Probability Distributions

1 **Binomial** (n,p), $\quad (x=0,1,\ldots,n)$ R:$x$binom

$f(x)=\binom{n}{x}p^x(1-p)^{n-x}$
$M(t)=(pe^t+1-p)^n$
$E(X)=np$, $\mathrm{Var}(X)=np(1-p)$

NOTE: $n\to\infty$ with $\mu=np \Rightarrow \mathrm{Poisson}(\mu)$
$F_{\mathrm{Bin}(n,p)}(x)=F_{\mathrm{Beta}(n-\lfloor x\rfloor,\lfloor x\rfloor+1)}(1-p)$
$F_{\mathrm{Bin}(n,p)}(r-1)=1-F_{\mathrm{NegBin}(r,p)}(n-r)$
$X_i\sim \mathrm{Bin}(n_i,p)\Rightarrow \sum X_i\sim \mathrm{Bin}(\sum n_i,p)$

2 **Geometric** (p) R:$x$geom

$f(x)=p(1-p)^x$, $\quad (x=0,1,\ldots)$
$F(t)=1-(1-p)^{\lfloor t\rfloor+1}$, where $\lfloor t\rfloor=\max\{m\in\mathbb{Z}: m\le t\}$
$S(t)=P[X\ge t]=(1-p)^{\lceil t\rceil}$ with $\lceil t\rceil=\min\{m\in\mathbb{Z}: m\ge t\}$
$R(t)=P[X>t]=(1-p)^{\lfloor t\rfloor+1}$
$M_X(t)=\dfrac{p}{1-(1-p)e^t}$, $\quad (t<-\log(1-p))$
$E(X)=\dfrac{1-p}{p}$, $\mathrm{Var}(X)=\dfrac{1-p}{p^2}$

$f(y)=p(1-p)^{y-a}$, $\quad (y=a,a+1,a+2\ldots)$
$F(t)=1-(1-p)^{\lfloor t\rfloor+1-a}$
$S(t)=P[Y\ge t]=(1-p)^{\lceil t\rceil-a}$
$R(t)=P[Y>t]=(1-p)^{\lfloor t\rfloor+1-a}$
$M_Y(t)=\dfrac{pe^{at}}{1-(1-p)e^t}$, $(t<-\log(1-p))$
$E(Y)=\dfrac{1-p}{p}+a$, $\mathrm{Var}(Y)=\dfrac{1-p}{p^2}$

NOTE: $\min_{1\le i\le n}(X_i)\sim \mathrm{Geo}\left(1-(1-p)^n\right)$: self-reproducing

3 **Hypergeometric** (N,M,n) R:$x$hyper

$f(x)=\dfrac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$,
$(\max(0,M-(N-n))\le x\le \min(n,M))$
$E(X)=n\left(\dfrac{M}{N}\right)$, $\mathrm{Var}(X)=n\dfrac{M}{N}\dfrac{(N-M)(N-n)}{N(N-1)}$

4 **Negative Binomial** (r,p) R:$x$nbinom

$f(x)=\binom{r+x-1}{x}p^r(1-p)^x$, $(x=0,1,\ldots)$
$M_X(t)=\left(\dfrac{p}{1-(1-p)e^t}\right)^r$
$E(X)=r\frac{1-p}{p}$, $\mathrm{Var}(X)=r\frac{1-p}{p^2}$

$f(y)=\binom{y-1}{r-1}p^r(1-p)^{y-r}$, $(y=r,r+1,\ldots)$
$M_Y(t)=\left(\dfrac{pe^t}{1-(1-p)e^t}\right)^r$
$E(Y)=r\frac{1}{p}$, $\mathrm{Var}(Y)=r\frac{1-p}{p^2}$

NOTE: $X=V_1+\cdots+V_r$, $(V_i\sim \mathrm{Geometric}(p))$
$(1-x)^{-n}=\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k \quad (|x|<1)$
$\mu=r(1-p), r\to\infty\Rightarrow \mathrm{Poisson}(\mu)$
$X_i\sim \mathrm{NB}(r_i,p)\Rightarrow \sum X_i\sim \mathrm{NB}(\sum r_i,p)$
$F_{\mathrm{NB}(r,p)}(x)=F_{\mathrm{Beta}(r,\lfloor x\rfloor+1)}(p)$
$F_{\mathrm{NB}(r,p)}(n-r)=1-F_{\mathrm{Bin}(n,p)}(r-1)$

