)=(pe^t+1-p)^n$
$E(X)=np$, $\mathrm{Var}(X)=np(1-p)$

NOTE: $n\to\infty$ with $\mu=np\Rightarrow$ Poisson(μ)
$F_{\mathrm{Bin}(n,p)}(x)=F_{\mathrm{Beta}(n-\lfloor x\rfloor,\lfloor x\rfloor+1)}(1-p)$
$F_{\mathrm{Bin}(n,p)}(r-1)=1-F_{\mathrm{NegBin}(r,p)}(n-r)$
$X_i\sim \mathrm{Bin}(n_i,p)\Rightarrow \sum X_i\sim \mathrm{Bin}(\sum n_i,p)$

2 **Geometric** (p) R:$x$geom

$f(x)=p(1-p)^x$, $\quad (x=0,1,\ldots)$
$F(t)=1-(1-p)^{\lfloor t\rfloor+1}$, where $\lfloor t\rfloor=\max\{m\in\mathbb{Z}:m\le t\}$
$S(t)=P[X\ge t]=(1-p)^{\lceil t\rceil}$ with $\lceil t\rceil=\min\{m\in\mathbb{Z}:m\ge t\}$
$R(t)=P[X>t]=(1-p)^{\lfloor t\rfloor+1}$
$M_X(t)=\dfrac{p}{1-(1-p)e^t}$, $\quad (t<-\log(1-p))$
$E(X)=\dfrac{1-p}{p}$, $\mathrm{Var}(X)=\dfrac{1-p}{p^2}$

$f(y)=p(1-p)^{y-a}$, $\quad (y=a,a+1,a+2\ldots)$
$F(t)=1-(1-p)^{\lfloor t\rfloor+1-a}$
$S(t)=P[Y\ge t]=(1-p)^{\lceil t\rceil-a}$
$R(t)=P[Y>t]=(1-p)^{\lfloor t\rfloor+1-a}$
$M_Y(t)=\dfrac{pe^{at}}{1-(1-p)e^t}$, $(t<-\log(1-p))$
$E(Y)=\dfrac{1-p}{p}+a$, $\mathrm{Var}(Y)=\dfrac{1-p}{p^2}$

NOTE: $\min\limits_{1\le i\le n}(X_i)\sim \mathrm{Geo}\left(1-(1-p)^n\right)$: self-reproducing

3 **Hypergeometric** (N,M,n) R:$x$hyper

$f(x)=\dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$,
$(\max(0,M-(N-n))\le x\le \min(n,M))$
$E(X)=n\left(\dfrac{M}{N}\right)$, $\mathrm{Var}(X)=n\dfrac{M}{N}\dfrac{(N-M)(N-n)}{N(N-1)}$

4 **Negative Binomial** (r,p) R:$x$nbinom

$f(x)=\binom{r+x-1}{x}p^r(1-p)^x$, $(x=0,1,\ldots)$
$M_X(t)=\left(\dfrac{p}{1-(1-p)e^t}\right)^r$
$E(X)=r\frac{1-p}{p}$, $\mathrm{Var}(X)=r\frac{1-p}{p^2}$

$f(y)=\binom{y-1}{r-1}p^r(1-p)^{y-r}$, $(y=r,r+1,\ldots)$
$M_Y(t)=\left(\dfrac{pe^t}{1-(1-p)e^t}\right)^r$
$E(Y)=r\frac{1}{p}$, $\mathrm{Var}(Y)=r\frac{1-p}{p^2}$

NOTE: $X=V_1+\cdots+V_r$, $(V_i\sim$ Geometric$(p))$
$(1-x)^{-n}=\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k\quad(|x|<1)$
$\mu=r(1-p), r\to\infty\Rightarrow$ Poisson(μ)
$X_i\sim \mathrm{NB}(r_i,p)\Rightarrow\sum X_i\sim \mathrm{NB}(\sum r_i,p)$
$F_{\mathrm{NB}(r,p)}(x)=F_{\mathrm{Beta}(r,\lfloor x\rfloor+1)}(p)$
$F_{\mathrm{NB}(r,p)}(n-r)=1-F_{\mathrm{Bin}(n,p)}(r-1)$

