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THREE-DIMENSIONAL SHAPE ANALYSIS OF THE SCOLIOTIC SPINE USING INVARIANT SHAPE PARAMETERS*

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Abstract—Use of calibrated stereoradiographs or other three-dimensional measuring systems enables a three-dimensional shape analysis of the spine. In the present study the shape of the spinal midline in the case of deformities such as scoliosis is investigated. The shape analysis is performed with reference to differential geometry of spatial curves. The resulting spatial shape parameters are invariant, i.e. independent of the coordinate system chosen. The relation of these parameters to conventional measures such as the Cobb angle is discussed.

1. INTRODUCTION

In present routine clinical examinations, the degree of a scoliosis is primarily assessed by inspection of conventional a.p. radiograms. Besides other parameters (such as the rib hump or the vertebral rotation), the angle according to Cobb (or Ferguson) is used as a simple measure for a quantitative description of the spinal shape and of shape changes (Lusted and Keats, 1972). However, use of the Cobb (Ferguson) angle has three major disadvantages:

(i) it is a global measure which does not give a description of the degree and distribution of curvature along the spine;

(ii) it does not take into account the three-dimensional extension of the scoliotic spine;

(iii) it depends on the position of the patient relative to the X-ray apparatus.

An improvement with respect to (i) might be achieved by a more detailed measurement of conventional radiograms. In practice, a semi-quantitative description is often used in curvature attributes such as 'upper thoracic', 'lumbar', 'double curved' etc. However, an objective quantification is very desirable, especially with respect to an estimation of therapeutic results.

Measurement errors with respect to (ii) and (iii) can be eliminated only by using a three-dimensional X-ray technique such as stereoradiography or 90° radiography. In order to take full advantage of two-plane radiography, a calibrated X-ray apparatus and a photogrammetric image evaluation technique should be employed (Selvik, 1974; Krataky, 1975; Brown *et al.*, 1976; Hindmarsh *et al.*, 1980). This procedure not only enables the spatial structure of the spine to be determined but also an exactly scaled model to be reconstructed. We are thus enabled to calculate absolute internal measures of the spine such as spatial angles and distances, which are independent of the patient's position relative to the X-ray apparatus. In mathematical terms we can calculate parameters which are *invariant* with respect to three-dimensional coordinate transformations (i.e. translations or rotations of the coordinate system). In other words, invariant parameters describe intrinsic shape properties of the spine independent of the accidental or abritrary choice of a coordinate system (defined e.g. by the geometry of the X-ray apparatus).

The purpose of the present study was to arrive at a quantitative description of scolioses (and other spinal deformities) which goes beyond the usual classification according to the two-dimensional Cobb angle or similar measures, and which also comprises spatial parameters.

For the sake of simplicity, the spine in its overall shape may be represented in a first stage by its midline, that is, by a spatial curve. Therefore, in the following considerations the rotation of the vertebral bodies around their longitudinal axes is neglected as well as the finer details of vertebral body shape such as torsion or wedge shape, even though these features are very important in the etiology of scoliosis. Likewise, the position of the spine relative to other parts of the skeleton such as the ribs or the pelvis has not yet been taken into account.

We define the midline of a spine as a three-dimensional polygon consisting of the axes of the vertebral bodies. A method for determining the vertebral axes using stereoradiograms is described in Section 3. The shape analysis of the spinal midline is performed with reference to differential geometry of spatial curves. However, since the methods of differential geometry apply only to smooth curves rather than to polygons such as the spinal midline, the formulae have to be modified appropriately.

2. INVARIANT SHAPE ANALYSIS

In our shape analysis the spinal midline is con-

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sidered as a spatial curve, or more strictly, as a spatial polygon. Any spatial curve can be completely described by a number of parameters which are related to its position, size and shape. In our analysis we are interested only in the intrinsic shape properties of the spine regardless of its position in space. That is, our shape description will be independent of the patient's position relative to the X-ray film or any other reference frame. The size and shape parameters are said to be invariant with respect to position.

In order to attain a plain description of the spinal midline, the size and shape parameters employed are required to be closely related to, or to be a direct extension of, the description in terms of conventional parameters such as the Cobb angle.

As already mentioned at the beginning, the Cobb angle depends on the patient's position and is thus not invariant. In addition, it is suited only for a description of a planar curve, i.e. a spine which is curved only in a plane and not in three dimensions. However, in certain cases the Cobb angle is 'approximately invariant'. That is, it is a relatively good measure of the spinal shape provided that the spine is nearly planar and that the patient can be positioned in a well defined and reproducible manner. Thus a three-dimensional generalization of the Cobb angle should obviously be possible.

The invariant parameters can be divided into size and shape parameters. Two geometrical objects are said to be of the same shape, i.e. geometrically similar, if they can be transformed into each other by a simple scaling. This is demonstrated in Fig. 1, where spines with equal Cobb angles are shown. The shapes of Fig. 1(a) and Fig. 1(b) are different, whereas those of Fig. 1(a) and Fig. 1(c) are the same. The Cobb angle is evidently independent of size and is thus a pure shape parameter. Generally, the degree of deformation of a spine is expressed with reference to an undeformed spine of equal size (length). Thus, the deformation of Fig. 1(a) is considered to be identical to that of Fig. 1(c). Mathematically, this is equivalent to introducing an individual unit length for each individual spine (or, even better, for each individual spinal segment). In other words, for the purpose of our shape analysis we consider the length of any spinal segment to be a natural local unit length. This enables a comparison of different spines in an easy manner using size independent pure shape parameters.

In Fig. 1(a) and Fig. 1(b) the Cobb angles (as well as the sizes) are equal, even though the shapes are different. The distribution of the partial angles between adjacent vertebral segments is obviously different in the two cases. Thus, for a complete shape description not only the total angle α, but also the angle distribution along the spine is essential. This also enables the spinal segments beyond the neutral vertebrae (i.e. outside the range of the primary curvature of the scoliosis) to be taken into account.

As the partial angle between adjacent segments is closely related to the curvature of a spatial curve (which is, however, an invariant depending on shape and size), we shall designate it the *curvature angle*. The curvature angle is defined between any two adjacent vertebrae as shown in Fig. 2. $\mathbf{s}_1$ and $\mathbf{s}_2$ are the axes of the vertebral bodies 1 and 2 which are calculated from the three-dimensional measurement (see next section). They have to be considered as polygon sides (or vectors) in three dimensions. Thus, in contrast to

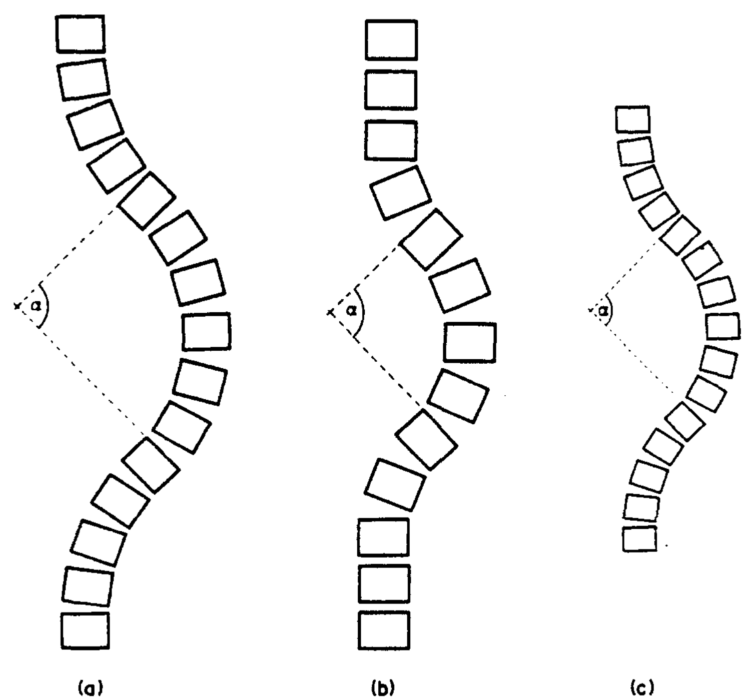


Fig. 1. Spines with equal Cobb angle and different sizes and shapes.

