Question

Let $n \geqslant 5$. Let $A^{(1)}, \ldots, A^{(n)} \in \mathbb{R}^{3\times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, construct $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ so that its (i, j, k, ℓ) entry for $1 \leqslant i, j, k, \ell \leqslant 3$ is given by $Q^{(\alpha\beta\gamma\delta)}_{ijk\ell} = \det\bigl[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)\bigr]$. Here $A(i, :)$ denotes the ith row of a matrix A, and semicolon denotes vertical concatenation. We are interested in algebraic relations on the set of tensors $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$.

More precisely, does there exist a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \to \mathbb{R}^N$ that satisfies the following three properties?

- The map $\mathbf{F}$ does not depend on $A^{(1)}, \ldots A^{(n)}$.
- The degrees of the coordinate functions of $\mathbf{F}$ do not depend on n.
- Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ satisfy $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for precisely $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical. Then $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$ holds if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical.

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Answer (from work by Daniel Miao, Gilad Lerman, Joe Kileel)

Yes, such algebraic relations do exist. Assemble the various tensors $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$ into one tensor $\mathbf{Q} \in \mathbb{R}^{3n\times 3n\times 3n\times 3n}$, thought of as an $n \times n \times n \times n$ block tensor where the $(\alpha, \beta, \gamma, \delta)$-block is $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$. Let $\mathbf{F}$ be the polynomial map sending $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$ to the 5×5 minors of the four $3n \times 27n^3$ matrix flattenings of $\mathbf{Q}$. We will prove that $\mathbf{F}$ satisfies the desired properties.

A key point is to discover the following algebraic identity.

Lemma 1. *Consider $\mathbf{Q} \in \mathbb{R}^{3n\times 3n\times 3n\times 3n}$ as above. It admits a Tucker tensor decomposition*

$$\mathbf{Q} = \mathcal{C} \times_1 \mathbf{A} \times_2 \mathbf{A} \times_3 \mathbf{A} \times_4 \mathbf{A}, \tag{1}$$

for $\mathcal{C} \in \mathbb{R}^{4\times 4\times 4\times 4}$ and $\mathbf{A} \in \mathbb{R}^{3n\times 4}$. Explicitly, we can take

$$\mathcal{C}_{abcd} = \begin{cases} \mathrm{sgn}(abcd) & \textit{if } a, b, c, d \in [4] \textit{ are distinct} \\ 0 & \textit{otherwise,} \end{cases}$$

where sgn is parity of a permutation, and $\mathbf{A}$ to be the vertical concatenation $[A^{(1)}; \ldots; A^{(n)}]$.

Proof. Let $[n] \times [3]$ stand for the indices of $\mathbf{Q}$ in each mode and for the row indices of $\mathbf{A}$. By definition of Tucker product, for all $(\alpha, i), (\beta, j), (\gamma, k), (\delta, \ell) \in [n] \times [3]$ we have

$$(\mathcal{C} \times_1 \mathbf{A} \times_2 \mathbf{A} \times_3 \mathbf{A} \times_4 \mathbf{A})_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)} = \sum_{a,b,c,d\in[4]} \mathcal{C}_{abcd} \mathbf{A}_{(\alpha,i),a} \mathbf{A}_{(\beta,j),b} \mathbf{A}_{(\gamma,k),c} \mathbf{A}_{(\delta,\ell),d}$$

$$= \sum_{a,b,c,d\in[4] \text{ distinct}} \mathrm{sgn}(abcd) A^{(\alpha)}_{ia} A^{(\beta)}_{jb} A^{(\alpha)}_{kc} A^{(\alpha)}_{\ell d} = \det\left[A^{(\alpha)}(i, :); A^{(\beta)}(j, :); A^{(\gamma)}(k, :); A^{(\delta)}(\ell, :)\right]$$

$$= Q^{(\alpha\beta\gamma\delta)}_{ijk\ell} \;=\; \mathbf{Q}_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)}. \qquad \square$$

The lemma explains why $\mathbf{F}$ captures algebraic relations between the tensors $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$. Indeed, the block tensor $\mathbf{Q}$ has multilinear rank bounded by $(4, 4, 4, 4)$ due to the Tucker decomposition in (1). Therefore, all 5×5 minors in $\mathbf{F}$ vanish.

