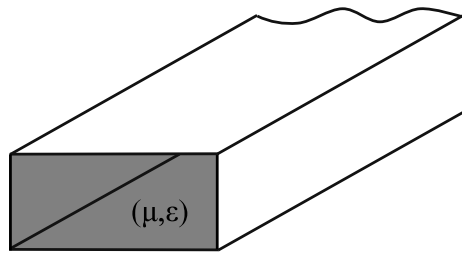


# Waveguides

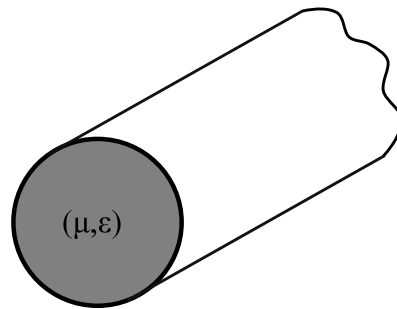
Waveguides, like transmission lines, are structures used to guide electromagnetic waves from point to point. However, the fundamental characteristics of waveguide and transmission line waves (*modes*) are quite different. The differences in these modes result from the basic differences in geometry for a transmission line and a waveguide.

Waveguides can be generally classified as either *metal waveguides* or *dielectric waveguides*. Metal waveguides normally take the form of an enclosed conducting metal pipe. The waves propagating inside the metal waveguide may be characterized by reflections from the conducting walls. The dielectric waveguide consists of dielectrics only and employs reflections from dielectric interfaces to propagate the electromagnetic wave along the waveguide.

## Metal Waveguides

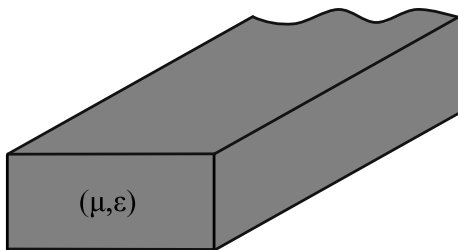


Rectangular Waveguide

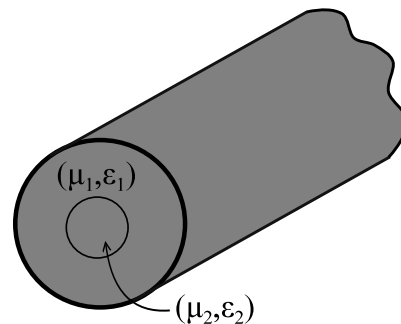


Circular Waveguide

## Dielectric Waveguides



Dielectric Slab Waveguide



Optical Fiber

## Comparison of Waveguide and Transmission Line Characteristics

### Transmission line

- Two or more conductors separated by some insulating medium (two-wire, coaxial, microstrip, etc.).
- Normal operating mode is the TEM or quasi-TEM mode (can support TE and TM modes but these modes are typically undesirable).
- No cutoff frequency for the TEM mode. Transmission lines can transmit signals from DC up to high frequency.
- Significant signal attenuation at high frequencies due to conductor and dielectric losses.
- Small cross-section transmission lines (like coaxial cables) can only transmit low power levels due to the relatively high fields concentrated at specific locations within the device (field levels are limited by dielectric breakdown).
- Large cross-section transmission lines (like power transmission lines) can transmit high power levels.

### Waveguide

- Metal waveguides are typically one enclosed conductor filled with an insulating medium (rectangular, circular) while a dielectric waveguide consists of multiple dielectrics.
- Operating modes are TE or TM modes (cannot support a TEM mode).
- Must operate the waveguide at a frequency above the respective TE or TM mode cutoff frequency for that mode to propagate.
- Lower signal attenuation at high frequencies than transmission lines.
- Metal waveguides can transmit high power levels. The fields of the propagating wave are spread more uniformly over a larger cross-sectional area than the small cross-section transmission line.
- Large cross-section (low frequency) waveguides are impractical due to large size and high cost.

## General Wave Characteristics as Defined by Maxwell's Equations

Given any time-harmonic source of electromagnetic radiation, the phasor electric and magnetic fields associated with the electromagnetic waves that propagate away from the source through a medium characterized by  $(\mu, \epsilon)$  must satisfy the source-free Maxwell's equations (in phasor form) given by

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega\epsilon\tilde{\mathbf{E}}$$

The source-free Maxwell's equations can be manipulated into wave equations for the electric and magnetic fields (as was shown in the case of plane waves). These wave equations are

$$\nabla^2 \tilde{\mathbf{E}} + k^2 \tilde{\mathbf{E}} = 0$$

$$\nabla^2 \tilde{\mathbf{H}} + k^2 \tilde{\mathbf{H}} = 0$$

$$k = \omega\sqrt{\mu\epsilon}$$

where the wavenumber  $k$  is real-valued for lossless media and complex-valued for lossy media. The electric and magnetic fields of a general wave propagating in the  $+z$ -direction (either unguided, as in the case of a plane wave or guided, as in the case of a transmission line or waveguide) through an arbitrary medium with a propagation constant of  $\gamma$  are characterized by a  $z$ -dependence of  $e^{-\gamma z}$ . The electric and magnetic fields of the wave may be written in rectangular coordinates as

$$\tilde{\mathbf{E}}(x, y, z) = \tilde{\mathbf{e}}(x, y) e^{-\gamma z}$$

$$\tilde{\mathbf{H}}(x, y, z) = \tilde{\mathbf{h}}(x, y) e^{-\gamma z}$$

$$\gamma = \alpha + j\beta$$

where  $\alpha$  is the wave attenuation constant and  $\beta$  is the wave phase constant. The propagation constant is purely imaginary ( $\alpha = 0, \gamma = j\beta$ ) when the wave travels without attenuation (no losses) or complex-valued when losses are present.

The transverse vectors  $[\tilde{\mathbf{e}}(x,y), \tilde{\mathbf{h}}(x,y)]$  in the general wave field expressions may contain both transverse field components and longitudinal field components. By expanding the curl operator of the source free Maxwell's equations in rectangular coordinates, we note that the derivatives of the transverse field components with respect to  $z$  are

$$\begin{aligned}\frac{\partial \tilde{E}_x}{\partial z} &= -\gamma \tilde{E}_x & \frac{\partial \tilde{E}_y}{\partial z} &= -\gamma \tilde{E}_y \\ \frac{\partial \tilde{H}_x}{\partial z} &= -\gamma \tilde{H}_x & \frac{\partial \tilde{H}_y}{\partial z} &= -\gamma \tilde{H}_y\end{aligned}$$

If we equate the vector components on each side of the two Maxwell curl equations, we find

$$j\omega\epsilon\tilde{E}_x = \frac{\partial \tilde{H}_z}{\partial y} + \gamma\tilde{H}_y \quad (1a)$$

$$j\omega\epsilon\tilde{E}_y = -\gamma\tilde{H}_x - \frac{\partial \tilde{H}_z}{\partial x} \quad (1b)$$

$$j\omega\epsilon\tilde{E}_z = \frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y} \quad (1c)$$

$$-j\omega\mu\tilde{H}_x = \frac{\partial \tilde{E}_z}{\partial y} + \gamma\tilde{E}_y \quad (2a)$$

$$-j\omega\mu\tilde{H}_y = -\gamma\tilde{E}_x - \frac{\partial \tilde{E}_z}{\partial x} \quad (2b)$$

$$-j\omega\mu\tilde{H}_z = \frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y} \quad (2c)$$

We may manipulate (1) and (2) to solve for the longitudinal field components in terms of the transverse field components.

solve (1a) and (2b) for  $\tilde{H}_y$   $\tilde{E}_x = \frac{1}{h^2} \left( -\gamma \frac{\partial \tilde{E}_z}{\partial x} - j\omega\mu \frac{\partial \tilde{H}_z}{\partial y} \right)$

solve (1b) and (2a) for  $\tilde{H}_x$   $\tilde{E}_y = \frac{1}{h^2} \left( -\gamma \frac{\partial \tilde{E}_z}{\partial y} + j\omega\mu \frac{\partial \tilde{H}_z}{\partial x} \right)$

solve (1b) and (2a) for  $\tilde{E}_y$   $\tilde{H}_x = \frac{1}{h^2} \left( j\omega\epsilon \frac{\partial \tilde{E}_z}{\partial y} - \gamma \frac{\partial \tilde{H}_z}{\partial x} \right)$

solve (1a) and (2b) for  $\tilde{E}_x$   $\tilde{H}_y = \frac{1}{h^2} \left( -j\omega\epsilon \frac{\partial \tilde{E}_z}{\partial x} - \gamma \frac{\partial \tilde{H}_z}{\partial y} \right)$

where the constant  $h$  is defined by

$$h^2 = \gamma^2 + \omega^2 \mu \epsilon = \gamma^2 + k^2 \quad \Rightarrow \quad \gamma = \sqrt{h^2 - k^2}$$

The equations for the transverse fields in terms of the longitudinal fields describe the different types of possible modes for guided and unguided waves.

Transverse electromagnetic (TEM modes)	$\tilde{E}_z = 0, \tilde{H}_z = 0$	plane waves, transmission line modes
Transverse electric (TE modes)	$\tilde{E}_z = 0, \tilde{H}_z \neq 0$	waveguide modes
Transverse magnetic (TM modes)	$\tilde{E}_z \neq 0, \tilde{H}_z = 0$	waveguide modes
Hybrid (EH or HE modes)	$\tilde{E}_z \neq 0, \tilde{H}_z \neq 0$	waveguide modes

For simplicity, consider the case of guided or unguided waves propagating through an ideal (lossless) medium where  $k$  is real-valued. For TEM modes, the only way for the transverse fields to be non-zero with  $\tilde{E}_z = \tilde{H}_z = 0$  is for  $h = 0$ , which yields

$$\gamma = \sqrt{-k^2} = jk = \alpha + j\beta \quad \Rightarrow \quad \beta = k \quad (\text{TEM modes})$$

Thus, for unguided TEM waves (plane waves) moving through a lossless medium or guided TEM waves (waves on a transmission line) propagating on an ideal transmission line, we have  $\gamma = jk = j\beta$ .