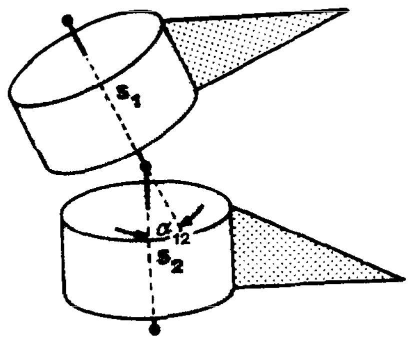


Fig. 2. Definition of the curvature angle α_{12}.

the Cobb angle, which is measured in a two-dimensional projection, the curvature angle is a spatial quantity which is measured directly in the three-dimensionally reconstructed model of the spine. From this definition it is obvious that the curvature angle is independent of the X-ray geometry and hence invariant. The curvature angle α_{12} may be calculated from the polygon sides s_1 and s_2 by means of vector algebraic formulae (see Appendix). These formulae provide a formal means of proving the invariance of the curvature angle.

Although the curvature angle is a quantity which is defined in three dimensions, it is not sufficient for a complete shape description of the spine. In general, the deformation of a scoliotic spine is three-dimensional: the midline is not contained in a plane. Although on the other hand, a healthy spine is curved, it lies completely in the sagittal plane. In scoliosis the spine does not only deviate from the sagittal plane. It is then not generally possible to find any other plane completely containing the spinal midline.

The curvature angle measures the deviation of the midline from a straight line. To comprehend the spatial shape the deviation from a plane must be taken into account, too. However, since two adjacent vertebral axes (s_1 and s_2 in Fig. 2) always lie in a plane, an appropriate shape parameter cannot be calculated from only two segments. Thus, we take another segment s_3 (Fig. 3) which, in general, is not coplanar with s_1 and s_2. The planes E_{12} (containing s_1 and s_2) and E_{23} (containing s_2 and s_3) include an angle γ_{123} which evidently is a measure for the deviation of the section $s_1s_2s_3$ from a plane. By association with differential geometry of spatial curves we designate γ_{123} the *torsion angle* (the term 'torsion' must not be confused with the torsion of a vertebra). The existence of a torsion angle $\gamma \neq 0$ is connected with a helical shape of the spine, where $\gamma > 0$ and $\gamma < 0$ indicates a right- or left-handed helix respectively.

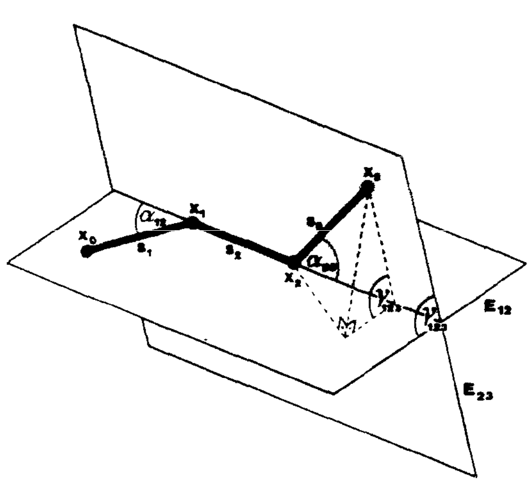


Fig. 3. Definition of the torsion angle γ_{123}.

As is the case with the curvature angle α the torsion angle γ as defined above is a pure shape parameter, independent of the lengths of the segments s_1, s_2 and s_3. Formulae for the calculation of the torsion angle are given in the Appendix. Again vector algebraic expressions are used from which the invariance of the torsion angle is directly evident.

In view of the magnitudes of the shape parameters α and γ a difficulty arises in the interpretation of the torsion angle γ. The curvature angle α is directly related to the degree of distortion as measured by the deviation from a straight line. This means that a small curvature angle entails little distortion. However, this is not the case for the torsion angle. Even for very low helical distortions, the torsion angle may take on any value between -180° and $+180^\circ$. Thus, the torsion angle itself is not very well suited for a clear description of the three-dimensional distortion of the spine.

To obtain a better representation we consider the curvature angle from a different point of view. Rather than by the angle α we may characterize the bending of the midline (Fig. 2) by its space requirements, e.g. by the area of the parallogram spanned by the axes s_1 and s_2 [Fig. 4(a)]. The greater the bending, the greater will be the area of the parallelogram (provided that the curvature angle does not exceed 90°, which it does not even in severe scolioses). In the following this area will be denoted as "curvature area"

$$A_{12} = |s_1 \times s_2|.$$

The extension of this consideration to three dimensions is straightforward. We now consider the parallelepipedon spanned by s_1, s_2 and s_3 [Fig. 4(b)], the volume of which is a measure for the space requirements in three dimensions. This volume is a perspicuous measure of the three-dimensional distortion of $s_1s_2s_3$. However, the latter is not only dependent on the torsion angle γ_{123}, but also on the curvature angles α_{12} (section s_2s_2) and α_{23} (section s_2s_3). If the torsion angle is zero the volume will be zero, too. On the other hand, for infinitely small curvature angles the volume will likewise be zero, even if the torsion angle is finite. This reflects the fact that a straight line (no curvature) has no torsional deformation whereas the torsion angle is indeterminate. The volume will be denoted as 'torsion volume':

$$V_{123} = \det(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3).$$

With the newly defined quantities A and V, a clear and detailed description of spinal deformations is possible. A final modification is necessary, however. As defined above, the curvature area and the torsion volume are dependent on the lengths S_1, S_2 and S_3 of the spinal segments $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$. They are therefore parameters describing shape *and* size. As already mentioned, we are mainly interested in pure shape parameters in order to compare spines of different sizes (e.g. during growth). This can be achieved by introducing the length of each spinal segment as a local unit length. The resulting quantities 'reduced curvature area',

$$a_{12} = A_{12}/(S_1 S_2) = \sin \alpha_{12}$$

and 'reduced torsion volume',

$$v_{123} = V_{123}/(S_1 S_2 S_3)$$
$$= \sin \alpha_{12} \sin \alpha_{23} \sin \gamma_{123}$$

are pure invariant shape parameters. In a similar way to the torsion angle, the torsion volume v may take on positive and negative values depending on the right or left helicity of the spine. To judge the amount of deformation, it is however sometimes sufficient to consider the absolute value of v only. Mathematical details are given in the Appendix.

The reduced curvature area a_{12} is a (differential) three-dimensional generalization of the Cobb angle (between two vertebral bodies only). There is, however, no analogue to the reduced torsion volume v_{123}, as the helicity of the spine is generally not taken into account in the conventional shape description of a scoliotic spine.

For the sake of completeness we note that, in addition to the two- and three-dimensional space requirements A and V, the length S of a spinal segment may be considered as a 'one-dimensional space requirement' which is a pure size parameter. Introducing a 'reduced length' s yields the trivial result $s = 1$.

The shape analysis in terms of curvature and torsion is limited to the intrinsic shape properties of the spine, i.e. those features which are independent of its position relative to any reference frame, be it the X-ray apparatus or other parts of the skeleton. Due to this fact some aspects of the conventional description of a scoliosis are not contained in the present shape analysis. For example, since the curvature of a spatial curve is always positive by definition, there is no distinction between 'right convex' and 'left convex' curves (except with reference to a fixed exterior coordinate system). Likewise, the spatial curvature does not differentiate between kyphosis, lordosis and scoliosis. Furthermore, one and the same curve (e.g. a helix) may appear as 'double curved' or 'single curved' depending on the projection plane. All these features of a scoliosis are dependent on the orientation of the spine relative to the frontal plane and cannot be described by internal shape parameters such as curvature and torsion.

