Correct derivations of formulas from lemma 4

1 Main errors in proof of Lemma 4 (Original in [1])

The main mistake in the proof of the formulas was the use of commutation of some matrices, however, it has been checked, that it isn't correct in general. In this section we will show you these places. In section 2 you can see the correct proof of this lemma.
So, here are incorrect places in lemma 4 (page $14-16$):

1. In the original paper we see (formula (10))
$$\hat{\mathcal{A}}^*\hat{\mathcal{A}} = \hat{\mathcal{A}_0}^*\hat{\mathcal{A}}_0 + aa^* = (\hat{\mathcal{A}_0}^*\hat{\mathcal{A}}_0)(I + (\hat{\mathcal{A}_0}^*\hat{\mathcal{A}}_0)^{-1}aa^*),$$
then for the inverse
$$(\hat{\mathcal{A}}^*\hat{\mathcal{A}})^{-1} = (\hat{\mathcal{A}_0}^*\hat{\mathcal{A}}_0)^{-1}(I + (\hat{\mathcal{A}_0}^*\hat{\mathcal{A}}_0)^{-1}aa^*)^{-1}$$
It isn't correct, because in general for matrices X and Y
$$(XY)^{-1} = Y^{-1}X^{-1} \neq X^{-1}Y^{-1}$$
2. Also, we can see this formula on page 14
$$(\hat{\mathcal{A}_0}^*\hat{\mathcal{A}}_0)^{-1}aa^* = aa^*(\hat{\mathcal{A}_0}^*\hat{\mathcal{A}}_0)^{-1}$$
We have checked it and it is also not correct.

2 Correct derivation of formulas for maxvol2 algorithm

In this section you can see correct proof of lemma 4. Despite the incorrect steps, all the final formulas turned out to be correct.

Lemma 2.1. *Let $\hat{\mathcal{A}}_0 \in \mathbb{C}^{n\times r}$ be a submatrix in the first n rows of the matrix $\mathcal{A} \in \mathbb{C}^{M\times r}$. Then adding i-th row of $\mathcal{A}$ to the submatrix $\hat{\mathcal{A}}_0$ changes the squared volume of $\hat{\mathcal{A}}_0$ as:*
$$\mathcal{V}_2(\hat{\mathcal{A}})^2/\mathcal{V}_2(\hat{\mathcal{A}}_0)^2 = 1 + (l_0)_i = 1 + ||(C_0)_{i,:}||_2^2,$$
where
$$C_0 = \mathcal{A}\hat{\mathcal{A}_0}^+ \in \mathbb{C}^{M\times n}$$

Proof. We deal with the matrix $\mathcal{A}$. It's submatrix $\hat{\mathcal{A}} \in \mathbb{C}^{(n+1)\times r}$ expands the submatrix $\hat{\mathcal{A}}_0 \in \mathbb{C}^{n\times r}$ by appending a row $a^* = \mathcal{A}_{i,:}$.

$$\hat{\mathcal{A}} = \begin{bmatrix} \hat{\mathcal{A}}_0 \\ a^* \end{bmatrix}$$

In order to add a row maximizing the volume, we should be able to update the matrix $C_0 \in \mathbb{C}^{M\times n}$ and the squared 2-norms of its rows. They are stored in a vector $l_0 \in \mathbb{C}^M$. For this we define a matrix $C \in \mathbb{C}^{M\times n}$ and a column $C' \in \mathbb{C}^M$ as follows:

$$\mathcal{A}\hat{\mathcal{A}}^+ = \begin{bmatrix} C & C' \end{bmatrix}, \quad \hat{\mathcal{A}} = \begin{bmatrix} \hat{\mathcal{A}}_0 \\ a^* \end{bmatrix}$$

They can also be expressed as

$$C = \mathcal{A}(\hat{\mathcal{A}}^*\hat{\mathcal{A}})^{-1}\hat{\mathcal{A}}_0{}^*, \quad C' = \mathcal{A}(\hat{\mathcal{A}}^*\hat{\mathcal{A}})^{-1}a \qquad (9)$$

