Numerical Solutions of Ordinary Differential Equations

A number of numerical methods are available for the solution of first-order differential equations of the form:

$$\frac{dy}{dx} = f(x, y) \text{ given } y(x_0) = y_0 \qquad (1)$$

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or a set of values of x and y.

The initial condition in (1) is specified at the point x_0. Such problems in which all the conditions are given at the initial point only are called **initial value problems**. However, there are problems involving second and higher-order differential equations in which the conditions may be given at two or more points. These are known as **boundary value problems**.

1. Taylor's Series Method

Consider the first order equation $\frac{dy}{dx} = f(x, y)$ (1)

Differentiating (1), we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

or **(2)**

$$y'' = f_x + f_y f'$$

Differentiating this successively, we can get y'', y''', y^{iv} etc. Putting $x=x_0$ and y =0, the values of $(y')_0, (y'')_0, (y''')_0$.

Can be obtained. Hence the Taylor's series

$$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \ldots.. \qquad \textbf{(3)}$$

Gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_i$ from (3), y′, y″ etc. can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3)

Problems

1. Solve y′ = x + y, y(0) = 1 by Taylor's series method. Hence find the values of y at x = 0.1 and x = 0.2.

Differentiating successively, we get

$$y' = x + y \qquad y'(0) = 1 \qquad [\because y(0) = 1]$$
$$y'' = 1 + y' \qquad y''(0) = 2$$
$$y''' = y'' \qquad y'''(0) = 2$$
$$y''' = y''' \qquad y'''(0) = 2, \text{ etc.}$$

Taylor's series is

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \cdots$$

Here $x_0 = 0$, $y_0 = 1$

$$\therefore \qquad y = 1 + x(1) + \frac{x^2}{2}(2) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(4)\cdots$$

Thus $$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{(0.1)^3}{3!} + \frac{(0.1)^4}{4!}\cdots$$
$$= 1.1103$$

and $$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{6} + \cdots$$
$$= 1.2427$$

2. Find by Taylor's series method, the values of y at x = 0.1 and x = 0.2 to five places of decimals from $\frac{dy}{dx} = x^2y - 1$ y(0) = 1.

Solution:

Differentiating successively, we get

$$y' = x^2y - 1, \qquad (y')_0 = -1 \qquad [\because y(0) = 1]$$
$$y'' = 2xy + x^2y', \qquad (y'')_0 = 0$$
$$y''' = 2y + 4xy' + x2y'', \qquad (y''')_0 = 2$$
$$y^{iv} = 6y' + 6xy'' + x2y''', \qquad (y^{iv})_0 = -6, \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{(x)^3}{3!}(2) + \frac{(x)^4}{4!}(-6) + \cdots$$
$$= 1 + -x + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Hence $y(0.1) = 0.90033$ and $y(0.21) = 0.80227$

3. Solve $\frac{dy}{dx} = 2y + 3e^x$, y (0)=0 using Taylor's series method and find y(0.1) & y(0.2) correct to 4 decimal places.

Solution: We have Taylor's series expansion of $y(x)$

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \frac{(x-x_0)^4}{4!}y^{iv}(x_0) + \cdots$$

Given $x_0 = 0,\ \ y_0 = 0\ \ and\ y'(x) = 2y + 3e^x$

$$\therefore\ \ y(x) = y(0) + xy'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) + \frac{x^4}{24}y^{iv}(0) + \cdots \qquad (1)$$

$\mathbf{y(0) = 0}$

$y'(x) = 2y + 3e^x \Rightarrow\ \ y'(0) = 2(0) + 3e^0 \Rightarrow\ \ \mathbf{y'(0) = 3}$

$y''(x) = 2y' + 3e^x \Rightarrow\ \ y''(0) = 2(3) + 3e^0 \Rightarrow\ \ \mathbf{y''(0) = 9}$

$y'''(x) = 2y'' + 3e^x \Rightarrow\ y'''(0) = 2(9) + 3e^0 \Rightarrow\ \ \mathbf{y'''(0) = 21}$

$y^{iv}(x) = 2y''' + 3e^x \Rightarrow\ y'''(0) = 2(21) + 3e^0 \Rightarrow\ \mathbf{y^{iv}(0) = 45}$

Substituting these values in (1), we get

$$y(x) = 0 + x(3) + \frac{x^2}{2}(9) + \frac{x^3}{6}(21) + \frac{x^4}{24}(45) + \cdots$$

$$\therefore\ \ \mathbf{y(x) = 3x + \frac{9x^2}{2} + \frac{7x^3}{2} + \frac{45x^4}{24} + \cdots}$$

Hence $y(0.1) = 3(0.1) + \frac{9(0.1)^2}{2} + \frac{7(0.1)^3}{2} + \frac{45(0.1)^4}{24} \Rightarrow\ \ \mathbf{y(0.1) = 0.34869}.$

$y(0.2) = 3(0.2) + \frac{9(0.2)^2}{2} + \frac{7(0.2)^3}{2} + \frac{45(0.2)^4}{24} \Rightarrow\ \ \mathbf{y(0.2) = 0.81100}.$

Practice problems

1. Employ Taylor's series method to find an approximate solution correct to fourth decimal places for the following initial value problem at $x = 0.1$, $\frac{dy}{dx} = x - y^2$, $y(0) = 1$.