5 **Poisson** (μ) R:$x$pois

$f(x)=\dfrac{e^{-\mu}\mu^x}{x!}$, $\quad (x=0,1,\ldots)$
$M(t)=\exp\left(\mu(e^t-1)\right)$
$E(X)=\mu$, $\mathrm{Var}(X)=\mu$

NOTE: $X_i\sim \mathrm{Poi}(\mu_i)\Rightarrow \sum X_i\sim \mathrm{Poi}(\sum\mu_i)$

$$P_{\mathrm{Poi}(\mu)}[X\ge n]=F_{\Gamma(n,1)}(\mu)=F_{\Gamma(n,\mu)}(1)=F_{\chi^2_{2n}}(2\mu)$$

1 **Beta** (α,β), $\quad (\alpha>0,\beta>0)$ R:$x$beta

$f(x)=\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $\;(0<x<1)$
$M(t)=1+\sum_{k=1}^{\infty}\left(\prod_{r=1}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$
$E(X)=\dfrac{\alpha}{\alpha+\beta}$, $\;\mathrm{Var}(X)=\dfrac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$

NOTE: $f_{\mathrm{Beta}(\alpha,\beta)}(x)=f_{\mathrm{Beta}(\beta,\alpha)}(1-x)$
$F_{\mathrm{Beta}(\alpha,\beta)}(x)=1-F_{\mathrm{Beta}(\beta,\alpha)}(1-x)$

2 **BS** (α,β), $(\alpha>0,\beta>0)$: Birnbaum-Saunders

$f(t)=\frac{1}{2\alpha\beta}\sqrt{\frac{\beta}{t}}\left(1+\frac{\beta}{t}\right)\phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)\right]$, $\;(t>0)$
$F(t)=\Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}}-\sqrt{\frac{\beta}{t}}\right)\right]$

$$F^{-1}(p)=\frac{1}{4}\left[\alpha\sqrt{\beta}\Phi^{-1}(p)+\sqrt{\alpha^2\beta\{\Phi^{-1}(p)\}^2+4\beta}\right]^2 =\beta\left\{1+\gamma(p)^2+\gamma(p)\sqrt{\gamma(p)^2+2}\right\},$$
where $\gamma(p)=\alpha\Phi^{-1}(p)/\sqrt{2}$

$E(T)=\beta(1+\frac{1}{2}\alpha^2)$, $\quad \mathrm{Var}(T)=(\alpha\beta)^2(1+\frac{5}{4}\alpha^2)$

NOTE: $\mathrm{median}(T)=\beta$, $cT\sim \mathrm{BS}(\alpha,c\beta)$, $T^{-1}\sim \mathrm{BS}(\alpha,\beta^{-1})$
$X=\left(\sqrt{\frac{T}{\beta}}-\sqrt{\frac{\beta}{T}}\right)\sim N(0,\alpha^2)$
$\log T\sim \text{sinh-Normal}(\log\beta,\alpha)$

3 **Cauchy** (α,β) R:$x$cauchy

$f(x)=\dfrac{\beta}{\pi}\dfrac{1}{\beta^2+(x-\alpha)^2}$
$F(x)=\dfrac{1}{\pi}\left[\arctan\left(\dfrac{x-\alpha}{\beta}\right)+\dfrac{\pi}{2}\right]$
$\phi(t)=\exp(it\alpha-\beta|t|)$

NOTE: $\mathrm{Cauchy}(0,1)=t(1)$
$X,Y\sim N(0,1)\Rightarrow X/Y\sim \mathrm{Cauchy}(0,1)$
$cX\sim \mathrm{Cauchy}(c\alpha,c\beta)$
$X_i\sim \mathrm{Cauchy}(\alpha,\beta)\Rightarrow \frac{1}{n}\sum X_i\sim \mathrm{Cauchy}(\alpha,\beta)$

4 **Chi-Square** (n) R:$x$chisq

$f(x)=\dfrac{1}{\Gamma(n/2)2^{n/2}}x^{n/2-1}e^{-x/2}$, $(x\ge 0)$
$E(X)=n$, $\mathrm{Var}(X)=2n$
$M(t)=\left(\frac{1}{1-2t}\right)^{n/2}$, $\quad (t<\frac{1}{2})$
$E(X^m)=2^m\Gamma(m+n/2)/\Gamma(n/2)$

