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Notes of

Real and Functional Analysis

for the Master in Mathematical Engineering
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Part I

Introduction

Chapter 0

Course structure

This course is splitted in two parts:

1. Real Analysis $\rightsquigarrow$ measure and integration theory, in particular:
    - Collections and sequences of sets
    - Measurable space, measure, outer measure
    - Generation of an outer measure
    - Carathéodory's condition, measure induced by an outer measure
    - Lebesgue's measure on $\mathbb{R}^n$
    - Measurable functions
    - The Lebesgue integral
    - Abstract integration
    - Monotone convergence theorem, Fatou's Lemma, Lebesgue's dominated convergence theorem
    - Comparison between the Lebesgue and Riemann integrals
    - Different types of convergence
    - Derivative of a measure and the Radon-Nikodym theorem
    - Product measures and the Fubini-Tonelli theorem
    - Functions of bounded variation and absolutely continuous functions
2. Functional Analysis $\rightsquigarrow$ infinte dimensional linear algebra, in particular:
    - Metric spaces, completeness, separability, compactness
    - Normed spaces and Banach spaces
    - Spaces of integrable functions
    - Linear operators
    - Uniform boundedness theorem, open mapping theorem, closed graph theorem
    - Dual spaces and the Hahn-Banach theorem
    - Reflexivity
    - Weak and weak* convergences
    - Banach-Alaoglu theorem
    - Compact operators
    - Hilbert spaces
    - Projection theorem, Riesz representation theorem
    - Orthonormal basis, abstract Fourier series
    - Spectral theorem for compact symmetric operators

- Fredholm alternativ

The foundation of this theory is the *Set Theory*, that is going to be explained in the next chapter. Enjoy!

NB: this page will be updated with more details and maybe the list of proofs.

Chapter 1

Set Theory

1.1 Equipotent, finite/infinite, countable/uncountable sets, cardinality of continuoum

Let X, Y be sets.

Def — Equipotent sets.
X, Y are equipotent if there exists a bijection $f : X \to Y$ (1-1 injective + onto surjective).

If X, Y are equipotent, then they have the same cardinality. On the other hand, X has cardinality $\geq$ than Y if there exists $f : X \to Y$ onto. For example, for

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

exists $f : X \to Y$ s.t. $\forall y \in Y \ \exists x \in X$ s.t. $f(x) = y$ (f *takes* all the elements of the codomain), but doesn't exist $g : Y \to X$ s.t. $\forall x \in X \ \exists y \in Y$ s.t. $g(y) = x$ (g doesn't *take* all the elements of the codomain).

Def — Finite/infinite sets.
X is finite if it is equipotent to $Y = \{1, 2, ..., k\}$ for some $k \in \mathbb{N}$. X is infinite otherwise.

Prop. X is infinite iff it is equipotent to a proper subset, i.e. if exists a bijection between X and one of his subsets.

For example, between the integers set $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ and the even integers set $\{0, \pm 2, \pm 4, ...\}$ there exists f s.t. $f(z) = 2z$ which is a bijection.

Def — Countable/uncountable (infinite) sets.
X inifinite is countable if it is equipotent to $\mathbb{N}$. It is uncountable otherwise, in which case is more than countable (countable sets are the "smallest" among infinite sets).

Def — Cardinality of continuoum.
X has the cardinality of continuoum if it is equipotent to $\mathbb{R}$. Any such set is uncountable.

For example:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable
- $\mathbb{R}, \mathbb{R}^N, (0, 1), (0, 1)^N$ have the cardinality of continuoum
- countable unions of countable sets are countable

1.2 Families of subsets

Let X be a set.

Def — Power set.
The power set of X, i.e. the set of all subsets of X, is

$$\mathcal{P}(X) = \{Y \,:\, Y \subset X\}$$

It is sometimes denoted as 2^X.

The power set has cardinality strictly bigger than X. For example, $\mathcal{P}(\mathbb{N})$ has the cardinality of continuoum.

