3.(50 points) Consider the following problem

$$\text{Minimize } \sum_{i=1}^{n} \frac{V_i}{x_i}$$

subject to

$$\sum_{i=1}^{n} x_i W_i \le B$$

$$x_i \ge 0 \text{ for all } i = 1, 2, \ldots, n,$$

where $W_1, \ldots, W_n$, $V_1, \ldots, V_n$, and B are positive constant parameters. Find the optimal solution of this problem using the Kuhn-Tucker conditions.

1 $\lambda \geq 0$ 2 $\lambda \left(\sum x_i W_i - B \right) = 0$

3 $\frac{V_i}{x_i^2} - \lambda W_i = 0 \quad i = 1, 2, \ldots, n$

$\Rightarrow \frac{V_i}{x_i^2} = \lambda W_i \Rightarrow x_i^2 = \frac{V_i}{\lambda W_i} \Rightarrow x_i = \frac{1}{\sqrt{\lambda}} \sqrt{\frac{V_i}{W_i}}$

$\lambda = 0 \Rightarrow x_i \to +\infty$ $\sum x_i W_i \le B$ not possible

$\lambda > 0 \Rightarrow \frac{1}{\sqrt{\lambda}} \sum_i \sqrt{\frac{V_i}{W_i}} \, W_i = B \Rightarrow \sqrt{\lambda} = \frac{\sum_{i=1}^{n} \sqrt{W_i V_i}}{B}$

$$\boxed{x_j^* = \frac{B}{\sum_i \sqrt{W_i V_i}} \sqrt{\frac{V_j}{W_j}}}$$

$$L = \sum_{i=1}^{n} \frac{V_i}{x_i} + \sum_{i=1}^{n} \mu (B - x_i w_i) + \sum_{i=1}^{n} \lambda_i (0 + x_i)$$

$x_i \geq 0$.

$-x_i \leq 0$.

$$\frac{\partial L}{\partial x_i} = -\frac{V_i}{x_i^2} + \mu w_i + \lambda_i = 0 \quad ①$$

$$\sum_{i=1}^{n} x_i w_i \leq B \quad ②$$

$$-x_i \leq 0 \quad ③$$

$$\mu (B - x_i w_i) = 0 \quad ④$$

$$\lambda_i x_i = 0 \quad ⑤$$

$$\mu \geq 0 \quad ⑥$$

$$\lambda_i \geq 0 \quad ⑦$$

from ① $\Rightarrow x_i \neq 0$. and ⑦ $\lambda_i \geq 0$, $\therefore$ from ⑤ $\Rightarrow \lambda_i = 0$.

(division by zero)

ii. $\lambda_i = 0$

$$\begin{cases} -\dfrac{v_i}{x_i^2} + \mu w_i = 0 & (1) \\ \sum_{i=1}^{n} x_i w_i \leq B & (2) \\ x_i \geq 0 & (3) \\ \mu(B - x_i w_i) = 0 & (4) \\ \mu \geq 0 & (5) \end{cases}$$

Case 1. $\mu = 0$.

$\dfrac{v_i}{x_i^2} = 0$. Not Possible.

Case 2. $\mu > 0$. $\dfrac{v_i}{x_i^2} = \mu w_i$. $\Rightarrow x_i^2 = \dfrac{v_i}{\mu w_i} \Rightarrow x_i = \dfrac{1}{\sqrt{\mu}} \cdot \sqrt{\dfrac{v_i}{w_i}}$.

(from (2) and (4))

let x_k a possible point $\Rightarrow x_i = 0$ for $\begin{cases} i = 1, \ldots, n \\ i \neq k. \end{cases}$

$\therefore \begin{cases} x_k = \dfrac{1}{\sqrt{\mu}} \cdot \sqrt{\dfrac{v_k}{w_k}} \\ x_k = \dfrac{B}{w_k}. \end{cases}$

$$\frac{1}{\mu} \cdot \frac{v_k}{w_k} = \frac{B}{w_k}$$

$$\frac{1}{\mu} = \frac{B}{w_k} \cdot \frac{w_k}{v_k} = \frac{B}{v_k}.$$

$$\Rightarrow \mu = \frac{v_k}{B}.$$

$\therefore x_k = \dfrac{B}{w_k} = \dfrac{v_k}{B} \sqrt{\dfrac{v_k}{w_k}}.$

obj. ✓

there is only one point x_k, minimizes original function.