Linear Stability Analysis of the Kuramoto model with noise

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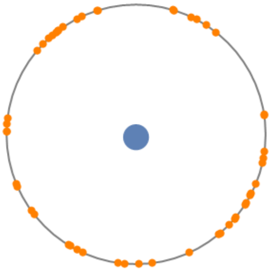
Kuramoto model

$$\theta'_i(t) = \omega_i + K r \sin(\psi - \theta_i)$$



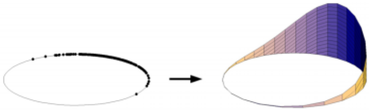
Adding noise to the model gives

$$\theta'_i(t) = \omega_i + \zeta_i(t) + K r \sin(\psi - \theta_i)$$

Where ζ is the white noise defined as

$$\langle \zeta_i(t) \rangle = 0$$
$$\langle \zeta_i(t) \zeta_j(t) \rangle = 2 D \delta_{ij} \delta(s - t)$$

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta} [\rho v]$$



We add a small perturbation to he incoherent state

$$\rho(\theta, t, \omega) \;=\; \epsilon\, \eta(\theta, t, \omega) + \frac{1}{2\,\pi}$$

Where the strength is set $\epsilon << 1$, the arbitrary function must obey for normalization of ρ

$$\int_0^{2\,\pi} \eta(\theta, t, \omega)\, d\,\theta \;=\; 0$$

$$\epsilon\, \frac{\partial\, \eta(\theta, t, \omega)}{\partial\, t} = \epsilon\, D\, \frac{\partial^2\, \eta(\theta, t, \omega)}{\partial\, \theta^2} - \frac{\partial}{\partial\, \theta}\left[\left(\frac{1}{2\,\pi} + \epsilon\, \eta\right) v\right]$$

$$r\, e^{i\,\psi} \;=\; \epsilon \int_0^{2\,\pi} \int_{-\infty}^{\infty} e^{i\,\theta}\, \eta(\theta, t, \omega)\, g(\omega)\, d\,\omega\, d\,\theta \qquad r \;=\; \epsilon\, r_1$$

$$v \;=\; \omega_{\,i} + \epsilon\, K\, r_1 \sin\left(\psi \,-\, \theta_i\right)$$

Linear stability analysis

Evolution of the perturbation η

$$\frac{\partial \eta}{\partial t} = D \frac{\partial^2 \eta}{\partial \theta^2} - \omega \frac{\partial \eta}{\partial \theta} + \frac{K}{2\pi} r_1 \cos(\psi - \theta_i)$$

We use Fourier method to analyze the solutions, solutions of the form

$$\eta(\theta, t, \omega) = c(t, \omega) e^{i\theta} + c^*(t, \omega) e^{-i\theta} + \eta^{\perp}(\theta, t, \omega)$$

$$r_1 \cos(\psi - \theta) = \pi\left[\left(\int_{-\infty}^{\infty} c^*(t, \omega) g(\omega) d\omega\right) e^{-i\theta} + \left(\int_{-\infty}^{\infty} c(t, \omega) g(\omega) d\omega\right) e^{i\theta}\right.$$

$$e^{i\theta}\left(\frac{\partial c}{\partial t} + D c + i\omega c - \frac{K}{2}\int_{-\infty}^{\infty} c\, g(v)\, dv\right) + e^{-i\theta}\left(\frac{\partial c^*}{\partial t} + D c^* - i\omega c^* - \frac{K}{2}\int_{-\infty}^{\infty} c^* g(v)\, dv\right) +$$
$$\left(\frac{\partial \eta^{\perp}}{\partial t} - D \frac{\partial^2 \eta^{\perp}}{\partial \theta^2} + \omega \frac{\partial \eta^{\perp}}{\partial \theta}\right) = 0$$

Linear stability analysis

$$\frac{\partial c}{\partial t} = -(D + i\omega) + \frac{K}{2}\int_{-\infty}^{\infty} c\, g(v)\, dv$$

$$\frac{\partial \eta^{+}}{\partial t} = D\frac{\partial^2 \eta^{+}}{\partial \theta^2} - \omega \frac{\partial \eta^{+}}{\partial \theta}$$

To study the evolution of the fundamental node

$$c(t,\omega) = b(\omega)\, e^{\lambda t} \qquad (L - \lambda \mathrm{I})\, c = (L - \lambda \mathrm{I})\, b\, e^{\lambda t} = 0$$

$$L\, c = -(D + i\omega)\, c + \frac{K}{2}\int_{-\infty}^{\infty} c\, g(v)\, dv$$

$$\lambda\, b = L\, b = -(D + i\omega)\, b + \frac{K}{2}\int_{-\infty}^{\infty} b(v)\, g(v)\, dv$$

$$1 = \frac{K}{2}\int_{-\infty}^{\infty} \frac{\lambda + D}{(\lambda + D)^2 + v^2}\, g(v)\, dv \qquad \lambda > -D$$

