Fowler's theorem for involutions.

Sylvain Cappell, S.Weinberger, and M. Yan

Fowler, in his Ph.D. thesis, proved that if Γ is a uniform lattice in a real semisimple group with odd torsion in Γ then there is no compact closed manifold M whose universal cover is rationally acyclic. A proof can be found in [W2]. We show that the same is true for Γ with 2-torsion.

Without loss of generality (by considering a normal subgroup of finite index), it suffices to prove this for the special case where $\Gamma = \pi \rtimes \mathbf{Z}_2$ for a torsion free group π, a lattice in G, for which there is an involution on M = K\G/π (by isometries with the locally symmetric metric) whose fixed set F is not empty. (F might be disconnected; for simplicity we will write what follows just for the connected case – there are no differences in the general case.)

Now suppose that $X^m$ is a manifold with fundamental group Γ, Y its 2-fold cover, and suppose that the universal cover of X (and therefore Y) are rationally acyclic. We will consider the symmetric signatures of Y in the (symmetric = quadratic L-group) $L(\mathbf{R}\pi)$, where **R** is the real numbers. There is an equivalence f:Y -> M which (while not degree one) gives an equivalence of symmetric signatures (because over **R**, all degrees have square roots, so the symmetric signature is only sensitive to the sign of the degree of the map). Since the Novikov conjecture is true for π, the assembly map from $H_m(B\pi; L(\mathbf{R})) \to L_m(\mathbf{R}\pi)$ is injective, and this detects in the degree m piece $H_m(B\pi; \mathbf{Z})$ the class that these manifolds represent in group homology. It follows that this map is degree one. $f_*[Y] = [M]$.

Now we use a cobordism argument from [W1]. We now consider the image of the fundamental class of any manifold Z with fundamental group π involution inducing this automorphism of π and the image of [Z] in $H_m(B\Gamma; \mathbf{Z}_2)$. It follows from standard equivariant homotopy theory that Z has an equivariant map, g, to M, and thus there is a map from its fixed set $Z^{Z_2} \to F$. We claim that $g_*[Z] = g_* [Z^{Z_2}]$ where we make use of the map from $Z_2 \times \pi_1 F \to \Gamma$ (and the periodicity on the group homology of $Z_2$ to raise the dimension from that of F to dim M).

This cobordism is between Z and a projective space bundle over $Z^{Z_2}$ - namely the projectivized normal bundle to $Z^{Z_2}$. (The fundamental class of the latter is the desired element by the Leray-Hirsch theorem.) It is explicitly Z x [0,1] and on Z x {1} mod out in the complement of the equivariant regular neighborhood of $Z^{Z_2}$ the Z/2 action.

Thus for Y, this image is 0, since the action is free. For M however, this is always nonzero. The action by $Z_2$ by isometries has fixed set which is aspherical and indeed the Borel construction for the action on M shows that $Z_2$x$F$ -> Γ induces an injection on homology in dimension dim($M/Z_2$) (and an isomorphism in higher dimensions, see [B]). Since the fundamental class of an aspherical manifold is always nontrivial in its group homology, we have a contradiction.

References


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[W1] S.Weinberger, Group actions and higher signatures II, CPAM 1987

[W2] S.Weinberger, Variations on a theorem of Borel, Cambridge University Press 2022