For the waveguide modes (TE, TM or hybrid modes),  $h$  cannot be zero since this would yield unbounded results for the transverse fields. Thus,  $\beta \neq k$  for waveguides and the waveguide propagation constant can be written as

$$\gamma = \sqrt{h^2 - k^2} = \sqrt{-k^2 \left( 1 - \frac{h^2}{k^2} \right)} = jk \sqrt{1 - \left( \frac{h}{k} \right)^2}$$

The propagation constant of a wave in a waveguide (TE or TM waves) has very different characteristics than the propagation constant for a wave on a transmission line (TEM waves). The ratio of  $h/k$  in the waveguide mode propagation constant equation can be written in terms of the *cutoff frequency*  $f_c$  for the given waveguide mode as follows.

$$\frac{h}{k} = \frac{h}{\omega \sqrt{\mu \epsilon}} = \frac{h}{2\pi f \sqrt{\mu \epsilon}} = \frac{f_c}{f}$$

$$f_c = \frac{h}{2\pi \sqrt{\mu \epsilon}} \quad (\text{waveguide cutoff frequency})$$

The waveguide propagation constant in terms of the waveguide cutoff frequency is

$$\gamma = jk \sqrt{1 - \left( \frac{f_c}{f} \right)^2}$$

An examination of the waveguide propagation constant equation reveals the cutoff frequency behavior of the waveguide modes.

If  $f < f_c$ ,  $\gamma = \alpha$  (purely real)

$$e^{-\gamma z} = e^{-\alpha z}$$

waves are attenuated  
(*evanescent* modes).

If  $f > f_c$ ,  $\gamma = j\beta$  (purely imaginary)

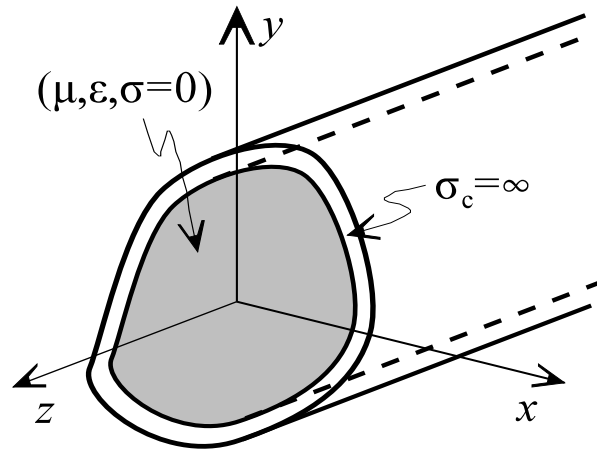
$$e^{-\gamma z} = e^{-j\beta z}$$

waves are unattenuated  
(*propagating* modes).

Therefore, in order to propagate a wave down a waveguide, the source must operate at a frequency higher than the cutoff frequency for that particular mode. If a waveguide source is operated at a frequency less than the cutoff frequency of the waveguide mode, then the wave is quickly attenuated in the vicinity of the source.

## TE and TM Modes in Ideal Waveguides (PEC tube, perfect insulator inside)

Waves propagate along the waveguide (+z-direction) within the waveguide through the lossless dielectric. The electric and magnetic fields of the guided waves must satisfy the source-free Maxwell's equations.



Assumptions:

- (1) the waveguide is infinitely long, oriented along the z-axis, and uniform along its length.
- (2) the waveguide is constructed from ideal materials [perfectly conducting pipe (PEC) is filled with a perfect insulator (lossless dielectric)].
- (3) fields are time-harmonic.

The cross-sectional size and shape of the waveguide dictates the discrete modes that can propagate along the waveguide. That is, there are only discrete electric and magnetic field distributions that will satisfy the appropriate boundary conditions on the surface of the waveguide conductor.

If the single non-zero longitudinal field component associated with a given waveguide mode can be determined ( $\tilde{E}_z$  for a TM mode,  $\tilde{H}_z$  for a TE mode), the remaining transverse field components can be found using the general wave equations for the transverse fields in terms of the longitudinal fields.



General waves in an arbitrary medium

$$\begin{aligned}\tilde{E}_x &= \frac{1}{h^2} \left( -\gamma \frac{\partial \tilde{E}_z}{\partial x} - j\omega\mu \frac{\partial \tilde{H}_z}{\partial y} \right) & \tilde{H}_x &= \frac{1}{h^2} \left( j\omega\epsilon \frac{\partial \tilde{E}_z}{\partial y} - \gamma \frac{\partial \tilde{H}_z}{\partial x} \right) \\ \tilde{E}_y &= \frac{1}{h^2} \left( -\gamma \frac{\partial \tilde{E}_z}{\partial y} + j\omega\mu \frac{\partial \tilde{H}_z}{\partial x} \right) & \tilde{H}_y &= \frac{1}{h^2} \left( -j\omega\epsilon \frac{\partial \tilde{E}_z}{\partial x} - \gamma \frac{\partial \tilde{H}_z}{\partial y} \right)\end{aligned}$$

TE modes in an ideal waveguide ( $\tilde{E}_z = 0, \gamma = j\beta$ )

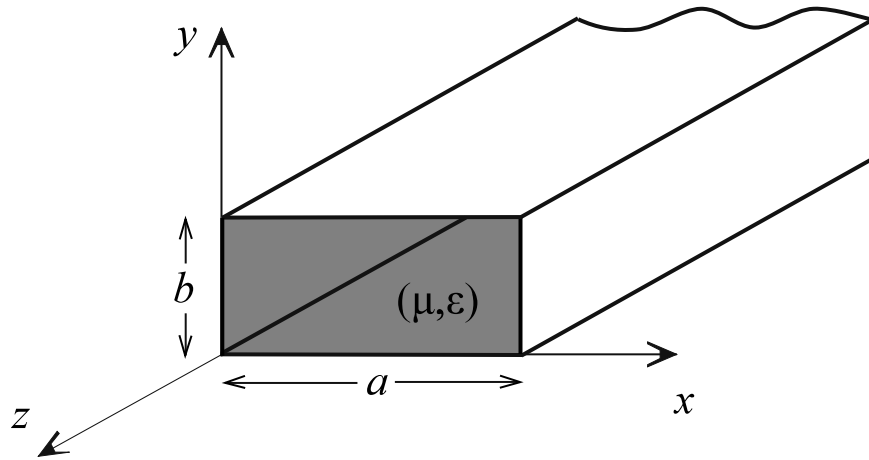
$$\begin{aligned}\tilde{E}_x &= -\frac{j\omega\mu}{h^2} \frac{\partial \tilde{H}_z}{\partial y} & \tilde{H}_x &= -\frac{j\beta}{h^2} \frac{\partial \tilde{H}_z}{\partial x} \\ \tilde{E}_y &= \frac{j\omega\mu}{h^2} \frac{\partial \tilde{H}_z}{\partial x} & \tilde{H}_y &= -\frac{j\beta}{h^2} \frac{\partial \tilde{H}_z}{\partial y}\end{aligned}$$

TM modes in an ideal waveguide ( $\tilde{H}_z = 0, \gamma = j\beta$ )

$$\begin{aligned}\tilde{E}_x &= -\frac{j\beta}{h^2} \frac{\partial \tilde{E}_z}{\partial x} & \tilde{H}_x &= \frac{j\omega\epsilon}{h^2} \frac{\partial \tilde{E}_z}{\partial y} \\ \tilde{E}_y &= -\frac{j\beta}{h^2} \frac{\partial \tilde{E}_z}{\partial y} & \tilde{H}_y &= -\frac{j\omega\epsilon}{h^2} \frac{\partial \tilde{E}_z}{\partial x}\end{aligned}$$

The longitudinal magnetic field of the TE mode and the longitudinal electric field of the TM mode are determined by solving the appropriate boundary value problem for the given waveguide geometry.

## Ideal Rectangular Waveguide



The rectangular waveguide can support either TE or TM modes. The rectangular cross-section ( $a > b$ ) allows for *single-mode operation*. Single-mode operation means that only one mode propagates in the waveguide over a given frequency range. A square waveguide cross-section does not allow for single-mode operation.

### Rectangular Waveguide TM modes

The longitudinal electric field of the TM modes within the rectangular waveguide must satisfy the wave equation

$$\nabla^2 \tilde{E}_z^{TM} + k^2 \tilde{E}_z^{TM} = 0$$

which expanded in rectangular coordinates is

$$\frac{\partial^2 \tilde{E}_z^{TM}}{\partial x^2} + \frac{\partial^2 \tilde{E}_z^{TM}}{\partial y^2} + \frac{\partial^2 \tilde{E}_z^{TM}}{\partial z^2} + k^2 \tilde{E}_z^{TM} = 0$$

The electric field function may be determined using the separation of variables technique by assuming a solution of the form

$$\tilde{E}_z^{TM} = X(x)Y(y)e^{-j\beta z} \quad (+\hat{z} \text{ traveling waves})$$

Inserting the assumed solution into the governing differential equation gives

$$Y(y) \frac{d^2 X(x)}{dx^2} e^{-j\beta z} + X(x) \frac{d^2 Y(y)}{dy^2} e^{-j\beta z} + (k^2 - \beta^2) X(x) Y(y) e^{-j\beta z} = 0$$

$$Y(y) \frac{d^2 X(x)}{dx^2} e^{-j\beta z} + X(x) \frac{d^2 Y(y)}{dy^2} e^{-j\beta z} + h^2 X(x) Y(y) e^{-j\beta z} = 0$$

where  $h^2 = \gamma^2 + k^2 = k^2 - \beta^2$ . Dividing this equation by the assumed solution gives

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + h^2 = 0 \quad (1)$$

Note that the first term in (1) is a function of  $x$  only while the second term is a function of  $y$  only. In order for (1) to be satisfied for every  $x$  and  $y$  within the waveguide, each of the first two terms in the equation must be constants.

