POLITECNICO
MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

Notes of

Real and Functional Analysis

for the Master in Mathematical Engineering
held by Prof. G. Verzini
a.a. 2023/2024

Edited by
Teo Bonfa

Disclaimer In questo testo sono raccolti gli appunti tratti dalle lezioni e dalle esercitazioni del corso di AAnalisi Reale e Funzionale, tenuto dal Professor Ginamaria Verzini e dall'esercitatrice Giulia Cavagnari per la LM in Mathematical Engineering, al Politecnico di Milano, nell'Anno Accademico 2023/24.

La proprietà intellettuale rimane ai docenti sopracitati, i quali non hanno revisionato la seguente opera, che si presenta unicamente come un supporto ulteriore alle lezioni, fatta da studenti per studenti, senza pretese di sostituire libri di testo ufficiali o la frequenza delle lezioni.

Il codice sorgente LaTeX è disponibile su
`https://github.com/BonfaTex/My-RFA-23`

Developed by:
Teo Bonfa - `math11235813piday@gmail.com`

Per segnalare eventuali errori o suggerimenti potete contattare gli autori o fare una Pull Request.

Contents

Part I

Introduction

Chapter 0

Course structure

This course is splitted in two parts:

1. Real Analysis $\rightsquigarrow$ measure and integration theory, in particular:
   - Collections and sequences of sets
   - Measurable space, measure, outer measure
   - Generation of an outer measure
   - Carathéodory's condition, measure induced by an outer measure
   - Lebesgue's measure on $\mathbb{R}^n$
   - Measurable functions
   - The Lebesgue integral
   - Abstract integration
   - Monotone convergence theorem, Fatou's Lemma, Lebesgue's dominated convergence theorem
   - Comparison between the Lebesgue and Riemann integrals
   - Different types of convergence
   - Derivative of a measure and the Radon-Nikodym theorem
   - Product measures and the Fubini-Tonelli theorem
   - Functions of bounded variation and absolutely continuous functions
2. Functional Analysis $\rightsquigarrow$ infinte dimensional linear algebra, in particular:
   - Metric spaces, completeness, separability, compactness
   - Normed spaces and Banach spaces
   - Spaces of integrable functions
   - Linear operators
   - Uniform boundedness theorem, open mapping theorem, closed graph theorem
   - Dual spaces and the Hahn-Banach theorem
   - Reflexivity
   - Weak and weak* convergences
   - Banach-Alaoglu theorem
   - Compact operators
   - Hilbert spaces
   - Projection theorem, Riesz representation theorem
   - Orthonormal basis, abstract Fourier series
   - Spectral theorem for compact symmetric operators

- Fredholm alternativ

The foundation of this theory is the *Set Theory*, that is going to be explained in the next chapter. Enjoy!

NB: this page will be updated with more details and maybe the list of proofs.

Chapter 1

Set Theory

1.1 Equipotent, finite/infinite, countable/uncountable sets, cardinality of continuoum

Let X, Y be sets.

Def — Equipotent sets.
X, Y are equipotent if there exists a bijection $f : X \to Y$ (1-1 injective + onto surjective).

If X, Y are equipotent, then they have the same cardinality. On the other hand, X has cardinality $\geq$ than Y if there exists $f : X \to Y$ onto. For example, for

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

exists $f : X \to Y$ s.t. $\forall y \in Y \; \exists x \in X$ s.t. $f(x) = y$ (f *takes* all the elements of the codomain), but doesn't exist $g : Y \to X$ s.t. $\forall x \in X \; \exists y \in Y$ s.t. $g(y) = x$ (g doesn't *take* all the elements of the codomain).

Def — Finite/infinite sets.
X is finite if it is equipotent to $Y = \{1, 2, ..., k\}$ for some $k \in \mathbb{N}$. X is infinite otherwise.

Prop. X is infinite iff it is equipotent to a proper subset, i.e. if exists a bijection between X and one of his subsets.

For example, between the integers set $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ and the even integers set $\{0, \pm 2, \pm 4, ...\}$ there exists f s.t. $f(z) = 2z$ which is a bijection.

Def — Countable/uncountable (infinite) sets.
X inifinite is countable if it is equipotent to $\mathbb{N}$. It is uncountable otherwise, in which case is more than countable (countable sets are the "smallest" among infinite sets).

Def — Cardinality of continuoum.
X has the cardinality of continuoum if it is equipotent to $\mathbb{R}$. Any such set is uncountable.

For example:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable
- $\mathbb{R}, \mathbb{R}^N, (0, 1), (0, 1)^N$ have the cardinality of continuoum
- countable unions of countable sets are countable

1.2 Families of subsets

Let X be a set.

Def — Power set.
The power set of X, i.e. the set of all subsets of X, is

$$\mathcal{P}(X) = \{Y : Y \subset X\}$$

It is sometimes denoted as 2^X.

The power set has cardinality strictly bigger than X. For example, $\mathcal{P}(\mathbb{N})$ has the cardinality of continuoum.

