Module 2
Numerical Interpolation, differentiation and Integration

Interpolation

Consider the statement $Y = f(x);\ x_0 \leq x \leq x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \leq x \leq x_n$. If the function $f(x)$ is single-valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, \dots, x_n$ can be calculated. The problem now is if we are given the set of tabular values

$\boldsymbol{X}$:	x_0	x_1	x_2	… … … … ….	x_n
$\boldsymbol{Y}$:	y_0	y_1	y_2	… … … … ….	y_n

Satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, it is possible to find a simple function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process to finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation.

(or)

Let $x_0, x_1, \dots, x_n$ be the values x and $y_0, y_1, \dots, y_n$ be the values of y and $y = f(x)$ be an unknown function. The process to find the value of the unknown function $y = f(x)$ when the given value of x and the value of x lies within the limits x_0 to x_n is called interpolation.

Extrapolation

Let $x_0, x_1, \dots, x_n$ be the values x and $y_0, y_1, \dots, y_n$ be the values of y and $y = f(x)$ be an unknown function. The process of finding the value of the unknown function $y = f(x)$ when the given value of x and the value of x lie outside the range of x_0 to x_n is called Extrapolation.

Note:

If the differences of x values are equal in the given data then it is called equal spaced points otherwise it is called unequal spaced points.

- Suppose a given value of x is nearer to starting value of x then we use Newton's forward interpolation formula.
- Suppose a given value of x is nearer to ending value of x then we use Newton's backward interpolation formula.
- Suppose a given value of x is nearer to middle value of x then we use Gauss interpolation formula.
- Suppose the given data has unequal spaced points then we use Newton's divided difference method or Lagrange's interpolation formula.

Forward Differences

The Forward Difference operator is denoted by Δ , The forward differences are usually arranged in tabular columns as shown in the following table called a Forward difference table

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x_0	$\boldsymbol{f_0}$				
		$f_1 - f_0 = \boldsymbol{\Delta f_0}$			
x_1	f_1		$\Delta f_1 - \Delta f_0 = \boldsymbol{\Delta^2 f_0}$		
		$f_2 - f_1 = \Delta f_1$		$\Delta^2 f_1 - \Delta^2 f_0 = \boldsymbol{\Delta^3 f_0}$	
x_2	f_2		$\Delta f_2 - f_1 = \Delta^2 f_1$		$\Delta^3 f_1 - \Delta^3 f_0 = \boldsymbol{\Delta^4 f_0}$
		$f_3 - f_2 = \Delta f_2$		$\Delta^2 f_2 - \Delta^2 f_1 = \Delta^3 f_1$	
x_3	f_3		$\Delta f_3 - \Delta f_2 = \Delta^2 f_2$		
		$f_4 - f_3 = \Delta f_3$			
x_4	f_4				

Backward Differences

The Backward Difference operator is denoted by ∇ , The backward differences are usually arranged in tabular columns as shown in the following table called a Backward difference table

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0				
		$f_1 - f_0 = \nabla f_1$			
x_1	f_1		$\nabla f_2 - \nabla f_1 = \nabla^2 f_2$		
		$f_2 - f_1 = \nabla f_2$		$\nabla^2 f_3 - \nabla^2 f_2 = \nabla^3 f_3$	
x_2	f_2		$\nabla f_3 - \nabla f_2 = \nabla^2 f_3$		$\nabla^3 f_4 - \nabla^3 f_3 = \boldsymbol{\nabla^4 f_4}$
		$f_3 - f_2 = \nabla f_3$		$\nabla^2 f_4 - \nabla^2 f_3 = \boldsymbol{\nabla^3 f_4}$	
x_3	f_3		$\nabla f_4 - \nabla f_3 = \boldsymbol{\nabla^2 f_4}$		
		$f_4 - f_3 = \boldsymbol{\nabla f_4}$			
x_4	$\boldsymbol{f_4}$				

1. Newton's Forward and Backward Interpolation

Given the set of $(n+1)$ values $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_n, y_n)$ of x and y. It is required to find a polynomial of n^{th} degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0 + ih$ $(i = 0, 1, 2 \ldots\ldots n)$ then,

The **Newton's forward interpolation formula** is given by

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots + \frac{p(p-1)(p-2)\ldots\ldots\ldots\ldots(p-(n-1))}{n!}\Delta^n y_0$$

Where, $p = \frac{x - x_0}{h}$

Note: This formula is used when value of x is located near beginning of tabular values.

The **Newton's backward interpolation formula** is given by

$$y = f(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots\ldots\ldots + \frac{p(p+1)(p+2)\ldots\ldots\ldots\ldots(p+(n-1))}{n!}\nabla^n y_n$$

Where, $p = \frac{x-x_n}{h}$

Note: This formula is used when value of x is located near end of tabular values.

