本科实验报告

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Consider a linear, position invariant image degradation system with impulse response

$$h(p,q) = e^{-(p^2+q^2)}$$

– Suppose that the input to the system is a binary image consisting of a white vertical line of infinite small width located at $x = a$, on a black background.

Such an image can be modeled as $f(x,y) = \delta(x-a)$.

Assume negligible noise and use $g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha, y-\beta)d\alpha d\beta$ to calculate the output image $g(x,y)$.

$$f(x,y) \longrightarrow \boxed{h(x,y)} \longrightarrow g(x,y)$$

(noise neglected)

∴退化图像 $g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\,h(x-\alpha, y-\beta)\,d\alpha\, d\beta$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\alpha-a)\,h(x-\alpha, y-\beta)\,d\alpha\, d\beta$$

由$\delta(t)$筛选性质：$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)dt = f(t_0)$

$$g(x,y) = \int_{-\infty}^{\infty} h(x-a, y-\beta)\,d\beta$$

$$= \int_{-\infty}^{\infty} e^{-[(x-a)^2+(y-\beta)^2]}\,d\beta$$

$$= e^{-(x-a)^2}\int_{-\infty}^{\infty} e^{-(y-\beta)^2}\,d\beta$$

令 $u = y-\beta$，则 $d\beta = -du$

由高斯积分性质，$\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$

$$g(x,y) = e^{-(x-a)^2}\left[-\int_{\infty}^{-\infty} e^{-u^2}du\right]$$

$$= \sqrt{\pi}\, e^{-(x-a)^2}$$

Assume that we have noisy image $g_i(x,y)$ from the image $f(x,y)$, i.e.

$$g_i(x,y) = f(x,y) + \eta_i(x,y)$$

where the noise η_i is zero-mean and at all point-pairs (x,y) are uncorrelated. Then reduce noise by taking the mean of all the noisy images:

$$\bar{g}(x,y) = \frac{1}{M}\sum_{i=1}^{M} g_i(x,y)$$

Prove that:

$$E\{\bar{g}(x,y)\} = f(x,y)$$

and:

$$\sigma^2_{\bar{g}(x,y)} = \frac{1}{M}\sigma^2_{\eta(x,y)}$$

where $\sigma^2_{\eta(x,y)}$ is the variance of η and $\sigma^2_{\bar{g}(x,y)}$ is the variance of $\bar{g}(x,y)$.

对于图像 $g_i(x,y) = f(x,y) + \eta_i(x,y)$

平均图像 $\bar{g}(x,y) = \frac{1}{M}\sum_{i=1}^{M}[f(x,y) + \eta_i(x,y)]$

$$= f(x,y) + \frac{1}{M}\sum_{i=1}^{M}\eta_i(x,y)$$

$\because$ noise η_i 是 zero-mean

$\therefore E\{\eta_i(x,y)\} = 0$，即 $E\{\frac{1}{M}\sum_{i=1}^{M}\eta_i(x,y)\} = 0$

$\therefore E\{\bar{g}(x,y)\} = f(x,y) + 0 = f(x,y)$

平均图像 $\bar{g}(x,y)$ 的方差 $\sigma^2_{\bar{g}(x,y)} = E\{(\bar{g}(x,y) - E\{\bar{g}(x,y)\})^2\}$

$$= E\{(\bar{g}(x,y) - f(x,y))^2\}$$

$$= E\{(\frac{1}{M}\sum_{i=1}^{M}\eta_i(x,y))^2\}$$

$$= \frac{1}{M^2}E\{\sum_{i=1}^{M}\eta_i(x,y)^2 + 2\sum_{i<j}\eta_i(x,y)\eta_j(x,y)\}$$

$\because$ (x,y)不相关

$\therefore$ 交叉项期望为0

$$\sigma^2_{\bar{g}(x,y)} = \frac{1}{M^2}\sum_{i=1}^{M}E\{\eta_i(x,y)^2\}$$

$$= \frac{1}{M^2}M\cdot\sigma^2_{\eta(x,y)}$$

$$= \frac{1}{M}\sigma^2_{\eta}(x,y)$$

Suppose an image's degradation function is the convolution of original image and $h(x,y) = \frac{x^2+y^2-2\sigma^2}{\sigma^4} e^{-(x^2+y^2)/2\sigma^2}$ and here assume x, y are continuous.

1. Prove that the degradation in frequency filed is
$$H(u,v) = -8\pi^4\sigma^2(u^2+v^2)e^{-2\pi^2\sigma^2(u^2+v^2)}$$

2. Assume the power spectrum ratio of noise and undegraded image (S_η/S_f) is the constant parameter K, give the transfer function expression of Wiener filter for this image.

(1) $H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$

其中 $h(x,y) = \frac{x^2+y^2-2\sigma^2}{\sigma^4} e^{-\frac{x^2+y^2}{2\sigma^2}}$

设二维高斯函数为 $g(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$

则注意到 $\nabla^2 g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = (\frac{x^2}{\sigma^4} - \frac{1}{\sigma^2}) e^{-\frac{x^2+y^2}{2\sigma^2}} + (\frac{y^2}{\sigma^4} - \frac{1}{\sigma^2}) e^{-\frac{x^2+y^2}{2\sigma^2}}$

$= h(x,y)$

由FT的微分性质，$F\{\frac{\partial^2 f}{\partial x^2}\} = (-j2\pi u)^2 F(u,v)$

$F\{\frac{\partial^2 f}{\partial y^2}\} = (-j2\pi v)^2 F(u,v)$

$\therefore F\{\nabla^2 g(x,y)\} = F\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\} = -4\pi^2(u^2+v^2)G(u,v)$

由高斯函数性质，$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-j2\pi ux} dx = \sqrt{2\pi\sigma^2}\, e^{-2\pi^2\sigma^2u^2}$

$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} e^{-j2\pi vy} dy = \sqrt{2\pi\sigma^2}\, e^{-2\pi^2\sigma^2v^2}$

$\therefore G(u,v) = 2\pi\sigma^2 e^{-2\pi^2\sigma^2(u^2+v^2)}$

$\therefore H(u,v) = -4\pi^2(u^2+v^2)G(u,v)$

$= -8\pi^3\sigma^2(u^2+v^2)e^{-2\pi^2\sigma^2(u^2+v^2)}$

(2) Wiener filter $H_w(u,v) = \frac{H^*(u,v)}{|H(u,v)|^2 + \frac{S_\eta}{S_f}}$，$\frac{S_\eta}{S_f} = \frac{\text{噪声功谱}}{\text{未退化功谱}} = K$

$\therefore H_w(u,v) = \frac{-8\pi^3\sigma^2(u^2+v^2)e^{-2\pi^2\sigma^2(u^2+v^2)}}{64\pi^6\sigma^4(u^2+v^2)^2 e^{-4\pi^2\sigma^2(u^2+v^2)} + K}$