Our task is to calculate C and C' on the basis of C_0. The expression (9) shows that it is sufficient to find $(\hat{\mathcal{A}}^*\hat{\mathcal{A}})^{-1}$. Since

$$\hat{\mathcal{A}}^*\hat{\mathcal{A}} = \hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0 + aa^* = (\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)(I + (\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}aa^*),$$

then for the inverse

$$(\hat{\mathcal{A}}^*\hat{\mathcal{A}})^{-1} = (I + (\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}aa^*)^{-1}(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1} = (I - \frac{(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}aa^*}{1 + a^*(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}a})(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1} \qquad (10)$$

The expression in the brackets can be simplified by introducing the notation $c^* = C_{0i,:}$. Indeed,

$$c^* = C_{0i,:} = \mathcal{A}_{i,:}\hat{\mathcal{A}}_0{}^+ = a^*(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}\hat{\mathcal{A}}_0{}^*,$$

$$c^*c = a^*(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}a = a^*(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}a$$

Substituting into (10) gives

$$(\hat{\mathcal{A}}^*\hat{\mathcal{A}})^{-1} = (I - \frac{(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}aa^*}{1 + c^*c})(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}$$

Taking into account

$$C_0c = \mathcal{A}(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}a = \mathcal{A}(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}a,$$

one can calculate C' from (9):

$$C' = \mathcal{A}(I - \frac{(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}aa^*}{1 + c^*c})(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}a = (\mathcal{A} - \frac{\mathcal{A}(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}aa^*}{1 + c^*c})(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}a = (\mathcal{A} - \frac{(C_0c)a^*}{1 + c^*c})(\hat{\mathcal{A}}_0{}^*\hat{\mathcal{A}}_0)^{-1}a =$$

$$= \mathcal{A}({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}a - \frac{(C_0c)a^*({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}a}{1+c^*c} = (C_0c)(1-\frac{c^*c}{1+c^*c}) = \frac{C_0c}{1+c^*c} = \frac{C_0C^*_{0i,:}}{1+l_i}$$

We compute C similarly:

$$C = \mathcal{A}(I - \frac{({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}aa^*}{1+c^*c})({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}{\hat{\mathcal{A}}_0}^* = (\mathcal{A} - \frac{\mathcal{A}({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}aa^*}{1+c^*c})({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}{\hat{\mathcal{A}}_0}^* =$$

$$= (\mathcal{A} - \frac{(C_0c)a^*}{1+c^*c})({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}{\hat{\mathcal{A}}_0}^* = \mathcal{A}({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}{\hat{\mathcal{A}}_0}^* - \frac{(C_0c)a^*({\hat{\mathcal{A}}_0}^*\hat{\mathcal{A}}_0)^{-1}{\hat{\mathcal{A}}_0}^*}{1+c^*c} = C_0 - C'c^* = C_0 - C'C_{0i,:}$$

Now we can directly calculate the lengths of the new rows. The updated l_0 is denoted by l.

$$l_j = C_{j,:}C^*_{j,:} + C'_jC'^*_j = (C_{0j,:} - C'_jC_{0i,:})(C_{0j,:} - C'_jC_{0i,:})^* + |C'_j|^2 = l_{0j} - 2C'^*_jC_{0j,:}C^*_{0i,:} + |C'_j|^2C_{0i,:}C^*_{0i,:} + |C'_j|^2 =$$

$$= l_{0j} - C'^*_jC'_j(1+l_{0i}) + |C'_j|^2l_{0i} + |C'_j|^2 = l_{0j} - |C'_j|^2(1+l_{0i})$$

$\square$

Corollary 2.1. *Let $\hat{\mathcal{A}}_0 \in \mathbb{C}^{n\times r}$ be a submatrix in the first n rows of the matrix $\mathcal{A} \in \mathbb{C}^{M\times r}$. Then adding i-th row of $\mathcal{A}$ to the submatrix $\hat{\mathcal{A}}_0$ changes matrix C_{new} from C_0 and vector of it's lengths l from l_0 as:*

$$C_{new} = \mathcal{A}\hat{\mathcal{A}}^+ = \begin{bmatrix} C & C' \end{bmatrix}, \quad l_j = l_{0j} - |C'_j|^2(1+l_{0i})$$

Where

$$C' = \frac{C_0C^*_{0i,:}}{1+l_i}, \quad C = C_0 - C'C_{0i,:}$$

References

[1] Alexander Osinsky, Rectangular maximum volume and projective volume search algorithms `https://arxiv.org/abs/1809.02334`