Problems

1. The area A of a circle corresponding to the diameter (D) is given below:

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the area corresponding to the diameter 105 by using appropriate interpolation formula.

Ans: $A(105) = 8666$

2. Find y at x=38 and x=85 given data

x	40	50	60	70	80	90
y	184	204	226	250	276	304

Ans). 180.24, 289.75

3. Find the polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	2	1	10

Also find f(4).

Ans) $f(x) = 2x^3 - 7x^2 + 6x + 1$, 41

4. The population of a town in the decimal census was given below. Estimate the population for the 1955.

$x\ (year)$	1951	1961	1971	1981	1991
Population in thousands	19.96	39.65	58.81	77.21	94.61

Ans: $Population\ at\ 1955\ is\ 27.89.$

5. From the following table, estimate the number of students who obtained marks between 40 and 45

$Marks$	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Ans) 48,17

Practice problems

6. In the table given below, the values of y are consecutive terms of a series of which 23.6 is the 6^{th} term. Find the first and tenth term of the series:

x	3	4	5	6	7	8	9
y	4.8	8.4	14.5	23.6	36.2	52.8	73.9

Ans: $y = 100$

7. The population of a town in the decimal census was given below. Estimate the population for the 1925.

$x\ (year)$	1891	1901	1911	1921	1931
Population in thousands	46	66	81	93	101

Ans: $Population\ at\ 1925\ is\ 96{,}837.$

8. Given $f(0) = 1$, $f(1) = 3$, $f(2) = 7$, $f(3) = 13$. Find $f(0.1)$ and $f(2.9)$ using Newton Interpolation formula.

Ans: $f(0.1) = 1.11 \; and \; f(2.9) = 12.31$

2. Newton's Divided difference Method

Newton's divided difference interpolation formula is an interpolation technique used when the interval difference is not same for all sequence of values. Suppose $f(x_0), f(x_1), f(x_2), \dots \dots \dots, f(x_n)$ be the $(n+1)$ values of the function $y = f(x)$ corresponding to the arguments $x = x_0, x_1, x_2 \dots x_n$, where interval differences are not same.

Then the first divided difference is given by

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The second divided difference is given by

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

and so on.

x	f		1st Order Difference	2nd Order Difference	3rd Order Difference
x_0	f_0				
		$\frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1]$			
x_1	f_1		$\frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2]$		
		$\frac{f_2 - f_1}{x_2 - x_1} = f[x_1, x_2]$		$\frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = f[x_0, x_1, x_2, x_3]$	
x_2	f_2		$\frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = f[x_1, x_2, x_3]$		$\frac{f[x_1, x_2, x_3, x_4] - f[x_0, x_1, x_2, x_3]}{x_4 - x_0} = f[x_0, x_1, x_2, x_3, x_4]$
		$\frac{f_3 - f_2}{x_3 - x_2} = f[x_2, x_3]$		$\frac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1} = f[x_1, x_2, x_3, x_4]$	
x_3	f_3		$\frac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2} = f[x_2, x_3, x_4]$		
		$\frac{f_4 - f_3}{x_4 - x_3} = f[x_3, x_4]$			
x_4	f_4				

Newton's divided difference formula is,

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + \{(x - x_0)(x - x_1)\}f(x_0, x_1, x_2) + \dots\dots\dots$$

$$\dots\dots\dots\dots + \{(x - x_0)(x - x_1) \dots \dots (x - x_{n-1})\}f(x_0, x_1, x_2, \dots, x_n)$$

Problems

1. Use Newton's divided difference formula to evaluate $f(4)$

x	0	2	3	6
$f(x)$	-4	2	14	158

Ans: $f(4) = 40$

2. Determine $f(x)$ as a polynomial in x for the following data using Newton's divided difference formula:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Also find f(8) and f(15)

Ans) **:** $f(x) = x^3 - x^2$,448,3150

3. Find the equation of the polynomial which pass through the point $(2,4),(4,56),(9,711),(10,980)$ by using Newton's divided difference interpolation and find y coordinate at $x =$ 3,5,7,11.

Ans: $f(x) = x^3 - 2x$, 21,115,329,1309

Practice problems

1. Using divided difference formula for unequal intervals find $f(2.5)$.

x	2	4	5	6	8	10
$f(x)$	10	96	196	350	868	1746

Ans: $f(2.5) = 21$

2. Using Use Newton's divided difference formula, find the missing value from the table

x	1	2	4	5	6
y	14	15	5		9

Ans: 3

3. Use Newton's divided difference formula to compute $f(5.5)$ from the following data

x	0	1	4	5	6
$f(x)$	1	14	15	6	3

Ans: $f(5.5) = 3.0969$

3. Lagrange's Interpolation Formula

Let (x) be continuous and differentiable $(n+1)$ times in the interval (a,b). Given the $(n+1)$ points as $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ where values of x not necessarily be equally spaced then the interpolating polynomial of degree 'n' say $f(x)$ is given by

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1) + \ldots\ldots$$

$$\ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n)$$

Note: This formula is used when values of x are unequally spaced and equally spaced.

