Linear Stability Analysis of the Kuramoto model with noise

Alexandre Sureda Croguennoc

Kuramoto model with noise

The Kuramoto model defined as

$$\theta'_i(t) = \omega_i + K r \sin(\psi - \theta_i) \quad (1)$$

Adding noise to the model gives

$$\theta'_i(t) = \omega_i + \zeta(t) + K r \sin(\psi - \theta_i) \quad (2)$$

Where ζ is the white noise defined as

$$< \zeta_i(t) > = 0$$
$$< \zeta_i(t) \zeta_j(t) > = 2 D \delta_{ij} \delta(s - t)$$

Where D is the strength of the noise . Analogously to the Wiener process we can also express the Kuramoto model with noise in terms of the evolution of a density function. This type of equation is called a Fokker-Planck equation.

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}[\rho v] \quad (3)$$

Where v = $\theta'_i(t)$. The order parameter r can be defined as . The incoherent state is

$$\rho(\theta, t, \omega) = \frac{1}{2\pi} \quad (4)$$

$$r e^{i\psi} = \int_0^{2\pi} \int_{-\infty}^{\infty} e^{i\theta} \rho(\theta, t, \omega) g(\omega) d\omega d\theta \quad (5)$$

Adding perturbation

We add a small perturbation to he incoherent state $\rho(\theta, t, \omega) = 1/2\pi$.

$$\rho(\theta, t, \omega) \;=\; \epsilon\,\eta(\theta, t, \omega) + \frac{1}{2\pi} \qquad (6)$$

Where the strength is set $\epsilon << 1$, the arbitrary function must obey for normalization of ρ

$$\int_0^{2\pi} \eta(\theta, t, \omega)\, d\theta \;=\; 0$$

Substitute this perturbation into the Fokker - Planck equation and find the first order [O(ϵ)] . Fom the Focker - Planck we have :

$$\epsilon\, \frac{\partial\, \eta(\theta, t, \omega)}{\partial\, t} = \epsilon\, D\, \frac{\partial^2\, \eta(\theta, t, \omega)}{\partial\, \theta^2} \;-\; \frac{\partial}{\partial\, \theta}\left[\left(\frac{1}{2\pi} + \epsilon\,\eta\right) v\right] \qquad (7)$$

We substitute equation (5) to the equation (4)

$$r\, e^{i\psi} \;=\; \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta}\left(\frac{1}{2\pi} + \epsilon\,\eta\right) g(\omega)\, d\omega\, d\theta \qquad (8)$$

Can be simplified, the term $e^{i\theta}/ 2\pi$ is 0 under the integral of theta, the equation gives

$$r\, e^{i\psi} \;=\; \epsilon \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta}\, \eta(\theta, t, \omega)\, g(\omega)\, d\omega\, d\theta \qquad (9)$$

We define r as

$$r \;=\; \epsilon\, r_1 \qquad (10)$$

By equation (1), v is given

$$v \;=\; \omega_i \;+ \epsilon\, K\, r_1 \sin(\psi \;-\; \theta_i) \qquad (11)$$

The last term of the equation (7) becomes

$$-\left(\frac{1}{2\pi} + \epsilon\,\eta\right) \frac{\partial\, v}{\partial\, \theta} \;-\; v\, \frac{\partial\, v}{\partial\, \theta}\left(\frac{1}{2\pi} + \epsilon\,\eta\right) \qquad (12)$$

Taking the derivative of v and simplify, gives

$$-\left(\frac{1}{2\pi} + \epsilon\,\eta\right) (-\,\epsilon\, K\, r_1 \cos(\psi \;-\; \theta_i)) \;-\epsilon\, v\, \frac{\partial\;\eta}{\partial\, \theta} \qquad (13)$$

We substitute equation (11) and taking only terms [O(ϵ)]

$$\epsilon \frac{K}{2\pi} r_1 \cos(\psi - \theta_i) - \epsilon\, \omega \frac{\partial \eta}{\partial \theta} \qquad (14)$$