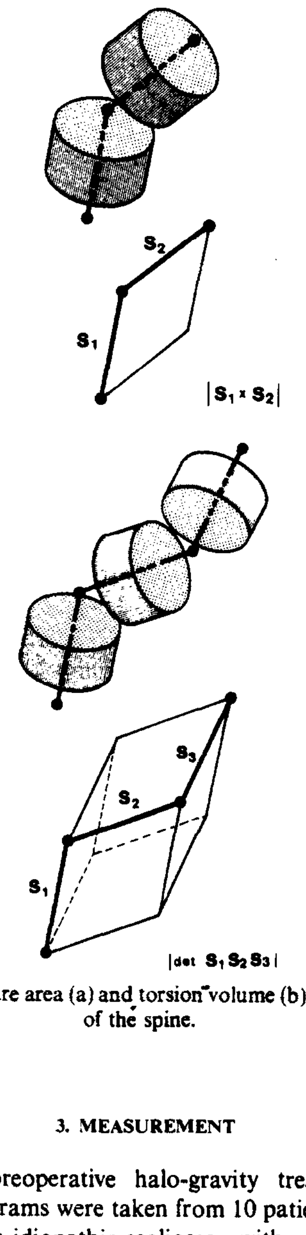


Fig. 4. Curvature area (a) and torsion volume (b) of a section of the spine.

3. MEASUREMENT

During preoperative halo-gravity treatment 23 stereoradiograms were taken from 10 patients—most of them with idiopathic scolioses—with and without an extension force or with different extension forces.

To obtain three-dimensional coordinate data of the spinal shape the stereoradiograms were evaluated with a stereocomparator which was connected on-line to a computer. Details of our stereoradiographic technique were reported in a previous paper (Hierholzer, 1977).

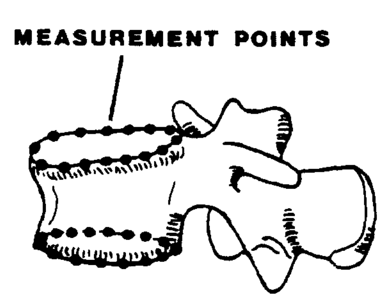


Fig. 5. Stereoscopic measurement of a vertebral body.

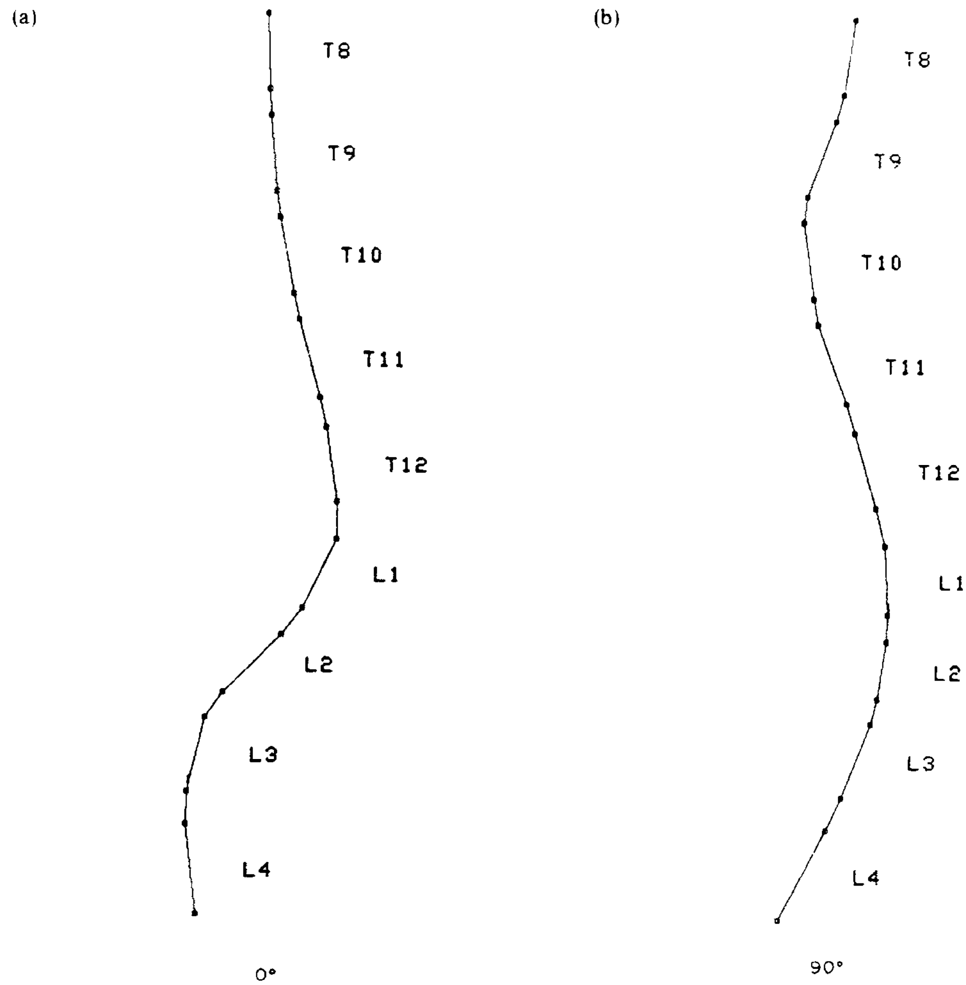


Fig. 6. Anterior–posterior (a) and lateral (b) view of the reconstructed spinal midline.

For a good definition of the position of the vertebral bodies a number of points at the rims of the endplates were measured stereoscopically (Fig. 5). In most cases the rims are readily discernible in the stereoscopic image. To enhance the measurement accuracy, which was mainly limited by subjective interpretation errors, each stereoradiogram was measured at least twice.

From the measured points the centroid of either endplate of a vertebra can be calculated. The axis of a vertebral body may then be defined by a line connecting the two endplate centroids. Alternatively, the vertebral axes can be defined by the interconnection of the midpoints of the intervertebral discs, which are defined in turn by the midpoint of a line connecting the centroids of adjacent endplates. By the latter procedure a certain degree of smoothing of the resulting spinal midline is achieved. This is appropriate with respect to the relatively large measuring errors due to possible misinterpretations of the X-ray structures. Due to this fact, the errors are, to some extent, not statistically distributed. Thus, an error analysis yielding standard deviations of the shape parameters would be questionable in the case of the present measuring system. An estimation of the possible uncertainties may, however, be obtained from Fig. 13 (see end of next section).

This method of calculating the vertebral axes works satisfactorily except for those cases where the contour of the endplate deviates heavily and unsymmetrically (with respect to the longitudinal axis) from the normal shape, e.g. in certain cases of congenital malformation. A reconstructed midline is shown in Fig. 6 in two orthogonal projections (a.p. and lateral).

As has already been mentioned, the rotation of the vertebral bodies has not been investigated in the present study. This is due to the fact that appropriate data, e.g. the tips of the spinous and transverse processes or the pedicles, could be measured with satisfactory accuracy only in a limited number of cases. In most cases, particularly in severe thoracic scolioses, these structures are obscured by heavy radiographic superimpositions. Thus a systematic evaluation of the rotation was not possible with the data available at present. It is to be hoped that with a new image evaluation technique using a display stereocomparator (Hierholzer, 1978) a general improvement of the measurement accuracy and thus a reliable measurement of the vertebral rotation will be possible. However, in the case of distorted vertebrae a reliable determination of rotation is basically questionable.

