Prophet Secretary

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Given that F_1,F_2,...F_n are the initial distribution and V_1,V_2,...V_n are the corresponding sampled values. For any t $\in [0, 1]$, define:
$\theta(t)$ = The probability that nothing is selected till time t.
$q_{-i}(t)$ = The probability that nothing is selected till time t conditioned that V_i arrives at time t.
α : [0,1] $\rightarrow$ [0,1] and β : [0,1] $\rightarrow \mathbb{R}$, both non-increasing such that,

$$Pr(\max_i\{V_1, V_2, ...V_n\} \leq \alpha(t)) = \beta(t)$$

LEMMA 1: $\forall$ i $\in$ [n], t $\in$ [0,1]

$$q_{-i}(t) \geq \frac{\theta(t)}{1 - t + \int_0^t Pr(V_i \leq \beta(x))dx}$$

Proof.

$$\begin{aligned}\theta(t) &= \int_0^1 Pr(T > t \mid t_i = x)dx \\ &= \int_0^t Pr(T > t \mid t_i = x)dx + \int_t^1 Pr(T > t \mid t_i = x)dx\end{aligned}$$

For any x greater than t,

$$Pr(T > t \mid t_i = x) = Pr(T > t \mid t_i > t) \leq q_{-i}(t)$$

For any x less than t and arbitary small ϵ,

$Pr(T > t \mid t_i = x) = Pr(V_i \leq \beta(x))$(Pr(Nothing is chosen before x) + Pr(Nothing is chosen in [x+ϵ,t]))

Since thresholds are non-increasing, for any V_k appearing at $t_k \in$ [x+ϵ,t], Pr($V_k < \beta(t_k)$) $\leq$ Pr($V_k < \beta(t_k - \epsilon)$). Therefore,

$$\begin{aligned}Pr(T > t \mid t_i = x) &\leq Pr(V_i \leq \beta(x))(\text{Pr(Nothing is chosen before x) + Pr(Nothing is chosen in [x,t-}\epsilon\text{]))} \\ &\leq Pr(V_i \leq \beta(x))(q_{-i}(t))\end{aligned}$$

Putting the inequalities and rearranging gives the result. □

LEMMA 2: $\forall$ t $\in$ [0,1/2]

$$\sum_{i=1}^{n} \frac{Pr(V_i > \beta(t))}{1 - \int_0^t Pr(V_i > \beta(x))dx} \geq \frac{Pr(\max_i\{V_i\} > \beta(t))}{1 - tPr(\max_i\{V_i\} > \beta(0))}$$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} \frac{Pr(V_i > \beta(t))}{1-\int_0^t Pr(V_i > \beta(x))dx} &= \sum_{i=1}^{n} \frac{Pr(V_i > \beta(t))}{1-t+\int_0^t Pr(V_i \leq \beta(x))dx} \\
&\geq \sum_{i=1}^{n} \frac{Pr(V_i > \beta(t))}{1-t+tPr(V_i \leq \beta(0))} \\
&= \sum_{i=1}^{n} \frac{1-F_i(\beta(t))}{1-t+tF_i(\beta(0))}
\end{aligned}$$

Using the following inequality (Proof in the paper)

$$\frac{1-F_1(\beta(t))}{1-t+tF_1(\beta(0))} + \frac{1-F_2(\beta(t))}{1-t+tF_2(\beta(0))} \geq \frac{1-F_1(\beta(t))F_2(\beta(t))}{1-t+tF_1(\beta(0))F_2(\beta(0))}$$

and repeating it n times, we get the required result. $\square$

For the competitive ratio of 0.669, we define $\alpha_1, \alpha_2, ...\alpha_m$ and set $\alpha(t) = \alpha_{ceil(tm)}$ We also define a non-decreasing function g, for $t \in \; [0,1]$,

$$g_p(t) = \begin{cases} \frac{1}{1-t(1-p)} & ; t \leq 1/2 \\ \frac{2}{1+p} & ; t > 1/2 \end{cases}$$