We put back the resulting equation to the equation (7) and divide by ϵ to find the final equation that describes the evolution of the perturbation η

$$\frac{\partial \eta}{\partial t} = D \frac{\partial^2 \eta}{\partial \theta^2} - \omega \frac{\partial \eta}{\partial \theta} + \frac{K}{2\pi} r_1 \cos(\psi - \theta_i) \qquad (15)$$

We use Fourier method to analyze the solutions, solutions of the form

$$\eta(\theta, t, \omega) = c(t, \omega)\, e^{i\theta} + c^*(t, \omega)\, e^{-i\theta} + \eta^{\perp}(\theta, t, \omega) \qquad (16)$$

Where c is the first Fourier coefficient, c^* is its complex conjugate, and $\eta^{\perp}$ contains all the higher harmonics. We can see that r depend only on the first harmonics by subsituting the equation (16) into equation (9) gives

$$r_1\ e^{i\psi} = \int_0^{2\pi} \int_{-\infty}^{\infty} e^{i\theta} \left(\sum_n c_n(w, t)\, e^{i n \theta} \right) g(\omega)\, d\omega\, d\theta$$
$$r_1\ e^{i\psi} = \int_{-\infty}^{\infty} \sum_n c_n(w, t) \left(\int_0^{2\pi} e^{i(n+1)\theta} \right) g(\omega)\, d\omega\, d\theta$$
$$r_1\ e^{i\psi} = \int_{-\infty}^{\infty} \sum_n c_n(w, t)\, \delta_{n,-1}\, g(\omega)\, d\omega$$
$$r_1\ e^{i\psi} = 2\pi \int_{-\infty}^{\infty} c_{-1}(w, t)\, g(\omega)\, d\omega$$

$$r_1\ e^{i\psi} = 2\pi \int_{-\infty}^{\infty} c^*(t, \omega)\, g(\omega)\, d\omega \qquad (17)$$

This results means that we can solve r with only the first harmonic . So we can rewritte this equation to insert in the equation (15) as

$$r_1\ \cos(\psi - \theta) = \pi\Big[\left(\int_{-\infty}^{\infty} c^*(t, \omega)\, g(\omega)\, d\omega \right) e^{-i\theta} + \left(\int_{-\infty}^{\infty} c(t, \omega)\, g(\omega)\, d\omega \right) e^{i\theta} \qquad (18)$$

The evolution equation with the equations (16) and (18) gives

$$\frac{\partial}{\partial t}\left(c(t,\omega)\, e^{i\theta}+c^{*}(t,\omega)\, e^{-i\theta}+\ \eta^{\perp}(\theta,t,\omega)\right)=$$
$$D\,\frac{\partial^{2}}{\partial\theta^{2}}\left(c(t,\omega)\, e^{i\theta}+c^{*}(t,\omega)\, e^{-i\theta}+\ \eta^{\perp}(\theta,t,\omega)\right)-\omega\,\frac{\partial}{\partial\theta}\left(c(t,\omega)\, e^{i\theta}+c^{*}(t,\omega)\, e^{-i\theta}+\ \eta^{\perp}(\theta,t,\omega)\right)\ +$$
$$\frac{K}{2}\ \left[\left(\int_{-\infty}^{\infty}c^{*}(t,\omega)\, g(\omega)\, d\omega\right)e^{-i\theta}+\left(\int_{-\infty}^{\infty}c(t,\omega)\, g(\omega)\, d\omega\right)e^{i\theta}\right]\qquad (19)$$

Taking the derivatives and collecting terms gives,

$$e^{i\theta}\left(\frac{\partial c}{\partial t}+D\,c\,+\,i\,\omega\,c-\frac{K}{2}\int_{-\infty}^{\infty}c\,g(v)\,dv\right)\,+\,e^{-i\theta}\left(\frac{\partial c^{*}}{\partial t}+D\,c^{*}\,-\,i\,\omega\,c^{*}-\frac{K}{2}\int_{-\infty}^{\infty}c^{*}\,g(v)\,dv\right)\,+$$
$$\left(\frac{\partial\,\eta^{\perp}}{\partial t}-\ D\,\frac{\partial^{2}\,\eta^{\perp}}{\partial\theta^{2}}\,+\,\omega\,\frac{\partial\,\eta^{\perp}}{\partial\theta}\right)=\ 0\qquad (20)$$