4. RESULTS AND DISCUSSION

The curvature area a and the torsion volume v may be calculated for any pair or triplet of consecutive vertebral bodies. Thus, if n vertebrae are measured, we can calculate $n - 1$ curvature areas $a_{12}, a_{23} \ldots, a_{n-1,n}$ and $n - 2$ torsion volumes $v_{123}, v_{234} \ldots, v_{n-2,n-1,n}$. We now consider these shape parameters to be a function of position in the spine, that is, as a function of the arc length of the spinal midline (summed length of the spinal segments) measured from some origin (e.g. L5) up to the vertebra under consideration. In order to obtain a size independent representation of our shape parameters we substitute the arc length by the number of vertebral bodies counted from the same origin (e.g. L5 = 1, L4 = 2,..., T12 = 6, etc.) This is equivalent to introducing a local unit length equal to the length of a spinal segment, as stated in Section 2.

In effect, since the two vertebral bodies 1 and 2 contribute to the curvature area a_{12}, this value is associated with a vertebral number (relative arc length) of 1.5. Similarly, the torsion volume v_{123} is associated with the vertebral number 2, and so on. It should be noted that these shape parameters are in fact functions of a discrete variable, namely the vertebral body number. Otherwise, the procedure is quite similar to that of differential geometry of smooth curves, which may be represented by both curvature and torsion as a function of arc length.

We are now in a position to plot a and v versus the vertebral body number. A still better representation is obtained, however, by summing up the individual contributions a_{ij} and v_{ijk} along the spine, starting from the origin (e.g. L5) up to the vertebra n under consideration. These partial sums which we call 'integral relative curvature area'

$$AI(n + 1/2) = \sum_{i=1}^{n-1} a_{i,i+1}$$

and 'integral relative torsion volume'

$$VI(n + 1) = \sum_{i=1}^{n-2} v_{i,i+1,i+2}$$

are again functions of the vertebral body number n. In the case of the torsion volume the absolute value of v is used preferentially (certain features of three-dimensional shape are, however, lost in this case).

As an example, these functions are shown in Fig. 8(a) and Fig. 8(b) for a patient with a 135° scoliosis (Fig. 7). In either graph two curves are plotted for an extension force of 0N and 150N respectively.

With the limitations discussed at the end of Section 2 all essential characteristics of shape and shape change can be extracted from these graphs. In Fig. 8(a) the slope of the curves is related to the local curvature of the spine. Steep slopes indicate heavily curved sections. Furthermore, the total height, i.e. the maximum ordinate of the curves (at T8/T7 or T7/T6 respectively) is related to the total spatial deflection. The total spatial deflection is defined as the accumulated local deflection obtained by summing up the curvature angles α_{ij} along the spine (similar to the sum AI, see above). Since the Cobb angle may be obtained in a similar way by summing up partial angles in the a.p. plane, the total spatial deflection is a three-dimensional generalization of the Cobb angle. Thus, a correlation between the Cobb angle and the total deflection is expected. However, since the latter contains contributions invisible in the standard a.p. projection (i.e. kyphotic and lordotic components), the total spatial deflection is generally higher than the Cobb angle.

In Fig. 9 the Cobb angle is plotted against the total deflection (calculated between the neutral segments) for all of our measurements. Since the helical distortion of the spine is not very much pronounced in most cases, a relatively good correlation between the Cobb angle and the total deflection is observed. As expected, in Fig. 9 the Cobb angle is systematically lower than the total deflection angle.

It should be noted that the neutral segments are defined as the points of inflection (points of zero curvature) in the a.p. projection. This does not necessarily mean a zero curvature in three dimensions, and these segments need not be neutral in a three-dimensional sense. For example, in Fig. 7 the upper neutral vertebra is T9. A zero curvature would entail a horizontal tangent of the curves in Fig. 8(a) in the region of T9 which, however, is not the case.

The effect of the halo-gravity extension can likewise be assessed from Fig. 8(a). The total extension effect (between L5/L4 and T8/T7) is represented by the ordinate difference of the two curves at T8/T7. On the other hand, local effects can be quantified by the slope changes in different regions. For example, the rectification of the section L2/L1–T11/T10 is relatively poor, whereas a maximum effect is obtained in the region of T9.

To judge the absolute deformation a comparison with a healthy spine having only physiological curvatures is necessary. In Fig. 8(a) the dashed curve is a reference curve based on data measured by Killus (1973) and Snijders (1970) which, however, did not take into account the positions of the individual vertebrae within the spine. These have been estimated from measurements of prepared vertebra specimens. The deformation as well as the extension effect can be expressed in a single figure by the mean square deviation between the appropriate curves in Fig. 8(a).

The interpretation of the torsion volume plot [Fig. 8(b)] is similar to that of Fig. 8(a), although less perspicuous. As mentioned earlier, the absolute values of v used in Fig. 8(b) yield monotonically increasing

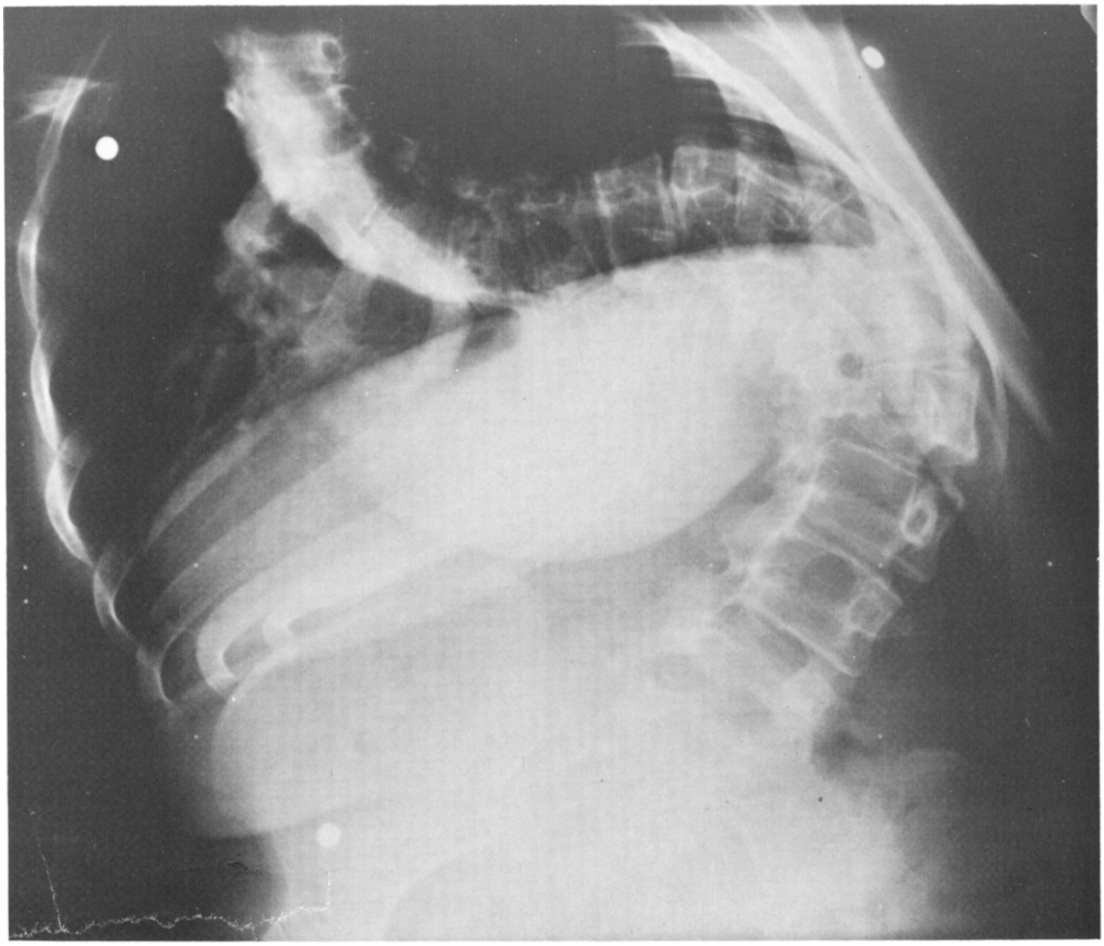


Fig. 7. 135° scoliosis (half-image of stereoradiograph).

curves. Again the total and local extension effects may be quoted from the ordinate or slope differences of the curves. Because a healthy spine is planar and has no torsion, the reference curve in Fig. 8(b) coincides with the abscissa.