5 **Poisson** (μ) R:$x$pois

$f(x)=\dfrac{e^{-\mu}\mu^x}{x!}$, $\quad (x=0,1,\ldots)$
$M(t)=\exp\left(\mu(e^t-1)\right)$
$E(X)=\mu$, $\mathrm{Var}(X)=\mu$

NOTE: $X_i\sim \mathrm{Poi}(\mu_i)\Rightarrow\sum X_i\sim\mathrm{Poi}(\sum\mu_i)$

$$P_{\mathrm{Poi}(\mu)}[X\ge n]=F_{\Gamma(n,1)}(\mu)=F_{\Gamma(n,\mu)}(1)=F_{\chi^2_{2n}}(2\mu)$$

1 **Beta** (α,β), $\quad(\alpha>0,\beta>0)$ R:$x$beta

$f(x)=\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $(0<x<1)$
$M(t)=1+\sum_{k=1}^{\infty}\left(\prod_{r=1}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$
$E(X)=\dfrac{\alpha}{\alpha+\beta}$, $\mathrm{Var}(X)=\dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

NOTE: $f_{\mathrm{Beta}(\alpha,\beta)}(x)=f_{\mathrm{Beta}(\beta,\alpha)}(1-x)$
$F_{\mathrm{Beta}(\alpha,\beta)}(x)=1-F_{\mathrm{Beta}(\beta,\alpha)}(1-x)$

2 **BS** (α,β), $(\alpha>0,\beta>0)$: Birnbaum-Saunders

$f(t)=\frac{1}{2\alpha\beta}\sqrt{\frac{\beta}{t}}\left(1+\frac{\beta}{t}\right)\phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)\right]$, $(t>0)$
$F(t)=\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)\right]$

$$\begin{aligned}F^{-1}(p)&=\frac{1}{4}\left[\alpha\sqrt{\beta}\Phi^{-1}(p)+\sqrt{\alpha^2\beta\{\Phi^{-1}(p)\}^2+4\beta}\right]^2\\&=\beta\left\{1+\gamma(p)^2+\gamma(p)\sqrt{\gamma(p)^2+2}\right\},\\&\quad\text{where } \gamma(p)=\alpha\Phi^{-1}(p)/\sqrt{2}\end{aligned}$$

$E(T)=\beta(1+\frac{1}{2}\alpha^2)$, $\mathrm{Var}(T)=(\alpha\beta)^2(1+\frac{5}{4}\alpha^2)$

NOTE: median$(T)=\beta$, $cT\sim\mathrm{BS}(\alpha,c\beta)$, $T^{-1}\sim \mathrm{BS}(\alpha,\beta^{-1})$
$X=\left(\sqrt{\frac{T}{\beta}}-\sqrt{\frac{\beta}{T}}\right)\sim N(0,\alpha^2)$
$\log T\sim$ sinh-Normal$(\log\beta,\alpha)$

3 **Cauchy** (α,β) R:$x$cauchy

$f(x)=\dfrac{\beta}{\pi}\dfrac{1}{\beta^2+(x-\alpha)^2}$
$F(x)=\dfrac{1}{\pi}\left[\arctan\left(\dfrac{x-\alpha}{\beta}\right)+\dfrac{\pi}{2}\right]$
$\phi(t)=\exp(it\alpha-\beta|t|)$

NOTE: Cauchy$(0,1)=t(1)$
$X,Y\sim N(0,1)\Rightarrow X/Y\sim$ Cauchy$(0,1)$
$cX\sim$ Cauchy$(c\alpha,c\beta)$
$X_i\sim$ Cauchy$(\alpha,\beta)\Rightarrow\frac{1}{n}\sum X_i\sim$ Cauchy(α,β)

4 **Chi-Square** (n) R:$x$chisq

$f(x)=\dfrac{1}{\Gamma(n/2)2^{n/2}}x^{n/2-1}e^{-x/2}$, $(x\ge 0)$
$E(X)=n$, $\mathrm{Var}(X)=2n$
$M(t)=\left(\frac{1}{1-2t}\right)^{n/2}$, $\quad(t<\frac{1}{2})$
$E(X^m)=2^n\Gamma(m+n/2)/\Gamma(n/2)$