Each bracket must be 0, we can get the evolution equation separated by two equations, because the second equation it’s just the complex conjugate of the first.

$$\frac{\partial c}{\partial t}=-(\,D\,+\,i\,\omega\,)+\frac{K}{2}\int_{-\infty}^{\infty}c\,g(v)\,dv\qquad (21)$$

$$\frac{\partial\,\eta^{\perp}}{\partial t}=\ D\,\frac{\partial^{2}\,\eta^{\perp}}{\partial\theta^{2}}\,-\,\omega\,\frac{\partial\,\eta^{\perp}}{\partial\theta}\qquad (22)$$

The first equation is the evolution of the fundamental node, wich tells about the synchronization behavior, and the second equation that describes the evolution of the higher harmonics.
To study the evolution of the fundamental node, we first look for separable solution of the form

$$c(t,\omega)\ =\ b\,(\omega)\,e^{\lambda t}\qquad (23)$$

The eigenvalue λ tells how c evolves in time . The linear operator L of the right hand side of the equation (21) is given by

$$L\,c=-(\,D\,+\,i\,\omega\,)\,c+\frac{K}{2}\int_{-\infty}^{\infty}c\,g(v)\,dv\qquad (24)$$

The eigenvalue equation is given by

$$(L\,-\,\lambda\,\mathrm{I}\,)\,c\,=\,(L\,-\,\lambda\,\mathrm{I}\,)\ b\ e^{\lambda t}\ =\ 0\qquad (25)$$

Where I is the identity operator. Dividing by $e^{\lambda t}$, gives

$$\lambda\,b\ =\ L\,b\ =-(\,D\,+\,i\,\omega\,)\,b+\frac{K}{2}\int_{-\infty}^{\infty}b(v)\,g(v)\,dv\qquad (26)$$

The integral is just a constant, we call it

$$A = \frac{K}{2} \int_{-\infty}^{\infty} b(v)\, g(v)\, dv \qquad (27)$$

Solving for b in the equation (26)

$$b(\omega) = \frac{A}{\lambda + D + i\omega} \qquad (28)$$

Using a sel-consistency argument by substitute b in the equation (27)

$$A = \frac{K}{2} \int_{-\infty}^{\infty} \frac{A}{\lambda + D + i v}\, g(v)\, dv \qquad (29)$$

Then the resulting equation for eigenvalue λ is

$$1 = \frac{K}{2} \int_{-\infty}^{\infty} \frac{g(v)}{\lambda + D + i v}\, dv \qquad (30)$$

multiplying and dividing by the complex conjugate of the denominator gives

$$1 = \frac{K}{2} \int_{-\infty}^{\infty} \frac{\lambda + D - i v}{(\lambda + D)^2 + v^2}\, g(v)\, dv \qquad (31)$$

The integral of the imaginary part is 0 because is an odd function

$$1 = \frac{K}{2} \int_{-\infty}^{\infty} \frac{\lambda + D}{(\lambda + D)^2 + v^2}\, g(v)\, dv \qquad (32)$$

Finding Kc with Noise

The eigenvalue λ tells the stability of c, and therefore the stability of r . From the equation (23) we see that if $\lambda > 0$, c grows exponentially, and therefore r grows exponentially, that means the incoherent state is unstable. If $\lambda < 0$, c decays and r decays to zero making the incoherent state stable. We can see that $\lambda > - D$ so that the right side of the equation (32) can be positive.

With no noise, D = 0, means that λ cannot be negative, but from the previous discussion λ must be negative for the incoherent state to be stable. This means, that the incoherent state cannot be linearly stable without the presence of noise.