Problems

1. Given the values

x	0	2	3	6
$f(x)$	-4	2	14	158

Evaluate f(4) using Lagrange's formula.

Ans:40

2. Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x	0	2	3	6
$f(x)$	-4	2	14	147

Ans: $f(x) = x^3 + x^2 - x + 2 \text{ and } f(3) = 35$

3. The following are the measurements T made on curve recorded by the oscillography representing a change of current I due to a change in condos of an electric current. Find the polynomial f(T).

T	1.2	2	2.5	3
I	1.36	0.58	0.34	0.2

Ans: $I = f(T) = -\,0.1004\,T^3 + 0.9532\,T^2 - 3.238\,T + 4.046$

4. Find the parabola passing through points $(0,1), (1,3)$ and $(3,55)$ using Lagrange's Interpolation Formula.

Ans: $f(x) = 8x^2 - 6x + 1$

Practice problems

5. Using Lagrange's interpolation formula, find the value of y when $x = 10$ from the following data

x	5	6	9	11
y	12	13	14	16

Ans: $y = 14.6667$

6. If $y(1) = 3,\ y(3) = 9,\ y(4) = 30,\ y(6) = 132$, find Lagrange's interpolation polynomial

Ans: $y = 8x^3 - 4x^2 - 5.8x + 8.4$

7. Using Lagrange's interpolation formula, find the value of y when $x = 2$ from the following table:

x	1	3	4	6
y	4	40	85	259

Ans: $y \text{ when } x = 2 \text{ is } 15$

8. Use Lagrange's interpolation formula to fit a polynomial for the data.

x	0	1	3	4
y	12	0	6	12

Ans: $y = -x^3 + 9x^2 - 20x + 12$

4. Numerical Integration Introduction

The process of evaluating a definite integral from a set of tabulated values of the integrand (x), which is not known explicitly is called Numerical Integration.

Newton –Cote's Quadrature Formula

We want to find the definite integral form $\int_a^b f(x)dx$, where $f(x)$ is unknown explicitly, then We replace $f(x)$ with interpolating polynomial.

Here we replace with Newton Forward Interpolation formula divide the interval (a, b) into n subintervals of width h so that

$$a = x_0 < x_1 = x_0 + h < x_2 = x_1 + h < \dots\dots\dots\dots < x_n = x_n + h = b.$$

Then,

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)(p-2)\dots(p-(n-1))}{n!}\Delta^n y_0$$

Where $p = \frac{x-x_0}{h}$ $hd\,p = d\,x\;\; a\,t\;\; x = x_0 \Rightarrow p = 0\;\; a\,n\,d\;\; x = x_n \Rightarrow p = n$

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} y_n(x)dx = h\int_{x_0}^{x_n}\left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots\right)dp$$

$$= h\int_0^n (y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots)dp$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

This is Newton Cotes Quadrature Formula.

Truncation error is the error that occurs when a mathematical process, often one that involves an infinite number of steps or terms (like an infinite series or a limiting process), is approximated by a finite one. It is the difference between the true, exact mathematical value and the value obtained from the finite approximation.

Derive Trapezoidal Rule for numerical integration of $\int_a^b f(x)dx$

Trapezoidal Rule

Substitute $n = 1$ in Newton Cotes Quadrature formula and taking the curve $y = f(x)$ passing through (x_0, y_0) and (x_1, y_1) as a straight line so that differences of order higher than first become zero (i.e., Δ^2, Δ^3 $e\,t\,c\;\; b\,e\,c\,o\,m\,e\;\; z\,e\,r\,o$) (n=number of intervals)

$$\int_{x_0}^{x_1} y_n(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h[y_0 + y_1]$$

Similarly, we get,

$$\int_{x_1}^{x_2} y_n(x)dx = h[y_1 + y_2]$$

. . . .

