MA 2051 Differential Equations Notes

Christopher Myers

June 13, 2020

Contents

1 Differential Equations Overview

Differential equations are equations that involve a derivative of something and any number of other terms. For example, $\frac{dy}{dx} + x + y + y^2$ is a first order differential equation. There are a number of different techniques for solving these, found in the below sections.

Differential equation order is given by the highest derivative found in the equation. The above example was first order because it had only a first order derivative in it. Others − such as $y'' + x + y' + y = 0$ − are of higher order.

The degree of a differential equation is given by the power on the Y variables in the equation. Linear equations have linear Y variables, i.e. they have an exponent of 1 (???).

Second order linear equations follow the form of $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$.

2 Techniques for Solving

2.1 Direct Integration

Direct integration is where a differential equation (an equation involving an integral) is directly integrated. This is a very simple technique, but unfortunately it is not applicable most of the time.

Example 1 Solve $\frac{dy}{dx} = x$.

$$\begin{aligned} \frac{dy}{dx} &= x \\ y &= \frac{x^2}{2} \end{aligned}$$

Example 2 Solve $\frac{dy}{dx} = x + y$. This problem cannot be solved using direct integration; y cannot be integrated here. This type of differential equaiton is called a first order linear equation. Future parts of these notes will outline a method for solving this problem.

2.2 Separation of Variables

Separation of variables is a technique to make a hard problem simpler so that it can be solved using other methods. Separable functions are functions with an X function multiplied by a Y function.

Example 1 - Moving projectile Suppose there is a projectile with both vertical and horizontal velocity. Find an equation for its path using differential equations.

Start with the equation $y' = f(x, y)$. Here, X and Y are separable (???) so it can be transformed into $X(x)Y(y)$ (note that the equation is only separable because X and Y are multiplied together and not added). Move Y(y) over to the other side to get $\frac{dy}{Y(y)} = X(x)dx$.

$$\begin{aligned} y' &= f(x, y) \\ &= X(x)Y(y) \\ \frac{y'}{Y(y)} &= X(x)dx \end{aligned}$$

This method is not guaranteed to find all solutions. Take, for instance, the function $y' = x^2y$. Separation of variables can be used here, but it does not yield the second solution of $y = 0$.

2.3 Decomposition Method

If you know the structure of a problem, you can solve each of its parts, then add each solution together to get the solution to the whole thing. Formally, this states that if y_1 and y_2 are solutions to * (where * is the linear first order equation of this section), then $y_2 = y_1 + cy_h$. y_h should be the solution to the homogenous solution, the one where we set $r = 0$. This means that we can obtain a second solution by adding the first solution to the homogenous solution multiplied by a constant. The proof is given below:

$$y_2 = y_1 + (y_2 - y_1)$$

This is necessity because $y_2 - y_1$ is a homogenous solution. This proof is unsettlingly simple. This leads to the theorem that $y_g = y_p + cy_h$, where y_g is a general soltution, y_p is a particular soltuion, c is a constant, and y_h is the aforementioned homogenous solution. After a lot of work, $y_h = ce^{-\int \frac{a_0(x)}{a_1(x)}dx}$.

This can be extended to cover second order differential equations as well: $y_g = y + p + c_1 y_{h1} + c_2 y_{h2}$.

2.4 Eigen Method

The characteristic equation can yield the homogenous solution to a differential equation, that can be used with the formula $y_g = y_p + cy_h$ (the decomposition method). wa_0, a_1 are constants below:

$$\begin{aligned} a_1 y' + a_0 y &= 0 \\ a_1 y' &= -a_0 y \\ y' &= -\frac{a_0}{a_1} y \end{aligned}$$

If the above function is differentiated, the function is not changed much at all. This is similar to the function $y = e^{2x}$, which is multiplied by two for every successive differentiation ($y' = 2e^{2x}, y'' = 4e^{2x}, \ldots$). The key step in the characteristic method is then to guess that a solution looks something like $y = e^{rx}$ (note that no constant on the front is needed because multiplying by any constant will still yield solution).

$$\begin{aligned} y &= e^{rx} \\ y' &= re^{rx} \\ a_1 re^{rx} + a_0 e^{rx} &= 0 \\ a_1 r + a_0 &= 0 \\ r &= -\frac{a_0}{a_1} \end{aligned}$$

A solution is thus $y_h = e^{-\frac{a_0}{a_1}x}$, a result that should be entirely unsurprising. The equation $a_1 r + a_0 = 0$ is known as the eigen-equation, and the value r is the eigen value.