NOTE: $\chi^2(n)=$ Gamma$(n/2,2)$
$\chi^2(2)=$ Exponential$(\beta=2)$
$X_i\sim\chi^2(n_i)\Rightarrow\sum X_i\sim\chi^2(\sum n_i)$
$X_i\sim N(0,1)\Rightarrow\sum_{i=1}^n X_i^2\sim\chi^2(n)$
$F_{\chi^2(p+2)}(x)=F_{\chi^2(p)}(x)-2xf_{\chi^2(p)}(x)/p$
$F_{\chi^2(1)}(x)=2\Phi(\sqrt{x})-1$
$F_{\chi^2(2)}(x)=1-\sqrt{2\pi}\phi(\sqrt{x})$
$F_{\chi^2(3)}(x)=2\Phi(\sqrt{x})-1-2\sqrt{x}\phi(\sqrt{x})$

5 **Exponential** (θ) R:$x$exp

$f(x)=\dfrac{1}{\theta}e^{-x/\theta}$ $(x\ge 0,\ \theta>0)$
$F(x)=1-e^{-x/\theta}$
$M(t)=\dfrac{1}{1-\theta t}$, $(t<\frac{1}{\theta})$
$E(X)=\theta$, $\mathrm{Var}(X)=\theta^2$

NOTE: Memoryless property
$cX\sim$ Exponential$(c\theta)$
$\sum X_i\sim$ Gamma(n,θ)
$\min\limits_{1\le i\le n}(X_i)\sim$ Exponential(θ/n): self-reproducing
$Y=X^{1/\alpha}\sim$ Weibull(α,θ)
$Y=\sqrt{2X/\theta}\sim$ Rayleigh(1)
$Y=\alpha-\gamma\log(X/\theta)\sim$ Gumbel(α,γ)

6 **F** (m,n) R:xf

$F(m,n)=\dfrac{\chi^2_m/m}{\chi^2_n/n}$

$E(X)=\dfrac{n}{n-2}\quad (n>2),$

$\mathrm{Var}(X)=2\left(\dfrac{n}{n-2}\right)^2\dfrac{m+n-2}{m(n-4)}\ (n>4)$

NOTE: $[F(m,n)]^{-1}=F(n,m)$
$F_{1-\alpha}(m,n)=[F_\alpha(n,m)]^{-1}$
$F(1,k)=t^2(k)$
If $X\sim F(m,n)$, $mX/(mX+n)\sim \mathrm{Beta}(\frac{m}{2},\frac{n}{2})$.

7 **Gamma** (α,θ) R:$x$gamma

$f(x)=\dfrac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}\exp(-x/\theta),\ (0<x<\infty)$

$M(t)=(1-\theta t)^{-\alpha}\ (t<1/\theta)$

$E(X)=\alpha\theta,\ \mathrm{Var}(X)=\alpha\theta^2$

NOTE: $X\sim \mathrm{Gamma}(n,\theta)=\mathrm{Erlang}(n,\theta)$
$X=V_1+\cdots+V_n,\ (V_i\sim \mathrm{Exponential}(\theta))$
$2X/\theta\sim\chi^2(2n)$

$$F_X(x)=1-\sum_{k=0}^{n-1}\left(\frac{x}{\theta}\right)^k\frac{e^{-x/\theta}}{k!}=\sum_{k=n}^{\infty}\left(\frac{x}{\theta}\right)^k\frac{e^{-x/\theta}}{k!}$$

$\mathrm{Gamma}(n/2,2)=\chi^2(n),\quad \Gamma(1/2)=\sqrt{\pi}$
$X_i\sim\mathrm{Gamma}(\alpha_i,\theta)\Rightarrow\sum_{i=1}^r X_i\sim\mathrm{Gamma}(\sum_{i=1}^r\alpha_i,\theta)$
$X/d\sim\mathrm{Gamma}(\alpha,\theta/d)$
$E[X^c]=\Gamma(\alpha+c)\theta^c/\Gamma(\alpha)\ (c>-\alpha)$

8 **Laplace** (μ,σ): Double Exponential

$f(x)=\frac{1}{2\sigma}e^{-|x-\mu|/\sigma}$

$$F(x)=\begin{cases}\frac12 e^{-|x-\mu|/\sigma} & (x<\mu)\\ 1-\frac12 e^{-|x-\mu|/\sigma} & (x\ge\mu)\end{cases}$$

