# Solving the Nonlinear Dyson Equation

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## 1 Problem

We solve the following Dyson-like equation:

$$G_i = g_i + g_i \Delta G_i, \quad \text{for } i \in \{1, 2\}, \tag{1}$$

where the *hybridization function* $\Delta$ is defined as:

$$\Delta(t, t') = J \frac{G_1(t, t') + G_2(t, t')}{2}. \tag{2}$$

The *free Green functions* correspond to the non-interacting Hamiltonians $h_1(t) = U(t)/2$ and $h_2(t) = -U(t)/2$, where $U(t)$ is a given function. The retarded free Green functions satisfy:

$$\frac{\mathrm{d}g_i^R(t, t')}{\mathrm{d}t} + ih_i(t)g_i^R(t, t') = -i\delta(t - t'). \tag{3}$$

In the real-time Keldysh formalism, we solve the following equations:

$$G^{\mathrm{R}} = g^{\mathrm{R}} + g^{\mathrm{R}}\Delta^{\mathrm{R}}G^{\mathrm{R}}, \tag{4}$$

$$G^{\mathrm{A}} = {G^{\mathrm{R}}}^\dagger, \tag{5}$$

$$G^{\mathrm{K}} = (\mathbb{1} + G^{\mathrm{R}}\Delta^{\mathrm{R}})g^{\mathrm{K}}(\mathbb{1} + \Delta^{\mathrm{A}}G^{\mathrm{A}}) + G^{\mathrm{R}}\Delta^{\mathrm{K}}G^{\mathrm{A}}, \tag{6}$$

where

$$\Delta^{\mathrm{R}}(t, t') = \frac{G_1^{\mathrm{R}}(t, t') + G_2^{\mathrm{R}}(t, t')}{2}, \tag{7}$$

$$\Delta^{\mathrm{K}}(t, t') = \frac{G_1^{\mathrm{K}}(t, t') + G_2^{\mathrm{K}}(t, t')}{2}. \tag{8}$$

## 2 Algorithm

### 2.1 Discretization

The kernels are discretized on a uniform time grid spanning from $t = 0$ to $t_f$:

$$\mathbf{A}[p, q] \equiv A(t_p, t_q), \qquad \text{where} \quad t_p = p\delta_t. \tag{9}$$

The number of time steps is given by $N \equiv t_f/\delta_t$. In the following, we consider the normalized Frobenius norm:

$$\|\cdot\| : A \mapsto \frac{1}{N}\sqrt{\sum_{p,q}\left(\mathbf{A}[p,q]\right)^2}. \tag{10}$$

This norm can be efficiently evaluated using matrix operations on the compressed representation, leveraging the relation:

$$\|\mathbf{A}\| = \sqrt{\mathrm{tr}(\mathbf{A}^\dagger\mathbf{A})}/N. \tag{11}$$

Integral operator are discritized by the Nystrom method using a trapezoidale quadrature.

## 2.2 Construction of $g_i$ and $J$

The operator $J$ is diagonal, making its construction trivial. To compress the free Green functions efficient implementation of the following maps are required

$$\mathbf{u} \mapsto \sum_q g_i(t_p, t_q)u_q, \tag{12}$$

$$\mathbf{u} \mapsto \sum_q g_i^\dagger(t_p, t_q)u_q, \tag{13}$$

$$(p,q) \mapsto g_i(t_p, t_q). \tag{14}$$

From Equation (3), we deduce:

$$g_i^R(t,t') = \Theta(t-t')g_i^R(t,0)g_i^R(t',0)^{-1}, \tag{15}$$

where $\Theta$ is the Heaviside function. Hence, the maps can be realized by solving $g_i^R(t,0)$ with a classical ODE solver and leveraging the low-rank structure of $g_i^R$.

