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E DELL'INFORMAZIONE

Notes of

REAL AND FUNCTIONAL ANALYSIS

for the Master in Mathematical Engineering

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Part I

Introduction

Chapter 0

Course structure

This course is splitted in two parts:

1. Real Analysis \leadsto measure and integration theory, in particular:
 - Collections and sequences of sets
 - Measurable space, measure, outer measure
 - Generation of an outer measure
 - Carathéodory's condition, measure induced by an outer measure
 - Lebesgue's measure on \mathbb{R}^n
 - Measurable functions
 - The Lebesgue integral
 - Abstract integration
 - Monotone convergence theorem, Fatou's Lemma, Lebesgue's dominated convergence theorem
 - Comparison between the Lebesgue and Riemann integrals
 - Different types of convergence
 - Derivative of a measure and the Radon-Nikodym theorem
 - Product measures and the Fubini-Tonelli theorem
 - Functions of bounded variation and absolutely continuous functions
2. Functional Analysis \leadsto infinite dimensional linear algebra, in particular:
 - Metric spaces, completeness, separability, compactness
 - Normed spaces and Banach spaces
 - Spaces of integrable functions
 - Linear operators
 - Uniform boundedness theorem, open mapping theorem, closed graph theorem
 - Dual spaces and the Hahn-Banach theorem
 - Reflexivity
 - Weak and weak* convergences
 - Banach-Alaoglu theorem
 - Compact operators
 - Hilbert spaces
 - Projection theorem, Riesz representation theorem
 - Orthonormal basis, abstract Fourier series
 - Spectral theorem for compact symmetric operators

- Fredholm alternativ

The foundation of this theory is the *Set Theory*, that is going to be explained in the next chapter. Enjoy!

NB: this page will be updated with more details and maybe the list of proofs.

Chapter 1

Set Theory

1.1 Equipotent, finite/infinite, countable/uncountable sets, cardinality of continuum

Let X, Y be sets.

DEF — Equipotent sets.

X, Y are equipotent if there exists a bijection $f : X \rightarrow Y$ (1-1 injective + onto surjective).

If X, Y are equipotent, then they have the same cardinality. On the other hand, X has cardinality \geq than Y if there exists $f : X \rightarrow Y$ onto. For example, for

$$X = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad Y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

exists $f : X \rightarrow Y$ s.t. $\forall y \in Y \exists x \in X$ s.t. $f(x) = y$ (f takes all the elements of the codomain), but doesn't exist $g : Y \rightarrow X$ s.t. $\forall x \in X \exists y \in Y$ s.t. $g(y) = x$ (g doesn't take all the elements of the codomain).

DEF — Finite/infinite sets.

X is finite if it is equipotent to $Y = \{1, 2, \dots, k\}$ for some $k \in \mathbb{N}$. X is infinite otherwise.

PROP. X is infinite iff it is equipotent to a proper subset, i.e. if exists a bijection between X and one of his subsets.

For example, between the integers set $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and the even integers set $\{0, \pm 2, \pm 4, \dots\}$ there exists f s.t. $f(z) = 2z$ which is a bijection.

DEF — Countable/uncountable (infinite) sets.

X infinite is countable if it is equipotent to \mathbb{N} . It is uncountable otherwise, in which case is more than countable (countable sets are the "smallest" among infinite sets).

DEF — Cardinality of continuum.

X has the cardinality of continuum if it is equipotent to \mathbb{R} . Any such set is uncountable.

For example:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable
- $\mathbb{R}, \mathbb{R}^N, (0, 1), (0, 1)^N$ have the cardinality of continuum
- countable unions of countable sets are countable

1.2 Families of subsets

Let X be a set.

DEF — Power set.

The power set of X , i.e. the set of all subsets of X , is

$$\mathcal{P}(X) = \{Y : Y \subset X\}$$

It is sometimes denoted as 2^X .