2. Evaluate y(0.1) correct to 6 decimal places by Taylor's series method if y(x) satisfies $\frac{dy}{dx} = xy + 1,\ \ y(0) = 1$.

3. Use Taylor series method to find at y at x=0.1, 0.2, 0.3 considering terms upto the term third degree given that $\frac{dy}{dx} = x^2 + y^2$ and y(0)=1

2. Euler's Method

Consider equation $\frac{dy}{dx} = f(x, y)$ given that $y(x_0) = y_0$ .Its curve of solution through $P(x_0, y_0)$is shown dotted in Figure.10.1. Now we have to find the ordinate of any other point Q on this curve.

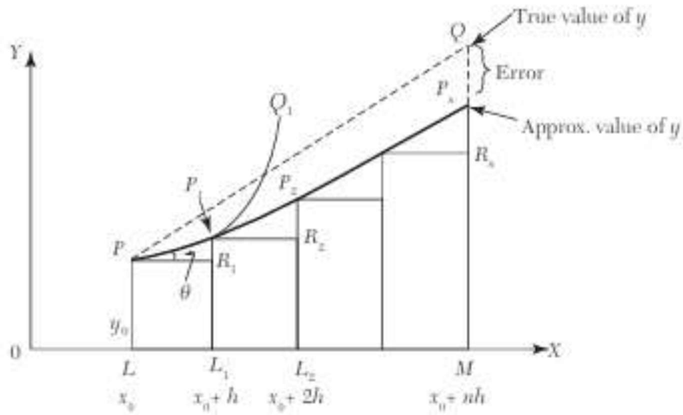

FIGURE 10.1

Let us divide LM into n sub-intervals each of width h at $L_1, L_2 \cdots$ so that h is quite small

In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0+h, y_1)$, then

$$y_1 = L_1P_1 = LP + R_1P_1 = y_0 + PR_1 \tan\theta$$

$$= y_0 + h\left(\frac{dy}{dx}\right)_P = y_0 + hf(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0+2h, y_2)$. Then

$$y_2 = y_1 + hf(x_0+h, y_1) \tag{1}$$

Repeating this process n times, we finally reach on an approximation MP_n of MQ given by

$$y_n = y_{n-1} + hf(x_0 + \overline{n-1}h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

Problems

1 Given $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$ with initial condition y = 1 at x = 0; find y for x = 0.1 by Euler's method

Solution:

We divide the interval (0, 0.1) in to five steps, *i.e.*, we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows:

x	y	dy/dx	$Old y + 0.02\,(dy/dx) = new\ y$
0.00	1.0000	1.0000	1.0000 + 0.02(1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02(0.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02(0.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02(0.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02(0.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

Ans: y = 1.0928.

3. Modified Euler's Method

Consider the initial value problem $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$. We need to find y at $x_1 = x_0 + h$. We first obtain $y(x_1) = y_1$ by applying Euler's formula and this value is regarded as the first approximation and is given by $y_1 = y_0 + hf(x_0, y_0)$.

Now by modified Euler's method, the first modified value of y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)].$$

The second modified value of y_1 is given by $y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$.

The third modified value of y_1 is given by $y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$ and so on.

Problems

1. Using Modified Euler's method, find an approximate value of y when $x = 0.3$ given that $\frac{dy}{dx} = x + y$, y(0) = 1. (carry out computations correct to 5 decimal places)

Solution: We need to find $y(0.3)$ by taking $h = 0.3$.

Given $\boldsymbol{x_0 = 0}$, $\boldsymbol{y_0 = 1}$, $\boldsymbol{f(x, y) = x + y}$. $x_1 = x_0 + h = 0 + 0.3 => \boldsymbol{x_1 = 0.3}$.

From Euler's formula, $y_1 = y_0 + hf(x_0, y_0)$

$$y_1 = 1 + 0.3f(0,1) => y_1 = 1 + 0.3(1) => \boldsymbol{y_1 = 1.3}$$

From modified Euler's formula, $y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] => y_1^{(1)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.3)]$$

$$y_1^{(1)} = 1 + \frac{0.3}{2}[1 + 1.6] => \boldsymbol{y_1^{(1)} = 1.39000}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] => y_1^{(2)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.39)]$$

$$y_1^{(2)} = 1 + \frac{0.3}{2}[1 + 1.69] => \boldsymbol{y_1^{(2)} = 1.40350}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] => y_1^{(3)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.4035)]$$

$$y_1^{(3)} = 1 + \frac{0.3}{2}[1 + 1.7035] => \boldsymbol{y_1^{(3)} = 1.40553}$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right] => y_1^{(4)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.40553)]$$

