Prophet Secretary

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Given that F_1,F_2,...F_n are the initial distribution and V_1,V_2,...V_n are the corresponding sampled values. For any $t \in [0,1]$, define:
$\theta(t)$ = The probability that nothing is selected till time t.
$q_{-i}(t)$ = The probability that nothing is selected till time t conditioned that V_i arrives at time t.
$\alpha : [0,1] \to [0,1]$ and $\beta : [0,1] \to \mathbb{R}$, both non-increasing such that,

$$\Pr[\max_i\{V_1, V_2, ...V_n\} \leq \alpha(t)] = \beta(t)$$

Lemma 0.1. $\forall i \in [n], t \in [0,1]$

$$q_{-i}(t) = \frac{\theta(t)}{1 - t + \int_0^t \Pr[V_i \leq \beta(x)]dx}$$

Proof.

$$\begin{aligned}
\theta(t) &= \int_0^1 \Pr[T > t \mid t_i = x]dx \\
&= \int_0^t \Pr[T > t \mid t_i = x]dx + \int_t^1 \Pr[T > t \mid t_i = x]dx
\end{aligned}$$

For any x greater than t,

$$\Pr[T > t \mid t_i = x] = \Pr[T > t \mid t_i > t] = q_{-i}(t)$$

For any x less than t,

$$\begin{aligned}
\Pr[T > t \mid t_i = x] &= \Pr[V_i \leq \beta(x)](\Pr[\text{Nothing is chosen before } x \mid V_i \text{ does not come before } x] \\
&\qquad + \Pr[\text{Nothing is chosen from } x \text{ to } t \mid V_i \text{ does not come between } x \text{ and } t]) \\
&= \Pr[V_i \leq \beta(x)](\Pr[\text{Nothing is chosen before } t \text{ and } V_i \text{ does not come before } t]) \\
&= \Pr[V_i \leq \beta(x)](q_{-i}(t))
\end{aligned}$$

Putting the inequalities and rearranging gives the result. □

Lemma 0.2. $\forall t \in [0, 1/2]$

$$\sum_{i=1}^{n} \frac{\Pr[V_i > \beta(t)]}{1 - \int_0^t \Pr[V_i > \beta(x)]dx} \geq \frac{\Pr[\max_i\{V_i\} > \beta(t)]}{1 - t\Pr[\max_i\{V_i\} > \beta(0)]}$$

Proof.

$$\sum_{i=1}^{n} \frac{\Pr[V_i > \beta(t)]}{1 - \int_0^t \Pr[V_i > \beta(x)]dx} = \sum_{i=1}^{n} \frac{\Pr[V_i > \beta(t)]}{1 - t + \int_0^t \Pr[V_i \leq \beta(x)]dx}$$
$$\geq \sum_{i=1}^{n} \frac{\Pr[V_i > \beta(t)]}{1 - t + t\Pr[V_i \leq \beta(0)]}$$
$$= \sum_{i=1}^{n} \frac{1 - F_i(\beta(t))}{1 - t + tF_i(\beta(0))}$$

Using the following inequality (Proof in the paper)

$$\frac{1 - F_1(\beta(t))}{1 - t + tF_1(\beta(0))} + \frac{1 - F_2(\beta(t))}{1 - t + tF_2(\beta(0))} \geq \frac{1 - F_1(\beta(t))F_2(\beta(t))}{1 - t + tF_1(\beta(0))F_2(\beta(0))}$$

and repeating it n times, we get the required result. □

For the competitive ratio of 0.669, we define $\alpha_1, \alpha_2, ...\alpha_m$ and set $\alpha(t) = \alpha_{ceil(tm)}$
We also define a non-decreasing function g, for $t \in [0, 1]$,

$$g_p(t) = \begin{cases} \frac{1}{1-t(1-p)} & ; t \leq 1/2 \\ \frac{2}{1+p} & ; t > 1/2 \end{cases}$$

Using the result from the last section, if z is the reward, for any $t \in [0, 1]$, we can write,

$$\Pr[z > \beta(t)] = \frac{1 - \theta(t)}{1 - \alpha(t)}(1 - \alpha(t)) + \sum_{i \in [n]} \Pr[V_i > \beta(t)] \int_t^1 q_{-1}(x)dx$$