The power set has cardinality strictly bigger than X . For example, $\mathcal{P}(\mathbb{N})$ has the cardinality of continuum.

DEF — Family of subsets.

A family, or collection, of subsets of X is just $\mathcal{C} \subset \mathcal{P}(X)$. Typically, a family of subsets (induced by $I \subset \mathbb{R}$ set of indexes) is $\mathcal{C} = \{E_i\}_{i \in I}$ where $E_i \subset X \forall i \in I$.

For example, $\{E_1, E_2, E_3\}$ is a family of subsets.

DEF — Union and intersection.

Given a family of sets $\{E_i\}_{i \in I} \subset \mathcal{P}(X)$, will often be considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists i \in I \text{ s.t. } x \in E_i\}$$

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i \forall i \in I\}$$

$\{E_i\}$ is said to be (pairwise) disjoint if $E_i \cap E_j = \emptyset \forall i \neq j$.

Ex — Standard topology of \mathbb{R} .

Given $X = \mathbb{R}$ (or \mathbb{R}^N), the standard/euclidian topology of \mathbb{R} (or \mathbb{R}^N) is $\mathcal{T} = \{E \subset X : E \text{ is open}\}$, i.e. it is the family of all open subsets of X .

More generally, this can be defined in metric spaces (X, d) where X is a set and d a distance between $x, y \in X$.

Some properties of \mathcal{T} :

- $\emptyset, X \in \mathcal{T}$
- finite intersection of open sets is open [⊗]
- any (finite/infinite, countable/uncountable, ...) union of open sets is open [⊙]

DEF — Covering and subcovering.

$\{E_i\}_{i \in I}$ is a covering of X if $X = \bigcup_{i \in I} E_i$. Any subfamily $\{E_i\}_{i \in J, J \subset I}$ is a subcovering if it is a covering.

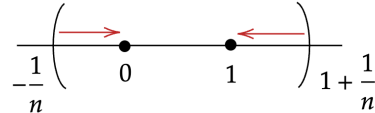
1.3 Sequences of sets

A sequence is just a family of subsets where $I \equiv \mathbb{N}$, e.g. $\{E_n\}_{n \in \mathbb{N}}$.

DEF — Monotone sequences.

$\{E_n\}$ is increasing (not decreasing), $\{E_n\} \nearrow$, if $E_n \subset E_{n+1} \forall n \in \mathbb{N}$. On the other hand, $\{E_n\}$ is decreasing (not increasing), $\{E_n\} \searrow$, if $E_{n+1} \subset E_n \forall n \in \mathbb{N}$. If $\{E_n\}$ is increasing/decreasing then it is monotone.

For example, given $X = \mathbb{R}$ and $E_n = \left(-\frac{1}{n}, 1 + \frac{1}{n}\right)$ for $n \geq 1$, we can say that E_n is a monotone decreasing sequence:



But what is $\bigcap_{n=1}^{\infty} E_n$? We know that

$$\bigcap_{n=1}^{\infty} E_n = [0, 1]$$

and this is an infinite intersection of open sets (this does not disagree with the prop \circledast). This type of intersection is called "G δ -set": a countable intersection of open sets.

Similarly, $E_n = \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$, $a < b$, is increasing and

$$\bigcup_{n=1}^{\infty} E_n = (a, b)$$

is called "F σ -set": a countable union of closed sets (doesn't disagree with \circledcirc).

DEF — lim sup and lim inf.

Let $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{P}$. We define

$$\limsup_n E_n := \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} E_k \right) \quad \liminf_n E_n := \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} E_k \right)$$

If these two sets are equal

$$\limsup_n E_n = \liminf_n E_n = \lim_n E_n = F$$

then F is the limit of the succession.

Take note that $\{E_n\} \nearrow$ (resp. \searrow) $\implies \exists \lim_n E_n = \bigcup_n E_n$ (resp. $\bigcap_n E_n$).