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = -k_x^2 \quad \Rightarrow \quad \frac{d^2 X(x)}{dx^2} + k_x^2 X(x) = 0$$

$$\frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} = -k_y^2 \quad \Rightarrow \quad \frac{d^2 Y(y)}{dy^2} + k_y^2 Y(y) = 0$$

$$h^2 = k_x^2 + k_y^2 \quad (\text{separation equation})$$

The original second order partial differential equation dependent on two variables has been separated into two second order ordinary differential equations each dependent on only one variable. The general solutions to the two separate differential equations are

$$X(x) = A \sin k_x x + B \cos k_x x$$

$$Y(y) = C \sin k_y y + D \cos k_y y$$

The resulting longitudinal electric field for a rectangular waveguide TM mode is

$$\tilde{E}_z^{TM}(x,y,z) = (A \sin k_x x + B \cos k_x x)(C \sin k_y y + D \cos k_y y) e^{-j\beta z}$$

The TM boundary conditions for the rectangular waveguide are

$$\tilde{E}_z^{TM}(0,y,z) = \tilde{E}_z^{TM}(a,y,z) = 0 \quad (\text{vertical walls})$$

$$\tilde{E}_z^{TM}(x,0,z) = \tilde{E}_z^{TM}(x,b,z) = 0 \quad (\text{horizontal walls})$$

The application of the boundary conditions yields

$$\tilde{E}_z^{TM}(0,y,z) = 0 \quad \Rightarrow \quad B = 0$$

$$\tilde{E}_z^{TM}(a,y,z) = 0 \quad \Rightarrow \quad k_x a = m\pi \quad (m = 1,2,3,\dots) \quad \Rightarrow \quad k_x = \frac{m\pi}{a}$$

$$\tilde{E}_z^{TM}(x,0,z) = 0 \quad \Rightarrow \quad D = 0$$

$$\tilde{E}_z^{TM}(x,b,z) = 0 \quad \Rightarrow \quad k_y b = n\pi \quad (n = 1,2,3,\dots) \quad \Rightarrow \quad k_y = \frac{n\pi}{b}$$

The resulting product of the constants  $A$  and  $C$  can be written as a single constant (defined as  $E_0$ ). The number of discrete TM modes is infinite based on the possible values of the indices  $m$  and  $n$ . An individual TM mode is designated as the  $TM_{mn}$  mode. The longitudinal electric field of the  $TM_{mn}$  mode in the rectangular waveguide is given by

$$\tilde{E}_z^{TM_{mn}}(x,y,z) = E_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

$$(m = 1,2,3,\dots)$$

$$(n = 1,2,3,\dots)$$

The transverse field components of the  $TM_{mn}$  mode are found by differentiating the longitudinal electric field as defined by the standard TM equations.

$$\tilde{E}_x^{TM_{mn}}(x,y,z) = -\frac{j\beta}{h^2} \frac{\partial \tilde{E}_z^{TM_{mn}}}{\partial x} = -\frac{j\beta}{h^2} \left( \frac{m\pi}{a} \right) E_o \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

$$\tilde{E}_y^{TM_{mn}}(x,y,z) = -\frac{j\beta}{h^2} \frac{\partial \tilde{E}_z^{TM_{mn}}}{\partial y} = -\frac{j\beta}{h^2} \left( \frac{n\pi}{b} \right) E_o \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$\tilde{H}_x^{TM_{mn}}(x,y,z) = \frac{j\omega\epsilon}{h^2} \frac{\partial \tilde{E}_z^{TM_{mn}}}{\partial y} = \frac{j\omega\epsilon}{h^2} \left( \frac{n\pi}{b} \right) E_o \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$\tilde{H}_y^{TM_{mn}}(x,y,z) = -\frac{j\omega\epsilon}{h_2} \frac{\partial \tilde{E}_z^{TM_{mn}}}{\partial x} = -\frac{j\omega\epsilon}{h^2} \left( \frac{m\pi}{a} \right) E_o \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

In general, the cutoff frequency will increase as the mode index increases. Thus, in practice, only the lower order modes are important as the waveguide is operated at frequencies below of the cutoff frequencies of the higher order modes.

## Rectangular Waveguide TE modes

The longitudinal magnetic field of the TE modes within the rectangular waveguide must satisfy the same wave equation as the longitudinal electric field of the TM modes:

$$\nabla^2 H_z^{TE} + k^2 H_z^{TE} = 0$$

which expanded in rectangular coordinates is

$$\frac{\partial^2 H_z^{TE}}{\partial x^2} + \frac{\partial^2 H_z^{TE}}{\partial y^2} + \frac{\partial^2 H_z^{TE}}{\partial z^2} + k^2 H_z^{TE} = 0$$

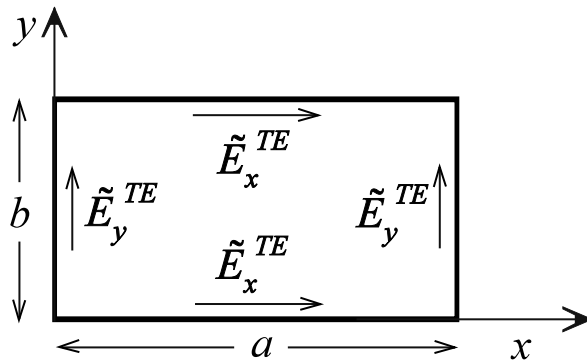
The same separation of variables technique used to solve for the longitudinal TM electric field applies to the longitudinal TE magnetic field. Thus, the longitudinal TE magnetic field may be written as

$$H_z^{TE}(x,y,z) = (A \sin k_x x + B \cos k_x x)(C \sin k_y y + D \cos k_y y) e^{-j\beta z}$$

To determine the unknown coefficients, we apply the TE boundary conditions. Given no longitudinal electric field for the TE case, the boundary conditions for the transverse electric field components on the walls of the waveguide must be enforced. The TE boundary conditions are:

$$\tilde{E}_y^{TE}(0,y,z) = \tilde{E}_y^{TE}(a,y,z) = 0 \quad (\text{vertical walls})$$

$$\tilde{E}_x^{TE}(x,0,z) = \tilde{E}_x^{TE}(x,b,z) = 0 \quad (\text{horizontal walls})$$



The transverse components of the TE electric field are related to longitudinal magnetic field by the standard TE equations.

$$\begin{aligned}
\tilde{E}_x^{TE} &= \frac{-j\omega\mu}{h^2} \frac{\partial \tilde{H}_z^{TE}}{\partial y} \\
&= \frac{-j\omega\mu}{h^2} k_y (A \sin k_x x + B \cos k_x x) (C \cos k_y y - D \sin k_y y) e^{-j\beta z} \\
\tilde{E}_y^{TE} &= \frac{j\omega\mu}{h^2} \frac{\partial \tilde{H}_z^{TE}}{\partial x} \\
&= \frac{j\omega\mu}{h^2} k_x (A \cos k_x x - B \sin k_x x) (C \sin k_y y + D \cos k_y y) e^{-j\beta z}
\end{aligned}$$

The application of the TE boundary conditions yields

$$\begin{aligned}
\tilde{E}_x^{TE}(x, 0, z) = 0 &\Rightarrow C = 0 \\
\tilde{E}_x^{TE}(x, b, z) = 0 &\Rightarrow k_y b = n\pi \quad (n = 0, 1, 2, \dots) \Rightarrow k_y = \frac{n\pi}{b} \\
\tilde{E}_y^{TE}(0, y, z) = 0 &\Rightarrow A = 0 \\
\tilde{E}_y^{TE}(a, y, z) = 0 &\Rightarrow k_x a = m\pi \quad (m = 0, 1, 2, \dots) \Rightarrow k_x = \frac{m\pi}{a}
\end{aligned}$$

Combining the constants  $B$  and  $D$  into the constant  $H_o$ , the resulting longitudinal magnetic field of the  $TE_{mn}$  mode is

$$\tilde{H}_z^{TE_{mn}}(x, y, z) = H_o \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$\begin{aligned}
(m = 0, 1, 2, \dots) & \quad (m = n \neq 0) \\
(n = 0, 1, 2, \dots) &
\end{aligned}$$

Note that the indices include  $m = 0$  and  $n = 0$  in the TE solution since these values still yield a non-zero longitudinal magnetic field. However, the case of  $n = m = 0$  is not allowed since this would make all of the transverse field components zero. The resulting transverse fields for the waveguide TE modes are

$$\tilde{E}_x^{TE_{mn}}(x,y,z) = \frac{-j\omega\mu}{h^2} \frac{\partial \tilde{H}_z^{TE_{mn}}}{\partial y} = \frac{j\omega\mu}{h^2} \left( \frac{n\pi}{b} \right) H_o \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

$$\tilde{E}_y^{TE_{mn}}(x,y,z) = \frac{j\omega\mu}{h^2} \frac{\partial \tilde{H}_z^{TE_{mn}}}{\partial x} = -\frac{j\omega\mu}{h^2} \left( \frac{m\pi}{a} \right) H_o \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$\tilde{H}_x^{TE_{mn}}(x,y,z) = -\frac{j\beta}{k_c^2} \frac{\partial \tilde{H}_z^{TE_{mn}}}{\partial x} = \frac{j\beta m\pi}{k_c^2 a} H_o \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$\tilde{H}_y^{TE_{mn}}(x,y,z) = -\frac{j\beta}{k_c^2} \frac{\partial \tilde{H}_z^{TE_{mn}}}{\partial y} = \frac{j\beta n\pi}{k_c^2 b} H_o \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

where ( $m = 0, 1, 2, \dots$ ) and ( $n = 0, 1, 2, \dots$ ) but  $m = n = 0$  for the  $TE_{mn}$  mode.