### 2.2.1 Dealing with relaxation in $g^R$

Even in presence of relaxation, one expects $\|g^R(t>0,0)\| > 0$ . Yet, simple relaxation models lead to $\log\left(\|g^R(t>0,0)\|\right) \propto -t$. Hence, when doing the computation in finite precision arithmetic eq. (15) quickly become useless. A very direct approach to bypass this issue would be to discretize the differential operator directly, and then inverse it.

## 2.3 Parmeters

We set :

$$U(t) = \alpha\sin(2t)\exp(-t/\tau)(1-\exp(-t/4\tau)) \tag{16}$$

$$J = 1 \tag{17}$$

## 2.4 Iteration Method

We iteratively compute:

$$G_{i,j+1} = g_i + g_i \Delta_j G_{i,j+1}, \tag{18}$$

$$\Delta_j = \frac{G_{1,j} + G_{2,j}}{2}. \tag{19}$$

## 2.5 Convergence Monitoring

We define the norm of the discretized kernel $A(t_p, t_q)$ with $t_{i,p}$ Convergence is monitored using the following error measures:

$$\mathcal{E}_{j,\mathrm{abs}} \equiv \sqrt{\|G^R_{1,j+1} - G^R_{1,j}\|^2 + \|G^R_{2,j+1} - G^R_{2,j}\|^2}, \tag{20}$$

$$\mathcal{E}_{j,\mathrm{rel}} \equiv \frac{\mathcal{E}_{j,\mathrm{abs}}}{\sqrt{\|G^R_{1,j+1}\|^2 + \|G^R_{2,j+1}\|^2}}, \tag{21}$$

## 2.6 Stopping Criterion

The iteration stops when:

- The number of iterations exceeds the preset maximum.
- $\mathcal{E}_{j,\mathrm{rel}}$ plateaus.
- The target accuracy is reached

# 3 Results

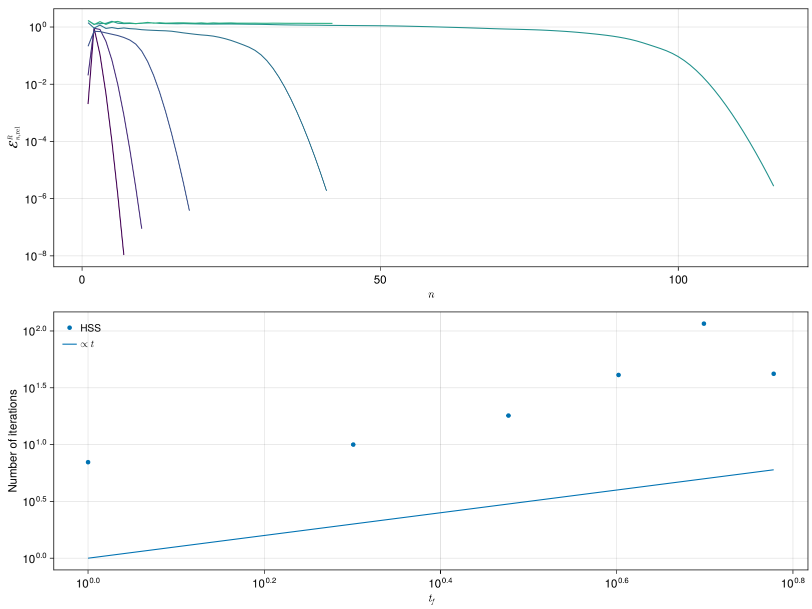


Fig. 1: Convergence toward self-consistent solution for different propagation times $t_f$. The compression accuracy is maintained at its nominal value throughout the computation. *a:* Real and imaginary part of $g_1^R$ *b:* Real and imaginary part of $g_2^R$ *c:* Evolution of relative difference between successive iterations. *d:* Number of iterations required to reach desired accuracy. *Parameters:* $\alpha = 20, \tau = 5$ and $\delta_t = 0.1$. Compression: $rtol_{\text{nominal}} = 10^{-8}$ and $atol_{\text{nominal}} = 10^{-9}$

## References