$M(t)=\dfrac{e^{\mu t}}{1-(\sigma t)^2},\ (|t|<\frac{1}{\sigma})$

$E(X)=\mathrm{median}(X)=\mu,\ \mathrm{Var}(X)=2\sigma^2$

9 **Logistic**,(μ,β) R:$x$logis

$$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}=\frac{1}{2\beta\left[1+\cosh\left(\frac{x-\alpha}{\beta}\right)\right]}$$

$F(x)=\dfrac{1}{1+e^{-(x-\mu)/\beta}}$

$M(t)=e^{\mu t}\Gamma(1-\beta t)\Gamma(1+\beta t),\ |t|<\frac{1}{\beta}$

$E(X)=\mathrm{median}(X)=\mu,\ \mathrm{Var}(X)=\frac{\pi^2\beta^2}{3}$

10 **Lognormal** (μ,σ^2) R:$x$lnorm

$f(x)=\dfrac{1}{\sigma\sqrt{2\pi}x}\exp(-\frac12(\frac{\log(x)-\mu}{\sigma})^2)$

$E(X^k)=e^{k\mu+k^2\sigma^2/2},\mathrm{Var}(X)=e^{2(\mu+\sigma^2)}-e^{2\mu+\sigma^2}$

NOTE: $F_{logN(\mu,\sigma^2)}(x)=F_{N(\mu,\sigma^2)}(\log x)=\Phi(\frac{\log x-\mu}{\sigma})$
Self-reproducing under multiplication and division

11 **Normal** (μ,σ^2) R:$x$norm

$f(x)=\dfrac{1}{\sigma\sqrt{2\pi}}\exp(-\frac12(\frac{x-\mu}{\sigma})^2)$

$M(t)=\exp(\mu t+\frac12\sigma^2t^2)$

NOTE: If $X\sim N(\mu,\sigma^2)$, $Y=e^X\sim logN(\mu,\sigma^2)$

$$\begin{aligned}F_{N(0,1)}(x)&=\tfrac12+\tfrac12\mathrm{sign}(x)F_{\Gamma(1/2,1)}(\tfrac12x^2)\\&=\tfrac12+\tfrac12\mathrm{sign}(x)F_{\chi^2(1)}(x)\end{aligned}$$

$\phi'(z)=-z\phi(z),\ \phi''(z)=(z^2-1)\phi(z)$
$E(X^3)=\mu^3+3\mu\sigma^2,\ E(X^4)=\mu^4+6\mu^2\sigma^2+3\sigma^4$

12 **Rayleigh** (β)

$f(x)=\dfrac{x}{\beta^2}\exp\left(-\dfrac{x^2}{2\beta^2}\right)$

$F(x)=1-\exp\left(-\dfrac{x^2}{2\beta^2}\right)$

$E(X)=\beta\sqrt{\pi/2},\ E(X^k)=(\sqrt{2}\beta)^k\frac{k}{2}\Gamma(\frac{k}{2})$

$\mathrm{median}(X)=\beta\sqrt{2\ln 2}$

$\mathrm{Var}(X)=(2-\pi/2)\beta^2$

NOTE: $\mathrm{Rayleigh}(\beta)=\mathrm{Weibull}(2,2\beta^2)$
$cX\sim\mathrm{Rayleigh}(c\beta)$
$(X/\beta)^2\sim\chi^2(2)$
$X^2\sim\mathrm{Exponential}(2\beta^2)=\mathrm{Gamma}(1,2\beta^2)$
$\min_{1\le i\le n}(X_i)\sim\mathrm{Rayleigh}(\beta/\sqrt{n})$: self-reproducing

13 **Slash** (α,β)

$f(x)=\frac{1}{\sqrt{2\pi(x-\alpha)^2/\beta^2}}(1-\exp(-\frac12(\frac{x-\alpha}{\beta})^2))$

$F(x)=\Phi(\frac{x-\alpha}{\beta})-(\frac{x-\alpha}{\beta})f_{\mathrm{Slash}(0,1)}(\frac{x-\alpha}{\beta})$

NOTE: $X=\alpha+\beta\dfrac{Z}{U}$,
where $Z\sim N(0,1)$ and $U\sim\mathrm{Uniform}(0,1)$.

