Indian Institute of Technology, Ropar
Department of Mathematics
MA202: Probability and Statistics
2nd Semester of Academic Year 2024-25
Tutorial Sheet 9: Conditional Moments and Function of Several Random Variables

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1. Let X and Y be two independent continuous random variables. Prove or disprove the following
$$P(X \le Y) = \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy$$
where $f_Y(\cdot)$ is the probability density function of Y and $F_X(\cdot)$ is the cumulative distribution function of X.

2. Let X and Y be two random variables such that X is standard normal. Further, $E(Y \mid X = x) = x^3$ for all $x \in \mathbb{R}$. Prove or disprove that correlation between X and Y is strictly positive, i.e., $corr(X, Y) > 0$.

3. Let X and Y be two random variables such that
$E(X + Y) = E(X - Y) = 0, Var(X + Y) = 3$ and $Var(X - Y) = 1$. Calculate $Cov(X, Y)$.

4. Let X and Y be independent random variables with respective moment generating functions
$$M_X(t) = \frac{(8 + e^t)^2}{81} \quad \text{and} \quad M_Y(t) = \frac{(1 + 3e^t)^3}{64}, \quad -\infty < t < \infty.$$
Then $P(X + Y = 1)$ equals __________

5. Let X and Y be two random variables such that X is uniformly distributed over $(0, 4)$ and the conditional distribution of Y given $X = x$ is uniform distribution over $(0, \frac{x}{2})$. Calculate $E(Y^2)$.

6. If X, Y are standard normal random variables and given that
$$\rho(aX + bY, bX + aY) = \frac{1 + 2ab}{a^2 + b^2}.$$
Find $\rho(X, Y)$ i.e the coefficient of correlation between X and Y.

7. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability mass function given by
$$P(X = x) = \left(\frac{\theta}{2}\right)^{|x|} (1 - \theta)^{1-|x|}, \ \ x = -1, 0, 1$$
where $\theta \in (0, 1)$ is a parameter. Find $E(\max(X_1, X_2, \ldots, X_n))$.

8. Let $X_1, X_2, \ldots, X_n$ be independent geometric RVs with parameters $p_1, p_2, \ldots, p_n$ respectively. Show that $\min(X_1, X_2, \ldots, X_n)$ is also a geometric RV with parameter
$$p = 1 - \prod_{i=1}^{n} (1 - p_i).$$

9. Let X and Y be iid $N(0, \sigma^2)$ RVs. Show that $\frac{X}{Y}$ is $Cauchy(1, 0)$.

10. Let X_i, $(i = 1, 2, \ldots, k)$ be independent uniform RVs with $X_i \sim U(0, 1)$. Show that $-2 \sum_{i=1}^{k} \log(X_i)$ is $\chi^2(2k)$.

11. Let X and Y be independent RVs with distribution $NB(r_1, p)$ and $NB(r_2, p)$, respectively. Show that the conditional PMF of X given $X + Y = t$ is

$$P(X = x \mid X + Y = t) = \frac{\binom{x+r_1-1}{x}\binom{t+r_2-x-1}{t-x}}{\binom{t+r_1+r_2-1}{t}}$$

If $r_1 = r_2 = 1$, conditional distribution is (discrete) uniform on $t + 1$ points.

12. Let X and Y be independent RVs with distribution $P(\lambda)$ and $P(\mu)$ respectively. Show that the conditional PMF of X given $X + Y$ is Binomial.

13. Let X and Y be independent and identically distributed RVs with exponential distribution with parameter β. Show that the distribution of $\frac{X}{X+Y}$ is $U(0, 1)$.

14. Suppose X, Y are independent r.v.s each having binomial distribution with parameters n and p, $(0 < p < 1)$. Find the joint PMF of $(X + Y, X - Y)$.

15. Let X, Y be i.i.d r.v.s with common PDF

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases}$$

Find the PDF of r.v.s $\min\{X, Y\}$, $\max\{X, Y\}$. Let $U = X + Y$ and $V = X - Y$. Find the conditional PDF of V, given $U = u$ for some fixed $u > 0$.

16. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution over unit interval, $(0, 1)$. Find mean and variance of $G = \left(\prod_{i=1}^{n} X_i\right)^{-\frac{1}{n}}$.

Best Wishes