$$y_1^{(4)} = 1 + \frac{0.3}{2}[1 + 1.70553] => \boldsymbol{y_1^{(4)} = 1.40583}$$

$$y_1^{(5)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(4)}\right)\right] => y_1^{(5)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.40583)]$$

$$y_1^{(5)} = 1 + \frac{0.3}{2}[1 + 1.70583] => \boldsymbol{y_1^{(2)} = 1.40587}$$

$$y_1^{(6)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(5)}\right)\right] => y_1^{(6)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.40587)]$$

$$y_1^{(6)} = 1 + \frac{0.3}{2}[1 + 1.70587] => \boldsymbol{y_1^{(6)} = 1.40588}$$

$$y_1^{(7)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(6)}\right)\right] => y_1^{(7)} = 1 + \frac{0.3}{2}[f(0,1) + f(0.3, 1.40588)]$$

$$y_1^{(7)} = 1 + \frac{0.3}{2}[1 + 1.70588] => \boldsymbol{y_1^{(7)} = 1.40588}$$

$$\therefore\ y(x_0 + h) = y(0 + 0.3) = \boldsymbol{y(0.3) = 1.40588}$$

Ans: $y(0.3) = 1.4004$

2. Using Modified Euler's method, find $y(0.2)$ and $y(0.4)$ given $y' = y + e^x, y(0) = 0$. (carry out computations correct to 4 decimal places)

Solution:

I Stage: We need to find $y(0.2)$ by taking $h = 0.2$.

Given $\boldsymbol{x_0 = 0}$, $\boldsymbol{y_0 = 0}$, $\boldsymbol{f(x, y) = y + e^x}$. $x_1 = x_0 + h = 0 + 0.2 \Rightarrow \boldsymbol{x_1 = 0.2}$.

From Euler's formula, $y_1 = y_0 + hf(x_0, y_0)$

$$y_1 = 0 + 0.2f(0,0) \Rightarrow y_1 = 0 + 0.2(1) \Rightarrow \boldsymbol{y_1 = 0.2}$$

From modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] \Rightarrow y_1^{(1)} = 0 + \frac{0.2}{2}[f(0,0) + f(0.2, 0.2)]$$

$$y_1^{(1)} = 0 + (0.1)[1 + 1.4214] \Rightarrow \boldsymbol{y_1^{(1)} = 0.2421}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] \Rightarrow y_1^{(2)} = 0 + \frac{0.2}{2}[f(0,0) + f(0.2, 0.2421)]$$

$$y_1^{(2)} = 0 + (0.1)[1 + 1.4635] \Rightarrow \boldsymbol{y_1^{(2)} = 0.2464}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] \Rightarrow y_1^{(3)} = 0 + \frac{0.2}{2}[f(0,0) + f(0.2, 0.2464)]$$

$$y_1^{(3)} = 0 + (0.1)[1 + 1.4678] \Rightarrow \boldsymbol{y_1^{(3)} = 0.2468}$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right] \Rightarrow y_1^{(4)} = 0 + \frac{0.2}{2}[f(0,0) + f(0.2, 0.2468)]$$

$$y_1^{(4)} = 0 + (0.1)[1 + 1.4682] \Rightarrow \boldsymbol{y_1^{(4)} = 0.2468}$$

$$\therefore \; y(x_0 + h) = y(0 + 0.2) = \boldsymbol{y(0.2) = 0.2468}$$

II Stage: We need to find $y(0.4)$ using $y(0.2) = 0.2468$ as the initial condition and taking $h = 0.2$. Now $\boldsymbol{x_0 = 0.2}$, $\boldsymbol{y_0 = 0.2468}$, $\boldsymbol{f(x, y) = y + e^x}$.

$$x_1 = x_0 + h = 0.2 + 0.2 \Rightarrow \boldsymbol{x_1 = 0.4}.$$

From Euler's formula, $y_1 = y_0 + hf(x_0, y_0)$

$y_1 = 0.2468 + 0.2f(0.2, 0.2468) \Rightarrow y_1 = 0.2468 + 0.2(1.4682) \Rightarrow \boldsymbol{y_1 = 0.5404}$

From modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$\Rightarrow y_1^{(1)} = 0.2468 + \frac{0.2}{2}[f(0.2, 0.2468) + f(0.4, 0.5404)]$$

$$\Rightarrow y_1^{(1)} = 0.2468 + (0.1)[1.4682 + 2.0322] \Rightarrow \boldsymbol{y_1^{(1)} = 0.5968}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$\Rightarrow y_1^{(2)} = 0.2468 + \frac{0.2}{2}[f(0.2, 0.2468) + f(0.4, 0.5968)]$$

$$\Rightarrow y_1^{(2)} = 0.2468 + (0.1)[1.4682 + 2.0886] \Rightarrow \boldsymbol{y_1^{(2)} = 0.6025}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