Both curvature and torsion graphs are essential for a complete description of the three-dimensional shape of the spine. This will be demonstrated by some examples. In Fig. 10(a) the curvature of two patients is compared. From these curves it might be concluded that the deformity of the two patients is quite similar, at least in the section L5/L4–L1/T12. However, in Fig. 10(b) considerable differences appear in the torsion and consequently in the spatial shape of the two spines. In fact, in the a.p. view the two scolioses looked similar between L5/L4 and L1/T12 despite the large difference in torsion. Beyond T12 larger differences appeared in the a.p. view which can mainly be attributed to curvature differences, i.e. slope differences in Fig. 10(a) (however, some of the effects discussed at the end of Section 2 might have an influence on the a.p. projection).

A similar example is shown in Fig. 11. A scoliotic patient was measured without and with an extension force of 320 N. From the curvatures [Fig. 11(a)] it might be inferred that the total extension effect (between L2/L3 and T4/T5) was minimal, with local improvements as well as even degradations (T6). The torsion [Fig. 11(b)] again indicates that an appreciable decrease of the three-dimensional deformation has nevertheless been achieved.

We consider the length of the spinal midline as a supplementary parameter. This is a pure size parameter. Similarly to Fig. 8, the summed length of the spinal segments is plotted in Fig. 12. It should be pointed out that the length can be reconstructed exactly to scale from calibrated stereoradiographs, i.e. without errors due to projection enlargement or perspective distortion.

As expected, the length of the spine remains equal, irrespective of whether or not an extension force is applied (except for measuring errors). On the other hand, in a follow-up study the spinal growth can easily be assessed from these graphs. In addition, deviations in overall or local growth can be recognised by comparison with a normal curve.

A complete analysis of all of our measurements is given by Lüxmann (1981). See also Hierholzer and Lüxmann (1979).

Some final remarks are necessary with respect to the accuracy and reliability of our results. The practical use of the shape parameters is, of course, greatly dependent on the statistical errors of the results. It can be shown that the arc length, curvature and torsion can be used for the description of a spatial curve in a zero, first and second order approximation respectively.

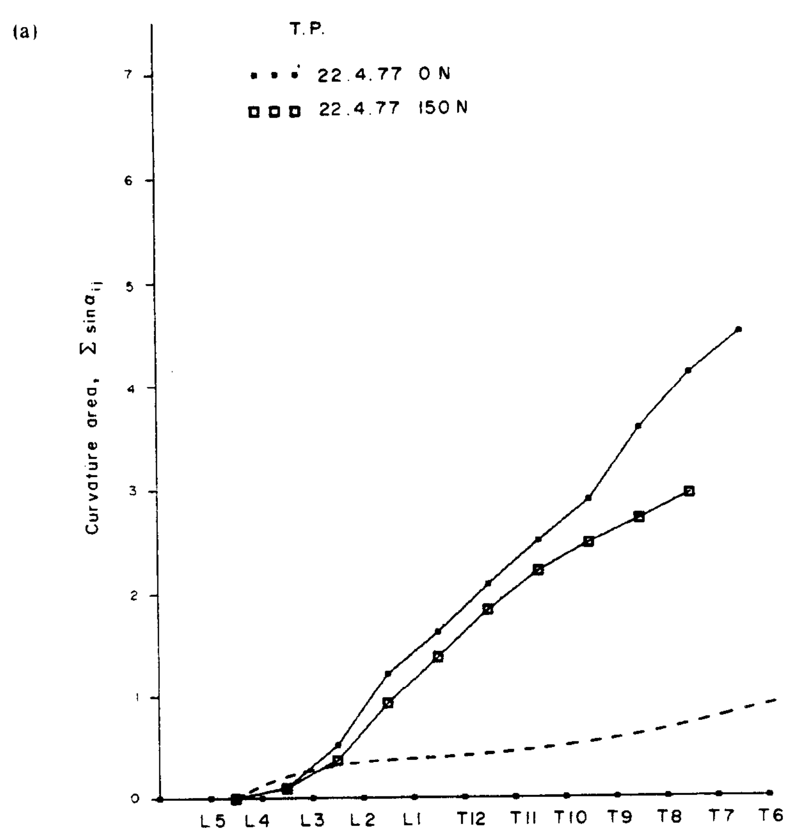


Fig. 8. (a) Curvature area of a 135° scoliosis (Fig. 7) (--- = estimated reference curve of a healthy spine).

(b)

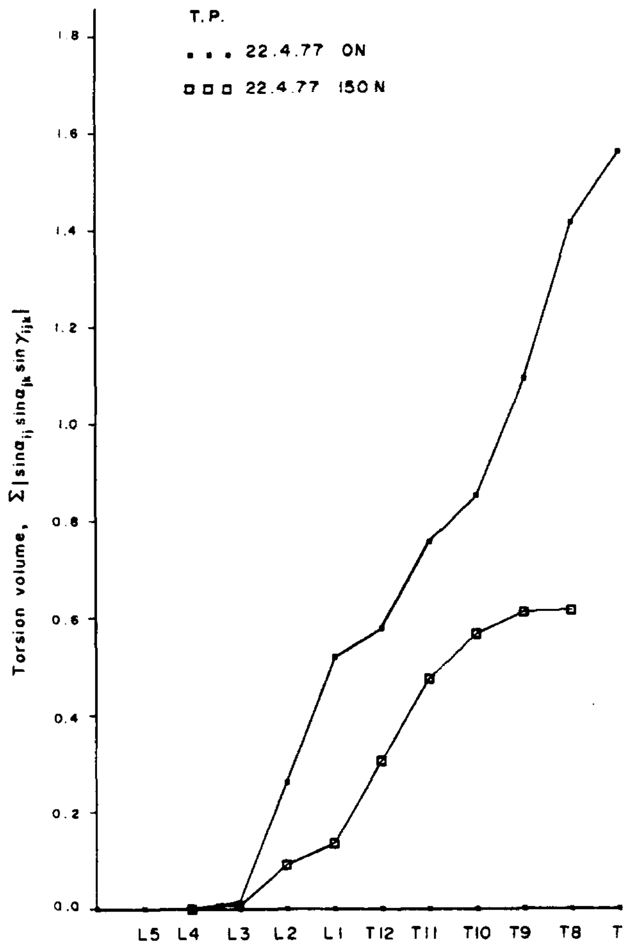


Fig. 8. (b) Torsion volume of a 135° scoliosis (Fig. 7) (--- = estimated reference curve of a healthy spine).