Def — Family of subsets.
A family, or collection, of subsets of X is just $\mathcal{C} \subset \mathcal{P}(X)$. Tipically, a family of subsets (induced by $I \subset \mathbb{R}$ set of indexes) is $\mathcal{C} = \{E_i\}_{i \in I}$ where $E_i \subset X\ \forall\, i \in I$.

For example, $\{E_1, E_2, E_3\}$ is a family of subsets.

Def — Union and intersection.
Given a family of sets $\{E_i\}_{i \in I} \subset \mathcal{P}(X)$, will often be considered

$$\bigcup_{i \in I} E_i = \{x \in X \,:\, \exists\, i \in I \text{ s.t. } x \in E_i\}$$
$$\bigcap_{i \in I} E_i = \{x \in X \,:\, x \in E_i\ \forall\, i \in I\}$$

$\{E_i\}$ is said to be (pairwise) disjoint if $E_i \cap E_j = \varnothing\ \forall\, i \neq j$.

Ex — Standard topology of $\mathbb{R}$.
Given $X = \mathbb{R}$ $\left(\text{or } \mathbb{R}^N\right)$, the standard/euclidian topology of $\mathbb{R}$ $\left(\text{or } \mathbb{R}^N\right)$ is $\mathcal{T} = \{E \subset X \,:\, E \text{ is open}\}$, i.e. it is the family of all open subsets of X.
More generally, this can be defined in metric spaces (X, d) where X is a set and d a distance between $x, y \in X$.

Some properties of $\mathcal{T}$:

- $\varnothing, X \in \mathcal{T}$
- finite intersection of open sets is open [$\circledast$]
- any (finite/infinite, countable/uncountable, ...) union of open sets is open [$\circledcirc$]

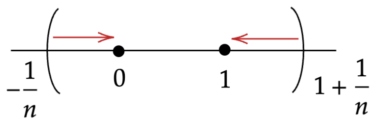
Def — Covering and subcovering.
$\{E_i\}_{i \in I}$ is a covering of X if $X = \bigcup_{i \in I} E_i$. Any subfamily $\{E_i\}_{i \in J, J \subset I}$ is a subcovering if it is a covering.

1.3 Sequences of sets

A sequence is just a family of subsets where $I \equiv \mathbb{N}$, e.g. $\{E_n\}_{n\in\mathbb{N}}$.

Def — Monotone sequences.
$\{E_n\}$ is increasing (not decreasing), $\{E_n\} \nearrow$, if $E_n \subset E_{n+1}\ \forall n \in \mathbb{N}$. On the other hand, $\{E_n\}$ is decreasing (not increasing), $\{E_n\} \searrow$, if $E_{n+1} \subset E_n\ \forall n \in \mathbb{N}$. If $\{E_n\}$ is increasing/decreasing then it is monotone.

For example, given $X = \mathbb{R}$ and $E_n = \left(-\frac{1}{n}, 1+\frac{1}{n}\right)$ for $n \geq 1$, we can say that E_n is a monotone decreasing sequence:



But what is $\bigcap_{n=1}^{\infty} E_n$? We know that

$$\bigcap_{n=1}^{\infty} E_n = [0,1]$$

and this is an infinite intersection of open sets (this does not disagree with the prop ⊛). This type of intersection is called "Gδ-set": a countable intersection of open sets.

Similarly, $E_n = \left[a+\frac{1}{n}, b-\frac{1}{n}\right]$, a<b, is increasing and

$$\bigcup_{n=1}^{\infty} E_n = (a,b)$$

is called "Fσ-set": a countable union of closed sets (doesn't disagree with ⊚).

Def — lim sup and lim inf.
Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}$. We define

$$\limsup_n E_n := \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} E_k\right) \qquad \liminf_n E_n := \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} E_k\right)$$

If these two sets are equal

$$\limsup_n E_n = \liminf_n E_n = \lim_n E_n = F$$

then F is the limit of the succession.