$$=> \ y_1^{(3)} = 0.2468 + \frac{0.2}{2}[f(0.2, 0.2468) + f(0.4, 0.6025)]$$

$$=> y_1^{(3)} = 0.2468 + (0.1)[1.4682 + 2.0943] => \boldsymbol{y_1^{(3)} = 0.6031}$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(3)}\right)\right]$$

$$=> \ y_1^{(4)} = 0.2468 + \frac{0.2}{2}[f(0.2, 0.2468) + f(0.4, 0.6031)]$$

$$=> y_1^{(4)} = 0.2468 + (0.1)[1.4682 + 2.0949] => \boldsymbol{y_1^{(4)} = 0.6031}$$

$$\therefore \ y(x_0 + h) = y(0.2 + 0.2) = \boldsymbol{y(0.4) = 0.6031}$$

Ans: $y(0.2) = 0.2468$ and $y(0.4) = 0.6031$

3. Use Modified Euler's method to solve $\frac{dy}{dx} = x + \left|\sqrt{y}\right|$, $y(0) = 1$, for the range $0 < x < 0.4$ taking $h = 0.2$. (carry out computations correct to 3 decimal places)

Solution:

I Stage: We need to find $y(0.2)$ by taking $h = 0.2$.

Given $\boldsymbol{x_0 = 0}$, $\boldsymbol{y_0 = 1}$, $\boldsymbol{f(x, y) = x + \sqrt{y}}$. $x_1 = x_0 + h = 0 + 0.2 => \boldsymbol{x_1 = 0.2}$.

From Euler's formula, $\quad y_1 = y_0 + hf(x_0, y_0)$

$$y_1 = 1 + 0.2f(0, 1) => y_1 = 1 + 0.2(1) => \ \boldsymbol{y_1 = 1.2}$$

From modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)] => \ y_1^{(1)} = 1 + \frac{0.2}{2}[f(0, 1) + f(0.2, 1.2)]$$

$$y_1^{(1)} = 1 + (0.1)[1 + 1.295] => \boldsymbol{y_1^{(1)} = 1.230}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right] => y_1^{(2)} = 1 + \frac{0.2}{2}[f(0, 1) + f(0.2, 1.230)]$$

$$y_1^{(2)} = 1 + (0.1)[1 + 1.309] => \boldsymbol{y_1^{(2)} = 1.231}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right] => y_1^{(3)} = 1 + \frac{0.2}{2}[f(0, 1) + f(0.2, 1.231)]$$

$$y_1^{(3)} = 1 + (0.1)[1 + 1.310] => \boldsymbol{y_1^{(3)} = 1.231}$$

$$\therefore \ y(x_0 + h) = y(0 + 0.2) = \boldsymbol{y(0.2) = 1.231}$$

II Stage: We need to find $y(0.4)$ using $y(0.2) = 1.231$ as the initial condition and taking $h = 0.2$. Now $\boldsymbol{x_0 = 0.2}$, $\boldsymbol{y_0 = 1.231}$, $\boldsymbol{f(x,y) = x + \sqrt{y}}$.

$$x_1 = x_0 + h = 0.2 + 0.2 => \boldsymbol{x_1 = 0.4}.$$

From Euler's formula, $y_1 = y_0 + hf(x_0, y_0)$

$$y_1 = 1.231 + 0.2f(0.2, 1.231) => y_1 = 1.231 + 0.2(1.310) => \boldsymbol{y_1 = 1.493}$$

From modified Euler's formula,

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

$$=> y_1^{(1)} = 1.231 + \tfrac{0.2}{2}[f(0.2, 1.231) + f(0.4, 1.493)]$$

$$=> y_1^{(1)} = 1.231 + (0.1)[1.310 + 1.622] => \boldsymbol{y_1^{(1)} = 1.524}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$=> y_1^{(2)} = 1.231 + \tfrac{0.2}{2}[f(0.2, 1.231) + f(0.4, 1.524)]$$

$$=> y_1^{(2)} = 1.231 + (0.1)[1.310 + 1.635] => \boldsymbol{y_1^{(2)} = 1.525}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

$$=> y_1^{(3)} = 1.231 + \tfrac{0.2}{2}[f(0.2, 1.231) + f(0.4, 1.525)]$$

$$=> y_1^{(3)} = 1.231 + (0.1)[1.310 + 1.635] => \boldsymbol{y_1^{(3)} = 1.525}$$

$$\therefore\ y(x_0 + h) = y(0.2 + 0.2) = \boldsymbol{y(0.4) = 1.525}$$

Ans: $(0.2) = 1.231$, $y(0.4) = 1.525$

Practice problems

1. Using modified Euler's method find $y\ at\ x = 0.2$, given $y' = 3x + \frac{y}{2}\ with\ y(0) = 1$, $h = 0.1$. (carry out computations correct to 4 decimal places)

   Ans: 1.1675

2. Using Modified Euler's method to find $y\ (0.1)$ given $\frac{dy}{dx} = x^2 + y$, $y(0) = 1$ by taking $h\ = 0.05$. (carry out computations correct to 4 decimal places)

   Ans: 1.1056

4. Runge Kutta Method of fourth order

The fourth order Runge Kutta method is often referred to as Runge Kutta method only. This method is used for finding the increment k of y corresponding to an increment h of x from the initial value problem $\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0$.