Def — Family of subsets.
A family, or collection, of subsets of X is just $\mathcal{C} \subset \mathcal{P}(X)$. Tipically, a family of subsets (induced by $I \subset \mathbb{R}$ set of indexes) is $\mathcal{C} = \{E_i\}_{i\in I}$ where $E_i \subset X \ \forall\, i \in I$.

For example, $\{E_1, E_2, E_3\}$ is a family of subsets.

Def — Union and intersection.
Given a family of sets $\{E_i\}_{i\in I} \subset \mathcal{P}(X)$, will often be considered

$$\bigcup_{i\in I} E_i = \{x \in X : \exists\, i \in I \text{ s.t. } x \in E_i\}$$
$$\bigcap_{i\in I} E_i = \{x \in X : x \in E_i \ \forall\, i \in I\}$$

$\{E_i\}$ is said to be (pairwise) disjoint if $E_i \cap E_j = \varnothing \ \forall\, i \neq j$.

Ex — Standard topology of $\mathbb{R}$.
Given $X = \mathbb{R}$ $\left(\text{or } \mathbb{R}^N\right)$, the standard/euclidian topology of $\mathbb{R}$ $\left(\text{or } \mathbb{R}^N\right)$ is $\mathcal{T} = \{E \subset X : E \text{ is open}\}$, i.e. it is the family of all open subsets of X.
More generally, this can be defined in metric spaces (X, d) where X is a set and d a distance between $x, y \in X$.

Some properties of $\mathcal{T}$:

- $\varnothing, X \in \mathcal{T}$
- finite intersection of open sets is open [$\circledast$]
- any (finite/infinite, countable/uncountable, ...) union of open sets is open [$\circledcirc$]

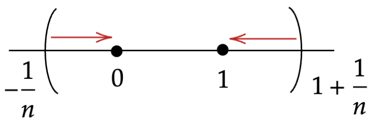
Def — Covering and subcovering.
$\{E_i\}_{i\in I}$ is a covering of X if $X = \bigcup_{i\in I} E_i$. Any subfamily $\{E_i\}_{i\in J, J\subset I}$ is a subcovering if it is a covering.

1.3 Sequences of sets

A sequence is just a family of subsets where $I \equiv \mathbb{N}$, e.g. $\{E_n\}_{n\in\mathbb{N}}$.

Def — Monotone sequences.
$\{E_n\}$ is increasing (not decreasing), $\{E_n\} \nearrow$, if $E_n \subset E_{n+1}\ \forall n \in \mathbb{N}$. On the other hand, $\{E_n\}$ is decreasing (not increasing), $\{E_n\} \searrow$, if $E_{n+1} \subset E_n\ \forall n \in \mathbb{N}$. If $\{E_n\}$ is increasing/decreasing then it is monotone.

For example, given $X = \mathbb{R}$ and $E_n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$ for $n \geq 1$, we can say that E_n is a monotone decreasing sequence:



But what is $\bigcap_{n=1}^{\infty} E_n$? We know that

$$\bigcap_{n=1}^{\infty} E_n = [0,1]$$

and this is an infinite intersection of open sets (this does not disagree with the prop ⊛). This type of intersection is called "Gδ-set": a countable intersection of open sets.

Similarly, $E_n = \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$, a<b, is increasing and

$$\bigcup_{n=1}^{\infty} E_n = (a,b)$$

is called "Fσ-set": a countable union of closed sets (doesn't disagree with ⊚).

Def — lim sup and lim inf.
Let $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{P}$. We define

$$\limsup_n E_n := \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} E_k \right) \qquad \liminf_n E_n := \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} E_k \right)$$

If these two sets are equal

$$\limsup_n E_n = \liminf_n E_n = \lim_n E_n = F$$

then F is the limit of the succession.

Take note that $\{E_n\} \nearrow$ (resp. $\searrow$) $\implies \exists \lim_n E_n = \bigcup_n E_n$ (resp. $\bigcap_n E_n$).

Exer. Looking at the previous definition of lim sup, we can easily proof that

$$x \in \limsup_n E_n \iff x \in \bigcup_{n=k}^{\infty} E_n\ \forall k \iff x \in E_n \text{ for } \infty\text{-ly many } n$$

A property that is true for infinitely many n is said to be true *frequently*. Similarly:

$$x \in \liminf_n E_n \iff x \in \bigcap_{n=k}^{\infty} E_n \text{ for at least one } \overline{k} \iff x \in E_n\ \forall n \geq \overline{k}$$

A property that is only true after n large enough is said to be true *eventually*.

1.4 Charateristic functions

Let X be a set.

Def — Charateristic function.
Given $E \subset X$, we define the charateristic (or indicator) function

$$\chi_E : X \to \mathbb{R} \quad \text{s.t.} \quad \chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \;\; (\text{or } x \in E^c = X \setminus E) \end{cases}$$

(another notation: $\mathbb{1}_E(x)$)

Prop.

