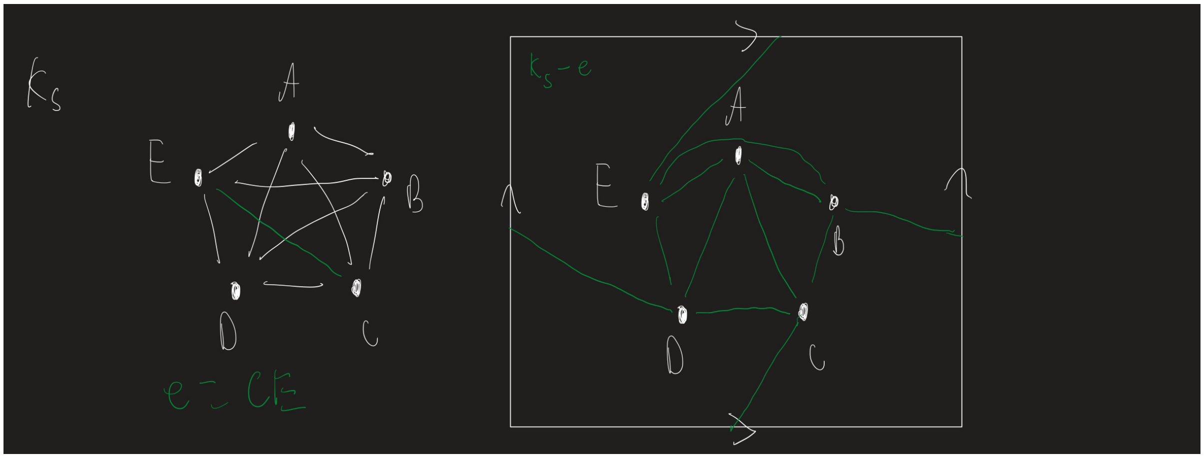
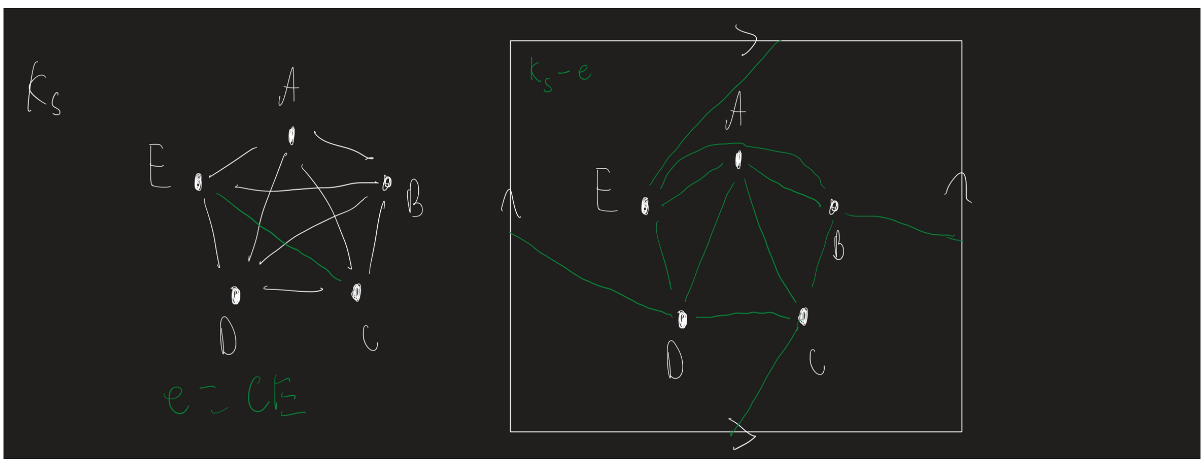
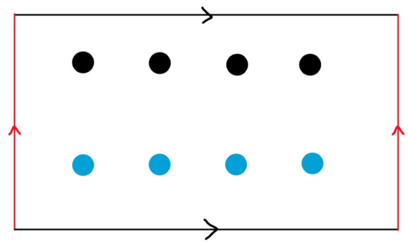
1. Let e be an edge of K_5. Show that $K_5 - e$ is planar. One way is to draw it in the plane without crossing the edges.