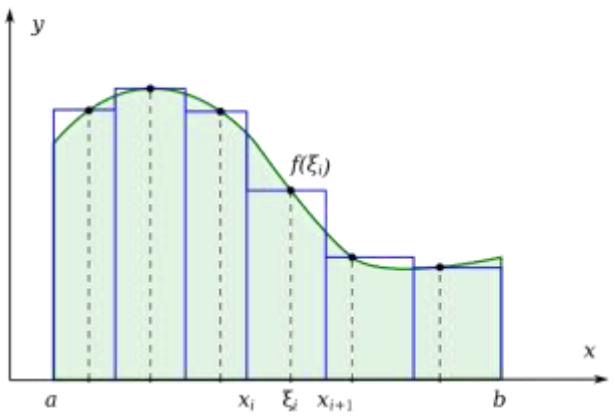
Adding the above we get

$$\int_a^b f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]$$

$$\int_a^b f(x)dx = \frac{h}{2}\ [(sum\ of\ the\ Ist\ \&\ last\ ordinates) + 2(sum\ of\ the\ remaining\ ord.)]$$

Geometrical interpretation of Trapezoidal Rule:

Here trapezoidal rule denotes sum of areas of above trapeziums.



Simpson's 1/3 Rule

Substitute $n = 2$ in Newton Cotes Quadrature Formula and taking the curve $y = f(x)$ passing through (x_0, y_0) , (x_1, y_1) and (x^2, y^2) as a parabola so that differences of order higher than second become zero (i.e., Δ^3 , Δ^4 $etc\ become\ zero$)

$$\int_{x_0}^{x_2} f(x)dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

We know $E = 1 + \Delta$. Then,

$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$$

Similarly,

$$\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

. . . . and so on.

Adding above we get

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$\int_a^b f(x)dx = \frac{h}{3}[(sum\ of\ the\ first\ and\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates) + 2(sum\ of\ the\ remaining\ even\ ordinates)]$$

This is known as Simpson's 1/3 Rule (or) Simply Simpson's Rule.

Simpson's 3/8th Rule

$$\int_a^b f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

Note:

1. Trapezoidal Rule is applicable for any number of subintervals
2. Simpson's 1/3rd rule is applicable when the number of subintervals must be even.
3. Simpson's 3/8th rule is applicable when the number of subintervals must be multiple of 3.
4. The trapezoidal rule is the simplest of the formulas for numerical integration, but it is also the least accurate. The accuracy of the result can be improved by decreasing the interval h.
5. Simpson's 1 / 3rd rule is also called a closed formula since the endpoints y_o and y_1 are also included in the formula.
6. Truncation error in trapezoidal rule is $\frac{(b-a)h^2}{12}f''(\xi)$ and it is of the order h^2.
7. Truncation error in Simpson's 1 / 3rd rule is $\frac{-(b-a)h^4}{180}f^{(iv)}(\xi)$ and it is of the order h^4.
8. Truncation error in Simpson's 3 / 8th rule is $\frac{-(b-a)h^4}{80}f^{(iv)}(\xi)$ and it is of the order h^4.

Comparison of Trapezoidal Rule and Simpson's 1/3 rule

In trapezoidal rule we take $n = 1$ (no of subintervals) between every two points we are taking a straight line (Linear) whereas in Simpson's rule, $n = 2$ means we are taking a parabola so the error is less compared to trapezoidal rule.

Weddle's Rule

Substitute $n = 6$ in Newton Cotes Quadrature formula and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)dx = 6h\left(y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0\right)$$

If we replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, the error made will be negligible.

$$\therefore \qquad \int_{x_0}^{x_0+6h} f(x)dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \cdots) \quad (6)$$

This is known as *Weddle's rule.*

Problems

1. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ taking seven ordinates by applying (i) Trapezoidal rule (ii) Simpson's 1/3rd rule (iii) Simpson's 3/8th rule (iv).Weddle's rule and Compare the results with its actual value.

   Ans: (i) 1.4108(ii) 1.3662 (iii) 1.3571(iv). 1.3735 and actual value is 1.4056

2. Evaluate $\int_0^1 \frac{dx}{1+x}$ by Simpson's 1/3rd rule taking seven ordinates and hence find $\log_e 2$.

   Ans: Simpson (1/3)rd Rule = 0.6931; $\log_e 2 = 0.6931$.

3. Evaluate $\int_0^{0.6} e^{-x^2} dx$ by taking seven ordinates using (i) Trapezoidal rule (ii) Simpson's 1/3rd rule (iii) Simpson's 3/8th rule (iv). Weddle's rule

   Ans: (i) 0.5344 (ii) 0.5351 (iii) 0.5351 (iv). 0.5351

4. The velocity v(km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows:

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

   Estimate approximately the distance covered in 20 minutes using Simpson's 1/3rd rule.

   Ans: 309.33km

Practice Problems

5. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ taking seven ordinates by applying (i) Trapezoidal rule (ii) Simpson's 1/3rd rule (iii) Simpson's 3/8th rule (iv).Weddle's rule

   Ans: (i) 0.78424 (ii) 0.7854 (iii) 0.7854 (iv).

6. A river is soft wide. The depth 'd' in feet at a distance $x\ ft.$ from one bank is given by the table

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

   Find approximately the area of cross-section.

   Ans: 710 $Sq.ft.$