- $\chi_{E_1 \bigcap E_2} = \chi_{E_1} \cdot \chi_{E_2}$
- $\chi_{E_1 \bigcup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1} \cdot \chi_{E_2}$
- $\chi_{\limsup_n E_n} = \limsup_n \chi_{E_n}$ and likewise $\chi_{\liminf_n E_n} = \liminf_n \chi_{E_n}$

! Take note that the first lim sup/inf refers to sets, the second one to (real) numbers.

1.5 Relations, Equivalence relations

Def — Cartesian product and Relations.
If X and Y are sets, their Cartesian product $X \times Y$ is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. A relation from X to Y is a subset of $X \times Y$. If R is a relation from X to Y, we shall sometimes write xRy to mean that $(x, y) \in R$.
If $Y = X$, we speak of a relation <u>on</u> X. For example, $xRy \iff y = x^2$.

The most important types of relations are the following:

Def — Equivalence relations.
An equivalence relation is a relation R s.t.

i)	$xRx \;\forall x \in X$	(reflexive)
ii)	$xRy \iff yRx \;\forall x, y$	(symmetric)
iii)	$xRy,\; yRz \implies xRz \;\forall x, y, z$	(transitive)

Equivalence relations allow us to introduce the following constructs which will be useful later on:

- **Equivalence class** of an element x: $E_x = \{y \in X \,:\, yRx\}$
- **Quotient set**: ${}^{X}/_{R} = \{E_x \,:\, x \in X\}$

For example, to define $\mathbb{Q}$ in a precise way we take the following relation R

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \ldots$$

i.e. the pairs $(1, 2), (2, 4), (3, 6), \ldots \in \mathbb{Z} \times \mathbb{Z}_0$ (without zero), therefore $\mathbb{Q} = {}^{\mathbb{Z}\times\mathbb{Z}_0}/_{R}$.

Part II

Real Analysis

Chapter 2

Measure Theory

2.1 Measure spaces

Let X be a set.

Def — σ-algebras.
A family $\mathcal{M} \subset \mathcal{P}(x)$ is called a σ-algebra if

i) $\varnothing \in \mathcal{M}$

ii) $E \subset \mathcal{M} \implies E^c = X \setminus E \in \mathcal{M}$

iii) $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{M} \Longrightarrow \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$ (infinite countable union)

If (iii) is replaced by "$E_1, E_2 \in \mathcal{M} \Longrightarrow E_1 \bigcup E_2 \in \mathcal{M}$" then $\mathcal{M}$ is just an algebra (finite union).

Trivial examples: $\mathcal{M} = \mathcal{P}(X)$ is the biggest σ-algebra, $\mathcal{M} = \{\varnothing, X\}$ is the smallest σ-algebra.

We say that

- $\mathcal{M}$ σ-algebra $\rightsquigarrow (X, \mathcal{M})$ is a **measurable space**
- $E \in \mathcal{M}$ are **measurable sets**

Basic properties of $\mathcal{M}$:

1. $X = \varnothing^c \in \mathcal{M}$ (by (i)+(ii))
2. $\mathcal{M}$ is an algebra (σ-alg. $\implies$ alg. but not the viceversa)

   To prove this you can take a finite union (e.g. $E_1 \cup E_2$) and then make infinite unions with $\varnothing$ to have an infinite union that still belongs to $\mathcal{M}$:

$$E_1 \cup E_2 = \underbrace{E_1 \cup E_2 \underbrace{\cup \varnothing \cup \ldots \cup \varnothing \cup \ldots}_{\in\mathcal{M} \text{ by (i)}}}_{\in\mathcal{M} \text{ by (iii)}}$$

3. $\{E_n\}_n \subset \mathcal{M} \Longrightarrow \bigcap_{n\in\mathbb{N}} E_n \in \mathcal{M}$
4. $E, F \in \mathcal{M} \Longrightarrow E \setminus F \in \mathcal{M}$

Now, we want to understand how to generate a σ-algebra.

Th 2.1. Take $\mathcal{S} \subset \mathcal{P}(X)$ any family. Then it is well defined $\sigma_0(\mathcal{S})$, the σ-algebra generated by $\mathcal{S}$ (the smallest σ-algebra containing $\mathcal{S}$):

i) $\sigma_0(\mathcal{S})$ is a σ-algebra

ii) $\mathcal{S} \subset \sigma_0(\mathcal{S})$

iii) if $\mathcal{M}$ is a σ-alg. and $\mathcal{S} \subset \mathcal{M}$ then $\sigma_0(\mathcal{S}) \subset \mathcal{M}$

Proof (Sketch).
We introduce a collection of collection of sets (we should be more strict: without knowing axiom choices we cannot properly prove this theorem, all we currently need is how to construct these σ-algebras):

$$\mathcal{V} = \{\mathcal{M} \subset \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-alg. and } \mathcal{S} \subset \mathcal{M}\}$$

(notice that $\mathcal{V}$ is not empty since $\mathcal{P}(X) \in \mathcal{V}$)

Then $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} : \mathcal{M} \in \mathcal{V}\}$ (to generate the smallest take the intersection of all).