### Summary of Rectangular Waveguide Modes

Rectangular waveguide $mn$ index pairs ( $TM_{mn}$ )					Rectangular waveguide $mn$ index pairs ( $TE_{mn}$ )				
	$n=1$	$n=2$	$n=3$			$n=0$	$n=1$	$n=2$	
$m=1$	11	12	13	...	$m=0$	×	01	02	...
$m=2$	21	22	23	...	$m=1$	10	11	12	...
$m=3$	31	32	33	...	$m=2$	20	21	22	...
	⋮	⋮	⋮	⋱		⋮	⋮	⋮	⋱



## Rectangular Waveguide TE and TM Mode Parameters

The propagation constant in the rectangular waveguide for both the TE<sub>mn</sub> and TM<sub>mn</sub> waveguide modes ( $\gamma_{mn}$ ) is defined by

$$\gamma_{mn} = \sqrt{h_{mn}^2 - k^2} = \sqrt{(k_x^2 + k_y^2) - k^2}$$

$$h_{mn} = \sqrt{k_x^2 + k_y^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\gamma_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \epsilon}$$

The equation for the waveguide propagation constant  $\gamma_{mn}$  can be used to determine the cutoff frequency for the respective waveguide mode. The propagation characteristics of the wave are defined by the relative sizes of the parameters  $h_{mn}$  and  $k$ . The propagation constant may be written in terms of the attenuation and phase constants as

$$\gamma_{mn} = \alpha_{mn} + j\beta_{mn}$$

so that,

$$\text{if } h_{mn} = k \Rightarrow \gamma_{mn} = 0 \ (\alpha_{mn} = \beta_{mn} = 0) \Rightarrow \text{cutoff frequency}$$

$$\text{if } h_{mn} > k \Rightarrow \gamma_{mn} \text{ (real), } [\gamma_{mn} = \alpha_{mn}] \Rightarrow \text{evanescent modes}$$

$$\text{if } h_{mn} < k \Rightarrow \gamma_{mn} \text{ (imag.), } [\gamma_{mn} = j\beta_{mn}] \Rightarrow \text{propagating modes}$$

Therefore, the cutoff frequencies for the TE and TM modes in the rectangular waveguide are found by solving

$$h_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = k_{c_{mn}} = 2\pi f_{c_{mn}} \sqrt{\mu \epsilon}$$

$$f_{c_{mn}} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad \left( \begin{array}{c} \text{rectangular waveguide} \\ \text{cutoff frequency} \end{array} \right)$$

Note that the cutoff frequency for a particular rectangular waveguide mode depends on the dimensions of the waveguide ( $a, b$ ), the material inside the waveguide ( $\mu, \epsilon$ ), and the indices of the mode ( $m, n$ ). The rectangular waveguide must be operated at a frequency above the cutoff frequency for the respective mode to propagate.

According to the cutoff frequency equation, the cutoff frequencies of both the  $TE_{10}$  and  $TE_{01}$  modes are less than that of the lowest order TM mode ( $TM_{11}$ ). Given  $a > b$  for the rectangular waveguide, the  $TE_{10}$  has the lowest cutoff frequency of any of the rectangular waveguide modes and is thus the dominant mode (the first to propagate). Note that the  $TE_{10}$  and  $TE_{01}$  modes are *degenerate modes* (modes with the same cutoff frequency) for a square waveguide. The rectangular waveguide allows one to operate at a frequency above the cutoff of the dominant  $TE_{10}$  mode but below that of the next highest mode to achieve single mode operation. A waveguide operating at a frequency where more than one mode propagates is said to be *overmoded*.

### Example (Rectangular waveguide propagating modes)

A rectangular waveguide ( $a = 2$  cm,  $b = 1$  cm) filled with deionized water ( $\mu_r = 1$ ,  $\epsilon_r = 81$ ) operates at 3 GHz. Determine all propagating modes and the corresponding cutoff frequencies.

$$\begin{aligned} f_{c_{mn}} &= \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{c}{2\sqrt{(1)(81)}} \sqrt{\left(\frac{m}{0.02}\right)^2 + \left(\frac{n}{0.01}\right)^2} \\ &= \frac{c}{18} \sqrt{\left(\frac{m}{0.02}\right)^2 + \left(\frac{n}{0.01}\right)^2} \end{aligned}$$

Cutoff frequencies - TM modes (GHz)

	$n=1$	$n=2$	...
$m=1$	1.863	3.436	...
$m=2$	2.357	3.727	...
$m=3$	3.005	∞	
	⋮		

Cutoff frequencies - TE modes (GHz)

	$n=0$	$n=1$	$n=2$	
$m=0$	×	1.667	3.333	...
$m=1$	0.833	1.863	3.436	...
$m=2$	1.667	2.357	3.727	...
$m=3$	2.500	3.005	∞	
$m=4$	3.333	⋮		
	⋮			

Mode	$f_c$ (GHz)
TE <sub>10</sub>	0.833
TE <sub>01</sub> , TE <sub>20</sub>	1.667
TE <sub>11</sub> , TM <sub>11</sub>	1.863
TE <sub>21</sub> , TM <sub>21</sub>	2.357
TE <sub>30</sub>	2.5

As previously shown, the propagation constant for a given mode can be defined in terms of the cutoff frequency for that mode by

$$\gamma_{mn} = jk \sqrt{1 - \left( \frac{f_{c_{mn}}}{f} \right)^2} = j\beta_{mn}$$

The field components, cutoff frequency and propagation constant associated with the dominant TE<sub>10</sub> mode (using the TE<sub>*mn*</sub> equations with  $m=1$ ,  $n=0$ , and  $\gamma_{mn} = j\beta_{10}$ ) are:

$$\begin{aligned}
\tilde{E}_x^{TE_{10}} &= 0 & \tilde{H}_x^{TE_{10}} &= \frac{j\beta_{10}a}{\pi} H_o \sin \frac{\pi x}{a} e^{-j\beta_{10}z} \\
\tilde{E}_y^{TE_{10}} &= -\frac{j\omega\mu a}{\pi} H_o \sin \frac{\pi x}{a} e^{-j\beta_{10}z} & \tilde{H}_y^{TE_{10}} &= 0 \\
\tilde{E}_z^{TE_{10}} &= 0 & \tilde{H}_z^{TE_{10}} &= H_o \cos \frac{\pi x}{a} e^{-j\beta_{10}z}
\end{aligned}$$

$$f_{c10} = \frac{1}{2a\sqrt{\mu\epsilon}} \quad \gamma_{10} = jk \sqrt{1 - \left(\frac{f_{c10}}{f}\right)^2} = j\beta_{10} \quad h_{10} = \frac{\pi}{a}$$

The corresponding instantaneous fields of the  $TE_{10}$  mode are determined by multiplying the phasor field components by  $e^{j\omega t}$  and taking the real part of the result.

$$\begin{aligned}
E_y^{TE_{10}} &= \frac{\omega\mu a}{\pi} H_o \sin \frac{\pi x}{a} \sin(\omega t - \beta_{10}z) \\
H_x^{TE_{10}} &= -\frac{\beta_{10}a}{\pi} H_o \sin \frac{\pi x}{a} \sin(\omega t - \beta_{10}z) \\
H_z^{TE_{10}} &= H_o \cos \frac{\pi x}{a} \cos(\omega t - \beta_{10}z) \\
E_x^{TE_{10}} &= E_z^{TE_{10}} = H_y^{TE_{10}} = 0
\end{aligned}$$

The waveguide wavelength is defined using the same definition as for unguided (TEM) waves [ $\lambda = 2\pi/\beta$ ]. However, the size of the waveguide wavelength can be quite different than that of an unguided wave at the same frequency. The wavelength of a TE or TM mode propagating in the rectangular waveguide can be written in terms of the wavelength for an unguided (TEM) wave propagating in the same medium ( $\mu, \epsilon$ ) as found inside the waveguide (designated as  $\lambda'$ ).

$$\lambda_{mn} = \frac{2\pi}{\beta_{mn}} = \frac{2\pi}{k \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} = \frac{\lambda'}{\sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\lambda' = \frac{2\pi}{k} \quad (\text{TEM wavelength})$$

The denominator of the rectangular waveguide wavelength equation becomes very small when the operating frequency is very close to the cutoff frequency. This yields a waveguide wavelength which is much longer than that of an unguided wave traveling through the same medium at the same frequency. Conversely, if the operating frequency is very large in comparison to the cutoff frequency, the denominator approaches a value of unity, and the waveguide wavelength is approximately equal to the TEM wavelength.

Just as the characteristic (wave) impedance for the TEM modes on a transmission line is defined by a ratio of the transverse electric field to the transverse magnetic field, the wave impedances of the TE and TM waveguide modes can be defined in the same manner. The waveguide wave impedance can be related to the wave impedance of a TEM wave traveling through the same medium (as that inside the waveguide) at the same frequency. The waveguide TE and TM wave impedances are defined by

$$\eta_{TE_{mn}} = \frac{\tilde{E}_x^{TE_{mn}}}{\tilde{H}_y^{TE_{mn}}} = -\frac{\tilde{E}_y^{TE_{mn}}}{\tilde{H}_x^{TE_{mn}}} = \frac{\omega\mu}{\beta_{mn}} = \frac{\omega\mu}{k \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\eta_{TM_{mn}} = \frac{\tilde{E}_x^{TM_{mn}}}{\tilde{H}_y^{TM_{mn}}} = -\frac{\tilde{E}_y^{TM_{mn}}}{\tilde{H}_x^{TM_{mn}}} = \frac{\beta_{mn}}{\omega\epsilon} = \frac{k}{\omega\epsilon \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

$$\frac{\omega \mu}{k} = \sqrt{\frac{\mu}{\epsilon}} = \eta' \qquad \frac{k}{\omega \epsilon} = \sqrt{\frac{\mu}{\epsilon}} = \eta' \qquad \left( \begin{array}{c} \text{TEM intrinsic} \\ \text{wave impedance} \end{array} \right)$$

$$\eta_{TE_{mn}} = \frac{\eta'}{\sqrt{1 - \left( \frac{f_{c_{mn}}}{f} \right)^2}}$$

$$\eta_{TM_{mn}} = \eta' \sqrt{1 - \left( \frac{f_{c_{mn}}}{f} \right)^2}$$

Note that the product of the TE and TM wave impedances is equal to the square of the TEM wave impedance.

$$\eta_{TM_{mn}} \eta_{TE_{mn}} = \eta'^2$$

## Waveguide Group Velocity and Phase Velocity

The velocity of propagation for a TEM wave (plane wave or transmission line wave) is referred to as the *phase velocity* (the velocity at which a point of constant phase moves). The phase velocity of a TEM wave is equal to the velocity of energy transport. The phase velocity of a TEM wave traveling in a lossless medium characterized by  $(\mu, \epsilon)$  is given by

$$u_p' = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \quad (\text{TEM phase velocity})$$

The phase velocity of TE or TM mode in a waveguide is defined in the same manner as that of a TEM wave (the velocity at which a point of constant phase moves). We will find, however, that the waveguide phase velocity is not equal to the velocity of energy transport along the waveguide. The velocity at which energy is transported down the length of the waveguide is defined as the *group velocity*.