Below we break up the proof of the third property into two directions. The other properties are clear. Throughout the proof, for $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ we let $\lambda \odot_b \mathbf{Q} \in \mathbb{R}^{3n\times 3n\times 3n\times 3n}$ denote blockwise scalar multiplication, i.e., the $(\alpha, \beta, \gamma, \delta)$-block of $\lambda \odot_b \mathbf{Q}$ is $\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$. Roughly speaking, we need to show that a blockwise scaling of $\mathbf{Q}$ preserves multilinear rank if and only if the scaling is a rank-1 tensor off the diagonal.

"If" Direction

This follows easily from Lemma 1. Assume $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ agrees off-diagonal with $u \otimes v \otimes w \otimes x$ for $u, v, w, x \in (\mathbb{R}^*)^n$ and is 0 on the diagonal. Then

$$\lambda \odot_b \mathbf{Q} = (u \otimes v \otimes w \otimes x) \odot_b \mathbf{Q},$$

because the diagonal blocks of $\mathbf{Q}$ vanish. That is, $Q^{(\alpha\alpha\alpha\alpha)} = 0$ since each entry of $Q^{(\alpha\alpha\alpha\alpha)}$ is the determinant of a matrix with a repeated row. Note that blockwise scalar product with a rank-1 tensor with nonzero entries is equivalent to Tucker product with invertible matrices:

$$(u \otimes v \otimes w \otimes w) \odot_b \mathbf{Q} = \mathbf{Q} \times_1 D_u \times_2 D_v \times_3 D_w \times_4 D_x.$$

Here $D_u \in \mathbb{R}^{3n\times 3n}$ is the diagonal matrix triplicating the entries of u and likewise for D_v, D_w, D_x. Thus $\lambda \odot_b \mathbf{Q}$ and $\mathbf{Q}$ have the same multilinear rank, and from the lemma $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$.

“Only If” Direction

The converse takes more work. Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ have nonzero entries precisely off the diagonal and assume $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta}Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$. We further assume $\lambda_{\alpha 111} = \lambda_{1\beta 11} = \lambda_{11\gamma 1} = \lambda_{111\delta} = 1$ for all $\alpha, \beta, \gamma, \delta \in \{2, \dots, n\}$. We reduce to this case by replacing λ by its entrywise product with $\bar{u} \otimes \bar{v} \otimes \bar{w} \otimes \bar{x}$, where

$$\bar{u}_\alpha = \begin{cases} 1 & \text{for } \alpha = 1 \\ \lambda_{\alpha 111}^{-1} & \text{for } \alpha \in \{2, \dots, n\}, \end{cases}$$

and $\bar{v}, \bar{w}, \bar{x}$ are defined similarly using the second, third and fourth modes respectively. The replacement preserves the multilinear rank of $\lambda \odot_b \mathbf{Q}$ and whether or not λ agrees off-diagonal with a rank-1 tensor. Hence it is without loss of generality.

Through some explicit calculations, we will prove there exists $c \in \mathbb{R}^*$ such that

- $\lambda_{\alpha\beta\gamma\delta} = c$ if exactly two of $\alpha, \beta, \gamma, \delta$ equal 1
- $\lambda_{\alpha\beta\gamma\delta} = c^2$ if exactly one of $\alpha, \beta, \gamma, \delta$ equals 1
- $\lambda_{\alpha\beta\gamma\delta} = c^3$ if none of $\alpha, \beta, \gamma, \delta$ equal 1 and $\alpha, \beta, \gamma, \delta$ are not identical.