The method is as follows:

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) \; and$$

$$k_4 = hf(x_0 + h, y_0 + k_3).$$

Finally compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ which gives the required approximate value $\boldsymbol{y_1 = y_0 + k}$.

Problems

1. Apply Runge-Kutta method of fourth order to solve $\frac{dy}{dx} = x + y, y(0) = 1$ at $x = 0.2$ with step length $h = 0.2$. (carry out computations correct to 4 decimal places)

Solution: Here $x_0 = 0,\quad y_0 = 1,\quad \boldsymbol{f(x, y) = x + y}$ and $h = 0.2$.

From Runge Kutta method,

$k_1 = hf(x_0, y_0) = 0.2f(0, 1) = 0.2(1) \Rightarrow \boldsymbol{k_1 = 0.2}$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right) = 0.2f(0.1, 1.1)$

$$\Rightarrow k_2 = 0.2(1.2) \Rightarrow \boldsymbol{k_2 = 0.24}$$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right) = 0.2f(0.1, 1.12)$

$$\Rightarrow k_3 = 0.2(1.22) \Rightarrow \boldsymbol{k_3 = 0.244}$$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0 + 0.2, 1 + 0.244) = 0.2f(0.2, 1.244)$

$$\Rightarrow k_4 = 0.2(1.444) \Rightarrow \boldsymbol{k_4 = 0.2888}$$

$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.2 + 2(0.24) + 2(0.244) + 0.2888)$

$$\Rightarrow \boldsymbol{k = 0.2428}$$

Hence the required approximate value of y is

$$y_1 = y(x_0 + h) = y_0 + k \Rightarrow y(0 + 0.2) = 1 + 0.2428$$

$$\therefore \boldsymbol{y(0.2) = 1.2428}$$

Ans: $y(0.2) = 1.2428$

2. Using Runge- Kutta 4th order method to solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0) = 1$ at $x = 0.2$, 0.4. (carry out computations correct to 4 decimal places)

Solution:

We have $f(x,y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

To find $y(0.2)$

Hence $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$$k_1 = hf(x_0, y_0) = 0.2f(0,1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = 0.2f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891] = 0.19599$$

Hence $y(0.2) = y_0 + k = 1.196.$

To find $y(0.4)$:

Here $x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2.$

$$k_1 = hf(x_1, y_1) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2f(0.4, 1.3753) = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$

Ans: $y(0.2) = 1.196 \ and \ y(0.4) = 1.3752$

3. Using Runge-Kutta method of order 4, Solve $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$ at the points $x = 0.1, 0.2$ by taking step length take h = 0.1. (carry out computations correct to 4 decimal places)

I Stage: First we need to find $y(0.1)$.

Here $x_0 = 0$, $y_0 = 1$, $\boldsymbol{f(x,y) = 3x + y/2}$ and $h = 0.1$.

From Runge Kutta method,

$k_1 = hf(x_0, y_0) = 0.1f(0, 1) = 0.1(0.5) => \boldsymbol{k_1 = 0.05}$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.05}{2}\right) = 0.1f(0.05, 1.025)$$

$$=> k_2 = 0.1(0.6625) => \boldsymbol{k_2 = 0.0663}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.0663}{2}\right) = 0.1f(0.05, 1.0331)$$

$$=> k_3 = 0.1(0.6666) => \boldsymbol{k_3 = 0.0677}$$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0 + 0.1, 1 + 0.0677) = 0.1f(0.1, 1.0677)$

$$=> k_4 = 0.1(0.8339) => \boldsymbol{k_4 = 0.0834}$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.05 + 2(0.0663) + 2(0.0677) + 0.0834)$$

$$=> \boldsymbol{k = 0.0669}$$

Hence the required approximate value of y is

$$y_1 = y(x_0 + h) = y_0 + k => y(0 + 0.1) = 1 + 0.0669$$

$$\therefore \ \boldsymbol{y(0.1) = 1.0669}$$

II Stage: We need to find $y(0.2)$ using $y(0.1) = 1.0669$ as the initial condition.

Here $x_0 = 0.1$, $y_0 = 1.0669$, $\boldsymbol{f(x, y) = 3x + y/2}$ and $h = 0.1$.