Example 3 Solve the second order equation $y'' + y' - 12y = 0$ for the homogenous solution.

$$\begin{aligned} r^2 + r - 12 &= 0 \\ (r+4)(r-3) &= 0 \\ r_1 &= -4 \\ r_2 &= 3 \\ y_h &= c_1 e^{-4t} + c_2 e^{3t} \end{aligned}$$

Example 4 Solve the second order equation $y'' + 2y' + y = 0$ for the homogenous solution.

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ r &= -1 \\ y &= c_1 e^{-t} \end{aligned}$$

This would work were it not for the fact that only one term is produced. Here, let's pull a magic one out of thin air (read: variation of parameters, sort of): $c_2 t e^{-t}$

$$\begin{aligned} y'_{h2} &= c_2 e^{-t} - c_2 t e^{-t} \\ y''_{h2} &= -c_2 e^{-t} + c_2 t e^{-t} - c_2 e^{-t} \\ &= -2c_2 e{-t} + c_2 t e^{-t} \\ &= What? \end{aligned}$$

Seriously, I lost it here. At some point it just ends in a solution of $y_h = c_1 e^{-t} + c_2 t e^{-t}$.

Example 5 Consider the second order differential equation $y'' - y = 0$. The eigen equation can be yielded by changing y to r, the degree of the derivative to the degree of r, and the last y to r^0 since y is a 0-order derivative: $r^2 - 1 = 0$, for $x = \pm 1$. So, $y = c_1 e^x + c_2 e^{-x}$.

Example 6 For $y'' + 3y' + 2y = 0$, the converted form is $r^2 + 3r + 2 = 0$. Solving this yields $(r+1)(r+2)$, or $r = -1, -2$. This yields the equation $y = c_1 e^{-x} + c_2 e^{-2x}$.

Example 7 Solve $y'' + 4y = 0$.

$$\begin{aligned} r^2 + 4 &= 0 \\ r^2 &= -4 \\ r &= \pm 2i \end{aligned}$$

This would lead to a solution of $y = c_1 e^{2it} + c_2 e^{-2it}$, but it needs to be made real first. Use Euler's formula $e^{ix} = \cos x + i \sin x$ to get $e^{2it} = \cos 2t + i \sin 2t$ and $e^{-2it} = \cos -2t + i \sin -2t$. A little trig can simplify the latter into $\cos 2t - i \sin 2t$, and adding the former to it will cancel out the imaginary component for a final answer of $2 \cos 2t$.

Example 8 Solve $y'' + 8y + 20y = 0$.

$$r^2 + 8r + 20 = 0$$

This equation cannot be factored, so the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ must be used: $\frac{-8 \pm \sqrt{64 - 80}}{2} = -4 \pm 2i$. There are two solutions here, so the final form is $y = c_1 e^{(-4+2i)t} + c_2 e^{(-4-2i)t}$:

$$\begin{aligned} y &= c_1 e^{(-4+2i)t} + c_2 e^{(-4-2i)t} \\ &= c_1 e^{-4t} e^{2it} + c_2 e^{-4t} e^{-2it} \\ &= c_1 e^{-4t} \cos 2t + c_2 e^{-4t} \sin 2t \end{aligned}$$

This technique is a useful method to use with the eigen method. Solutions that involve an imaginary number generally end in sine and/or cosine.

2.5 Undetermined Coefficient Method

Also known as the "guessing method". This method has the requirement that in $a_1y' + a_0y = r(x)$, a_0 and a_1 are constants.

Take the equation $y' + 2y = 5$. Try to take a particular solution y_p. In this case, we can assume that y_p is a constant A, so $2A = 5$, $A = \frac{5}{2}$, and $y_p = \frac{5}{2}$ as well. That's a particular solution, so now we need a homogenous solution using the eigen method: $y_h = e^{-2x}$. So, $y_h = ce^{-2x} + \frac{5}{2}$.

Example 1 Solve $y' + 2y = x$. Guess a solution that involves an X with an arbitrary constant tacked on, like $y_p = Ax$. This won't work, though, because substituting it in for y in the original equation won't validate. There must be a second constant added on, for $y_p = AX + B\ldots$

$$\begin{aligned}
y + y' &= A + 2(AX + B) = x \\
x &= 2A = 1 \\
A &= \frac{1}{2} \\
A + 2B &= 0 \\
2B &= -A \\
B &= -\frac{A}{2} = -\frac{1}{4} \\
y_p &= \frac{1}{2}x - \frac{1}{4}
\end{aligned}$$

Now that we have a particular solution, we just need to add that to the homogenous solution ce^{-2x} to get $ce^{-2x} + \frac{1}{2}x - \frac{1}{4}$. There's probably a lot of material missing from this, not all of this example will make sense on its own.