The differences between the waveguide phase velocity and group velocity can be illustrated using the field equations of the TE or TM rectangular waveguide modes. It can be shown that the field components of general TE and TM waveguide modes can be written as sums and differences of TEM waves. Consider the equation for the  $y$ -component of the TE mode electric field in a rectangular waveguide.

$$\tilde{E}_y^{TE_{10}} = -\frac{j\omega\mu a}{\pi} H_o \sin \frac{\pi x}{a} e^{-j\beta_{10} z}$$

By applying the trigonometric identity:

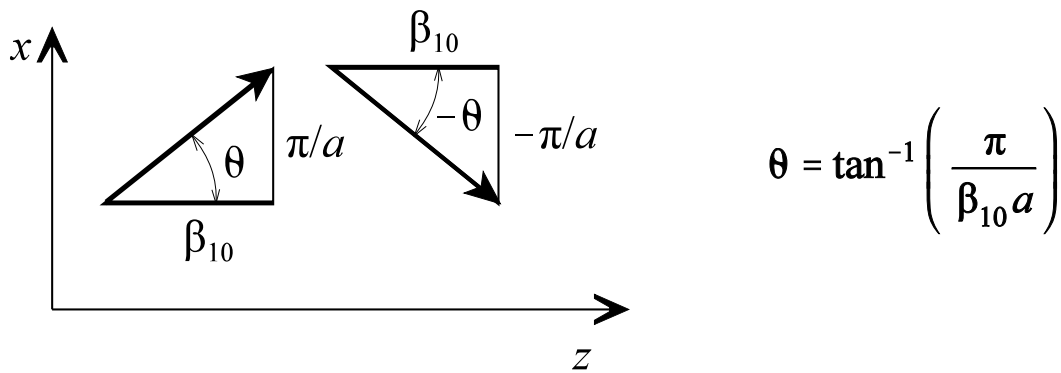
$$\sin \frac{\pi x}{a} = \frac{1}{2j} [e^{j\pi x/a} - e^{-j\pi x/a}]$$

this component of the waveguide electric field can be written as

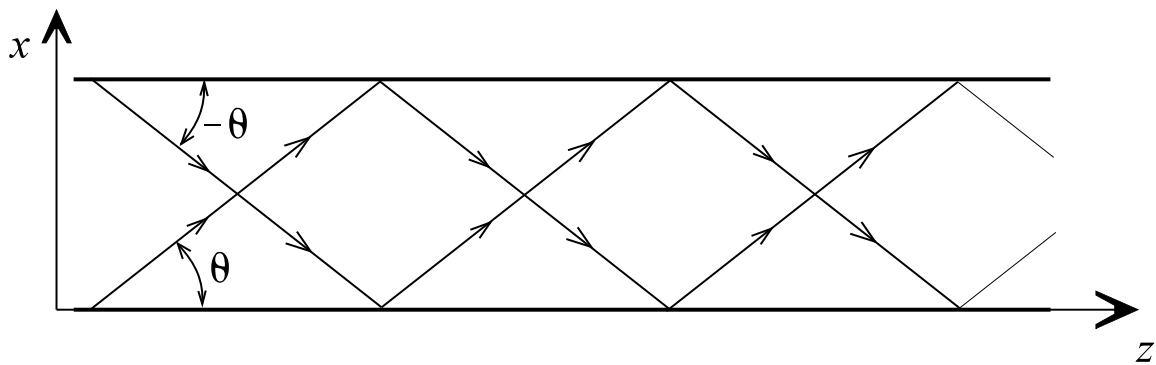
$$\tilde{E}_y^{TE_{10}} = \frac{\omega \mu a}{2\pi} H_o \left[ e^{-j(\beta_{10}z + \pi x/a)} - e^{-j(\beta_{10}z - \pi x/a)} \right]$$

$$\beta_{10} = k \sqrt{1 - \left( \frac{f_{c10}}{f} \right)^2}$$

The two terms in the TE field equation above represent TEM waves moving in the directions shown below.



Thus, the TE wave in the rectangular waveguide can be represented as the superposition of two TEM waves reflecting from the upper and lower waveguide walls as they travel down the waveguide.



For the general  $TE_{mn}$  or  $TM_{mn}$  waves, the phase velocity of the TEM component is given by

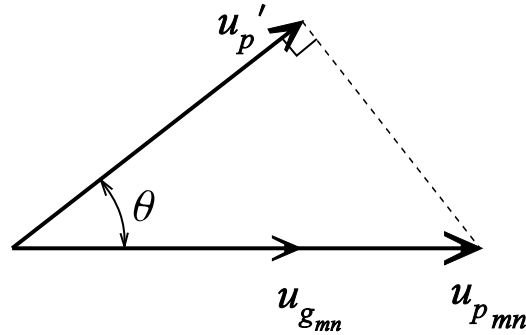
$$u_{p_{mn}} = \frac{\omega}{\beta_{mn}}$$



Inserting the equation for the waveguide phase constant  $\beta_{mn}$  gives

$$u_{p_{mn}} = \frac{\omega}{\beta_{mn}} = \frac{\omega}{k \sqrt{1 - \left( \frac{f_{c_{mn}}}{f} \right)^2}} = \frac{u_p'}{\sqrt{1 - \left( \frac{f_{c_{mn}}}{f} \right)^2}}$$

The waveguide phase velocity represents the speed at which points of constant phase of the component TEM waves travel down the waveguide. The waveguide phase velocity is larger than the TEM wave phase velocity given that the square root in the denominator of the waveguide phase velocity equation is less than unity. The relationship between the waveguide phase velocity, waveguide group velocity, and the TEM component wave velocity is shown below.



$$u_p' = u_{p_{mn}} \cos \theta \quad \Rightarrow \quad u_{p_{mn}} = \frac{u_p'}{\cos \theta} \quad \Rightarrow \quad \cos \theta = \sqrt{1 - \left( \frac{f_{c_{mn}}}{f} \right)^2}$$

$$u_{g_{mn}} = u_p' \cos \theta = u_p' \sqrt{1 - \left( \frac{f_{c_{mn}}}{f} \right)^2}$$

$$u_{g_{mn}} u_{p_{mn}} = u_p'^2$$

The waveguide group velocity (the velocity of energy transport) is always smaller than the TEM wave phase velocity given the square root term in the numerator of the group velocity equation.

### Example

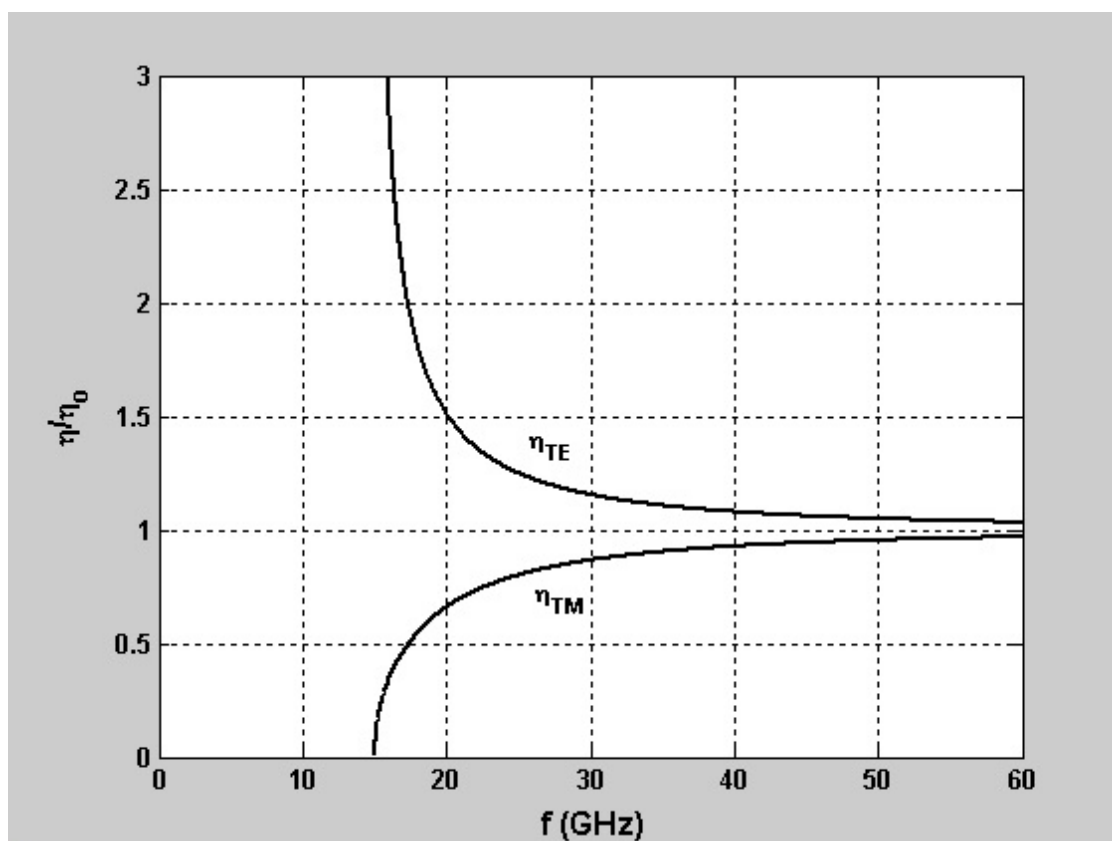
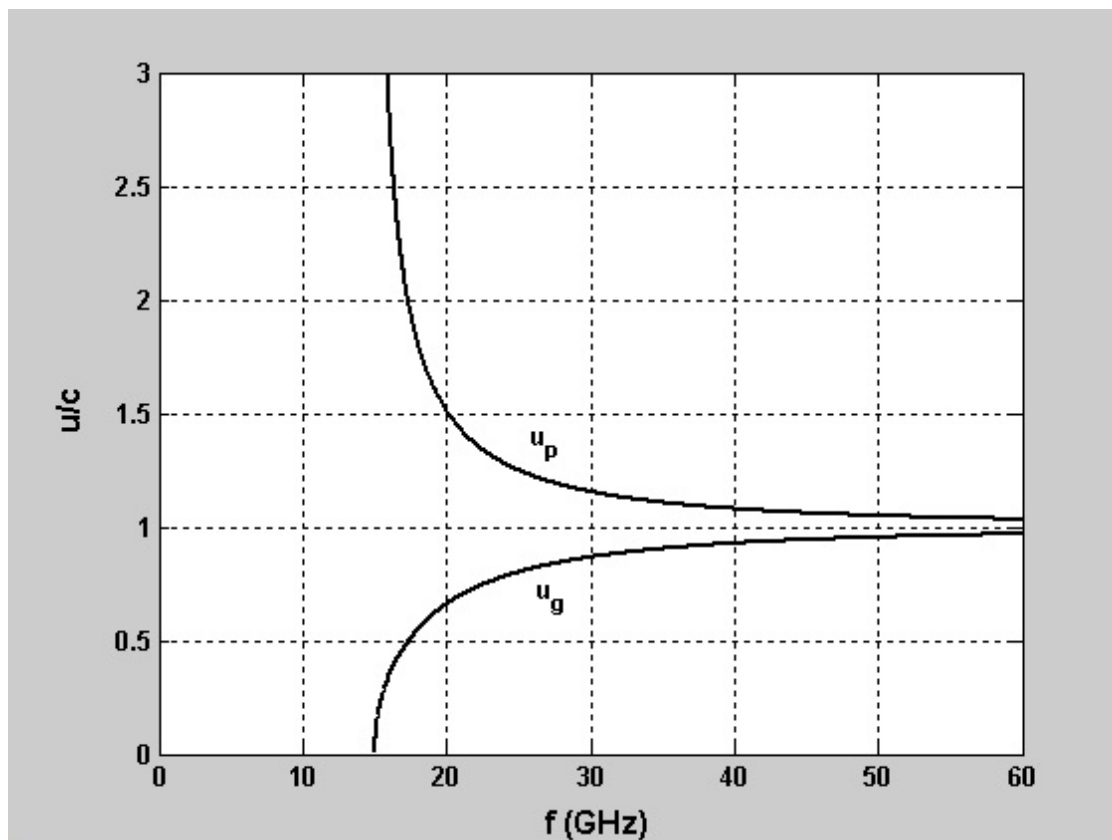
Given a pair of degenerate modes ( $TE_{mn}$  and  $TM_{mn}$ ) in an air-filled rectangular waveguide with a cutoff frequency of 15 GHz, plot the following parameters as a function of frequency: phase velocity and group velocity, TE wave impedance and TM wave impedance, TEM wavelength and mode wavelength, TEM phase constant and mode phase constant.