This will establish the “only if” direction, as setting $u = v = w = (1, c, \dots, c)$ and $x = (\frac{1}{c}, 1, \dots, 1)$ gives $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ whenever $\alpha, \beta, \gamma, \delta$ are not identical. Our proof strategy is to examine appropriate coordinates of $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta}Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$ in order to constrain λ. Equivalently, we will consider the vanishing of the determinants of certain well-chosen 5×5 submatrices of the flattenings of $\lambda \odot_b \mathbf{Q}$. Write $\mathbf{Q}_{(1)}$ and $(\lambda \odot_b \mathbf{Q})_{(1)}$ for mode-1 flattenings in $\mathbb{R}^{3n \times 27n^3}$. Rows correspond to the first tensor mode and are indexed by $(\alpha, i) \in [n] \times [3]$, while columns correspond to the other modes and are indexed by $((\beta, j), (\gamma, k), (\delta, \ell)) \in ([n] \times [3])^3$.

<u>Step 1</u>: The first submatrix of $(\lambda \odot_b \mathbf{Q})_{(1)}$ we consider has column indices $((\alpha, 1), (1, 3), (1, 2))$, $((1, 2), (\beta, 2), (1, 1))$, $((1, 2), (\beta, 3), (1, 1))$, $((1, 3), (\beta, 3), (1, 2))$, $((1, 1), (\beta, 1), (1, 3))$ and row indices $(1, 1)$, $(1, 2)$, $(1, 3)$, $(\alpha, 1)$, $(\alpha, 2)$, where $\alpha, \beta \in \{2, \dots, n\}$. Explicitly, the submatrix is

$$\begin{bmatrix} Q^{(1\alpha 11)}_{1132} & Q^{(11\beta 1)}_{1221} & Q^{(11\beta 1)}_{1231} & Q^{(11\beta 1)}_{1332} & Q^{(11\beta 1)}_{1113} \\ Q^{(1\alpha 11)}_{2132} & Q^{(11\beta 1)}_{2221} & Q^{(11\beta 1)}_{2231} & Q^{(11\beta 1)}_{2332} & Q^{(11\beta 1)}_{2113} \\ Q^{(1\alpha 11)}_{3132} & Q^{(11\beta 1)}_{3221} & Q^{(11\beta 1)}_{3231} & Q^{(11\beta 1)}_{3332} & Q^{(11\beta 1)}_{3113} \\ \lambda_{\alpha\alpha 11}Q^{(\alpha\alpha 11)}_{1132} & \lambda_{\alpha 1\beta 1}Q^{(\alpha 1\beta 1)}_{1221} & \lambda_{\alpha 1\beta 1}Q^{(\alpha 1\beta 1)}_{1231} & \lambda_{\alpha 1\beta 1}Q^{(\alpha 1\beta 1)}_{1332} & \lambda_{\alpha 1\beta 1}Q^{(\alpha 1\beta 1)}_{1113} \\ \lambda_{\alpha\alpha 11}Q^{(\alpha\alpha 11)}_{2132} & \lambda_{\alpha 1\beta 1}Q^{(\alpha 1\beta 1)}_{2221} & \lambda_{\alpha 1\beta 1}Q^{(\alpha 1\beta 1)}_{2231} & \lambda_{\alpha 1\beta 1}Q^{(\alpha 1\beta 1)}_{2332} & \lambda_{\alpha 1\beta 1}Q^{(\alpha 1\beta 1)}_{2113} \end{bmatrix},$$

which we abbreviate as

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ \lambda_{\alpha\alpha 11}* & \lambda_{\alpha 1\beta 1}* & \lambda_{\alpha 1\beta 1}* & \lambda_{\alpha 1\beta 1}* & \lambda_{\alpha 1\beta 1}* \\ \lambda_{\alpha\alpha 11}* & \lambda_{\alpha 1\beta 1}* & \lambda_{\alpha 1\beta 1}* & \lambda_{\alpha 1\beta 1}* & \lambda_{\alpha 1\beta 1}* \end{bmatrix}, \tag{2}$$