■

2.2 Borel sets

We now want to define between measurable and open sets, we do this by constructing the borel σ-algebra.

Def — Borel σ-algebras and Borel sets.
Let (X, d) be a metric space, so that open subsets of X are defined (a topological space is enough) and let $\mathcal{T} = \{E \subset X : E \text{ is open}\}$. The σ-algebra generated by $\mathcal{T}$, $\sigma_0(\mathcal{T})$, is the Borel σ-algebra of X, and we write $\mathcal{B}(X) = \sigma_0(\mathcal{T})$.

Furthermore, any $E \in \mathcal{B}(X)$ is a Borel (measurable) set.

All the followings are Borel sets:

- all open sets
- all closed sets (because the σ-algebra is closed under complements, and the complements of open sets are naturally closed sets)
- all countable intersections of open sets (G_δ-sets)
- All countable unions of closed sets (F_σ-sets)

We will deal with two main cases:

- real numbers $X = \mathbb{R} = (-\infty, +\infty)$
- *extended* real numbers $X = \overline{\mathbb{R}} = [-\infty, \infty]$

Subtleties. Defining a measure in $\overline{\mathbb{R}}$ isn't trivial, we therefore define how to extend the following operations to $\overline{\mathbb{R}}$.

Operations: let $a \in \mathbb{R}$. Then:

- $a > 0 \implies a \cdot \pm\infty = \pm\infty$
- $a < 0 \implies a \cdot \mp\infty = \pm\infty$
- $a \pm \infty = \pm\infty$
- $0 \cdot \pm\infty = 0$ (note that we are not taking any limits in this assumption, we define it this way because we want the zero function to have a null integral in an unbounded interval)
- $+\infty - \infty$ is not defined

Open intervals: let $a, b \in \mathbb{R}$, $a < b$. Then:

- (a, b) is open
- $[-\infty, b)$ is open
- $(a, +\infty]$ is open

We will deal with $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}\left(\overline{\mathbb{R}}\right)$.

!

Note that

$$\begin{aligned}\mathcal{B}(\mathbb{R}) :&= \sigma_0\left(\{\text{open sets}\}\right)\\ &= \sigma_0\left(\{\text{open intervals}\}\right)\\ &= \sigma_0\left(\{(a,+\infty)\}\right)\end{aligned} \tag{2.1}$$

This means we can get open sets and open intervals by taking complements, unions and intersections, starting from sets in the form of (2.1). This is very useful for proving properties of open intervals and sets.
The properties are generally proven more easily when the set of generators is smaller.
Moreover:

$$\begin{aligned}\mathcal{B}(\overline{\mathbb{R}}) :&= \sigma_0\left(\{\text{open sets}\}\right)\\ &= \sigma_0\left(\{(a,+\infty)\}\right)\\ \mathcal{B}(\mathbb{R}^n) :&= \sigma_0\left(\{\text{open rectangles}\}\right)\\ &= \sigma_0\left(\{\text{closed rectangles}\}\right)\end{aligned}$$

2.3 Measures

Let $(X,\mathcal{M})$ be a measurable space.

Def — Measure.
A measure on $\mathcal{M}$ is a function

$$\mu : \mathcal{M} \longrightarrow [0,+\infty] \quad \text{s.t.} \quad \begin{array}{ll}\text{i)} & \mu(\varnothing) = 0\\ \text{ii)} & \{E_n\}_n \subset \mathcal{M} \text{ disjoint } \implies \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n) \qquad (\sigma\text{-additivity})\end{array}$$

Def — Measure space.
Take $X,\mathcal{M},\mu$ as above. Then $(X,\mathcal{M},\mu)$ is a measure space.

In particular:

- if $\mu(X)=1$ then $(X,\mathcal{M},\mu)$ is a probability space and μ is a probability measure
- if $\mu(X)<+\infty$ then μ is a finite measure
- if $\exists\ \{E_n\}_n\ :\ \mu\big(E_n\big) < +\infty$ and $X = \bigcup_n E_n$ then μ is a σ-finite measure

Some examples:

1) for any $(X,\mathcal{M}) \longrightarrow\ \mu\big(E\big)=0 \ \ \forall E$ is the *trivial measure*

2) for any $(X,\mathcal{M}) \longrightarrow \begin{cases}\mu(E)=+\infty, & \forall E\neq\varnothing\\ \mu(\varnothing)=0\end{cases}$ is a measure

3) for $(X,\mathcal{P}(X)) \longrightarrow \mu_{\#}(E) = \begin{cases}\#\left\{\text{elements of } E\right\}, & \text{if } E \text{ is finite}\\ +\infty, & \text{otherwise}\end{cases}$ is the *counting measure*

4) for $(X,\mathcal{P}(X))$ with X nonempty, pick $x_0\in X \longrightarrow \delta_{x_0}(E) = \begin{cases}1, & \text{if } x_0\in X\\ 0, & \text{otherwise}\end{cases}$ is the *Dirac measure*

Chapter 3

Chapter 4

Chapter 5

Chapter 6

Chapter 7

Chapter 8

Chapter 9

Chapter 10

Part III

Functional Analysis

Chapter 11

Chapter 12

Chapter 13

Chapter 14

Chapter 15

Chapter 16

Chapter 17

Chapter 18

Part IV

Esercitazioni

Chapter 19

Exercise session 18/09

Today's **aim**: we want to get the essence of the notion of "being closed" in order to deal with continuity (so this lesson will be a little more theoretical than the following ones).