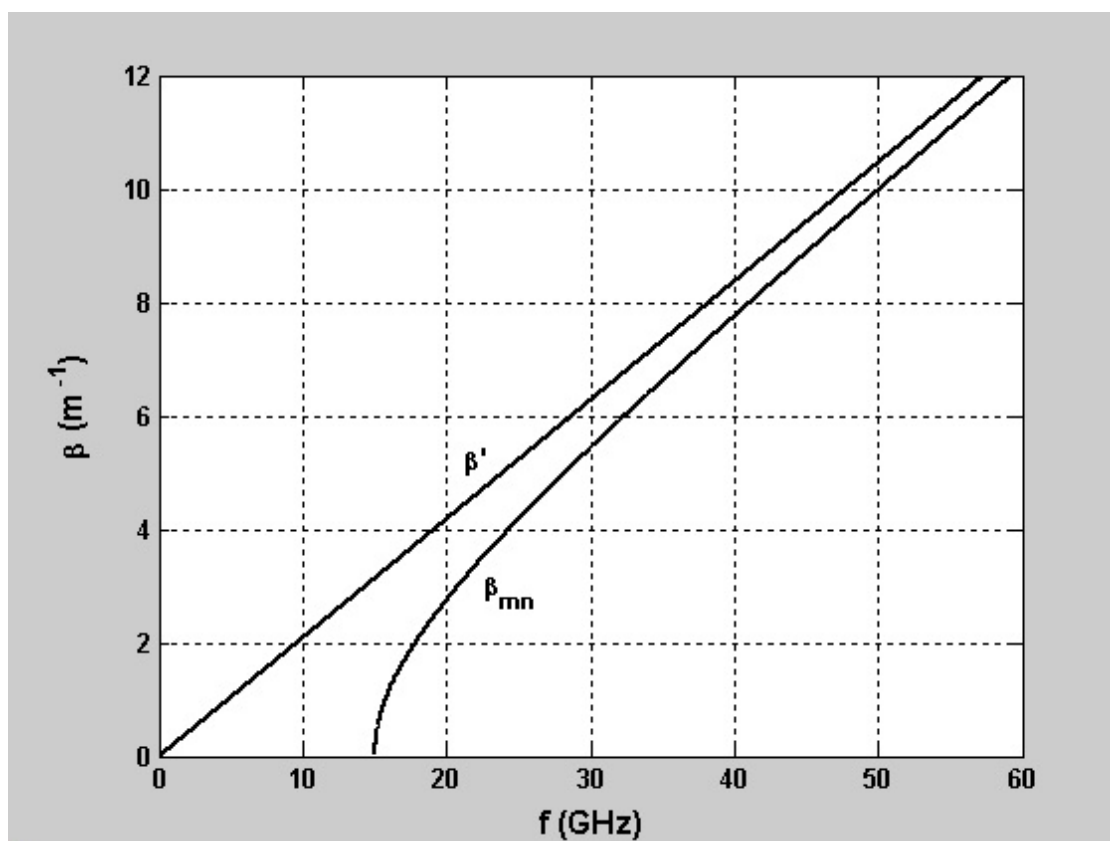
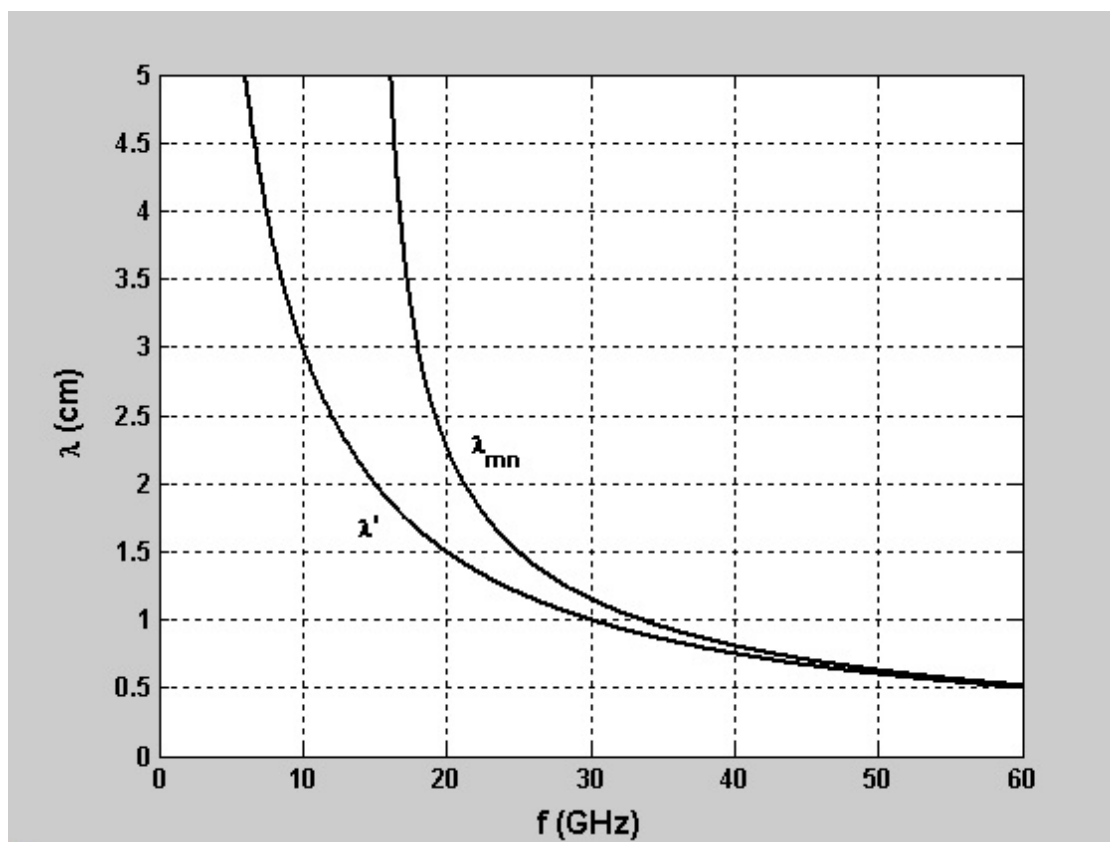
$$u_p' = \frac{1}{\sqrt{\mu_o \epsilon_o}} = c$$

$$\eta' = \sqrt{\frac{\mu_o}{\epsilon_o}} = \eta_o$$

$$\lambda' = \frac{c}{f}$$

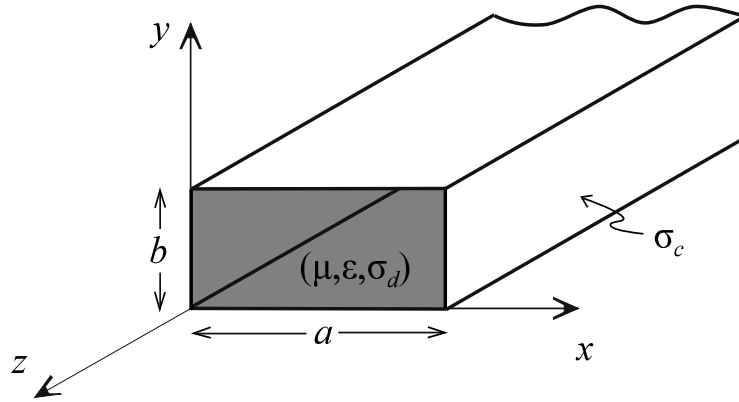
$$\beta' = k = \omega \sqrt{\mu_o \epsilon_o} = \frac{\omega}{c}$$





## Attenuation in Waveguides

Only ideal waveguides have been considered thus far (characterized by a perfect conductor filled with a perfect insulator). The propagating waves in an ideal waveguide suffer no attenuation as they travel down the waveguide. Two loss mechanisms exist in a realistic waveguide: conductor loss and dielectric loss. The fields associated with the propagating waveguide modes produce currents that flow in the walls of the waveguide. Given that the waveguide walls are constructed from an imperfect conductor ( $\sigma_c < \infty$ ), the walls act like resistors and dissipate energy in the form of heat. Also, the dielectric within the waveguide is not ideal ( $\sigma_d > 0$ ) so that dielectric also dissipates energy in the form of heat.