From Runge Kutta method,

$k_1 = hf(x_0, y_0) = 0.1f(0.1, 1.0699) = 0.1(0.8335) => \boldsymbol{k_1 = 0.0833}$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1f\left(0.1 + \frac{0.1}{2}, 1.0669 + \frac{0.0833}{2}\right) = 0.1f(0.15, 1.1086)$$

$$=> k_2 = 0.1(1.0043) => \boldsymbol{k_2 = 0.1004}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1f\left(0.1 + \frac{0.1}{2}, 1.0669 + \frac{0.1004}{2}\right) = 0.1f(0.15, 1.1171)$$

$$=> k_3 = 0.1(1.0086) => \boldsymbol{k_3 = 0.1009}$$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1 + 0.1, 1.0669 + 0.1009) = 0.1f(0.2, 1.1678)$

$$=> k_4 = 0.1(1.1839) => \boldsymbol{k_4 = 0.1184}$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.0835 + 2(0.1004) + 2(0.1009) + 0.1184)$$

$$=> \boldsymbol{k = 0.1007}$$

Hence the required approximate value of y is

$$y_1 = y(x_0 + h) = y_0 + k => y(0.1 + 0.1) = 1.0669 + 0.1008$$

$$\therefore \ \boldsymbol{y(0.2) = 1.1676}$$

Ans: $y(0.1) = 1.0669$ and $y(0.2) = 1.1676$

Practice problems

1. Apply Runge-Kutta method of order 4, to compute $y(0.2)$ given $10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ taking h = 0.1. (carry out computations correct to 4 decimal places)

Ans: 1.0207

2. Use Runge-Kutta method of 4th order for $y(0.1), y(0.2)$ given that $\frac{dy}{dx} = y(x+y), y(0) = 1$. (carry out computations correct to 4 decimal places)

Ans: $y(0.1) = 1.1169, y(0.2) = 1.2774$

3. Using Runge-Kutta method of order 4, find y(0.2) for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$ taking h =0.1. (carry out computations correct to 4 decimal places)

Ans: 1.1678

5. Predictor-Corrector Methods

Consider the differential equation $y' = \frac{dy}{dx} = f(x, y)$ with a set of 4 determined values of $y: y(x_0) = y_0, y(x_1) = y_1, y(x_2) = y_2$ and $y(x_3) = y_3$.

Here x_0, x_1, x_2, x_3 are equally spaced values of x width h.

Also $x_4 = x_3 + h = x_0 + 4h$

Predictor and Corrector formulae to compute $y(x_4) = y_4$ are as follows.

(a) Milne's Predictor-Corrector formulae

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)\text{......} \qquad \text{Predictor formula}$$

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)\text{......} \qquad \text{Corrector formula}$$

General form

$$y_{n+1}^{(P)} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}' - y_{n-1}' + 2y_n'\right]$$

$$y_{n+1}^{(C)} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right]$$

(b) Adams-Bashforth Corrector formulae

$$y_4^{(P)} = y_3 + \frac{h}{24}\left(55y_3' - 59y_2' + 37y_1' - 9y_0'\right) \qquad \text{Predictor formula}$$

$$y_4^{(C)} = y_3 + \frac{h}{24}\left(9y_4' + 19y_3' - 5y_2' + y_1'\right) \qquad \text{Corrector formula}$$

Problems

1. Apply (a) Milne's method and (b) Adams-Bashforth method, to compute y at x=0.8 for the given $\frac{dy}{dx}=x-y^2$ and the data y(0)=0, y(0.2)=0.02, y(0.4)=0.0795, y(0.6)=0.1762.
2. Apply Milne's method to Compute y(1.4) correct to four decimal places given $\frac{dy}{dx}=x^2+\frac{y}{2}$ and following data: y(1)=2, y(1.1)=2.2156, y(1.2)=2.4649, y(1.3)=2.7514
   Ans) 3.4997, 3.0794
3. If $\frac{dy}{dx}=2e^x-y$, y(0)=2, y(0.1)=2.010, y(0.2)=2.040 and y(0.3)=2.090, find y(0.4) correct to four decimal places by using (a) Milne's method and (b) Adams-Bashforth method (Apply the corrector formula twice)

Practice Problems

1. Use Taylor's series method (upto third derivative term) to find y at x=0.1, 0.2, 0.3 given that $\frac{dy}{dx}=x^2+y^2$ with y(0)=1 . Apply Milne's predictor-corrector formulae to find y(0.4) using the generated set of initial values.

Solution of One Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (i)$$

where $c^2 = k/s\rho$ is the diffusivity of the substance (cm²/sec.)

also known as *diffusion equation*.

We can solve this by Schmidt method and Crank-Nicolson method.

Schmidt method

or $$u_{i,j+1} = \alpha u_{i-1,j} + (1 - 2\alpha)\, u_{i,j} + \alpha u_{i+1,j} \qquad (2)$$

where $\alpha = kc^2/h^2$ is the mesh ratio parameter.

This formula enables us to determine the value of u at the $(i, j + 1)$th mesh point in terms of the known function values at the points x_{i-1}, x_i, and x_{i+1} at the instant t_j. It is a relation between the function values at the two time levels $j + 1$ and j and is therefore, called a *two-level formula*

In particular when $\alpha = 1/2$, reduces to

$$u_{i,j+1} = 1/2, (u_{i-1,j} + u_{i+1,j})$$

This is Bendre Schmidt formula.