NOTE: $\chi^2(n)=\mathrm{Gamma}(n/2,2)$
$\chi^2(2)=\mathrm{Exponential}(\beta=2)$
$X_i\sim\chi^2(n_i)\Rightarrow \sum X_i\sim\chi^2(\sum n_i)$
$X_i\sim N(0,1)\Rightarrow \sum_{i=1}^n X_i^2\sim\chi^2(n)$
$F_{\chi^2(p+2)}(x)=F_{\chi^2(p)}(x)-2xf_{\chi^2(p)}(x)/p$
$F_{\chi^2(1)}(x)=2\Phi(\sqrt{x})-1$
$F_{\chi^2(2)}(x)=1-\sqrt{2\pi}\phi(\sqrt{x})$
$F_{\chi^2(3)}(x)=2\Phi(\sqrt{x})-1-2\sqrt{x}\phi(\sqrt{x})$

5 **Inverse-Chi-Square** (n)

$f(x)=\dfrac{1}{\Gamma(n/2)2^{n/2}}x^{-n/2-1}e^{-1/(2x)}$, $(x\ge 0)$
$E(X)=1/(n-2)$ for $n>2$
$\mathrm{Var}(X)=2/[(n-2)^2(n-4)]$ for $n>4$

NOTE: If $X\sim\chi^2(n)$ and $Y=1/X$, then $Y\sim \text{Inverse-}\chi^2(n)$.
$F_{\text{Inv-}\chi^2(n)}(x)=1-F_{\chi^2(n)}(1/x)$

6 **Exponential** (θ) R:$x$exp

$f(x)=\dfrac{1}{\theta}e^{-x/\theta}$ $(x\ge 0,\ \theta>0)$
$F(x)=1-e^{-x/\theta}$
$M(t)=\dfrac{1}{1-\theta t}$, $(t<\frac{1}{\theta})$
$E(X)=\theta$, $\mathrm{Var}(X)=\theta^2$

NOTE: Memoryless property
$cX \sim \text{Exponential}(c\theta)$
$\sum X_i \sim \text{Gamma}(n, \theta)$
$\min_{1 \le i \le n} (X_i) \sim \text{Exponential}(\theta/n)$: self-reproducing
$Y = X^{1/\alpha} \sim \text{Weibull}(\alpha, \theta)$
$Y = \sqrt{2X/\theta} \sim \text{Rayleigh}(1)$
$Y = \alpha - \gamma \log(X/\theta) \sim \text{Gumbel}(\alpha, \gamma)$

7 **F** (m, n) R:xf

$$F(m,n) = \frac{\chi^2_m/m}{\chi^2_n/n}$$
$$E(X) = \frac{n}{n-2} \quad (n > 2),$$
$$\text{Var}(X) = 2\left(\frac{n}{n-2}\right)^2 \frac{m+n-2}{m(n-4)} \quad (n > 4)$$

NOTE: $[F(m,n)]^{-1} = F(n,m)$
$F_{1-\alpha}(m,n) = [F_\alpha(n,m)]^{-1}$
$F(1,k) = t^2(k)$
If $X \sim F(m,n)$, $mX/(mX+n) \sim \text{Beta}(\frac{m}{2}, \frac{n}{2})$.

8 **Gamma** (α, θ) R:$x$gamma

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} \exp(-x/\theta), \ (0 < x < \infty)$$
$$M(t) = (1 - \theta t)^{-\alpha} \ (t < 1/\theta)$$
$$E(X) = \alpha\theta, \ \text{Var}(X) = \alpha\theta^2$$

NOTE: $X \sim \text{Gamma}(n, \theta) = \text{Erlang}(n, \theta)$
$X = V_1 + \cdots + V_n$, $(V_i \sim \text{Exponential}(\theta))$
$2X/\theta \sim \chi^2(2n)$
$$F_X(x) = 1 - \sum_{k=0}^{n-1} \left(\frac{x}{\theta}\right)^k \frac{e^{-x/\theta}}{k!} = \sum_{k=n}^{\infty} \left(\frac{x}{\theta}\right)^k \frac{e^{-x/\theta}}{k!}$$
$\text{Gamma}(n/2, 2) = \chi^2(n)$, $\Gamma(1/2) = \sqrt{\pi}$
$X_i \sim \text{Gamma}(\alpha_i, \theta) \Rightarrow \sum_{i=1}^r X_i \sim \text{Gamma}(\sum_{i=1}^r \alpha_i, \theta)$
$X/d \sim \text{Gamma}(\alpha, \theta/d)$
$E[X^c] = \Gamma(\alpha + c)\theta^c/\Gamma(\alpha) \ (c > -\alpha)$