with asterisk denoting the corresponding entry in $\mathbf{Q}_{(1)}$. We view the determinant of (2) as a polynomial with respect to λ. It has degree $\leqslant 2$ in the variables $\lambda_{\alpha\alpha 11}, \lambda_{\alpha 1\beta 1}$. Observe that if $\lambda_{\alpha 1\beta 1} = 0$, the bottom two rows of the matrix are linearly independent. Also if $\lambda_{\alpha 1\beta 1} - \lambda_{\alpha\alpha 11} = 0$, then (2) equals a 5×5 submatrix of $\mathbf{Q}_{(1)}$ with rows operations performed; therefore (2) is rank-deficient. It follows that the determinant of (2) takes the form

$$s\lambda_{\alpha 1\beta 1}(\lambda_{\alpha 1\beta 1} - \lambda_{\alpha\alpha 11}).$$

Here the scale $s = s(A^{(1)}, A^{(\alpha)}, A^{(\beta)})$ is a polynomial in the A-matrices. Due to polynomiality, s is nonzero Zariski-generically if we can exhibit a *single* instance of matrices $A^{(1)}, A^{(\alpha)}, A^{(\beta)}$ where the determinant of (2) does not vanish identically for all $\lambda_{\alpha 1 \beta 1}, \lambda_{\alpha\alpha 11}$. Furthermore, we just need an instance with $\alpha = \beta$, as this corresponds to a specialization of the case $\alpha \neq \beta$. Computational verification with a random numerical instance of $A^{(1)}, A^{(\alpha)}$ proves the non-vanishing (see attached code). Recalling the standing assumptions, we deduce $\lambda_{\alpha 1 \beta 1} = \lambda_{\alpha\alpha 11}$.

We apply the same argument to modewise permutations of $\lambda \odot_b \mathbf{Q}$ and $\mathbf{Q}$, and obtain

$$\lambda_{\pi(\alpha 1 \beta 1)} = \lambda_{\pi(\alpha\alpha 11)} \quad \text{for all } \alpha, \beta \in \{2, \ldots, n\} \text{ and permutations } \pi.$$

The argument goes through as $\pi \cdot \mathbf{Q}$ and $\pi \cdot (\lambda \odot_b \mathbf{Q})$ have multilinear ranks bounded by $(4, 4, 4, 4)$ and $\pi \cdot \mathbf{Q} = \operatorname{sgn}(\pi)\mathbf{Q}$. So (2) looks the same but with indices permuted and possibly a sign flip.

We now see that λ-entries with two 1-indices agree. Indeed, taking $\alpha = \beta$ above gives $\lambda_{\pi_1(\alpha 1 \alpha 1)} = \lambda_{\pi_2(\alpha\alpha 11)}$ for all π_1 and π_2 that fix $(\alpha\alpha 11)$ and $(\alpha 1 \alpha 1)$ respectively. So, $\lambda_{\alpha\alpha 11} = \lambda_{\pi(\alpha\alpha 11)}$ for all π. Taking $\alpha \neq \beta$ gives $\lambda_{\alpha\alpha 11} = \lambda_{\pi(\alpha 1 \beta 1)} = \lambda_{\beta\beta 11}$ for all π. Together, there exists $c \in \mathbb{R}^*$ such that $c = \lambda_{\pi(\alpha\beta 11)}$ for all $\alpha, \beta \in \{2, \ldots, n\}$ and permutations π.

Step 2: Next we consider the submatrix of $(\lambda \odot_b \mathbf{Q})_{(1)}$ with column indices $((\beta, 1), (\gamma, 3), (1, 2))$, $((1, 2), (\beta, 2), (1, 1))$, $((1, 2), (\beta, 3), (1, 1))$, $((1, 3), (\beta, 3), (1, 2))$, $((1, 1), (\beta, 1), (1, 3))$ and row indices $(1, 1)$, $(1, 2)$, $(1, 3)$, $(\alpha, 1)$, $(\alpha, 2)$, where $\alpha, \beta, \gamma \in \{2, \ldots, n\}$. It looks like