19.1 Recall on $\mathbb{R}^n$

Given $x, y \in \mathbb{R}$ a possible distance between x and y is

$$d(x, y) := |x - y|$$

(we will analyze its properties in a moment)

Given $x, y \in \mathbb{R}^2$ a possible distance between $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is

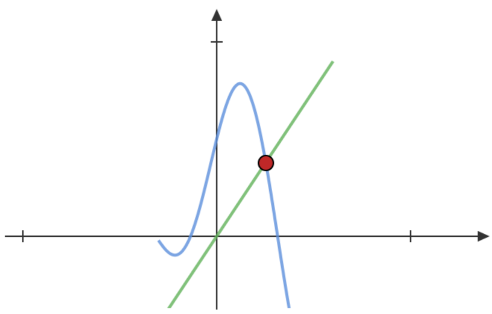
$$d_E(x, y) := \sqrt{\sum_{i=1}^{2} |x_i - y_i|^2}$$

the **Euclidian/canonical distance**. For $\mathbb{R}^n$ is just the same.

Given $f, g \in \mathcal{C}^0([a, b])$, i.e. $f, g : [a, b] \to \mathbb{R}$ continuous, the distance

$$d(f, g) := \min_{x \in [a,b]} |f(x) - g(x)|$$

cannot be a proper distance because $d = 0 \not\Rightarrow f = g$ (see the definition of distance below).



19.2 Recall on metric spaces

Let X be a nonempty set (no need of any algebraic structure such as vector spaces).

Def. A function $d : X \times X \to [0, +\infty)$ is said to be a **metric/distance** on X if all the followings are satisfied:

(a) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$

(b) $d(x,y) = d(y,x)$ (symmetry)

(c) $d(x,y) \leq d(x,z) + d(z,y)$ (**triangular inequality**)

$\forall\, x, y, z \in X$. If d is a distance on X, then (X, d) is called **metric space**.

Examples:

1. $(\mathbb{R}^n, d_E)$ is a metric space

Homework. Prove it.

Hint: in order to prove the tr. inequality use the **Cauchy-Schwarz inequality**

$$\left|\sum_{i=1}^n a_i b_i\right| \leq \sqrt{\sum_{i=1}^n |a_i|^2} \cdot \sqrt{\sum_{i=1}^n |b_i|^2}$$

(a partial solution is in the Cavagnari's slides `2023-09-18.pdf` on Webeep)

2. in $\mathbb{R}^n$ $(n \geq 2)$ we can also define

$$d_1(x,y) = \textstyle\sum_{i=1}^n |x_i - y_i| \qquad \textbf{Manhattan distance}$$

$$d_\infty(x,y) = \max_{i=1,\dots,n} |x_i - y_i| \qquad \textbf{uniform/maximum distance}$$

Homework. Prove that both d_1 and d_∞ are distances on $\mathbb{R}^n$.

Hint: use that for $a, b \in \mathbb{R}$, $d(a,b) := |a - b|$ is a distance on $\mathbb{R}$ and so $|a - b| \leq |a - c| + |c - b|$ $\forall\, a, b, c \in \mathbb{R}$

3. for $p > 1$ (but also for $0 < p < 1$)

$$d_p(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}$$

is a metric on $\mathbb{R}^n$ $(p = 2 \rightsquigarrow d_E)$

4. let X be a nonempty set. Can we also define a distance on X? Yes, with the **discrete distance**

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Homework. Prove that is a metric on X.

Def. Given (X, d) metric space, $x_0 \in X$, $r > 0$,

$$B_r(x_0) = B_d(x_0, r) := \{x \in X \,:\, d(x, x_0) < r\}$$

is a **open ball** of center x_0 and radius r, or a open spherical neighborhood of x_0.

Balls can have different shapes. For examples, let be $X = \mathbb{R}^n$ ($n = 2$ for drawing purposes):