Example 2 Solve the equation $y' + 2y = x + 5$. The first thing to do is to guess what kind of function ca be put in for y to get something that looks like $x + 5$. In this case, it's $y = Ax + B$. Since we already have a solution for $y' + 2y$ (from the last example, it's $y = \frac{1}{2}x - \frac{1}{4}$). $y' + 2y = 5$, so $y = \frac{5}{2}$. Add the two solutions together in order to get the soltution to $y' + 2y = x + 5$, which is just $y = \frac{1}{2}x + \frac{1}{4} + \frac{5}{2}$. This whole solution relies on the superposition principle.

Alternatively, the long route could be taken to get $A + 2(Ax + B) = x + 5$. $x = 2A = 1$ so $A = \frac{1}{2}$ (again), and solving for B in $A + 2B = 5$ yields $\frac{9}{4}$ (also known as $\frac{1}{4} + \frac{5}{2}$.

Example 3 Solve $y' + 2y = e^x$. A guess of $y = e^x$ does not work here as there's no undetermined coefficient, so the actual "guess" should be $y = Ae^x$. Substitute the guess in, and you get $Ae^x + 2Ae^x = e^x$. e^x can be cancelled out and A can be solved to $A = \frac{1}{3}$. The final answer is then $y_p = \frac{1}{3}e^x$.

Note that an equation involving the eigen value will generate nothing in the "guess" stage, so the undetermined coefficient method cannot be used. For example, $y' + 2y = e^{-2x}$ is inoperable because -2 is the eigen value. A trick involving the derivative can be done to get $Ae^{-2x} - 2Axe^{-2x} + 2Axe{-}2x = e^{-2x}$. This can cancel out and be simplified to get $A = 1$ and thus $y_p = xe^{-2x}$.

Example 3 Solve $y' + 2y = \sin x$. The "guess" is $y_p = A\sin x + B\cos x$. The cosine at the end is required because the derivative of sine is involved. Now take $A\cos x - B\sin x + 2(A\sin x + B\cos x) = \sin x$ and collect the terms to $-B + 2A = 1$ for $\sin x$ and $A + 2B = 0$ for $\cos x$. When solved, $B = \frac{=1}{5}$ and $A = \frac{2}{5}$. The final solution is then $y = \frac{2}{5}\sin x - \frac{1}{5}\cos x$.

Example 4 Solve $y' + 2y = \tan x$. If you try to put a tangent in for y, you get a $\sec^2$ at some point, so part of the guess is $y = A\tan x + B\tan\sec^2$. This will of course involve the derivative of $\sec^2$, so it's useful to use a theorem that says that the series of derivatives involved must be finite (i.e. either terminate or loop). Since this derivative series does not terminate, the undetermined coefficient method does not work here.

Example 5 Solve the uber-complex problem $y' + 2y = \sin x + e^{2x} + x^2$. This is a linear problem, so just use the superposition principle to get $y = y_{p1} + y_{p2} + y_{p3}$:

$$A\sin x + B\cos x + Ce^{2x} + Dx^2 + Ex + F$$

This is extremely very time consuming, so the process of solving for 6 unknown constants will not be shown here.

2.6 Variation of Parameters

This method is by far the most complicated method, although it is straightforward when done step by step. It can be used to solve equations of the form $a_1y' + a_0y = r(x)$ where a_1 and a_0 aren't constants (unlike in the undetermined coefficients method from above). The homogenous solution is relatively simple for $y' + g(x)y = 0$, since the function on y' isn't needed (???): $y_h = ce^{-\int g(x)dx}$.

To get the particular solution, make the constant on the homogenous solution a function instead: $y_p = u(x)e^{-\int g(x)dx} = u(x)y_h(x)$. Then just find what $u(x)$ should be! To do *that*, use the original equation $y' + g(x)y = f(x)$ and substitute $u(x)$ in to get $(u(x)y_h)' + g(x)(u(x)y_h) = f(x)$. Simplify to get $u'y_h + uy_h' + g(x)uy_h$, and finally down to $u'y_h + (u(-gy_h)) + g(x)uy_h$. The final form of the equation is
$u = \int f(x)e^{\int g(x)dx}dx$.