Crank-Nicolson method

$$-\alpha u_{i-1,j+1} + (2 + 2\alpha)u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j} + (2 - 2\alpha)u_{i,j} + \alpha u_{i+1,j}$$

Where

$$\alpha = kc^2/h^2.$$

This is a two level implicit relation and is known as Crank-Nicolson formula.

Problems

Solve the boundary value problem $u_t = u_{xx}$ under the conditions $u(0, t) = u(1, t) = 0$ and $u(x, 0) = \sin px$, $0 \le x \le 1$ using the Schmidt method (Take $h = 0.2$ and $\alpha = 1/2$).

Solution:

Since $\quad h = 0.2 \text{ and } \alpha = ½$

$\therefore \quad \alpha = \dfrac{k}{h^2}$ gives $k = 0.02$

Since $\alpha = 1/2$, we use the Bendre-Schmidt relation

$$u_{i,j+1} = \frac{1}{2}(i_{i-1,j} + u_{i+1,j}) \qquad (i)$$

We have $u(0, 0) = 0$, $u(0.2, 0) = \sin \pi/5 = 0.5875$

$u(0.4, 0) = \sin 2\pi/5 = 0.9511$, $u(0.6, 0) = \sin 3\pi/5 = 0.9511$

$u(0.8, 0) = \sin 4\pi/5 = 0.5875$, $u(1, 0) = \sin \pi = 0$

The values of u at the mesh points can be obtained by using the recurrence relation (i) as shown in the table below:

$x \rightarrow$		0	0.2	0.4	0.6	0.8	1.0
t ↓ 0	j \ i	0	1	2	3	4	5
	0	0	0.5878	0.9511	0.9511	0.5878	0
0.02	1	0	0.4756	0.7695	0.7695	0.4756	0
0.04	2	0	0.3848	0.6225	0.6225	0.3848	0
0.06	3	0	0.3113	0.5036	0.5036	0.3113	0
0.08	4	0	0.2518	0.4074	0.4074	0.2518	0
0.1	5	0	0.2037	0.3296	0.3296	0.2037	0

Q) Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ in $0 < x < 5$, $t \geq 0$ given that $u(x,0) = 20$, $u(0,t) = 0$, $u(5,t) = 100$. Compute u for the time-step with $h = 1$ by Crank-Nicholson method.

Ans) Compare given eq $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$c^2 = 1$ & given that $h = 1$

Taking $\alpha = 1$,

$$\alpha = \frac{kc^2}{h^2}$$

$$1 = 1 \cdot k / 1^2 \quad \therefore k = 1$$

$x : 0, 1, 2, 3, 4, 5$

$u(x,0) = 20$, $\boxed{u(0,t) = 0, \quad u(5,t) = 100}$, $u(3,0) = 20$
$u(1,0) = 20$, $u(2,0) = 20$,
$u(4,0) = 20$

$u(0,0) = 0$ & $u(5,0) = 100$

$t \backslash x$	0	1	2	3	4	5
0	0	20	20	20	20	100
1	0	u_1	u_2	u_3	u_4	100

Crank Nicholson formula

$$-\alpha u_{i-1,j+1} + (2+2\alpha) u_{i,j+1} - \alpha u_{i+1,j+1} = \alpha u_{i-1,j} + (2-2\alpha) u_{i,j} + \alpha u_{i+1,j} \quad \text{(1)}$$

where $\alpha = kc^2/h^2$

Here $\alpha = 1$

From ① $4u_{i,j+1} = u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}$

$4u_1 = 0 + 20 + 0 + u_2 \rightarrow (2)$

$4u_2 = 20 + 20 + u_1 + u_3 \rightarrow (3)$

$4u_3 = 20 + 20 + u_2 + u_4 \rightarrow (4)$

$4u_4 = 20 + 100 + u_3 + 100 \rightarrow (5)$

From (2) $u_1 = 5 + \frac{u_2}{4} \rightarrow (6)$

substitute (6) in (3)

$4u_2 = 20 + 20 + \left(5 + \frac{u_2}{4}\right) + u_3$

$4u_2 = 45 + \frac{u_2}{4} + u_3$

$\frac{u_2}{4} + u_3 - 4u_2 = -45$ (or)

$u_2\left(\frac{1}{4} - 4\right) + u_3 = -45$

$\boxed{-\frac{15}{4}u_2 + u_3 = -45} \rightarrow (7)$

$\frac{1}{4} - 4$
$= \frac{1-16}{4} = -\frac{15}{4}$

From (4), $\boxed{u_2 - 4u_3 + u_4 = -40} - (8)$

From (5), $\boxed{u_3 - 4u_4 = -220} - (9)$

On solving (7), (8) & (9), $u_2 = 20.2$, $u_3 = 30.7$, $u_4 = 62.7$

From (6) $u_1 = 5 + \frac{20.2}{4} = 10.05$

(3) Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the conditions $u(x,0) = \sin \pi x$, $0 \le x \le 1$; $u(0,t) = u(1,t) = 0$, using (a) Schmidt method (b) Crank Nicolson method. Carry out computation for two levels, taking $h = 1/3$, $k = 1/36$