The overall attenuation constant  $\alpha$  (in units of Np/m) for a realistic waveguide can be written in terms of the two loss components as

$$\alpha_{mn} = \alpha_{c_{mn}} + \alpha_{d_{mn}}$$

where  $\alpha_c$  is the attenuation constant due to conductor loss and  $\alpha_d$  is the attenuation constant due to dielectric loss. For either TE or TM modes in a rectangular waveguide, the attenuation constant due to dielectric loss is given by

$$\alpha_{d_{mn}} = \frac{\sigma_d \eta'}{2 \sqrt{1 - \left( \frac{f_{c_{mn}}}{f} \right)^2}} \quad \eta' = \sqrt{\frac{\mu}{\epsilon}}$$

The attenuation constant due to conductor loss in a rectangular waveguide depends on the mode type (TE or TM) due to the different components of field present in these modes. The attenuation constant due to conductor losses for the  $TM_{mn}$  mode in a rectangular waveguide is given by

$$\alpha_{c_{mn}}^{TM} = \frac{2R_s}{b\eta' \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} \left[ \frac{b^3 m^2 + a^3 n^2}{ab^2 m^2 + a^3 n^2} \right]$$

where

$$R_s = \frac{1}{\sigma_c \delta}$$

is the surface resistance of the waveguide walls and

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}}$$

is the skin depth of the waveguide walls at the operating frequency. It is assumed that the waveguide wall thickness is several skin depths such that the wall currents are essentially surface currents. This is an accurate assumption at the typical operating frequencies of waveguides ( $\sim$ GHz) where the skin depth of common conductors like aluminum and copper are on the order of  $\mu\text{m}$ .

The attenuation constant due to conductor losses for the  $TE_{mn}$  mode in a rectangular waveguide with ( $n \neq 0$ ) is given by

$$\alpha_{c_{mn}}^{TE} = \frac{2R_s}{b\eta' \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} \left\{ \left[ 1 + \frac{b}{a} \right] \left( \frac{f_{c_{mn}}}{f} \right)^2 + \frac{b^2 m^2 + abn^2}{b^2 m^2 + a^2 n^2} \left[ 1 - \left( \frac{f_{c_{mn}}}{f} \right)^2 \right] \right\}$$

For the special case of ( $n = 0$ ), the attenuation constant due to conductor losses for the  $TE_{m0}$  mode in a rectangular waveguide is

$$\alpha_{c_{m0}}^{TE} = \frac{2R_s}{b\eta' \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} \left\{ \frac{1}{2} + \frac{b}{a} \left(\frac{f_{c_{mn}}}{f}\right)^2 \right\}$$

The equation above applies to the dominant rectangular waveguide mode [ $TE_{10}$ ].

### Example (Waveguide attenuation)

An aluminum waveguide ( $a = 4.2$  cm,  $b = 1.5$  cm,  $\sigma_c = 3.5 \times 10^7$   $\Omega/\text{m}$ ) filled with polystyrene ( $\mu_r = 1$ ,  $\epsilon_r = 2.6$ ,  $\sigma_d = 10^{-15}$   $\Omega/\text{m}$ ) operates at 4 GHz. Determine (a.)  $\alpha_c$  and  $\alpha_d$  for the  $TE_{10}$  mode (b.) the waveguide loss in dB over a distance of 1.5 m.

$$(a.) \quad f_{c_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{1}{2(0.042)\sqrt{\mu_o(2.6\epsilon_o)}} = 2.213 \text{ GHz}$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}} = 1.35 \text{ } \mu\text{m}$$

$$\eta' = \sqrt{\frac{\mu}{\epsilon}} = 233.6 \text{ } \Omega$$

$$\alpha_{c_{10}}^{TE} = 8.868 \times 10^{-3} \text{ Np/m}$$

$$\alpha_{d_{10}} = 1.403 \times 10^{-13} \text{ Np/m}$$

$$(b.) \quad \alpha_{mn} = \alpha_{c_{mn}} + \alpha_{d_{mn}} = 8.868 \times 10^{-3} + 1.403 \times 10^{-13}$$

For this problem, we see that the dielectric losses are negligible in comparison to the conductor losses.

$$\alpha_{mn} \approx \alpha_{c_{mn}} = 8.868 \times 10^{-3} \text{ Np/m}$$

The waves propagating in the +z direction in the rectangular waveguide vary as

$$e^{-\gamma_{mn}z} = e^{-(\alpha_{mn} + j\beta_{mn})z} = e^{-\alpha_{mn}z} e^{-j\beta_{mn}z}$$

Thus, over a distance of 1.5 m, the fields associated with the wave decay according to

$$e^{-\alpha_{mn}z} = e^{-(8.868 \times 10^{-3} \text{ Np/m})(1.5 \text{ m})} = 0.9868$$

In terms of dB, we find

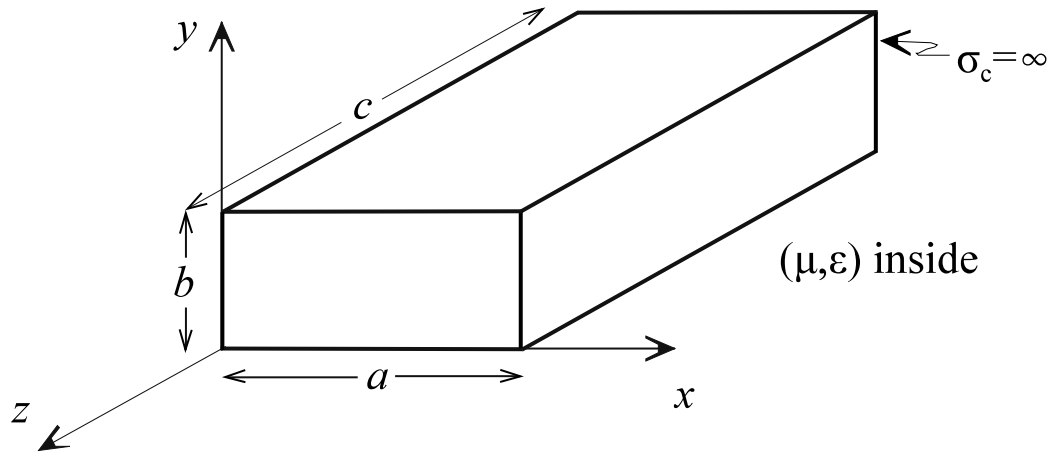
$$20 \log_{10} (0.9868) = -0.1154 \text{ dB}$$

[a loss of 0.1154 dB in 1.5m].



## Cavity Resonators

At high frequencies where waveguides are used, lumped element tuned circuits (RLC circuits) are very inefficient. As the element dimensions become comparable to the wavelength, unwanted radiation from the circuit occurs. Waveguide resonators are used in place of the lumped element RLC circuit to provide a tuned circuit at high frequencies. The rectangular waveguide resonator is basically a section of rectangular waveguide which is enclosed on both ends by conducting walls to form an enclosed conducting box. We assume the same cross-sectional dimensions as the rectangular waveguide ( $a, b$ ) and define the longitudinal length of the resonator as  $c$ . Given the conducting walls on the ends of the waveguide, the resonator modes may be described by waveguide modes which are reflected back and forth within the resonator ( $+z$  and  $-z$  directions) to form standing waves.



Waveguide (waves in one direction)

$$\tilde{H}_z(x, y, z) = H_o \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z} \quad (\text{TE modes})$$

$$\tilde{E}_z(x, y, z) = E_o \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z} \quad (\text{TM modes})$$

Cavity (waves in both directions, standing waves)

$$\tilde{H}_z(x,y,z) = H_o \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} (A \sin k_z z + B \cos k_z z) \quad (\text{TE modes})$$

$$\tilde{E}_z(x,y,z) = E_o \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (C \sin k_z z + D \cos k_z z) \quad (\text{TM modes})$$

The separation equation for the cavity modes is

$$k^2 = k_x^2 + k_y^2 + k_z^2 \quad (\text{separation equation})$$

The cavity boundary conditions (in addition to the boundary conditions satisfied by the rectangular waveguide wave functions) are

$$\tilde{E}_x(x,y,0) = \tilde{E}_y(x,y,0) = 0 \quad (\text{end wall})$$

$$\tilde{E}_x(x,y,c) = \tilde{E}_y(x,y,c) = 0 \quad (\text{end wall})$$

From the source-free Maxwell's curl equations, the TE and TM boundary conditions on the end walls of the cavity are satisfied if

$$\tilde{H}_z(x,y,0) = \tilde{H}_z(x,y,c) = 0 \quad (\text{TE modes})$$

$$\left. \frac{\partial \tilde{E}_z(x,y,z)}{\partial z} \right|_{z=0} = \left. \frac{\partial \tilde{E}_z(x,y,z)}{\partial z} \right|_{z=c} = 0 \quad (\text{TM modes})$$

Application of the TE and TM boundary conditions yields

$$\tilde{H}_z(x,y,0) = 0 \quad \Rightarrow \quad B = 0$$

$$\tilde{H}_z(x,y,c) = 0 \quad \Rightarrow \quad k_z c = p\pi \quad (p = 1, 2, 3, \dots) \quad \Rightarrow \quad k_z = \frac{p\pi}{c}$$

$$\tilde{E}_z(x,y,0) = 0 \quad \Rightarrow \quad C = 0$$

$$\tilde{E}_s(x,y,c) = 0 \quad \Rightarrow \quad k_z c = p\pi \quad (p = 0, 1, 2, \dots) \quad \Rightarrow \quad k_z = \frac{p\pi}{c}$$

The TE and TM modes in the rectangular cavity are then

$$\tilde{H}_z(x,y,z) = H_o \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sin \frac{p\pi z}{c} \quad (\text{TE}_{\text{mnp}} \text{ modes})$$

$$(m = 0, 1, 2, \dots)$$

$$(n = 0, 1, 2, \dots) \quad m = n \neq 0$$

$$(p = 1, 2, 3, \dots)$$

$$\tilde{E}_z(x,y,z) = E_o \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \frac{p\pi z}{c} \quad (\text{TM}_{\text{mnp}} \text{ modes})$$

$$(m = 1, 2, 3, \dots)$$

$$(n = 1, 2, 3, \dots)$$

$$(p = 0, 1, 2, \dots)$$

The resonant frequency associated with the TE<sub>mnp</sub> or TM<sub>mnp</sub> mode is found from the separation equation to be

$$k = \omega \sqrt{\mu\epsilon} = 2\pi f_r \sqrt{\mu\epsilon} = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2}$$

$$f_r = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2}$$

The lowest order modes in a rectangular cavity are the  $TM_{110}$ ,  $TE_{101}$ , and  $TE_{011}$  modes. Which of these modes is the dominant mode depends on the relative dimensions of the resonator.