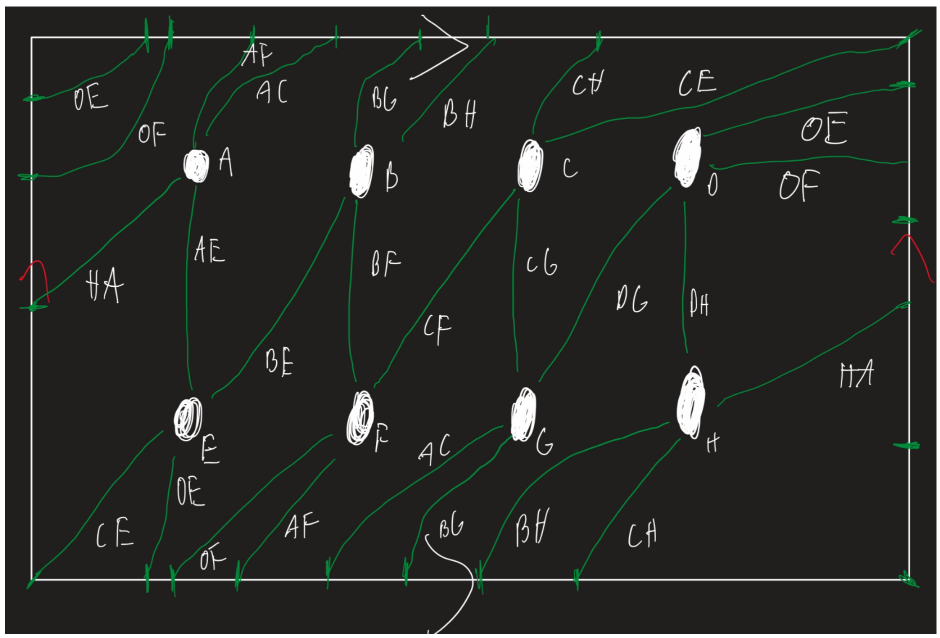


2. Let e be an edge of $K_{3,3}$. Show that $K_{3,3} - e$ is planar.



3. Show how to draw $K_{4,4}$ on a one-holed torus without edges crossing. The diagram below has the vertices strategically lined up on the torus.





4. What is the Euler equation for the following surface (algebraically, $abca^{-1}b^{-1}c^{-1}$)? Here, you need to determine h for the equation $n - m + r = 2 - h$.

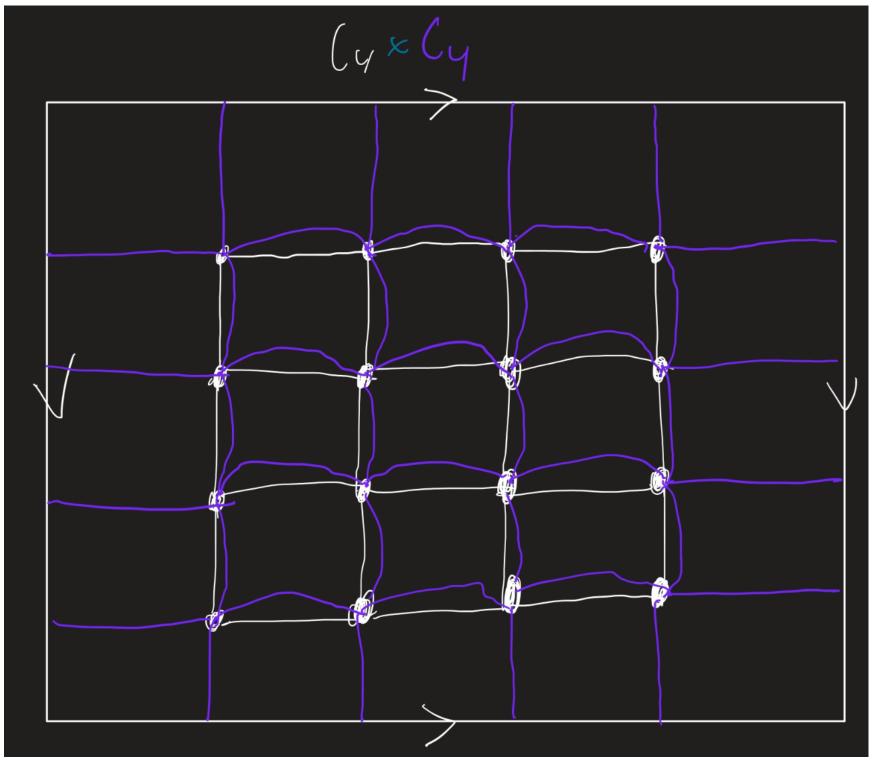
$abca^{-1}b^{-1}c^{-1}$ is a closed loop formed by paths abc followed by their inverses $a^{-1}b^{-1}c^{-1}$. The inverse pathing implies that there exists only 1 boundary. Because the surface has 1 boundary, the existence of the inverses means it's more like a sphere with 3 holes that are still a part of the sphere.