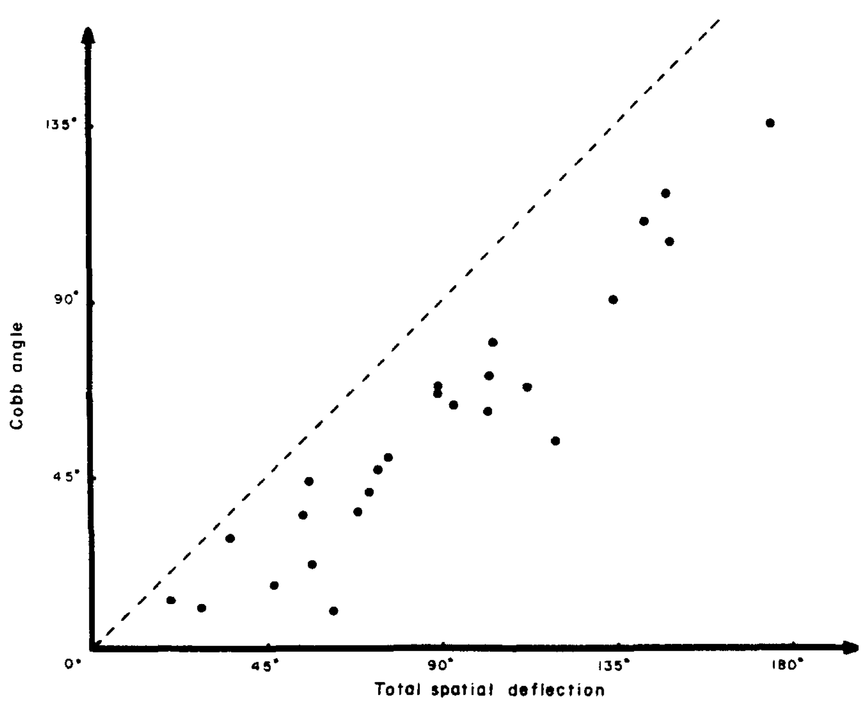


Fig. 9. Correlation of Cobb angle and total spatial deflection.

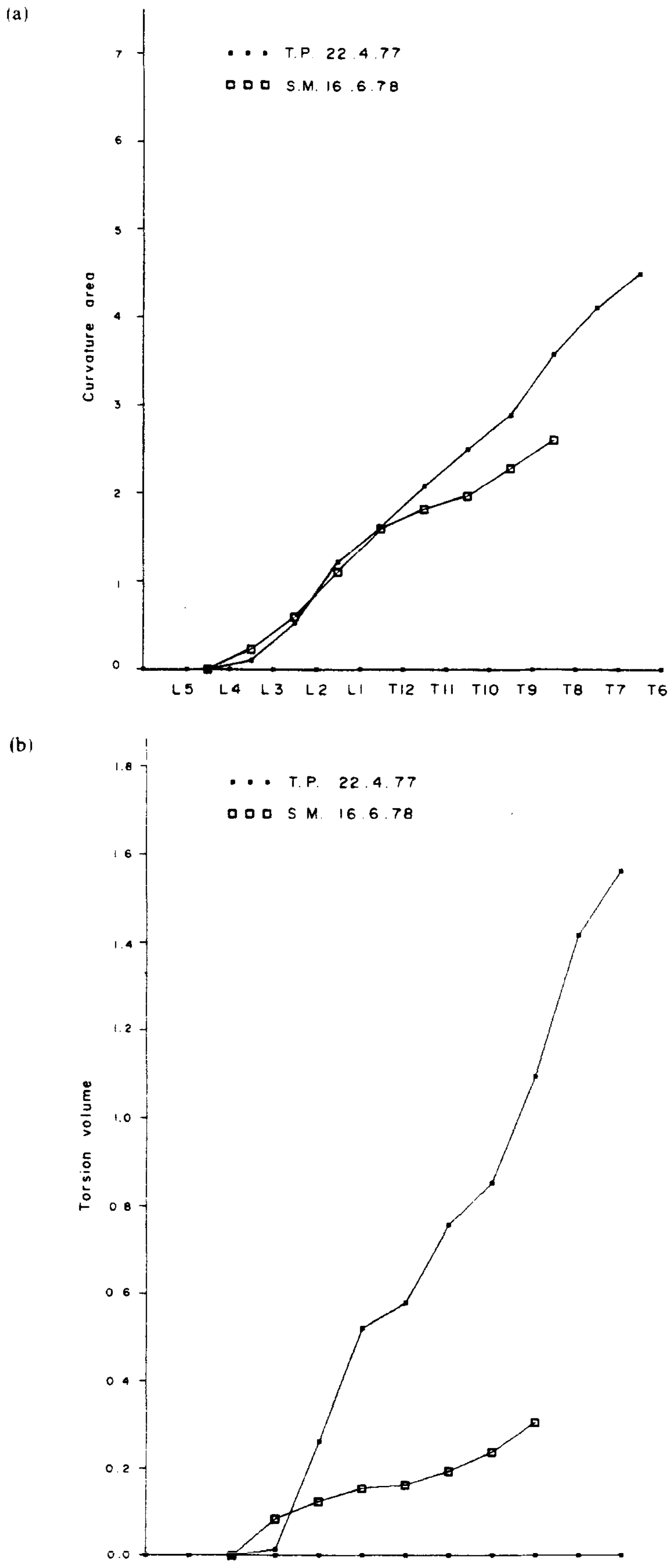


Fig. 10. Comparison of the shape of two scoliotic spines.

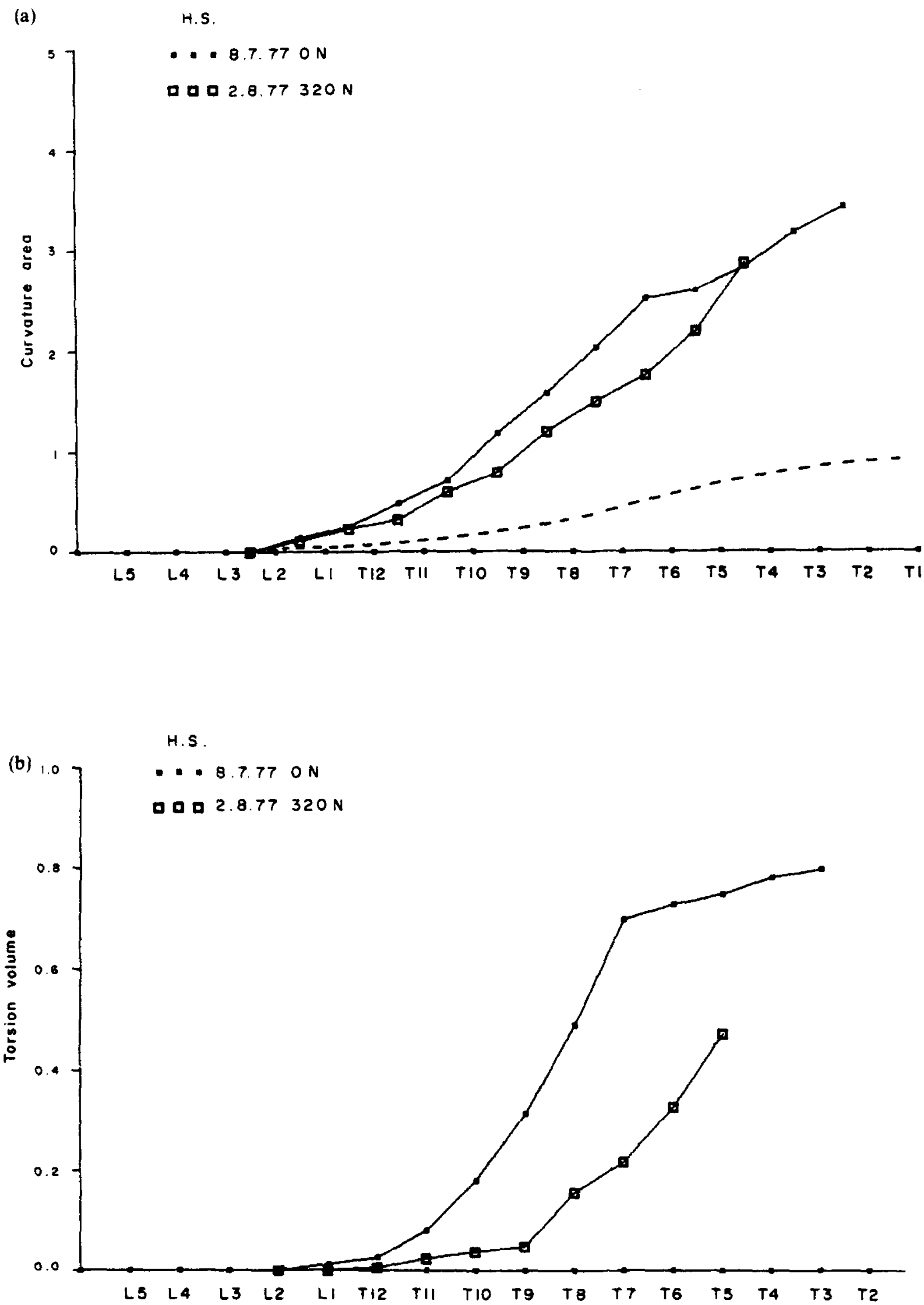


Fig. 11. Influence of halo-gravity traction on curvature (a) and torsion (b) of a scoliotic spine (--- = estimated reference curve of a healthy spine).