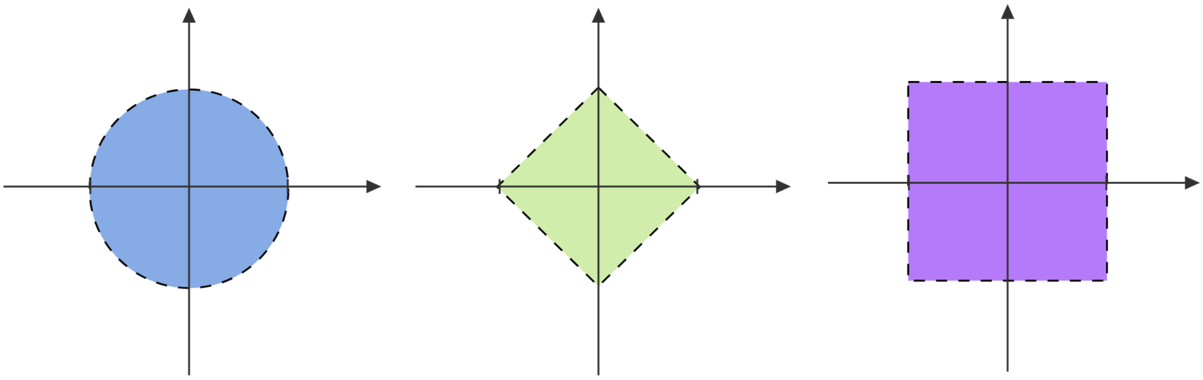
(a) with d_E we have that $B_{d_E}(x_0, r)$ is the open disk of center x_0 and radius r

(b) with d_1 we have

$$\begin{aligned} B_{d_1}(0, r) &= \left\{(x, y) \in \mathbb{R}^2 \,:\, d_1\left((x, y), (0, 0)\right) < r\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 \,:\, |x| + |y| < r\right\} \end{aligned}$$

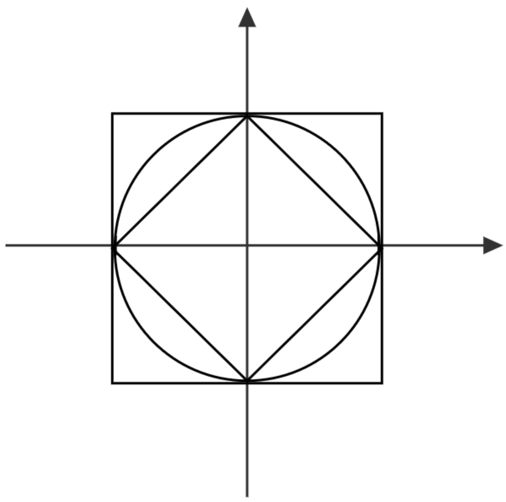
(c) with d_∞ we have

$$\begin{aligned} B_{d_\infty}(0, r) &= \left\{(x, y) \in \mathbb{R}^2 \,:\, d_\infty\left((x, y), (0, 0)\right) < r\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 \,:\, \max\{|x| + |y|\} < r\right\} \end{aligned}$$

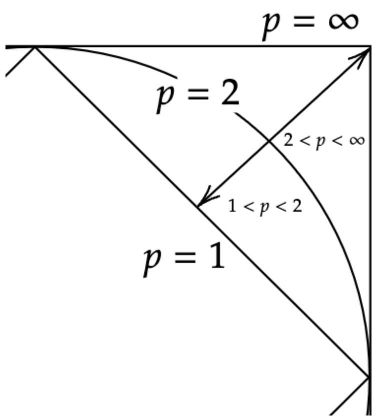


(a) *Ball in* $(\mathbb{R}^n, d_E)$ (b) *Ball in* $(\mathbb{R}^n, d_1)$ (c) *Ball in* $(\mathbb{R}^n, d_\infty)$

It is remarkable that these balls are equivalent (we will explain it better in due time).



If you generalize to every p, you could say that



$$d_{p,\, p \to \infty} \to d_\infty$$
$$d_{p,\, p=2} \equiv d_E$$
$$d_{p,\, p \to 1} \to d_1$$

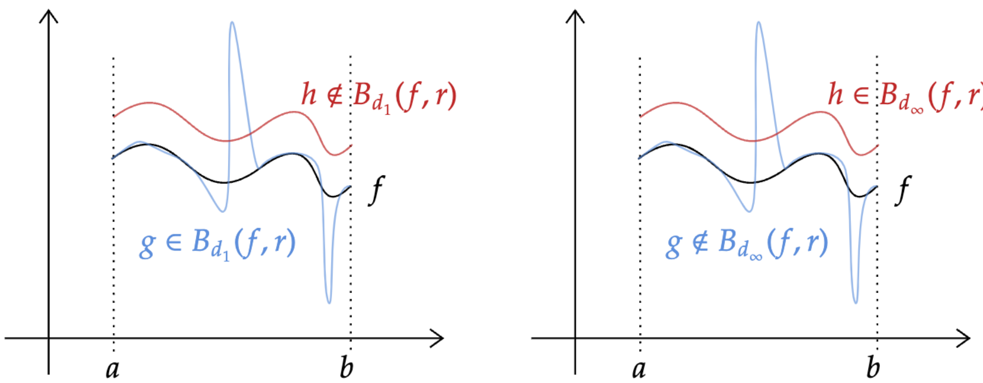
Furthermore, with (X, d) where d is the discrete metric, we have

$$B_d(x_0, r) = \begin{cases} \{x_0\}, & \text{if } r \leq 1 \\ X, & \text{if } r > 1 \end{cases}$$