n is 1 since it's all connected
m is 6 since there's 6 edges in the whole surface.
r is 1 since there's only 1 boundary component.

The Euler formula is $\chi = n - m + r = 2 - h$.
$1 - 6 + 1 = 2 - h$
$-4 = 2 - h$
$-6 = -h$, so $h = 6$

5. Recall that Q_4 and $C_4 \times C_4$ are isomorphic graphs. Use this fact to show that Q_4 can be successfully drawn on a one-holed torus.

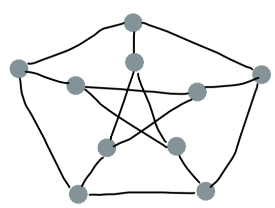
   So Q_4 is composed of 16 vertices and 32 edges. The vertices can also correspond to binary strings of length 4. The edges connect vertices that differ by 1 bit
   Because Q_4 is isomorphic to $C_4 \times C_4$, you can redraw Q_4 as $C_4 \times C_4$, since they both have the same vertices and edge connections.
   From here we can draw $C_4 \times C_4$ to have a graph with lines crossing, just the outer edges crossing over. To deal with this we can visualize the graph to show it wrapping from all sides to show it's planar.
   We'll end up with the following drawing:

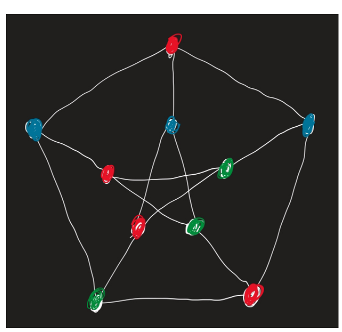


6. Prove that $C_4 \times C_4$ cannot be drawn on a sphere. To do this, compute what n and m are and then what r would have to be if it were drawn on a sphere. Then use a total edge count argument—recall that $C_4 \times C_4$ is bipartite—to arrive at a contradiction.

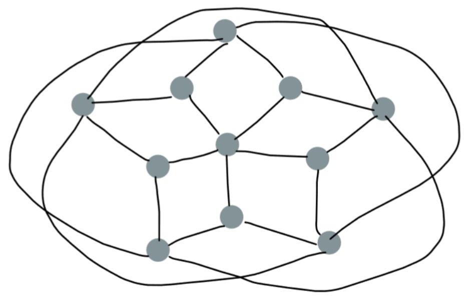
   Proof. Let's assume that we have a graph $C_4 \times C_4$ that can be drawn on a sphere, G
   Given G, we know n is 16 due to being the Cartesian product of 4 cycles, $4 \times 4 = 16$.
   We know m is 32 because since each vertex in G has a degree of 4, we can use an equation $m = \frac{n*deg}{2}$.
   $m = \frac{4*16}{2} = \frac{64}{2} = 32$
   For r, we can use the equation $n - m + r = 2$ to find r, substituting for n and m.
   $16 - 32 + r = 2 \Rightarrow r = 18$
   Let's find r in another way.
   G is a bipartite graph.
   In G, every face corresponds to a 4-cycle, because it's constructed as the Cartesian product of 4-cycles.
   So all faces in G are squares (4-cycles).
   Cycles with an odd number of edges can't occur in a bipartite graph, so G can only have faces with even lengths.
   Since each face brings 4 edges, and each edge is shared by 2 faces, the total number of edges can be found as $m = 2r$
   In substituting m, we can get $32 = 2r \Rightarrow r = 16$
   Earlier we computed that r should be 18 with regards to what n and m are, but we just got 16 by using a total edge count argument.
   This is a contradiction.
   Therefore, $C_4 \times C_4$ cannot be drawn on a sphere. □