EXER. Looking at the previous definition of lim sup, we can easily proof that

$$x \in \limsup_n E_n \iff x \in \bigcup_{n=k}^{\infty} E_n \forall k \iff x \in E_n \text{ for } \infty\text{-ly many } n$$

A property that is true for infinitely many n is said to be true *frequently*. Similarly:

$$x \in \liminf_n E_n \iff x \in \bigcap_{n=k}^{\infty} E_n \text{ for at least one } \bar{k} \iff x \in E_n \forall n \geq \bar{k}$$

A property that is only true after n large enough is said to be true *eventually*.

1.4 Charateristic functions

Let X be a set.

DEF — Charateristic function.

Given $E \subset X$, we define the charateristic (or indicator) function

$$\chi_E : X \rightarrow \mathbb{R} \quad \text{s.t.} \quad \chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \text{ (or } x \in E^c = X \setminus E) \end{cases}$$

(another notation: $\mathbb{1}_E(x)$)

PROP.

- $\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$
- $\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1} \cdot \chi_{E_2}$
- $\chi_{\limsup_n E_n} = \limsup_n \chi_{E_n}$ and likewise $\chi_{\liminf_n E_n} = \liminf_n \chi_{E_n}$



Take note that the first lim sup/inf refers to sets, the second one to (real) numbers.

1.5 Relations, Equivalence relations

DEF — Cartesian product and Relations.

If X and Y are sets, their Cartesian product $X \times Y$ is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. A relation from X to Y is a subset of $X \times Y$. If R is a relation from X to Y , we shall sometimes write xRy to mean that $(x, y) \in R$.

If $Y = X$, we speak of a relation on X . For example, $xRy \iff y = x^2$.

The most important types of relations are the following:

DEF — Equivalence relations.

An equivalence relation is a relation R s.t.

- i) $xRx \forall x \in X$ (reflexive)
- ii) $xRy \iff yRx \forall x, y$ (symmetric)
- iii) $xRy, yRz \implies xRz \forall x, y, z$ (transitive)

Equivalence relations allow us to introduce the following constructs which will be useful later on:

- **Equivalence class** of an element x : $E_x = \{y \in X : yRx\}$
- **Quotient set**: $X/R = \{E_x : x \in X\}$

For example, to define \mathbb{Q} in a precise way we take the following relation R

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$$

i.e. the pairs $(1, 2), (2, 4), (3, 6), \dots \in \mathbb{Z} \times \mathbb{Z}_0$ (without zero), therefore $\mathbb{Q} = \mathbb{Z} \times \mathbb{Z}_0 / R$.

Part II

Real Analysis

Chapter 2

Measure Theory

2.1 Measure spaces

Let X be a set.

DEF — σ -algebras.

A family $\mathcal{M} \subset \mathcal{P}(X)$ is called a σ -algebra if

- i) $\emptyset \in \mathcal{M}$
- ii) $E \in \mathcal{M} \implies E^c = X \setminus E \in \mathcal{M}$
- iii) $\{E_n\}_{n \in \mathbb{N}} \subset \mathcal{M} \implies \bigcup_{n=1}^{\infty} E_n \in \mathcal{M}$ (infinite countable union)

If (iii) is replaced by " $E_1, E_2 \in \mathcal{M} \implies E_1 \cup E_2 \in \mathcal{M}$ " then \mathcal{M} is just an algebra (finite union).

Trivial examples: $\mathcal{M} = \mathcal{P}(X)$ is the biggest σ -algebra, $\mathcal{M} = \{\emptyset, X\}$ is the smallest σ -algebra.