Using the result from the last section, if z is the reward, for any $t \in [0,1]$, we can write,

$$Pr(z > \beta(t)) = \frac{1-\theta(t)}{1-\alpha(t)}(1-\alpha(t)) + \sum_{i \in [n]} Pr(V_i > \beta(t)) \int_t^1 q_{-1}(x)dx$$

Using the bound for $\theta(t)$ derived in the last part, the first term can be written as,

$$\frac{1-\theta(t)}{1-\alpha(t)}(1-\alpha(t)) \geq \frac{\int_0^t 1-\alpha(x)dx}{1-\alpha(t)}(Pr(\max_i\{V_i\} > \beta_t)$$

For the second term, first we use Lemma 1 to write,

$$q_{-i}(x) \geq \frac{\theta(x)}{1-\int_0^x Pr(V_i > \beta(y))dy}$$

Then interchanging the order of sums,

$$\sum_{i \in [n]} Pr(V_i > \beta(t)) \int_t^1 q_{-1}(x)dx \geq \int_t^1 \theta(x) \sum_{i \in [n]} \frac{Pr(V_i > \beta(t))}{1-\int_0^x Pr(V_i > \beta(y))dy} dx$$

Now, using Lemma 2, for x $\leq 1/2$

$$\begin{aligned}
\sum_{i \in [n]} \frac{Pr(V_i > \beta(t))}{1-\int_0^x Pr(V_i > \beta(y))dy} &\geq \frac{Pr(\max_i\{V_i\} > \beta(x))}{1-xPr(\max_i\{V_i\} > \beta(0))} \\
&= g_{\alpha(0)}(x) Pr(\max_i\{V_i\} > \beta(x))
\end{aligned}$$

Now, for x > 1/2, it can be observed that

$$\sum_{i\in[n]} \frac{Pr(V_i > \beta(t))}{1-\int_0^x Pr(V_i > \beta(y))dy} \geq \frac{Pr(\max_i\{V_i\} > \beta(x))}{1-\frac{1}{2}Pr(\max_i\{V_i\} > \beta(0))}$$
$$= g_{\alpha(0)}(x)Pr(\max_i\{V_i\} > \beta(x))$$

Finally, we divide x according to the m thresholds, and use the bound for $\theta(t)$ derived in the last section to get the final result.

$$\begin{aligned}
\int_t^1 \theta(x)g_{\alpha(0)}(x)dx &= \sum_{i=tm}^{m} \int_{i/m}^{(i+1)/m} \theta(x)g_{\alpha(0)}(x)dx \\
&\geq \sum_{i=tm}^{m} \int_{i/m}^{(i+1)/m} \theta(i/m)Pr(V_{time=x} < \beta_i)^{x-i/m} g_{\alpha(0)}(i/m)dx \\
&\geq \sum_{i=tm}^{m} g_{\alpha(0)}(i/m)e^{\int_0^{i/m} ln(\alpha(y))dy} \int_{i/m}^{(i+1)/m} \alpha_i^{x-i/m}dx \\
&= \sum_{i=tm}^{m} g_{\alpha(0)}(i/m)(\prod_{k=1}^{i} \alpha_k)^{1/m} \int_{i/m}^{(i+1)/m} \alpha_i^{x-i/m}dx \\
&= \sum_{i=tm}^{m} g_{\alpha(0)}(i/m)(\prod_{k=1}^{i} \alpha_k)^{1/m} \frac{1-\alpha_i^{1/m}}{-ln(\alpha_i)}
\end{aligned}$$

Hence, we proved that:

$$Pr(z > \beta(t)) \geq (\frac{\int_0^t 1-\alpha(x)dx}{1-\alpha(t)} + \sum_{i=tm}^{m} g_{\alpha(0)}(i/m)(\prod_{k=1}^{i} \alpha_k)^{1/m} \frac{1-\alpha_i^{1/m}}{-ln(\alpha_i)})Pr(\max_i\{V_i\} > \beta_t)$$

Where optimization of values for α lead to a competitive ratio of 0.669