7. Find a successful 3-coloring of Pete:





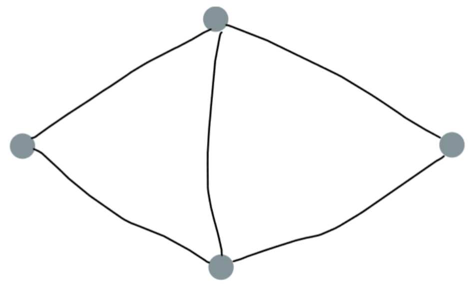
8. We constructed a family of triangle-free graphs G_c with chromatic number $\chi(G_c) = c$ where $G_3 = C_5$ has 5 vertices and G_4 is the Grötzsch graph drawn below:



When we used G_c to construct $G_(c+1)$, we doubled the number of vertices in G_c and added 1 to arrive at the number of vertices in $G_(c+1)$. Use an induction proof to show that the number of vertices in G_c is $(3 \cdot 2^{c-2}) - 1$.

Proof. We'll prove that the number of vertices in G_c is $(3 \cdot 2^{c-2}) - 1$.
Base Case: c = 3
G_3 leads to the cycle C_5, which has 5 vertices.
Plugging it into the formula, we end up with:
$3 * 2^{3-2} - 1 = 3 * 2^1 - 1 = 3 * 2 - 1 = 5$
The given info is accurate
Inductive Step
Inductive Hypothesis: Assume the formula is true when $c = k$, so $n_k = 3 * 2^{k-2} - 1$
Now, let's show this works when $c = k + 1$
When making G_{k+1} from G_k, The new number of vertices will be $n_{k+1} = 2 * n_k + 1$
Using the inductive hypothesis, we can substitute in for n_k,
$n_{k+1} = 2 * (3 * 2^{k-2} - 1) + 1 = 3 * 2 * 2^{k-2} - 2 + 1$
$3 * 2^1 * 2^{k-2} - 1 = 3 * 2^{k-1} - 1$
$k - 1 = (k + 1) - 2$, so we have $3 * 2^{(k+1)-2} - 1$, which is equivalent to n_{k+1}.
We have proven our inductive step is true Therefore, the number of vertices in G_c is $(3 \cdot 2^{c-2}) - 1$ ■

9. Find the chromatic polynomial of the graph drawn below:



For this graph, we'll have $P(G,x) = P(G-e,x) - P(G|e,x)$, in which $G-e$ is the graph with the edge removed, and $G|e$ is the graph with the edge e merging the vertices at the ends of e.
The graph given is very close to C_4, just missing one diagonal edge e, so we have a graph $G-e$ that can be called $C_4 - e$.
The chromatic polynomial of a cycle C_n is given by: $P(C_n,x) = (x-1)^n + (-1)^n * (x-1)$
For C_4, we have $P(C_4,x) = (x-1)^4 + (x-1)$
Next for $(G|e)$, merging the diagonal e's vertices will result with a triangle graph K_3
The chromatic polynomial of K_3 is $P(K_3,x) = x(x-1)(x-2)$
An isolated vertex contributes x to the whole thing,
$P(G|e,x) = x * x(x-1)(x-2) = x^2(x-1)(x-2)$
Since we now have the two parts we need, we can get the complicated version of the chromatic polynomial of the original graph:
$P(G,x) = P(G-e,x) - P(G|e,x)$
$P(G,x) = ((x-1)^4 + (x-1)) - x^2(x-1)(x-2)$
That's huge and complicated, so let's break stuff down into addition and subtraction.
First, $(x-1)^4$
$(x-1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$
Next, $x^2(x-1)(x-2)$
$x^2(x-1)(x-2) = x^4 - 3x^3 + 2x^2$
Let's combine everything now
$P(G,x) = (x^4 - 4x^3 + 6x^2 - 4x + 1 + x - 1) - (x^4 - 3x^3 + 2x^2)$
$= x^4 - 4x^3 + 6x^2 - 3x - x^4 + 3x^3 + 2x^2$
$P(G,x) = -x^3 + 4x^2 - 3x$

So the chromatic polynomial will be $-x^3 + 4x^2 - 3x$.