For D > 0 , the funademental node can be stable, by setting $\lambda = 0$ in the equation (32) we find the critical point of the incoherent state

$$K_c = 2\left[\int_{-\infty}^{\infty} \frac{D}{D^2 + v^2}\, g(v)\, dv\right]^{-1} \qquad (33)$$

For som distributions of natural frequencies $g(\omega)$, the eigenvalue can be found. For example, a uniform distribution given by

$$\begin{cases} g(\omega) = \frac{1}{2\gamma} & -\gamma \leq \omega \leq \gamma \\ 0 & \text{else} \end{cases} \qquad (34)$$

Performing the integral gives the eigenvalue and a critical coupling constant

$$\lambda = \gamma \cot \frac{2\gamma}{K} - D \qquad (35)$$

$$K_c = \frac{2\gamma}{\arctan \frac{2\gamma}{K}} \qquad (36)$$

For a Lorentzian distribution ,

$$g(\omega) = \frac{\gamma}{\pi\left(\gamma^2 + \omega^2\right)} \qquad (37)$$

Giving the eigenvalue and the critical coupling constant as

$$\lambda = \frac{K}{2} - D - \gamma \qquad (38)$$

$$K_c = 2\,(D + \gamma) = \beta^2 + 2\gamma \qquad (39)$$

Second order perturbation analysis

In the Strogatz and Mirollo article suggest one problem that remain unfinished, the linear stability analysis to second order to see if a higher-order term will make affect the stability.
Adding the second term of perturbation [O(ϵ^2)]

$$\rho(\theta, t, \omega) = \frac{1}{2\pi} + \epsilon\,\eta(\theta, t, \omega) + \epsilon^2\,\mu(\theta, t, \omega) \qquad (40)$$

Imposing the same condition for normalize

$$\int_0^{2\pi} \mu(\theta, t, \omega)\, d\theta = 0$$

Substituting the equation (40) into the Fokker - Planck equation gives

$$\epsilon \frac{\partial \eta}{\partial t} + \epsilon^2 \frac{\partial \mu}{\partial t} = \epsilon D \frac{\partial^2 \eta}{\partial \theta^2} + \epsilon^2 D \frac{\partial^2 \mu}{\partial \theta^2} - \frac{\partial}{\partial \theta}\left[\left(\frac{1}{2\pi} + \epsilon\eta + \epsilon^2 \mu\right) v\right] \qquad (41)$$

Inserting the equation (40) to solve r, like in the first order case gives

$$r\, e^{i\psi} = \epsilon \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta}\, \eta(\theta, t, \omega)\, g(\omega)\, d\omega\, d\theta \;+$$
$$\epsilon^2 \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta}\, \mu(\theta, t, \omega)\, g(\omega)\, d\omega\, d\theta \qquad (42)$$

Defining r_1 and r_2 gives

$$r_1\, e^{i\psi} = \epsilon \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta}\, \eta(\theta, t, \omega)\, g(\omega)\, d\omega\, d\theta \qquad (43)$$
$$r_2\, e^{i\psi} = \epsilon^2 \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta}\, \mu(\theta, t, \omega)\, g(\omega)\, d\omega\, d\theta \qquad (44)$$
$$r = \epsilon\, r_1 + \epsilon^2\, r_2 \qquad (45)$$

The expression for v gives

$$v = \omega_i + K\left(\epsilon\, r_1 + \epsilon^2\, r_2\right) \sin(\psi - \theta_i) \qquad (46)$$

Substituting the expression (46) into the Fokker - Planck equation, and taking only terms up to [O(ϵ^2)] gives

$$\epsilon \frac{\partial \eta}{\partial t} + \epsilon^2 \frac{\partial \mu}{\partial t} = \epsilon D \frac{\partial^2 \eta}{\partial \theta^2} - \omega \frac{\partial \eta}{\partial \theta} + \frac{K}{2\pi} r_1 \cos(\psi - \theta) + \epsilon^2 D \frac{\partial^2 \mu}{\partial \theta^2} +$$
$$\epsilon^2 K\left[\frac{r_2}{2\pi} + \eta\, r_1\right] \cos(\psi - \theta_i) - \epsilon^2 K\, r_1 \sin(\psi - \theta) \frac{\partial \eta}{\partial \theta} - \omega \frac{\partial \mu}{\partial \theta} \qquad (47)$$

The first order terms are the same as those found in the first analysis, we can substract them out. Dividing the equation by ϵ^2 we obtain the evolution of the second order - perturbation

$$\frac{\partial \mu}{\partial t} = D \frac{\partial^2 \mu}{\partial \theta^2} + K\left[\frac{r_2}{2\pi} + \eta\, r_1\right] \cos(\psi - \theta_i) - K\, r_1 \sin(\psi - \theta) \frac{\partial \eta}{\partial \theta} - \omega \frac{\partial \mu}{\partial \theta} \qquad (48)$$