- Critical point $\lambda = 0$

$$K_c = 2\left[\int_{-\infty}^{\infty} \frac{D}{D^2 + \nu^2} g(\nu)\, d\nu\right]^{-1}$$

For a Lorentzian distribution ,

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)}$$

$$\lambda = \frac{K}{2} - D - \gamma$$

$$K_c = 2(D + \gamma) = \beta^2 + 2\gamma$$

Linear stability analysis

- Adding the second term of perturbation [O(ϵ^2)]

$$\rho(\theta, t, \omega) = \frac{1}{2\pi} + \epsilon\,\eta(\theta, t, \omega) + \epsilon^2\,\mu(\theta, t, \omega)$$

$$\epsilon \frac{\partial \eta}{\partial t} + \epsilon^2 \frac{\partial \mu}{\partial t} = \epsilon D \frac{\partial^2 \eta}{\partial \theta^2} + \epsilon^2 D \frac{\partial^2 \mu}{\partial \theta^2} - \frac{\partial}{\partial \theta}\left[\left(\frac{1}{2\pi} + \epsilon\,\eta + \epsilon^2\,\mu\right) v\right]$$

$$r = \epsilon r_1 + \epsilon^2 r_2 \qquad v = \omega_i + K(\epsilon r_1 + \epsilon^2 r_2)\sin(\psi - \theta_i)$$

$$\frac{\partial \mu}{\partial t} = D \frac{\partial^2 \mu}{\partial \theta^2} + K\left[\frac{r_2}{2\pi} + \eta\, r_1\right]\cos(\psi - \theta_i) - K\, r_1 \sin(\psi - \theta)\frac{\partial \eta}{\partial \theta} - \omega \frac{\partial \mu}{\partial \theta}$$

Linear stability analysis

$$e^{i\theta}\left[\frac{\partial s}{\partial t} + D s + i\omega s - \frac{K}{2}\int_{-\infty}^{\infty} s\, g(v)\, dv - \pi K\left(\eta^{\perp} - i\frac{\partial \eta^{\perp}}{\partial t}\right)\int_{-\infty}^{\infty} c\, g(v)\, dv\right] +$$

$$e^{-i\theta}\left[\frac{\partial s^*}{\partial t} + D s^* - i\omega s^* - \frac{K}{2}\int_{-\infty}^{\infty} s^*\, g(v)\, dv - \pi K\left(\eta^{\perp} - i\frac{\partial \eta^{\perp}}{\partial t}\right)\int_{-\infty}^{\infty} c^*\, g(v)\, dv\right] +$$

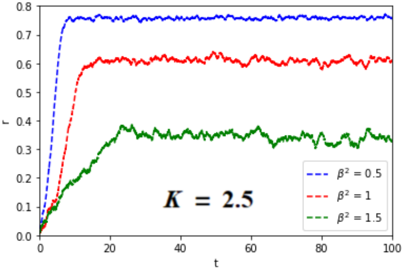
$$\left(\frac{\partial \mu^{\perp}}{\partial t} - D\frac{\partial^2 \mu^{\perp}}{\partial \theta^2} + \omega\frac{\partial \mu^{\perp}}{\partial \theta}\right) - 2\pi K e^{2i\theta}\left[c\int_{-\infty}^{\infty} c\, g(v)\, dv\right] - 2\pi K e^{-2i\theta}\left[c\int_{-\infty}^{\infty} c^*\, g(v)\, dv\right] = 0$$

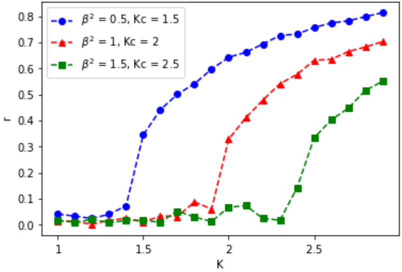
$$\frac{\partial s}{\partial t} = -(D + i\omega) + \frac{K}{2}\int_{-\infty}^{\infty} s\, g(v)\, dv$$

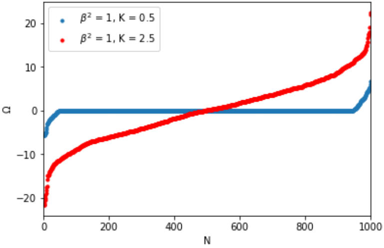
$$\frac{\partial \mu^{\perp}}{\partial t} = D\frac{\partial^2 \mu^{\perp}}{\partial \theta^2} - \omega\frac{\partial \mu^{\perp}}{\partial \theta}$$

Test results

- $K_c = 2(D+\gamma) = \beta^2 + 2\gamma \qquad \gamma = 0.5 \qquad N = 5000$







Conclusions

- Powerful way to study synchronization mathematically
- Random noise can be easily dealt with in the Kuramoto model
- Numerical results show that the simulations agree with the model's predictions