Using the bound for $\theta(t)$ derived in the last part, the first term can be written as,

$$\frac{1 - \theta(t)}{1 - \alpha(t)}(1 - \alpha(t)) \geq \frac{\int_0^t 1 - \alpha(x)dx}{1 - \alpha(t)}(\Pr[\max_i\{V_i\} > \beta_t]$$

For the second term, first we use Lemma 0.1 to write,

$$q_{-i}(x) \geq \frac{\theta(x)}{1 - \int_0^x \Pr[V_i > \beta(y)]dy}$$

Then interchanging the order of sums,

$$\sum_{i \in [n]} \Pr[V_i > \beta(t)] \int_t^1 q_{-1}(x)dx \geq \int_t^1 \theta(x) \sum_{i \in [n]} \frac{\Pr[V_i > \beta(t)]}{1 - \int_0^x \Pr[V_i > \beta(y)]dy} dx$$

Now, using Lemma 2, for $x \leq 1/2$

$$\sum_{i \in [n]} \frac{\Pr[V_i > \beta(t)]}{1 - \int_0^x \Pr[V_i > \beta(y)]dy} \geq \frac{\Pr[\max_i\{V_i\} > \beta(x)]}{1 - x\Pr[\max_i\{V_i\} > \beta(0)]}$$
$$= g_{\alpha(0)}(x)\Pr[\max_i\{V_i\} > \beta(x)]$$

Now, for $x > 1/2$, it can be observed that

$$\sum_{i\in[n]} \frac{\Pr[V_i > \beta(t)]}{1 - \int_0^x \Pr[V_i > \beta(y)]dy} \geq \frac{\Pr[\max_i\{V_i\} > \beta(x)]}{1 - \frac{1}{2}\Pr[\max_i\{V_i\} > \beta(0)]}$$
$$= g_{\alpha(0)}(x) \Pr[\max_i\{V_i\} > \beta(x)]$$

Finally, we divide x according to the m thresholds, and use the bound for $\theta(t)$ derived in the last section to get the final result.

$$\begin{aligned}
\int_t^1 \theta(x) g_{\alpha(0)}(x)dx &= \sum_{i=tm}^{m} \int_{i/m}^{(i+1)/m} \theta(x) g_{\alpha(0)}(x)dx \\
&= \sum_{i=tm}^{m} \int_{i/m}^{(i+1)/m} \theta(i/m) \Pr[V_{time=x} < \beta_i]^{x-i/m} g_{\alpha(0)}(x)dx \\
&\geq \sum_{i=tm}^{m} \int_{i/m}^{(i+1)/m} \theta(i/m) \Pr[V_{time=x} < \beta_i]^{x-i/m} g_{\alpha(0)}(i/m)dx \\
&\geq \sum_{i=tm}^{m} g_{\alpha(0)}(i/m) e^{\int_0^{i/m} ln(\alpha(y))dy} \int_{i/m}^{(i+1)/m} \alpha_i^{x-i/m} dx \\
&= \sum_{i=tm}^{m} g_{\alpha(0)}(i/m) (\prod_{k=1}^{i} \alpha_k)^{1/m} \int_{i/m}^{(i+1)/m} \alpha_i^{x-i/m} dx \\
&= \sum_{i=tm}^{m} g_{\alpha(0)}(i/m) (\prod_{k=1}^{i} \alpha_k)^{1/m} \frac{1 - \alpha_i^{1/m}}{-ln(\alpha_i)}
\end{aligned}$$

Hence, we proved that, for each $t \in [0, 1]$:

$$\Pr[z > \beta(t)] \geq \min_t \left(\frac{\int_0^t 1 - \alpha(x)dx}{1 - \alpha(t)} + \sum_{i=tm}^{m} g_{\alpha(0)}(i/m) (\prod_{k=1}^{i} \alpha_k)^{1/m} \frac{1 - \alpha_i^{1/m}}{-ln(\alpha_i)} \right) \Pr[\max_i\{V_i\} > \beta_t]$$

Where optimization of values for α lead to a competitive ratio of 0.669