$$\begin{bmatrix} c* & * & * & * & * \\ c* & * & * & * & * \\ c* & * & * & * & * \\ \lambda_{\alpha\beta\gamma 1}* & c* & c* & c* & c* \\ \lambda_{\alpha\beta\gamma 1}* & c* & c* & c* & c* \end{bmatrix}, \tag{3}$$

where asterisks denote corresponding entries in $\mathbf{Q}_{(1)}$. As a polynomial in c and $\lambda_{\alpha\beta\gamma 1}$, the determinant of (3) is a scalar multiple of $c(c^2 - \lambda_{\alpha\beta\gamma 1})$. This is because the polynomial has degree $\leqslant 3$, if $c = 0$ then the bottom two rows of (3) are linearly dependent, and if $c^2 = \lambda_{\alpha\beta\gamma 1}$ then (3) is a 5×5 submatrix of $\mathbf{Q}_{(1)}$ with row and column operations performed. The scale is a polynomial in $A^{(1)}, A^{(\alpha)}, A^{(\beta)}, A^{(\gamma)}$. It is Zariski-generically nonzero if we exhibit one instance of A-matrices such that the determinant of (2) does not vanish for all $c, \lambda_{\alpha\beta\gamma 1}$. Further, it suffices to find an instance where $\alpha = \beta = \gamma$, as all other cases specialize to this. Computational verification with a random numerical instance of $A^{(1)}, A^{(\alpha)}$ proves the non-vanishing. It follows that $c^2 = \lambda_{\alpha\beta\gamma 1}$. Appealing to symmetry like before, $c^2 = \lambda_{\pi(\alpha\beta\gamma 1)}$ for all $\alpha, \beta, \gamma \in \{2, \ldots, n\}$ and permutations π. Summarizing, all λ-entries with a single 1-index equal c^2.

Step 3: Consider the submatrix of $(\lambda \odot \mathbf{Q})_{(1)}$ with columns $((\beta, 1), (\gamma, 3), (\delta, 2))$, $((1, 2), (\alpha, 2), (1, 1))$, $((1, 2), (\alpha, 3), (1, 1))$, $((1, 3), (\alpha, 3), (1, 2))$, $((1, 1), (\alpha, 1), (1, 3))$ and rows $(1, 1)$, $(1, 2)$, $(1, 3)$, $(\alpha, 1)$, $(\alpha, 2)$, where $\alpha, \beta, \gamma, \delta \in \{2, \ldots, n\}$ and α, δ are distinct. The submatrix looks like

$$\begin{bmatrix} c^2* & * & * & * & * \\ c^2* & * & * & * & * \\ c^2* & * & * & * & * \\ \lambda_{\alpha\beta\gamma\delta}* & c* & c* & c* & c* \\ \lambda_{\alpha\beta\gamma\delta}* & c* & c* & c* & c* \end{bmatrix}. \tag{4}$$

The determinant of (4) is $c(c^3 - \lambda_{\alpha\beta\gamma\delta})$ multiplied by a polynomial in $A^{(1)}, A^{(\alpha)}, A^{(\beta)}, A^{(\gamma)}, A^{(\delta)}$. The most specialized case is $\alpha = \beta = \gamma$. Computer verification with a random numerical instance proves the polynomial is not identically zero. We deduce that $c^3 = \lambda_{\alpha\beta\gamma\delta}$. By symmetry, $c^3 = \lambda_{\pi(\alpha\beta\gamma\delta)}$ for all $\alpha, \beta, \gamma, \delta \in \{2, \ldots, n\}$ with α, δ distinct and all permutations π. In other words, λ-entries with no 1-indices and non-identical indices equal c^3.

Steps 1, 2 and 3 show that λ takes the announced form. So, λ is rank-1 off the diagonal. This finishes the "only if" direction. Overall, we have proven that the 5×5 minors of the $3n \times 27n^3$ flattenings of $\mathbf{Q}$ give algebraic relations on $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$ with the desired properties.