Take note that $\{E_n\} \nearrow$ (resp. $\searrow$) $\Longrightarrow \exists \lim_n E_n = \bigcup_n E_n$ (resp. $\bigcap_n E_n$).

Exer. Looking at the previous definition of lim sup, we can easily proof that

$$x \in \limsup_n E_n \Longleftrightarrow x \in \bigcup_{n=k}^{\infty} E_n\ \forall k \Longleftrightarrow x \in E_n \text{ for } \infty\text{-ly many } n$$

A property that is true for infinitely many n is said to be true *frequently*. Similarly:

$$x \in \liminf_n E_n \Longleftrightarrow x \in \bigcap_{n=k}^{\infty} E_n \text{ for at least one } \bar{k} \Longleftrightarrow x \in E_n\ \forall n \geq \bar{k}$$

A property that is only true after n large enough is said to be true *eventually*.

1.4 Charateristic functions

Let X be a set.

Def — Charateristic function.
Given $E \subset X$, we define the charateristic (or indicator) function

$$\chi_E : X \to \mathbb{R} \quad \text{s.t.} \quad \chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \ \text{(or } x \in E^c = X \setminus E) \end{cases}$$

(another notation: $\mathbb{1}_E(x)$)

Prop.

- $\chi_{E_1 \bigcap E_2} = \chi_{E_1} \cdot \chi_{E_2}$
- $\chi_{E_1 \bigcup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1} \cdot \chi_{E_2}$
- $\chi_{\limsup_n E_n} = \limsup_n \chi_{E_n}$ and likewise $\chi_{\liminf_n E_n} = \liminf_n \chi_{E_n}$

! Take note that the first lim sup/inf refers to sets, the second one to (real) numbers.

1.5 Relations, Equivalence relations

Def — Cartesian product and Relations.
If X and Y are sets, their Cartesian product $X \times Y$ is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. A relation from X to Y is a subset of $X \times Y$. If R is a relation from X to Y, we shall sometimes write xRy to mean that $(x, y) \in R$.
If $Y = X$, we speak of a relation <u>on</u> X. For example, $xRy \iff y = x^2$.

The most important types of relations are the following:

Def — Equivalence relations.
An equivalence relation is a relation R s.t.

i) $xRx \ \forall x \in X$ (reflexive)

ii) $xRy \iff yRx \ \forall x, y$ (symmetric)

iii) $xRy,\ yRz \implies xRz \ \forall x, y, z$ (transitive)

Equivalence relations allow us to introduce the following constructs which will be useful later on:

- **Equivalence class** of an element x: $E_x = \{y \in X \,:\, yRx\}$
- **Quotient set**: $X/R = \{E_x \,:\, x \in X\}$

For example, to define $\mathbb{Q}$ in a precise way we take the following relation R

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$$

i.e. the pairs $(1, 2), (2, 4), (3, 6), \dots \in \mathbb{Z} \times \mathbb{Z}_0$ (without zero), therefore $\mathbb{Q} = \mathbb{Z}\times\mathbb{Z}_0/R$.

Part II

Real Analysis

Chapter 2

Measure Theory

2.1 Measure spaces

Let X be a set.

Def — σ-algebras.
A family $\mathcal{M} \subset \mathcal{P}(x)$ is called a σ-algebra if

i) $\varnothing \in \mathcal{M}$

ii) $E \subset \mathcal{M} \implies E^c = X \setminus E \in \mathcal{M}$

iii) $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M} \Longrightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$ (infinite countable union)

If (iii) is replaced by "$E_1, E_2 \in \mathcal{M} \Longrightarrow E_1 \bigcup E_2 \in \mathcal{M}$" then $\mathcal{M}$ is just an algebra (finite union).

Trivial examples: $\mathcal{M} = \mathcal{P}(X)$ is the biggest σ-algebra, $\mathcal{M} = \{\varnothing, X\}$ is the smallest σ-algebra.