9 **Laplace** (μ, σ): Double Exponential

$$f(x) = \tfrac{1}{2\sigma} e^{-|x-\mu|/\sigma}$$
$$F(x) = \begin{cases} \frac{1}{2} e^{-|x-\mu|/\sigma} & (x < \mu) \\ 1 - \frac{1}{2} e^{-|x-\mu|/\sigma} & (x \ge \mu) \end{cases}$$
$$M(t) = \frac{e^{\mu t}}{1 - (\sigma t)^2}, \ (|t| < \tfrac{1}{\sigma})$$
$$E(X) = \text{median}(X) = \mu, \ \text{Var}(X) = 2\sigma^2$$

10 **Logistic**,(μ, β) R:$x$logis

$$f(x) = \frac{e^{-(x-\mu)/\beta}}{\beta[1 + e^{-(x-\mu)/\beta}]^2} = \frac{1}{2\beta\left[1 + \cosh\left(\frac{x-\mu}{\beta}\right)\right]}$$
$$F(x) = \frac{1}{1 + e^{-(x-\mu)/\beta}}$$
$$M(t) = e^{\mu t}\Gamma(1 - \beta t)\Gamma(1 + \beta t), \ |t| < \tfrac{1}{\beta}$$
$$E(X) = \text{median}(X) = \mu, \text{Var}(X) = \tfrac{\pi^2\beta^2}{3}$$

11 **Lognormal** (μ, σ^2) R:$x$lnorm

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}x} \exp(-\tfrac{1}{2}(\tfrac{\log(x)-\mu}{\sigma})^2)$$
$$E(X^k) = e^{k\mu + k^2\sigma^2/2}, \text{Var}(X) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$$

NOTE: $F_{logN(\mu,\sigma^2)}(x) = F_{N(\mu,\sigma^2)}(\log x) = \Phi(\frac{\log x - \mu}{\sigma})$
Self-reproducing under multiplication and division

12 **Normal** (μ, σ^2) R:$x$norm

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\tfrac{1}{2}(\tfrac{x-\mu}{\sigma})^2)$$
$$M(t) = \exp(\mu t + \tfrac{1}{2}\sigma^2 t^2)$$

NOTE: If $X \sim N(\mu, \sigma^2)$, $Y = e^X \sim logN(\mu, \sigma^2)$
$$F_{N(0,1)}(x) = \tfrac{1}{2} + \tfrac{1}{2}\text{sign}(x) F_{\Gamma(1/2,1)}(\tfrac{1}{2}x^2)$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\text{sign}(x) F_{\chi^2(1)}(x)$$
$$\phi'(z) = -z\phi(z), \ \phi''(z) = (z^2 - 1)\phi(z)$$
$$E(X^3) = \mu^3 + 3\mu\sigma^2, \ E(X^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$$

13 **Rayleigh** (β)

$$f(x) = \frac{x}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right)$$
$$F(x) = 1 - \exp\left(-\frac{x^2}{2\beta^2}\right)$$
$$E(X) = \beta\sqrt{\pi/2}, \ E(X^k) = (\sqrt{2}\beta)^k \tfrac{k}{2}\Gamma(\tfrac{k}{2})$$
$$\text{median}(X) = \beta\sqrt{2\ln 2}$$
$$\text{Var}(X) = (2 - \pi/2)\beta^2$$

NOTE: $\text{Rayleigh}(\beta) = \text{Weibull}(2, 2\beta^2)$
$cX \sim \text{Rayleigh}(c\beta)$
$(X/\beta)^2 \sim \chi^2(2)$
$X^2 \sim \text{Exponential}(2\beta^2) = \text{Gamma}(1, 2\beta^2)$
$\min_{1 \le i \le n} (X_i) \sim \text{Rayleigh}(\beta/\sqrt{n})$: self-reproducing

14 **Slash** (α, β)

$$f(x) = \frac{1}{\sqrt{2\pi}(x-\alpha)^2/\beta^2}\left(1 - \exp(-\tfrac{1}{2}(\tfrac{x-\alpha}{\beta})^2)\right)$$
$$F(x) = \Phi\left(\tfrac{x-\alpha}{\beta}\right) - \left(\tfrac{x-\alpha}{\beta}\right) f_{\text{Slash}(0,1)}\left(\tfrac{x-\alpha}{\beta}\right)$$

NOTE: $X = \alpha + \beta\dfrac{Z}{U}$,
where $Z \sim N(0,1)$ and $U \sim \text{Uniform}(0,1)$.