14 **Student** $t(k)$ R:xt

$f(x)=\dfrac{\Gamma((k+1)/2)}{\Gamma(k/2)}\dfrac{1}{\sqrt{k\pi}}\left(1+\dfrac{x^2}{k}\right)^{-(k+1)/2},\ (k\ge1)$

$E(X)=0\ (k>1),\ \mathrm{Var}(X)=k/(k-2)\ (k>2)$

NOTE: $X\sim\dfrac{N(0,1)}{\sqrt{\chi^2_k/k}},\qquad F_\alpha(1,k)=t_{\alpha/2}(k)^2$

$$\begin{aligned}F_{t(k)}(x)&=1-\tfrac12F_{\mathrm{Beta}(k/2,1/2)}(\tfrac{k}{k+x^2}) &&(x\ge0)\\&=\tfrac12F_{\mathrm{Beta}(k/2,1/2)}(\tfrac{k}{k+x^2})&&(x<0)\end{aligned}$$

15 **Uniform** (a,b) R:$x$unif

$f(x)=\dfrac{1}{b-a}$

$M(t)=\dfrac{e^{tb}-e^{ta}}{t(b-a)},\ (t\ne0)$

$E(X)=\mathrm{median}(X)=\dfrac{a+b}{2},\ \mathrm{Var}(X)=\dfrac{(b-a)^2}{12}$

NOTE: $X\sim\mathrm{Uniform}(0,1),\ -\log X\sim\mathrm{Exponential}(1)$

16 **Wald** (μ,λ): Inverse Gaussian (IG) R:$x$invgauss{statmod}

$f(x;\mu,\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}\exp\left[-\lambda\frac{(x-\mu)^2}{2\mu^2x}\right]=\sqrt{\frac{\lambda}{x^3}}\phi\left(\sqrt{\frac{\lambda}{x}}\frac{x-\mu}{\mu}\right)$

$E(X)=\mu,\ \mathrm{Var}(X)=\mu^3/\lambda$

$F(x)=\Phi\left(\sqrt{\frac{\lambda}{x}}\frac{x-\mu}{\mu}\right)+\exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{x}}\frac{x+\mu}{\mu}\right)$

$M(t)=\exp\left[\frac{\lambda}{\mu}(1-\sqrt{1-2\mu^2t/\lambda})\right]$

$\hat\mu_{\mathrm{mle}}=\bar X,\ \hat\lambda_{\mathrm{mle}}=\left[\frac1n\sum\{X_i^{-1}-\bar X^{-1}\}\right]^{-1}.$

NOTE: $\lambda(X-\mu)^2/(\mu^2X)\sim\chi^2(1)$
$X_i\sim\mathrm{IG}(\mu,\lambda)\Rightarrow kX_i\sim\mathrm{IG}(k\mu,k\lambda)$,
$\sum X_i\sim\mathrm{IG}(n\mu,n^2\lambda),\ n\lambda/\hat\lambda_{\mathrm{mle}}\sim\chi^2(n-1)$

17 **Weibull** (α,θ) R:$x$weibull

$f(x)=\dfrac{\alpha}{\theta}\left(\dfrac{x}{\theta}\right)^{\alpha-1}\exp\left(-(x/\theta)^\alpha\right),\ (x\ge0,\alpha>0,\theta>0)$

$F(x)=1-\exp\left(-(x/\theta)^\alpha\right)$

$E(X^k)=\theta^k\Gamma(1+\frac{k}{\alpha}),\quad \mathrm{median}(X)=\theta\,(\ln2)^{1/\alpha}$

$\mathrm{Var}(X)=\theta^2\left[\Gamma(1+\frac{2}{\alpha})-\Gamma^2(1+\frac{1}{\alpha})\right]$

NOTE: $\theta=1-e^{-1}\approx63.2\%$ percentile
$Y=X^\alpha\sim\mathrm{Exponential}(\theta^\alpha)$
$\mathrm{Weibull}(1,\theta)=\mathrm{Exponential}(\theta)$
$\mathrm{Weibull}(2,\sqrt{2}\theta)=\mathrm{Rayleigh}(\theta)$
$\min_{1\le i\le n}(X_i)\sim\mathrm{Weibull}(\alpha,\theta/n^{1/\alpha})$: self-reproducing
$Y=1/X\sim\mathrm{Inv.\,Wei}(\alpha,1/\theta),\ F_Y(y)=\exp\left(-(\frac{1/\theta}{y})^\alpha\right)$