Therefore, many possible distances can be introduced on a set. In some cases they lead to the same *structure* but not in genersal. For example, on $X = \mathcal{C}^0([a, b])$, $f, g \in X$, we have:

- discrete distance (we've got it folks)
- $d_1(f, g) := \displaystyle\int_a^b |f(x) - g(x)| \, \mathrm{d}x$
- $d_\infty(f, g) := \max\limits_{x \in [a,b]} |f(x) - g(x)|$

They are both distances on $\mathcal{C}^0([a, b])$ but they lead to different structures:



19.3 Topology in metric spaces

Def. Let (X, d) be a metric space and let $A \subset X$. For $x_0 \in X$, we say that x_0 is:

(i) an **interior point** of A if
$$\exists\, r > 0 \quad \text{s.t.} \quad B(x_0, r) \subset A$$

(ii) an **exterior point** of A if
$$\exists\, r > 0 \quad \text{s.t.} \quad B(x_0, r) \subset A^c := X \setminus A$$

(iii) a **boundary point** of A if it is neither interior nor exterior, i.e.
$$B(x_0, r) \cap A \neq \varnothing \quad \text{and} \quad B(x_0, r) \cap A^c \neq \varnothing \quad \forall\, r > 0$$

(iv) an **adherence point** of A if it is either interior or boundary, i.e.
$$B(x_0, r) \cap A \neq \varnothing \quad \forall\, r > 0$$

(v) an **accumulation point** of A if
$$(B(x_0, r) \cap A) \setminus \{x_0\} \neq 0$$

(vi) an **isolated point** of A if
$$x_0 \in A \quad \text{and} \quad \exists\, r > 0 \quad \text{s.t.} \quad B(x_0, r) \cap A = \{x_0\}$$

> ! By definition, an isolated point cannot be an accumulation point.

We also define:

$$\overset{o}{A} := \{x_o \in X \,:\, x_0 \text{ is an interior point of } A\} \qquad \textbf{interior of } A$$
$$ext(A) := \{x_o \in X \,:\, x_0 \text{ is an exterior point of } A\} \qquad \textbf{exterior of } A$$
$$\partial A := \{x_o \in X \,:\, x_0 \text{ is a boundary point of } A\} \qquad \textbf{boundary of } A$$
$$\overline{A} := \{x_o \in X \,:\, x_0 \text{ is an adherence point of } A\} \qquad \textbf{closure of } A$$

> ! The definitions (i)-(ii)-(iii) are mutual disjoint, thus $\overset{o}{A}, ext(A), \partial A$ are pairwise disjoint sets (see the property below).

Def. A set $A \subset X$ is **open** if $A = \overset{o}{A}$. Similarly, A is **closed** if A^c is open.

Let's take a look at some basic properties:

- $\overset{o}{A}, ext(A), \partial A$ form a partition of X
- A open iff $A \cap \partial A = \varnothing$
- $\overline{A} = \overset{o}{A} \cup \partial A = A \cup \partial A$
- A closed iff $A \equiv \overline{A}$
- $\overset{o}{A}$ is the largest (w.r.t. the inclusion order) open subset of A
- $\overline{A}$ is the smallest closed subset of X containing A
- Let I be a family of indexes (may be uncountable) and A_i be an open set $\forall\, i \in I$. Then $\bigcup\limits_{i \in I} A_i$ is open.
- Let $A_1, ..., A_m$ for $m \in \mathbb{N}$ be a finite number of open sets. Then $\bigcap\limits_{i=1}^{m} A_i$ is open.

Homework. Prove these last two properties.

- Let I be a family of indexes (may be uncountable) and C_i be closed $\forall\, i \in I$. Then $\bigcap_{i\in I} C_i$ is closed.
- Let $C_1, ..., C_m$ for $m \in \mathbb{N}$ be closed. Then $\bigcup_{i=1}^{m} A_i$ is closed.

Subtleties. These last two properties are deduced from the Morgan's laws.

19.4 Limits for sequences

Let (X, d) metric space, $\{x_n\}_{n\in\mathbb{N}} \subset X$ sequence, $x^* \in X$ point.

Def. We say that $x_n \xrightarrow{d} x^*$ as $n \to +\infty$ if

$$\underbrace{d(x_n, x^*) \to 0}_{\text{limit in real numbers}} \quad \text{as } n \to +\infty$$

i.e.

$$\forall\, \mathcal{E} > 0 \quad \exists\, \overline{n} = \overline{n}(\mathcal{E}) \in \mathbb{N} \quad \text{s.t.} \quad \underbrace{d(x_n, x^*) < \mathcal{E}}_{\text{or } x_n \in B_d(x^*, \mathcal{E})} \quad \forall\, n > \overline{n}$$

Prop. The limit (it it exists) is unique in metric space.

Homework. Prove it.

Hint: assuming by contraddiction that exist $x^*, y^*, x^* \neq y^*$ and then verify the triangular inequality.

(a sketch is in the Cavagnari's slides `2023-09-18.pdf` on Webeep)

Prop. If $\{x_n\}_n$ converges to x^*, then any subsequence of $\{x_n\}_n$ converges and it converges to x^*.