So now $y_p = u \cdot y_h = \int f(x)e^{\int g(x)dx}dx \cdot ce^{-\int g(x)dx}$. This is a completely understandable, non-spaghetti formula that can surely be understood by looking at it long enough, although eye bleeding is a serious possibility. Quote: "This doesn't look too bad, but it's only when you start to do the problems that you get stopped".

Example 1 I'm not bothering with any text here, just jump straight to the align* already:

$$
\begin{aligned}
y' + xy &= 2x, y(0) = 5 \\
f &= 2x \\
g &= x \\
y_h &= e^{\int -g(x)dx} \\
&= e^{\frac{-1}{2}x^2} \\
u &= \int 2xe^{\int xdx}dx \\
&= \int 2xe^{\frac{-1}{2}x^2}dx \\
v &= \frac{1}{2}x^2 \\
dv &= xdx \\
u &= \int 2e^v dv \\
&= 2e^v \\
&= 2e^{\frac{1}{2}x^2} \\
y_p &= u \cdot y_h \\
&= 2e^{\frac{1}{2}x^2} \cdot e^{-\frac{1}{2}x^2} \\
&= 2
\end{aligned}
$$

Since we have an initial value, we must now get the general solution using it:

$$
\begin{aligned}
y &= ce^{-\frac{1}{2}x^2} + 2 \\
y(0) &= 5 \\
5 &= c + 2 \\
c &= 3
\end{aligned}
$$

2.7 Numerical Methods

Numerical Methods are ways of solving differential equations by solving for a set of points from which a curve can be inferred.

2.7.1 Euler Integration

Euler integration sums small changes over time using tangent lines:

$$
\begin{aligned}
y_1 - y_0 &= f(x_0, y_0)(x_1 - x_0) \\
y_1 &= y_0 + f(x_0, y_0)(x_1 - x_0)
\end{aligned}
$$

This procedure can be repeated recursively to yield more and more points, although error will accumulate over time. The general formula is as follows:

$$y_{k+1} = y_k + f(x_k, y_k)(x_{k+1} - x_k)$$

Note that the difference between X values is almost always kept as a constant. That is to say, $\Delta x = x_{k+1} - x_k$ where Δx is a constant.

Example Use the Euler method to find $P(2000)$ of a population model $\frac{dP}{dt} = 0.02P, P(1990) = 2.1$. Use 2 steps.

$$\begin{aligned}
P_0 &= 2.1, t_0 = 1990 \\
P_1 &= P_0 + f(t_0, P_0) \cdot 5 \\
&= 2.1 + [0.02 \cdot 2.1] \cdot 5 \\
&= 2.31 \\
P_2 &= P_1 + f(t_1, P_1) \cdot 5 \\
&= 2.31 + [0.02 \cdot 2.31] \cdot 5 \\
&= 2.541
\end{aligned}$$

Now compare this to the actual solution, found using the exact solution of $y_{exact} = 2.1e^{0.02(t-1990)}$. $y_{exact}(2000) = 2.5649$. With a difference of only 0.0239, the Euler's method was fairly accurate to this equation.

3 Stability

Steady state is where the output from the equation does not change. For example, in $\frac{dP}{dt} = kP$, 0 is a steady state solution as it yields a derivative that is always 0. In the heat loss model, $T = T_{out}$ is a steady state solution as well. For a more complicated example, in the equation $\frac{dx}{dx} = 3y^2 - 4$, solving the right side for 0 yields a steady state solution of $\pm\sqrt{\frac{4}{3}}$. In short, steady state is a solution where the derivative is 0.

Stable solutions are solutions that return to their original value (or steady state) if disturbed. For a physical example, consider a pendulum that returns to its resting place if bumped. For a mathematical example, again consider a population model. The steady state is defined as $y = 0$, but any deviation from that results in a population that quickly runs away from the steady state. In the heat loss model, however, all paths lead to the steady state instead; if temperature is increased suddenly, it will slowly cool, and vice versa. Thus, the population model is unstable while the heat loss model is stable. To prove whether something is stable or unstable, a sketched directional field should suffice.

The formal definition is as follows: y_{ss} is asymptotically stable if $|y - y_{ss}| \to 0$ as $t \to \infty$. On the other hand, y_{ss} is unstable if it is not asymptotically stable.

Example Take the model $\frac{dy}{dx} = y^2 - 5y + 4$ and determine if it is stable. The first step is to find the steady state:

$$\begin{aligned}
y^2 - 5y + 4 &= 0 \\
(y-4)(y-1) &= 0
\end{aligned}$$

The steady states are then at y=4 and y=1. The directional field is based on the parabola that the equation forms out of the Y axis: zero at the steady state values, negative between the two, and positive elsewhere. Since the directional field leads away from the steady state y=4 , y=4 is unstable. However, the directional field leads toward the steady state y=1, so y=1 is stable.