Example (Cavity resonator)

Find the first five resonances of an air-filled rectangular cavity with dimensions of  $a = 5$  cm,  $b = 4$  cm and  $c = 10$  cm ( $c > a > b$ ).

$$f_r = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2}$$

$$(m = 0, 1, 2, \dots)$$

$$(\text{TE}_{mnp} \text{ modes}) \quad (n = 0, 1, 2, \dots) \quad m = n \neq 0$$

$$(p = 1, 2, 3, \dots)$$

$$(m = 1, 2, 3, \dots)$$

$$(\text{TM}_{mnp} \text{ modes}) \quad (n = 1, 2, 3, \dots)$$

$$(p = 0, 1, 2, \dots)$$

$$f_{r_{101}} = 3.335 \text{ GHz} \quad TE_{101}$$

$$f_{r_{011}} = 4.040 \text{ GHz} \quad TE_{011}$$

$$f_{r_{102}} = 4.243 \text{ GHz} \quad TE_{102}$$

$$f_{r_{110}} = 4.800 \text{ GHz} \quad TM_{110}$$

$$f_{r_{111}} = 5.031 \text{ GHz} \quad TE_{111}, TM_{111}$$

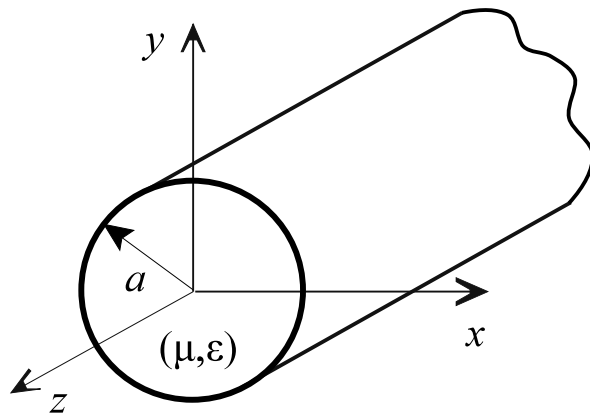
The quality factor (Q) of a waveguide resonator is defined the same way as that for an RLC network.

$$Q = 2\pi \frac{\text{energy stored/cycle}}{\text{energy lost/cycle}}$$

where the energy lost per cycle is that energy dissipated in the form of heat in the waveguide dielectric and the cavity walls (ohmic losses). The resonator quality factor is inversely proportional to its bandwidth. Given a resonator made from a conductor such as copper or aluminum, the ohmic losses are very small and the quality factor is large (high Q, small bandwidth). Thus, resonators are used in applications such as oscillators, filters, and tuned amplifiers. Comparing the modes of the rectangular resonator with the propagating modes in the rectangular waveguide, we see that the waveguide modes exist over a wide band (the rectangular waveguide acts like a high-pass filter) while the rectangular resonator modes exist over a very narrow band (the rectangular resonator acts like a band-pass filter).

### Circular Waveguide

The same techniques used to analyze the ideal rectangular waveguide may be used to determine the modes that propagate within an ideal circular waveguide [radius =  $a$ , filled with dielectric ( $\mu, \epsilon$ )] The separation of variables technique yields solutions for the circular waveguide TE and TM propagating modes in terms of Bessel functions. The cutoff frequencies for the circular waveguide can be written in terms of the zeros associated with Bessel functions and derivatives of Bessel functions.



The cutoff frequencies of the TE and TM modes in a circular waveguide are given by

$$f_{c_{mn}}^{TE} = \frac{\chi'_{mn}}{2\pi a \sqrt{\mu\epsilon}}$$

$$f_{c_{mn}}^{TM} = \frac{\chi_{mn}}{2\pi a \sqrt{\mu\epsilon}}$$

where  $\chi_{mn}$  and  $\chi'_{mn}$  define the  $n^{\text{th}}$  zero of the  $m^{\text{th}}$ -order Bessel function and Bessel function derivative, respectively. The values of these zeros are shown in the tables below.

#### TE modes

$\chi'_{mn}$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	...
$n=1$	3.8318	1.8412	3.0542	4.2012	5.3175	6.4155	7.5013	...
$n=2$	7.0156	5.3315	6.7062	8.0153	9.2824	10.5199	11.7349	...
$n=3$	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872	15.2682	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

#### TM modes

$\chi_{mn}$	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	...
$n=1$	2.4049	3.8318	5.1357	6.3802	7.5884	8.7715	9.9361	...
$n=2$	5.5201	7.1056	8.4173	9.7610	11.0647	12.3386	13.5893	...
$n=3$	8.6537	10.1735	11.6199	13.0152	14.3726	15.7002	17.0038	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Note that the dominant mode in a circular waveguide is the TE<sub>11</sub> mode, followed in order by the TM<sub>01</sub> mode, the TE<sub>21</sub> mode and the TE<sub>01</sub> mode.

### Example (Circular waveguide)

Design an air-filled circular waveguide yielding a frequency separation of 1 GHz between the cutoff frequencies of the dominant mode and the next highest mode.

The cutoff frequencies of the  $TE_{11}$  mode (dominant mode) and the  $TM_{01}$  mode (next highest mode) for an air-filled circular waveguide are

$$f_{c11}^{TE} = \frac{\chi'_{11}}{2\pi a \sqrt{\mu_o \epsilon_o}} = \frac{1.8412}{2\pi a \sqrt{\mu_o \epsilon_o}}$$

$$f_{c01}^{TM} = \frac{\chi_{01}}{2\pi a \sqrt{\mu_o \epsilon_o}} = \frac{2.4049}{2\pi a \sqrt{\mu_o \epsilon_o}}$$

For a difference of 1 GHz between these frequencies, we write

$$f_{c01}^{TM} - f_{c11}^{TE} = \frac{2.4049 - 1.8412}{2\pi a \sqrt{\mu_o \epsilon_o}} = 10^9$$

Solving this equation for the waveguide radius gives

$$a = \frac{2.4049 - 1.8412}{2\pi 10^9 \sqrt{\mu_o \epsilon_o}} = 2.6896 \text{ cm}$$

The corresponding cutoff frequencies for this waveguide are

$$f_{c11}^{TE} = \frac{\chi'_{11}}{2\pi a \sqrt{\mu_o \epsilon_o}} = \frac{1.8412}{2\pi (0.026896) \sqrt{\mu_o \epsilon_o}} = 3.266 \text{ GHz}$$

$$f_{c01}^{TM} = \frac{\chi_{01}}{2\pi a \sqrt{\mu_o \epsilon_o}} = \frac{2.4049}{2\pi (0.026896) \sqrt{\mu_o \epsilon_o}} = 4.266 \text{ GHz}$$

One unique feature of the circular waveguide is that some of the higher order modes ( $TE_{0n}$ ) have particularly low loss. The magnetic field distribution for these modes generates lower current levels on the walls of the waveguide than the other waveguide modes. Therefore, a circular waveguide carrying this mode is commonly used when signals are sent over relatively long distances (microwave antennas on tall towers).

The general equations for the circular waveguide  $TE_{mn}$  and  $TM_{mn}$  mode attenuation constants due to conductor loss are given by

$$\alpha_{c_{mn}}^{TE} = \frac{R_s}{a\eta' \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} \left[ \left(\frac{f_{c_{mn}}}{f}\right)^2 + \frac{m^2}{(\chi'_{mn})^2 - m^2} \right]$$

$$\alpha_{c_{mn}}^{TM} = \frac{R_s}{a\eta' \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}}$$

Example (Circular waveguide attenuation)

An air-filled copper waveguide ( $a = 5$  mm,  $\sigma_c = 5.8 \times 10^7$  U/m) is operated at 30 GHz. Determine the loss in dB/m for the  $TM_{01}$  mode.

$$f_{c_{01}}^{TM} = \frac{\chi_{01}}{2\pi a \sqrt{\mu_o \epsilon_o}} = \frac{2.4049}{2\pi(0.005) \sqrt{\mu_o \epsilon_o}} = 22.95 \text{ GHz}$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}} = 0.38154 \text{ } \mu\text{m}$$

$$R_s = \frac{1}{\sigma_c \delta} = 0.045189 \text{ } \Omega \quad \eta' = \eta_o = 377 \text{ } \Omega$$



$$\alpha_{c_{mn}}^{TM} = \frac{R_s}{a\eta' \sqrt{1 - \left(\frac{f_{c_{mn}}}{f}\right)^2}} = \frac{0.045189}{(0.005)(377) \sqrt{1 - \left(\frac{22.95}{30}\right)^2}}$$

$$= 0.0372 \text{ Np/m}$$

The attenuation in terms of dB/m is

$$20 \log_{10} (e^{-0.0372}) = -0.3231 \text{ dB/m}$$

[a loss of 0.3231 dB/m]