THEORY

$X_n \mid \mu,\tau \sim \mathcal{N}\left(\mu, \frac{1}{\tau}\right) \qquad (\mu,\tau) \sim NG(\mu_0, \lambda, \alpha, \beta)$

AS BEFORE, $P(\theta|D) = \frac{P(D|\theta)\,P(\theta)}{P(D)} \propto P(D|\theta)\,P(\theta)$

① $P(D|\theta) \overset{IID}{=} \prod_{i=1}^{N} f_\theta(x_i) = \prod_{i=1}^{N} \sqrt{\frac{\tau}{2\pi}}\, e^{-\frac{\tau}{2}(x_i-\mu)^2}$

② $P(\theta) = P(\mu,\tau) = \underset{?}{P(\tau)}\,\underset{?}{P(\mu|\tau)} \overset{PDFs}{=} \frac{\beta^\alpha}{\Gamma(\alpha)}\tau^{\alpha-1}e^{-\beta\tau}\sqrt{\frac{\lambda\tau}{2\pi}}\, e^{-\frac{\lambda\tau}{2}(\mu-\mu_0)^2}$

BY THEORY $\quad \text{Gamma}(\alpha,\beta) \quad \mathcal{N}\left(\mu_0, \frac{1}{\lambda\tau}\right)$

$\underset{\tau,\mu}{\propto} \tau^{\alpha-\frac{1}{2}} e^{-\frac{\lambda\tau}{2}[2\beta + (\mu-\mu_0)^2]}$

$$\log\left(P(\mu,\tau|X)\right) = \log\left(\prod_{n=1}^{N} P(x_n|\theta)\,P(\theta)\right) = \sum_{n=1}^{N}\log\left(P(x_n|\theta)\right) + \log \overbrace{P(\mu,\tau)}^{P(\theta)}$$

THEORY (PREVIOUSLY INTRODUCED)

$$= \sum_{n=1}^{N}\log\left(\frac{\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{\tau}{2}(x_n-\mu)^2}\right) + \log\left(\frac{\beta^\alpha\sqrt{\lambda}}{\Gamma(\alpha)\sqrt{2\pi}}\tau^{(\alpha-\frac{1}{2})}e^{-\beta\tau}e^{-\frac{\lambda\tau}{2}(\mu-\mu_0)^2}\right)$$

$$\underset{\tau,\mu}{\propto} \sum_{n=1}^{N}\left\{\frac{1}{2}\log\tau - \frac{\tau}{2}(x_n-\mu)^2\right\} + \left(\alpha-\frac{1}{2}\right)\log\tau - \beta\tau - \frac{\lambda\tau}{2}(\mu-\mu_0)^2$$

$$= \frac{N}{2}\log\tau - \frac{\tau}{2}\sum_{n=1}^{N}\left(x_n^2+\mu^2-2\mu x_n\right) + \left(\alpha-\frac{1}{2}\right)\log\tau - \beta\tau - \frac{\lambda\tau}{2}\left(\mu^2+\mu_0^2-2\mu\mu_0\right)$$

$$= \left(\frac{N}{2}+\alpha-\frac{1}{2}\right)\log\tau - \beta\tau - \frac{\tau}{2}\left[\mu^2(N+\lambda) - 2\mu\left(\sum_{n=1}^{N}x_n + \mu_0\lambda\right) + \sum_{n=1}^{N}x_n^2 + \lambda\mu_0^2\right]$$

$$= \left(\frac{N}{2}+\alpha-\frac{1}{2}\right)\log\tau - \beta\tau - \frac{\tau}{2}(N+\lambda)\left[\mu^2 - \frac{2\mu}{N+\lambda}\left(\sum_{n=1}^{N}x_n + \mu_0\lambda\right) + \underbrace{\frac{1}{N+\lambda}\left(\sum_{n=1}^{N}x_n^2 + \lambda\mu_0^2\right)}_{C}\right]$$

$$= \left(\frac{N}{2}+\alpha-\frac{1}{2}\right)\log\tau - \beta\tau - \frac{\tau}{2}(N+\lambda)\left[\mu^2 - \frac{2\mu}{N+\lambda}\left(\sum_{n=1}^{N}x_n + \mu_0\lambda\right) + C\right]$$

$$= \left(\frac{N}{2}+\alpha-\frac{1}{2}\right)\log\tau - \tau\left(\beta+\frac{C}{2}(N+\lambda)\right) - \frac{\tau}{2}(N+\lambda)\left[\mu^2 - 2\mu\underbrace{\frac{1}{N+\lambda}\left(\sum_{n=1}^{N}x_n + \mu_0\lambda\right)}_{D}\right] + \frac{\tau}{2}(N+\lambda)D^2 - \frac{\tau}{2}(N+\lambda)D^2$$

GOAL: COMPLETE THE SQUARES

$$= \underbrace{\left(\frac{N}{2}+\alpha-\frac{1}{2}\right)}_{\alpha^*}\log\tau - \tau\underbrace{\left(\beta+\frac{N+\lambda}{2}C - \frac{N+\lambda}{2}D^2\right)}_{\beta^*} - \frac{\tau}{2}\underbrace{(N+\lambda)}_{\lambda^*}\underbrace{\left[\mu^2 - 2\mu D + D^2\right]}_{=(\mu-D)^2}$$

$D = \mu_0^*$

$$= \left(\alpha^*-\frac{1}{2}\right)\log\tau - \beta^*\tau - \frac{\tau}{2}\lambda^*\left(\mu-\mu_0^*\right)^2$$

$$= \text{NORMAL GAUSSIAN}\left(\mu_0^*, \lambda^*, \alpha^*, \beta^*\right)$$

PARAMETERS:

$\alpha^* = \frac{N}{2}+\alpha$

$\lambda^* = N+\lambda$

$\mu^* = D = \frac{1}{N+\lambda}\left(\sum_{n=1}^{N}x_n + \mu_0\lambda\right) = \frac{1}{\lambda^*}\left(\sum_{n=1}^{N}x_n + \mu_0\lambda\right)$

RECALL

$C = \frac{1}{N+\lambda}\left(\sum_{n=1}^{N}x_n^2 + \lambda\mu_0^2\right)$

$$\beta^* = \beta + \frac{N+\lambda}{2}\left(C - D^2\right) = \beta + \frac{\lambda^*}{2}\left(\frac{1}{\lambda^*}\left(\sum_{n=1}^{N}x_n^2 + \lambda\mu_0^2\right) - \mu_0^{*2}\lambda^*\right)$$

$$= \beta + \frac{1}{2}\left(\sum_{n=1}^{N}x_n^2 + \lambda\mu_0^2 - \lambda^*\mu^{*2}\right)$$