Again we do fourier methods for analyze the equation

$$\mu(\theta, t, \omega) = s(t, \omega)\, e^{i\theta} + s^*(t, \omega)\, e^{-i\theta} + \mu^{\perp}(\theta, t, \omega) \qquad (49)$$

By analogy with the first order analysis

$$r_1\, e^{i\psi} = 2\pi \int_{-\infty}^{\infty} c^*(t, \omega)\, g(\omega)\, d\omega \qquad (50)$$
$$r_2\, e^{i\psi} = 2\pi \int_{-\infty}^{\infty} s^*(t, \omega)\, g(\omega)\, d\omega \qquad (51)$$

With the same logic used later, we can compute the terms of the evolution equation

$$r_2\ \cos(\psi-\theta) = \pi\left[\left(\int_{-\infty}^{\infty} s^*(t,\omega)\,g(\omega)\,d\omega\right)e^{-i\theta} + \left(\int_{-\infty}^{\infty} s(t,\omega)\,g(\omega)\,d\omega\right)e^{i\theta}\right] \quad (52)$$

$$\eta\, r_1\ \cos(\psi-\theta) = \pi\left[\left(\int_{-\infty}^{\infty} c^*(t,\omega)\,g(\omega)\,d\omega\right)e^{-i\theta} + \left(\int_{-\infty}^{\infty} c(t,\omega)\,g(\omega)\,d\omega\right)e^{i\theta}\right] \quad (53)$$

$$r_1\ \sin(\psi-\theta) = -i\,\pi\left[\left(\int_{-\infty}^{\infty} c^*(t,\omega)\,g(\omega)\,d\omega\right)e^{-i\theta} + \left(\int_{-\infty}^{\infty} c(t,\omega)\,g(\omega)\,d\omega\right)e^{i\theta}\right] \quad (54)$$

Substituting this equations in the evoution equation equation gives

$$e^{i\theta}\left[\frac{\partial s}{\partial t} + D s + i\omega s - \frac{K}{2}\int_{-\infty}^{\infty} s\,g(v)\,dv - \pi K\left(\eta^{\perp} - i\,\frac{\partial \eta^{\perp}}{\partial t}\right)\int_{-\infty}^{\infty} c\,g(v)\,dv\right] +$$
$$e^{-i\theta}\left[\frac{\partial s^*}{\partial t} + D s^* - i\omega s^* - \frac{K}{2}\int_{-\infty}^{\infty} s^*\,g(v)\,dv - \pi K\left(\eta^{\perp} - i\,\frac{\partial \eta^{\perp}}{\partial t}\right)\int_{-\infty}^{\infty} c^*\,g(v)\,dv\right] +$$
$$\left(\frac{\partial \mu^{\perp}}{\partial t} - D\,\frac{\partial^2 \mu^{\perp}}{\partial \theta^2} + \omega\,\frac{\partial \mu^{\perp}}{\partial \theta}\right) - 2\pi K\, e^{2i\theta}\left[c\int_{-\infty}^{\infty} c\,g(v)\,dv\right] - 2\pi K\, e^{-2i\theta}\left[c\int_{-\infty}^{\infty} c^*\,g(v)\,dv\right] = 0$$
(55)

All the brackets must be 0, the terms $e^{\pm 2i\theta}$ terms force that c^* and c are 0, giving r1 equal to 0. So in the second order analysis, the first perturbation has no effect on the order paramter r in the second order analysis. Like the last analysis, we give two equations

$$\frac{\partial s}{\partial t} = -(D + i\omega) + \frac{K}{2}\int_{-\infty}^{\infty} s\,g(v)\,dv \quad (56)$$

$$\frac{\partial \mu^{\perp}}{\partial t} = D\,\frac{\partial^2 \mu^{\perp}}{\partial \theta^2} - \omega\,\frac{\partial \mu^{\perp}}{\partial \theta} \quad (57)$$

This shows that these are equivalent tot the evolution equations for the first order perturbation.