Ans) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ compare with $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$c^2 = 1$ and given that $h = 1/3$ $k = 1/36$

$$\alpha = \frac{kc^2}{h^2} = \frac{\frac{1}{36}(1)}{(1/3)^2} = 1/4$$

Schmidt formula $u_{i,j+1} = \alpha u_{i-1,j} + (1-2\alpha)u_{i,j} + \alpha u_{i+1,j}$

$$u_{i,j+1} = \frac{1}{4} u_{i-1,j} + \frac{1}{2} u_{i,j} + \frac{1}{4} u_{i+1,j}$$

$1 - 2\alpha$
$1 - 2 \cdot \frac{1}{4}$
$= 1/2$

$x = 0, 1/3, 2/3, 1$ & $t = 0, 1/36, 2/36$

$u(x,0) = \sin \pi x$

$u(1/3, 0) = \sin \pi/3 = \sqrt{3}/2$

$u(2/3, 0) = \sin \frac{2\pi}{3} = \sqrt{3}/2$

$u(0,t) = u(1,t) = 0$

t \ x	0	1/3	2/3	1
0	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0
1/36	0	0.65	0.65	0
2/36	0	0.49	0.49	0

(b) Crank - Nicolson formula

$$-\alpha u_{i-1,j+1} + (2+2\alpha)u_{i,j+1} - \alpha u_{i-1,j+1} = \alpha u_{i-1,j} + (2-2\alpha)u_{i,j} + \alpha u_{i-1,j} \quad \text{(1)}$$

$$\alpha = kc^2/h^2$$

Here $\alpha = 1/4$

From (1)

$$\left.\begin{aligned} -\frac{1}{4}u_{i-1,j+1} + \frac{5}{2}u_{i,j+1} - \frac{1}{4}u_{i-1,j+1} \\ = \frac{1}{4}u_{i-1,j} + \frac{3}{2}u_{i,j} + \frac{1}{4}u_{i-1,j} \end{aligned}\right\} \text{(2)}$$

t \ x	0	1/3	2/3	1
0	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0
1/36	0	u_{11}	u_{12}	0
2/36	0	u_{21}	u_{22}	0

From (2)

$$\frac{5}{2}u_{11} = 0 + \frac{\sqrt{3}}{2}\cdot\frac{1}{4} + 0 + \frac{1}{4}u_{12} + \frac{3}{2}\cdot\frac{\sqrt{3}}{2}$$

$$\frac{5}{2}u_{11} = \frac{u_{12}}{4} + \sqrt{3}\left(\frac{1}{8} + \frac{3}{4}\right)$$

$$\frac{5}{\cancel{2}}u_{11} = \frac{u_{12}}{\cancel{4}\,2} + \sqrt{3}\left(\frac{1+6}{\cancel{8}\,4}\right)$$

$$5u_{11} = \frac{u_{12}}{2} + \frac{7\sqrt{3}}{4} = \frac{2u_{12} + 7\sqrt{3}}{4}$$

$$20u_{11} = 2u_{12} + 7\sqrt{3}$$

$$20u_{11} - 2u_{12} = 7\sqrt{3} \longrightarrow (3)$$

Again From (2)

$$\frac{5}{2}u_{12} = \frac{\sqrt{3}}{8} + 0 + \frac{1}{4}u_{11} + 0 + \frac{3\sqrt{3}}{4}$$

$$\frac{5}{\cancel{2}}u_{12} = \frac{\sqrt{3} + 2u_{11} + 6\sqrt{3}}{\cancel{8}\,4}$$

$$20u_{12} = 2u_{11} + 7\sqrt{3}$$

$$-2u_{11} + 20u_{12} = 7\sqrt{3} \longrightarrow (4)$$

On solving (3) & (4), $u_{11} = 0.67$ &
$u_{12} = 0.67$

From (2)

$$\frac{5}{2}u_{21} = 0 + (0.67)\frac{1}{4} + 0 + \frac{1}{4}u_{22} + \frac{3}{2}(0.67)$$

$$\frac{5}{2}u_{21} = \frac{0.67 + u_{22} + 6(0.67)}{4}$$

$$\frac{5}{\cancel{2}}u_{21} = \frac{u_{22} + 4.69}{\cancel{4}\,2}$$

$$10u_{21} = u_{22} + 4.69 \text{ (or) } \boxed{10u_{21} - u_{22} = 4.69} \rightarrow (5)$$

From (2) $\frac{5}{2}u_{22} = \frac{1}{4}(0.67) + 0 + \frac{1}{4}u_{21} + 0 + (0.67)\frac{3}{2}$

$$\frac{5}{\cancel{2}}u_{22} = \frac{0.67 + u_{21} + 6(0.67)}{\cancel{4}\,2}$$

$$10u_{22} = u_{21} + 4.69 \text{ (or) } \boxed{-u_{21} + 10u_{22} = 4.69} \rightarrow (6)$$

On solving (5) & (6), $u_{21} = u_{22} = 0.52$