Indian Institute of Technology, Ropar
Department of Mathematics
MA202: Probability and Statistics
2nd Semester of Academic Year 2024-25
Tutorial Sheet 10: Convergence of Random Variables and Parameter Estimation

1. Let X be a random variable and $X_n = X + Y_n$, where

$$E(Y_n) = \frac{1}{n} \text{ and } Var(Y_n) = \frac{\sigma^2}{n},$$

for some constant $\sigma > 0$. Show that X_n converges to X in probability.

2. Let X be a discrete random variable with PMF given by

$$f_X(x) = \begin{cases} 1/3 & \text{if } x = 1, \\ 2/3 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consider a sequence of random variable $X_n := (1 + \frac{1}{n})X$. Prove or disprove that X_n converges in probability to X.

3. Consider $X_2, X_3, X_4, \ldots$ be a sequence of random variables such that

$$F_{X_n}(x) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{nx} & \text{for } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Show that X_n converges in distribution to $Exponential(1)$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from $U(0, \theta)$, $\theta > 0$. Find the probability distribution functions of $\min(X_1, X_2, \ldots, X_n)$ and $\max(X_1, X_2, \ldots, X_n)$.

5. Let $X_1, X_2, \cdots$ be a sequence of iid Uniform $[0, 1]$ random variables. Let $Y_n = \max\{1 + X_1, 1 + X_2 \cdots, 1 + X_n\}$. Show that Y_n converges in probability.

6. Consider the sequence of rvs $X_1, X_2, \ldots$ where the PDF of X_n is given by

$$f_{X_n}(x) = \begin{cases} (n-1)/2 & \text{for } -1/n < x < 1/n, \\ 1/n & \text{for } n < x < 1/n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Check if the sequence $X_n, n \geq 1$ converges in probability to 0.

7. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables following uniform distribution, $U(0, 1)$. Find

$$\lim_{n \to \infty} P\left(- \sum_{i=1}^{n} \ln(X_i) \leq n + \sqrt{n} \right).$$

8. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta, \theta)$. Find MLE of θ.

9. Let X_n be a random variable that follows Poisson distribution with mean n. Find

(a)

$$\lim_{N\to\infty} P\Big(\sum_{i=1}^{N} X_i \le N + \sqrt{N}\Big).$$

(b)

$$\lim_{N\to\infty} P\Big(\sum_{i=1}^{N} X_i \le N\Big).$$

10. Let $X_1, X_2, \dots,$ be a sequence of iid uniform random variables over unit interval. Then,

$$\lim_{n\to\infty} P\Big(\frac{-1}{n}\sum_{i=1}^{n} \ln(X_i) \le 1 + \frac{1}{\sqrt{n}}\Big)$$

equals

11. (a) Let $X_1, X_2, \dots, X_n$ be a random sample from uniform distribution, $U(a, b)$. Find the method of moments estimator (MME) of a and b.

    (b) Let $X_1, X_2, \dots, X_n$ be a random sample from uniform distribution $U(-\theta, \theta)$, $\theta > 0$. Find the method of moments estimator (MME) and maximum likelihood estimator (MLE) of θ.

12. (a) Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with pdf

$$f(x|\beta) = \frac{1}{2\beta}\exp\left(\frac{-|x|}{\beta}\right), \qquad -\infty < x < \infty, \qquad \beta > 0$$

    Find the method of moments estimator (MME) and maximum likelihood estimator (MLE) of β.

    (b) Let {0, 0, 1, 1, 0} be a random sample from Bernoulli(θ), where $\theta \in \{2/10,\ 5/10,\ 7/10\}$. Find the maximum likelihood estimate of θ.

13. Let $X_1, X_2, \dots, X_n$ be a random sample from $Binomial(m, \theta)$ where m is known. Find the maximum likelihood estimator of θ.

14. Let $X_1, X_2, \dots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. Find the maximum likelihood estimators of μ and σ^2.

Best Wishes