Def. A set $A \subset X$ is **sequentially closed** if for any converging sequence $\{x_n\}_n \subset A$ its limit x^* belongs to A.

Prop. $A \subset X$ closed iff A is sequentially closed.

Homework. Prove it.

Hint: for $(\Rightarrow)$ use the def. of closed, for $(\Leftarrow)$ prove that $A = \overline{A}$ (i.e. $A \subset \overline{A}$ and $\overline{A} \subset A$).

(a solution is in the Cavagnari's slides `2023-09-18.pdf` on Webeep)

19.5 Closure, Boundedness, Compactness

Let (X, d) metric space and $E \subset X$ set.

Def. The **diameter** of E is
$$\operatorname{diam}(E) := \sup_{x,y \in E} d(x, y) \quad \in [0, +\infty]$$

Def. E is **bounded** if its diameter is finite: $\operatorname{diam}(E) < +\infty$, or equivalently if $\exists\, R > 0$ and $x \in X$ s.t. $E \subset B(x, R)$.

Def. $\{E_i\}_{i \in I} \subset \mathcal{P}(X)$ is a **cover/covering** of E if
$$E \subset \bigcup_{i=I} E_i$$
A subfamily of $\{E_i\}$ which is still a covering for E is called **subcover/subcovering** of $\{E_i\}_{i \in I}$.

Def. E is **compact** if for any open cover $\{E_i\}_{i \in I}$ (i.e. all E_i are open sets) of E there exists a finite subcover (i.e. a finite subset of indexes $J \subset I$ s.t. $E \subset \bigcup_{i \in J} E_i$).

For example, $E \subset (\mathbb{R}, d_E)$, $E = (0, 1)$ is not compact, indeed if you take $E_n = (1/n, 1)$ then $\{E_n\}_{n \in \mathbb{N}}$ is an open cover for E but it doesn't admit any finite subcover.

! Let's consider (X, d) with d the discrete metric. We notice that any subset of X is an open set, hence only finite sets (made of a limite number of elements) are compact.

Th 19.1. Let (X, d) be a metric space and $E \subset X$ be a compact set. Then $S \subset E$ closed $\Longrightarrow$ S compact.

Homework. Prove it.

(a solution is in the Cavagnari's slides `2023-09-18.pdf` on Webeep)

Def. E is **sequentially compact** if any sequence $\{x_n\}_n \subset E$ admits a convergent subsequence whose limit is in E.

Th 19.2. Let (X, d) be a metric space and $E \subset X$ be a set. Then:

(i) E is compact $\Longrightarrow$ E closed and bounded

(ii) E compact $\Longleftrightarrow$ E is sequentially compact

Th 19.3 — Heine-Borel. In $(\mathbb{R}^n, d_E)$ (actually any distance), we have $E \subset \mathbb{R}^n$ compact iff E closed and bounded.

Its is remarkable that the $\Longleftarrow$ is not true in general, for example it is false in ∞-dim. spaces.

19.6 Continuous functions in metric spaces

Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ a function (if $X = \mathbb{N}$ then f is a sequence).

! You can have fun proving the followings:

$$f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$$
$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$
$$f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$$

Def. $y_0 \in Y$ is the **limit** of $f(x)$ as $x \to x_0 \in X$ if

$$\forall\, \mathcal{E} > 0 \quad \exists\, \delta = \delta(\mathcal{E}) > 0 \quad \text{s.t.} \quad \underbrace{\underbrace{0 <}_{x \neq x_0} d_X(x, x_0) < \delta \implies d_Y(f(x), y_0) < \mathcal{E}}_{\text{i.e. } f\left(B_{d_X}(x_0, \delta) \setminus \{x_0\}\right) \subset B_{d_Y}(y_0, \mathcal{E})}$$

Then we write $\lim\limits_{x \to x_0} f(x) = y_0$.

As for sequences, the limit (if exists) is unique.

Def.

1. f is **continuous** at $x_0 \in X$ if
   (i) x_0 is an isolated point
   or
   (ii) x_0 is an accumulation point for X and $f(x_0) = \lim\limits_{x \to x_0} f(x)$
2. f is **sequentially continuous** at $x_0 \in X$ if for any sequence $\{x_n\}_n \subset X$ s.t. $x_n \xrightarrow{d_X} x_0$, we have that

$$\text{i.e. } f(x_0) = \lim_{n \to +\infty} f(x_n) \quad \text{i.e. } \overbrace{f(x_n) \xrightarrow{d_Y} f(x_0) \quad \text{as } n \to +\infty}^{} \; d_Y(f(x_n), f(x_0)) \to 0 \text{ as } n \to +\infty$$

Th 19.4. Let $f : (X, d_X) \to (Y, d_Y)$ be a function. Then, f is continuous at $x_0 \in X$ iff f is sequentially continuous at x_0.

Chapter 20

Chapter 21

Chapter 22

Chapter 23

Chapter 24

Chapter 25

Chapter 26

Chapter 27

Chapter 28