Indian Institute of Technology, Ropar
Department of Mathematics
MA202: Probability and Statistics
2nd Semester of Academic Year 2024-25
Tutorial Sheet 12: Sampling Distributions-II

1. Let $X_1, X_2, \ldots, X_{20}$ be a random sample from the $N(5, 2)$ distribution. Define
$$Y_i = X_{2i} - X_{2i-1}, \quad i = 1, 2, \ldots, 10.$$
Find the distribution of $W = \frac{1}{4}\sum_{i=1}^{10} Y_i^2$.

2. Let X_1, X_2, X_3 be $i.i.d.$ random variables, each having the $N(0, 1)$ distribution. Prove or disprove: $\frac{\sqrt{2}(X_1 - X_2)}{\sqrt{(X_1+X_2)^2+2X_3^2}} \sim t_1$

3. Let $X_1, X_2, \ldots, X_{10}$ be a random sample from $N(1, 2)$ distribution. If
$$\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i \quad \text{and} \quad S^2 = \frac{1}{9}\sum_{i=1}^{10}(X_i - \bar{X})^2,$$
Find $\text{Var}(S^2)$.

4. Let $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$ be i.i.d. (independent and identically distributed) random variables following $N(\mu, \sigma^2)$. Define
$$\overline{X} = \frac{1}{3}\sum_{i=1}^{3} X_i, \quad \overline{Y} = \frac{1}{4}\sum_{j=1}^{4} Y_j.$$
Find $n - A$ if
$$A\frac{\overline{X} - \overline{Y}}{\sqrt{\sum_{i=1}^{3}(X_i - \overline{X})^2 + \sum_{i=j}^{4}(Y_i - \overline{Y})^2}}$$
follows Student's-t distribution with n degrees of freedom.

5. Let $X_1, \ldots, X_{10}$ be a random sample from a $N(3, 12)$ population. Suppose
$$Y_1 = \frac{1}{6}\sum_{i=1}^{6} X_i \quad \text{and} \quad Y_2 = \frac{1}{4}\sum_{i=7}^{10} X_i.$$
If
$$\frac{(Y_1 - Y_2)^2}{\alpha}$$
has a χ_1^2 distribution. Find α.

6. Let $X_1, X_2, \ldots$ be i.i.d. $\mathcal{N}(1, 1)$ random variables. Let
$$S_n = X_1^2 + X_2^2 + \cdots + X_n^2 \quad \text{for } n \geq 1.$$
Find
$$\lim_{n\to\infty} \frac{\text{Var}(S_n)}{n}$$

7. Let $X_1, X_2, \ldots, X_{100}$ be a random sample from a $N(2,4)$ population. Let

$$\bar{X} = \frac{1}{99}\sum_{i=1}^{99} X_i, \quad S = \sqrt{\frac{1}{98}\sum_{i=1}^{99}(X_i - \bar{X})^2}, \quad \text{and} \quad W = \frac{X_{100} - 2}{S}.$$

Find the distribution of W.

8. Let $X \sim \chi^2(61)$. Find $P(X > 50)$.

9. Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathcal{N}(\mu, \sigma^2)$ and $\overline{X}$ and S^2, respectively, be the sample mean and the sample variance. Let $X_{n+1} \sim \mathcal{N}(\mu, \sigma^2)$, and assume that $X_1, X_2, \ldots, X_n, X_{n+1}$ are independent. Find the sampling distribution of

$$\left(\frac{X_{n+1} - \overline{X}}{S}\right)\sqrt{\frac{n}{n+1}}.$$

10. If X_1, X_2, X_3 and X_4 are independent observations from N(0,1). State giving reasons, the sampling distributions of

   (a) $U = \frac{\sqrt{2}X_3}{\sqrt{X_1^2+X_2^2}}$

   (b) $V = \frac{3X_4^2}{X_1^2+X_2^2+X_3^2}$

11. Let $X_1, X_2, \cdots, X_5$ be a random sample of size 5 from a population having standard normal distribution. Let $\bar{X} = \frac{\sum_{i=1}^{5} X_i}{5}$ and $T = \sum_{i=1}^{5}(X_i - \bar{X})^2$. Find expectation and variance of $T^2 + \bar{X}^2$.

12. Using Central Limit Theorem, prove the following

$$\lim_{n\to\infty} e^{-n} \sum_{i=0}^{n+\sqrt{n}} \frac{n^i}{i!} = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-\frac{x^2}{2}} dx$$

Best Wishes