We say that

- $\mathcal{M}$ σ-algebra $\rightsquigarrow$ $(X, \mathcal{M})$ is a **measurable space**
- $E \in \mathcal{M}$ are **measurable sets**

Basic properties of $\mathcal{M}$:

1. $X = \varnothing^c \in \mathcal{M}$ (by (i)+(ii))
2. $\mathcal{M}$ is an algebra (σ-alg. $\implies$ alg. but not the viceversa)

   To prove this you can take a finite union (e.g. $E_1 \cup E_2$) and then make infinite unions with $\varnothing$ to have an infinite union that still belongs to $\mathcal{M}$:

$$E_1 \cup E_2 = \underbrace{E_1 \cup E_2 \underbrace{\cup \varnothing \cup \ldots \cup \varnothing \cup \ldots}_{\in \mathcal{M} \text{ by (i)}}}_{\in \mathcal{M} \text{ by (iii)}}$$

3. $\{E_n\}_n \subset \mathcal{M} \Longrightarrow \bigcap_{n\in\mathbb{N}} E_n \in \mathcal{M}$
4. $E, F \in \mathcal{M} \Longrightarrow E \setminus F \in \mathcal{M}$

Now, we want to understand how to generate a σ-algebra.

Th 2.1. Take $\mathcal{S} \subset \mathcal{P}(X)$ any family. Then it is well defined $\sigma_0(\mathcal{S})$, the σ-algebra generated by $\mathcal{S}$ (the smallest σ-algebra containing $\mathcal{S}$):

i) $\sigma_0(\mathcal{S})$ is a σ-algebra

ii) $\mathcal{S} \subset \sigma_0(\mathcal{S})$

iii) if $\mathcal{M}$ is a σ-alg. and $\mathcal{S} \subset \mathcal{M}$ then $\sigma_0(\mathcal{S}) \subset \mathcal{M}$

Proof (Sketch).
We introduce a collection of collection of sets (we should be more strict)

$$\mathcal{V} = \{\mathcal{M} \subset \mathcal{P}(X) \,:\, \mathcal{M} \text{ is a } \sigma\text{-alg. and } \mathcal{S} \subset \mathcal{M}\}$$

(notice that $\mathcal{V}$ is not empty since $\mathcal{P}(X) \in \mathcal{V}$)

Then $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} \,:\, \mathcal{M} \in \mathcal{V}\}$ (to generate the smallest take the intersection of all).

■

2.2 Borel sets

2.3 Measures

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Part III

Functional Analysis

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Part IV

Esercitazioni

Chapter 19

Exercise session 18/09

Today's **aim**: we want to get the essence of the notion of "being closed" in order to deal with continuity (so this lesson will be a little more theoretical than the following ones).

19.1 Recall on $\mathbb{R}^n$

Given $x, y \in \mathbb{R}$ a possible distance between x and y is

$$d(x, y) := |x - y|$$

(we will analyze its properties in a moment)

Given $x, y \in \mathbb{R}^2$ a possible distance between $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is

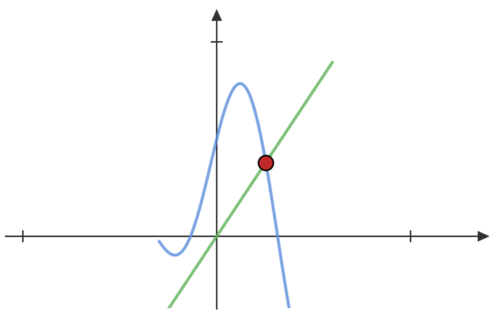
$$d_E(x, y) := \sqrt{\sum_{i=1}^{2} |x_i - y_i|}$$

the **Euclidian/canonical distance**. For $\mathbb{R}^n$ is just the same.

Given $f, g \in \mathcal{C}^0([a, b])$, i.e. $f, g : [a, b] \to \mathbb{R}$ continuous, the distance

$$d(f, g) := \min_{x \in [a,b]} |f(x) - g(x)|$$

cannot be a proper distance because $d = 0 \not\Longrightarrow f = g$ (see the definition of distance in the next page).



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