We say that

- \mathcal{M} σ -algebra $\leadsto (X, \mathcal{M})$ is a **measurable space**
- $E \in \mathcal{M}$ are **measurable sets**

Basic properties of \mathcal{M} :

1. $X = \emptyset^c \in \mathcal{M}$ (by (i)+(ii))
2. \mathcal{M} is an algebra (σ -alg. \implies alg. but not the viceversa)

To prove this you can take a finite union (e.g. $E_1 \cup E_2$) and then make infinite unions with \emptyset to have an infinite union that still belongs to \mathcal{M} :

$$E_1 \cup E_2 = E_1 \cup E_2 \underbrace{\cup \emptyset \cup \dots \cup \emptyset \cup \dots}_{\in \mathcal{M} \text{ by (i)}} \underbrace{}_{\in \mathcal{M} \text{ by (iii)}}$$

3. $\{E_n\}_n \subset \mathcal{M} \implies \bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$
4. $E, F \in \mathcal{M} \implies E \setminus F \in \mathcal{M}$

Now, we want to understand how to generate a σ -algebra.

TH 2.1. Take $\mathcal{S} \subset \mathcal{P}(X)$ any family. Then it is well defined $\sigma_0(\mathcal{S})$, the σ -algebra generated by \mathcal{S} (the smallest σ -algebra containing \mathcal{S}):

- i) $\sigma_0(\mathcal{S})$ is a σ -algebra
- ii) $\mathcal{S} \subset \sigma_0(\mathcal{S})$
- iii) if \mathcal{M} is a σ -alg. and $\mathcal{S} \subset \mathcal{M}$ then $\sigma_0(\mathcal{S}) \subset \mathcal{M}$

PROOF (Sketch).

We introduce a collection of collection of sets (we should be more strict)

$$\mathcal{V} = \{\mathcal{M} \subset \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-alg. and } \mathcal{S} \subset \mathcal{M}\}$$

(notice that \mathcal{V} is not empty since $\mathcal{P}(X) \in \mathcal{V}$)

Then $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} : \mathcal{M} \in \mathcal{V}\}$ (to generate the smallest take the intersection of all).

■

2.2 Borel sets

2.3 Measures

Chapter 3

Chapter 4

Chapter 5

Chapter 6

Chapter 7

Chapter 8

Chapter 9

Chapter 10

Part III

Functional Analysis

Chapter 11

Chapter 12

Chapter 13

Chapter 14

Chapter 15

Chapter 16

Chapter 17

Chapter 18

Part IV

Esercitazioni

Chapter 19

Exercise session 18/09

Today's **aim**: we want to get the essence of the notion of "being closed" in order to deal with continuity (so this lesson will be a little more theoretical than the following ones).

19.1 Recall on \mathbb{R}^n

Given $x, y \in \mathbb{R}$ a possible distance between x and y is

$$d(x, y) := |x - y|$$

(we will analyze its properties in a moment)

Given $x, y \in \mathbb{R}^2$ a possible distance between $x = (x_1, x_2)$ and $y = (y_1, y_2)$ is

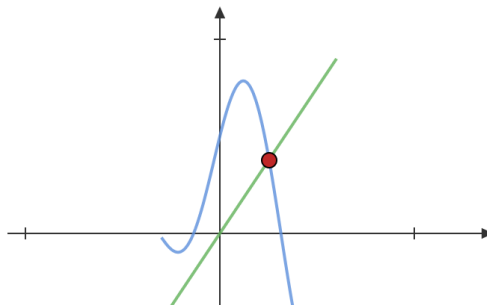
$$d_E(x, y) := \sqrt{\sum_{i=1}^2 |x_i - y_i|^2}$$

the **Euclidian/canonical distance**. For \mathbb{R}^n is just the same.

Given $f, g \in \mathcal{C}^0([a, b])$, i.e. $f, g : [a, b] \rightarrow \mathbb{R}$ continuous, the distance

$$d(f, g) := \min_{x \in [a, b]} |f(x) - g(x)|$$

cannot be a proper distance because $d = 0 \not\Rightarrow f = g$ (see the definition of distance in the next page).



Chapter 20

Chapter 21

Chapter 22

Chapter 23

Chapter 24

Chapter 25

Chapter 26

Chapter 27

Chapter 28