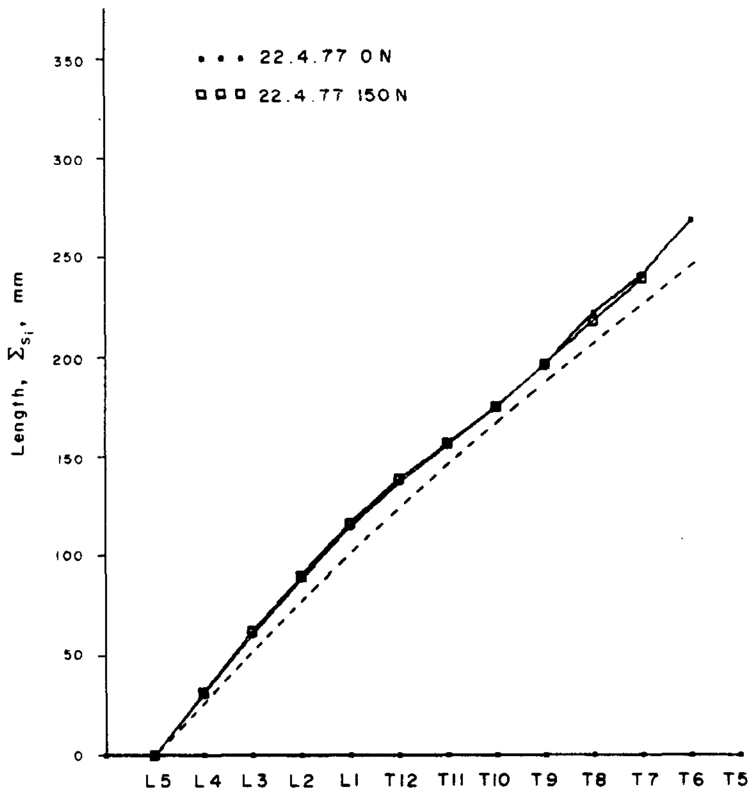


Fig. 12. Arc length of the spinal midline (Fig. 7) (--- = estimated reference curve of a healthy spine).

(a)

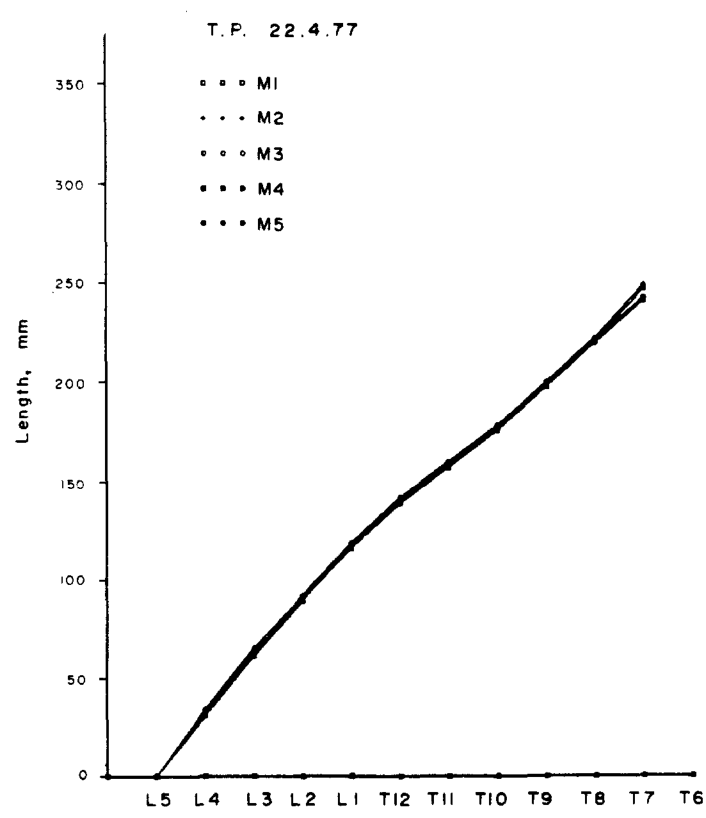
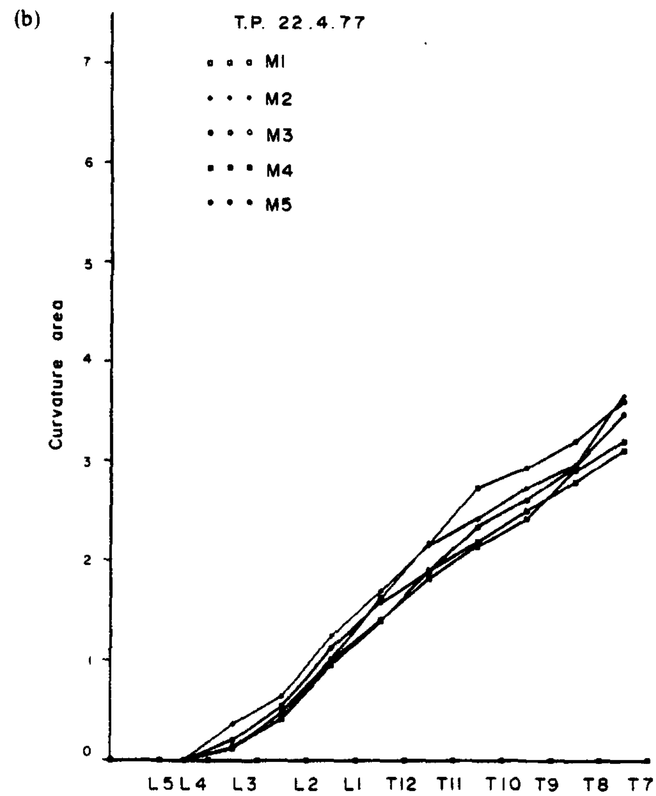


Fig. 13. (a) Reproducibility of arc length of the spine as determined from stereoradiographs.



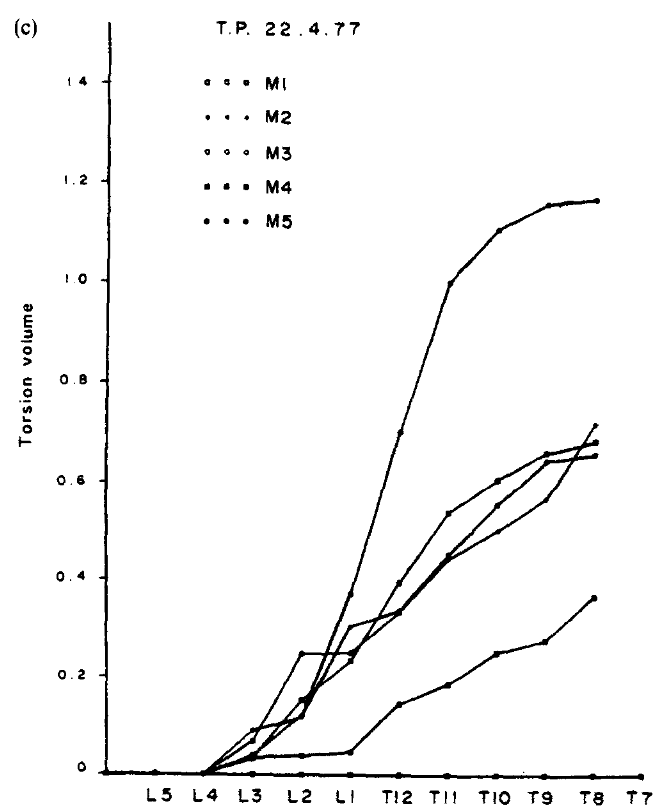


Fig. 13. Reproducibility of curvature (b) and torsion (c) of the spine as determined from stereoradiographs.