15 **Student** $t(k)$ R:xt

$$f(x) = \frac{\Gamma((k+1)/2)}{\Gamma(k/2)} \frac{1}{\sqrt{k\pi}} \left(1 + \frac{x^2}{k}\right)^{-(k+1)/2}, \ (k \ge 1)$$
$$E(X) = 0 \ (k > 1), \ \text{Var}(X) = k/(k-2) \ (k > 2)$$

NOTE: $X \sim \dfrac{N(0,1)}{\sqrt{\chi^2_k/k}}$, $\quad F_\alpha(1,k) = t_{\alpha/2}(k)^2$
$$F_{t(k)}(x) = 1 - \tfrac{1}{2} F_{\text{Beta}(k/2,1/2)}(\tfrac{k}{k+x^2}) \qquad (x \ge 0)$$
$$= \tfrac{1}{2} F_{\text{Beta}(k/2,1/2)}(\tfrac{k}{k+x^2}) \qquad (x < 0)$$

16 **Uniform** (a, b) R:$x$unif

$$f(x) = \frac{1}{b-a}$$
$$M(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}, \ (t \ne 0)$$
$$E(X) = \text{median}(X) = \frac{a+b}{2}, \ \text{Var}(X) = \frac{(b-a)^2}{12}$$

NOTE: $X \sim \text{Uniform}(0,1)$, $-\log X \sim \text{Exponential}(1)$

17 **Wald** (μ, λ): Inverse Gaussian (IG) R:$x$invgauss{statmod}

$$f(x; \mu, \lambda) = \sqrt{\tfrac{\lambda}{2\pi x^3}} \exp\left[-\lambda\tfrac{(x-\mu)^2}{2\mu^2 x}\right] = \sqrt{\tfrac{\lambda}{x^3}}\phi\left(\sqrt{\tfrac{\lambda}{x}}\tfrac{x-\mu}{\mu}\right)$$
$$E(X) = \mu, \ \text{Var}(X) = \mu^3/\lambda$$
$$F(x) = \Phi\left(\sqrt{\tfrac{\lambda}{x}}\tfrac{x-\mu}{\mu}\right) + \exp\left(\tfrac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\tfrac{\lambda}{x}}\tfrac{x+\mu}{\mu}\right)$$
$$M(t) = \exp\left[\tfrac{\lambda}{\mu}\left(1 - \sqrt{1 - 2\mu^2 t/\lambda}\right)\right]$$
$$\hat{\mu}_{\text{mle}} = \bar{X}, \ \hat{\lambda}_{\text{mle}} = \left[\tfrac{1}{n}\sum\{X_i^{-1} - \bar{X}^{-1}\}\right]^{-1}.$$

NOTE: $\lambda(X - \mu)^2/(\mu^2 X) \sim \chi^2(1)$
$X_i \sim \text{IG}(\mu, \lambda) \Rightarrow kX_i \sim \text{IG}(k\mu, k\lambda)$,
$\sum X_i \sim \text{IG}(n\mu, n^2\lambda)$, $n\lambda/\hat{\lambda}_{\text{mle}} \sim \chi^2(n-1)$

18 **Weibull** (α, θ) R:$x$weibull

$$f(x) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1} \exp\left(-(x/\theta)^\alpha\right), \ (x \ge 0, \alpha > 0, \theta > 0)$$
$$F(x) = 1 - \exp\left(-(x/\theta)^\alpha\right)$$
$$E(X^k) = \theta^k\Gamma(1 + \tfrac{k}{\alpha}), \quad \text{median}(X) = \theta\,(\ln 2)^{1/\alpha}$$
$$\text{Var}(X) = \theta^2\left[\Gamma(1 + \tfrac{2}{\alpha}) - \Gamma^2(1 + \tfrac{1}{\alpha})\right]$$

NOTE: $\theta = 1 - e^{-1} \approx 63.2\%$ percentile
$Y = X^\alpha \sim \text{Exponential}(\theta^\alpha)$
$\text{Weibull}(1, \theta) = \text{Exponential}(\theta)$
$\text{Weibull}(2, \sqrt{2}\theta) = \text{Rayleigh}(\theta)$
$\min_{1 \le i \le n} (X_i) \sim \text{Weibull}(\alpha, \theta/n^{1/\alpha})$: self-reproducing
$Y = 1/X \sim \text{Inv. Wei}(\alpha, 1/\theta)$, $F_Y(y) = \exp\left(-(\frac{1/\theta}{y})^\alpha\right)$