Consequently, the determination of these parameters is influenced by measuring errors in increasing order. It is thus expected that a curvature measurement is less accurate than that of length, and that a torsion measurement is less accurate than that of curvature (see Appendix). This is verified by the results shown in Fig. 13. One stereoradiograph was measured five times. In Fig. 13(a) the spinal length is plotted for each measurement. The measurement errors are quite low. In Fig. 13(b) the curvature area is plotted. The reproducibility is significantly lower but still satisfactory. The torsion volume [Fig. 13(c)], however, can be determined only with poor reliability. We conclude that with the present measurement technique the helicity of a spine can be estimated only approximately.

Nevertheless the description of a spine in terms of curvature and torsion is quite useful. The gross deformity is represented by the curvature which can be measured with sufficient reliability. The torsion, however, may be considered as a 'second curvature' giving an additional characterization of the deformation in the third dimension.

5. CONCLUSION

Use of calibrated stereoradiographs enables a thorough size and shape analysis of the deformed spine. However, for a complete investigation of the skeletal geometry as far as the spine is concerned, the analysis presented here has to be extended in two opposite directions. Firstly, not only the midline but also the finer details of the local structure, such as shape and rotation of the individual vertebrae, must be considered. Secondly, in a more global view the orientation of the spine as a whole in relation to other parts of the skeleton (e.g. the pelvis) has to be accounted for. For example, in the present analysis the curvature cannot be differentiated with respect to kyphosis, lordosis or scoliosis. This distinction is related to the orientation of the spine within the patient's body which is beyond the scope of the present study. Only intrinsic shape properties can be described by curvature and torsion (helicity).

On the other hand, it seems to be sensible to separate the description of a geometrical structure into that of shape, size and position (Bookstein, 1978). This may be effected in a hierarchical manner: in the case of the spine, the shape and size properties of a single vertebra are considered in a first stage. Next, the position of each vertebra within the spine is determined, yielding size and shape parameters of the whole spine. Finally, the position of the whole spine relative tother parts of the skeleton is investigated.

The clinical applicability of the shape analysis presented here is dependent on the existence of a simple and accurate system for a three-dimensional measurement of the spinal midline. In our opinion conventional stereoradiographic techniques seem to be too cumbersome for routine applications. Thus, at present this shape analysis would essentially be restricted to applications in basic research. In addition, the practical use of the torsion parameter is presently limited by our measurement accuracy.

The significance of a shape analysis using invariant parameters consists in its independence of the patient's position. For example, the total length of the spine can be calculated irrespective of projectional distortion. Furthermore, the apex of a scoliosis can be determined without the need to rotate the patient (as with Stagnara's method). Local and global shape characteristics as well as shape changes can be recognized from the curvature and torsion curves. Whereas the curvature graph is a three-dimensional extension or generalization of the Cobb angle, there is no conventional analogue to torsion.

The completion of the shape analysis including the vertebral rotation and the orientation of the spine relative to the pelvis is a task which has to be solved in further studies.

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APPENDIX

Generalization of curvature and torsion for spatial polygons

A smooth curve is generally represented as a vector function of its arc length s:

$$\mathbf{x}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix} \tag{1}$$

If the derivatives of $\mathbf{x}$ with respect to arc length are denoted by superscript points, then curvature κ and torsion τ of this curve as functions of s are given by the equations (Lipschutz, 1969):

$$\kappa^2(s) = (\dot{\mathbf{x}} \times \ddot{\mathbf{x}})^2 \tag{2}$$

$$\kappa^2(s)\,\tau(s) = \det(\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \dddot{\mathbf{x}}). \tag{3}$$

These quantities are evidently invariant with respect to three-dimensional coordinate transformations.

If, instead of a smooth curve, a polygon is considered, the derivatives with respect to s in equations (2) and (3) may be formally replaced by difference quotients. These are most simply given by the so-called 'divided differences' (Zurmühl, 1965):

$$\dot{\mathbf{x}}_1 = [\mathbf{x}_1, \mathbf{x}_0] = (\mathbf{x}_1 - \mathbf{x}_0)/S_1 = \mathbf{s}_1/S_1 = \mathbf{e}_1 \tag{4}$$

$$\ddot{\mathbf{x}}_{12} = 2[\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_0] = 2(\dot{\mathbf{x}}_2 - \dot{\mathbf{x}}_1)/(S_1 + S_2) \tag{5}$$

$$\begin{aligned} \dddot{\mathbf{x}}_{123} &= 6[\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_0] \\ &= 3(\ddot{\mathbf{x}}_{23} - \ddot{\mathbf{x}}_{12})/(S_1 + S_2 + S_3). \end{aligned} \tag{6}$$

Under certain assumptions (equal and uncorrelated statistical errors of each component of $\mathbf{x}_i$, equal segment lengths S_i) the ratios of the statistical errors of equations (4), (5) and (6) are as $\frac{\sqrt{2}}{s} : \frac{\sqrt{6}}{s^2} : \frac{\sqrt{20}}{s^3}$. Consequently, torsion [equation (3)] is more affected by measuring errors than curvature [equation (2)].

For the definition of $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ see Fig. 3. If these expressions, which may be interpreted as generalized difference quotients for non-equidistant intervals, are introduced into equations (2) and (3) we obtain in analogy to curvature

$$\begin{aligned} K_{12} &= 2|\mathbf{e}_1 \times \mathbf{e}_2|/(S_1 + S_2) \\ &= \sin\alpha_{12}/S_{12} \end{aligned} \tag{7}$$

where S_{12} is the mean length of $\mathbf{s}_1$ and $\mathbf{s}_2$. Similarly, from equation (3) we obtain in analogy to the torsion formula

$$\begin{aligned} U_{123} &= (K^2T)_{123} \\ &= 12\det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)/(S_1+S_2+S_3)(S_1+S_2)(S_2+S_3) \\ &= \sin\alpha_{12}/S_{12} \cdot \sin\alpha_{23}/S_{23} \cdot \sin\gamma_{123}/S_{123}. \end{aligned} \tag{8}$$

Using equation (7) we may write

$$U_{123} = K_{12}K_{23}T_{123} \tag{9}$$

with

$$T_{123} = \sin\gamma_{123}/S_{123}. \tag{10}$$

S_{123} is the mean length of segments $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$. Introducing reduced length units we obtain

$$a_{12} = \sin\alpha_{12} \tag{11}$$

$$v_{123} = \sin\alpha_{12} \cdot \sin\alpha_{23} \cdot \sin\gamma_{123}. \tag{12}$$

From equation (7) it is evident that a_{12} is the area of a parallelogram spanned by the two unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ parallel to the polygon sides $\mathbf{s}_1$ and $\mathbf{s}_2$. Similarly, from equation (8) it can be seen that v_{123} is the volume of a parallelepipedon spanned by the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in the direction of $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$.

The complete mathematical analysis is given elsewhere (Hierholzer and Lüxmann, 1979).