4 Superposition Principle

Linear equations are relatively easy to solve, but nonlinear equations are much harder and cannot always be solved. This is because of the superposition principle:

$$\begin{aligned}
a_1y_1' + a_2y_1 &= r \\
a_1y_2' + a_2y_2 &= s
\end{aligned}$$

If the above two equations hold, then $y_1 + y_2$ is a solution to $a_1 y' + a_2 y = r + s$. In plain english, you can add two solutions together and get another solution. This only holds for linear equations. Below is a proof, based on substitution:

$$a_1(y_1 + y_2)' + a_2(y_1 + y_2) = r + s$$
$$a_1 y_1' + a_1 y_2' + a_2 y_1 + a_2 y_2 =$$

The terms $a_1 y_2'$ and $a_2 y_2$ and the two remaining terms can be grouped together to form the earlier two starting conditions (which yielded r and s), resulting in the original two starting conditions. This completes the proof.

Second order differential equations can use the superposition principle in the form of $y_g = y_p + c_1 y_{h1} + c_2 y_{h2}$

5 Linear Independence

For two solutions y_1 and y_2 to be linearly independent, $c_1 y_1 + c_2 y_2 = 0$, then $c_1 = 0$ and $c_2 = 0$. This means that the two solutions must be independent from the other; one function cannot be obtained using the other function. If $c_1 \neq 0$, for instance, then $y_1 = -\frac{c_2}{c_1} y_2$, which would violate independence. There are a number of useful theorems to help with this:

5.1 Theorem 1

The Wronskian of y_1 and y_2, $W(y_1, y_2)$, is defined as the determinate of a 2x2 matrix y_1, y_2, y_1', y_2' (lr-tb order). y_1 and y_2 are linearly independent if $W \neq 0$.

Example 1 Prove that e^{-4t} and e^{-3t} are linearly independent.

$$W(y_1, y_2) = [e^{-4t}, e^{3t}], [-4e^{-4t}, 3e^{3t}]$$
$$= 3e^{-t} + 4e^{-t}$$

The above can never be equal to zero, so these equations are linearly indpendent.

5.2 Theorem 2

If y_1 and y_2 are solutions of $a_2 y'' + a_1 y' + a_0 y$, then either the Wronskian is never zero or it's always zero. In other words, if you want to check if the Wronskian is equal to zero, you only need to check one point. If it's zero, then it will always be zero; if nonzero, it will never be zero.

6 Directional Fields

Directional fields are created by functions of X and Y that yield a slope at a point. Every point in the field has its own slope. The general form is given by $\frac{dy}{dx} = f(x, y)$. For example, $\frac{dy}{dx} = 2y$. In this equation, every point along thel in $y = 1$ has a slope of 2, and thus an angle of $\arctan 2 \approx 63^o$. The same process can be repeated for every point in the field; they will all have a slope given by twice their Y value.

7 Models

Models are systems that can produce behavior using defined rules (usually equations). Models always require some kind of base assumption so that they can be worked with, such as an initial condition or a given parameter.

7.1 Population

Populations can be modeled in different ways using different assumptions. For example, one assumption could be a constant population! The most common assumption is that rate of growth is proportional to the current state, i.e. a larger population will grow faster than a smaller population. This yields the ODE $\frac{dP}{dt} = kP$

Solving

$$\begin{aligned}\frac{dP}{dt} &= kP \\ \frac{dP}{P} &= kdt \\ \int \frac{dP}{P} &= \int kdt \\ \ln|P| &= kt + c \\ P &= e^{kt+c} \\ &= ce^{kt}\end{aligned}$$

Note that $P = 0$ is also a valid solution, despite that it does not show up in the above solution using separable variables. The general solution is then $P = P_0 e^{k(t-t_0)}$. This equation can also be reversed to yield the growth rate constant: $k = \frac{1}{P_1} \frac{P_2 - P_1}{t_2 - t_1}$.

Finding doubling time To find the doubling time for an arbitrary growth constant k, use the formula $2P_0 = P_0 e^{k(t-t_0)}$. The below example will assume $k = 0.02$:

$$\begin{aligned}\ln 2 &= 0.02(t - t_0) \\ t - t_0 &= \frac{\ln 2}{0.02}\end{aligned}$$

This means that with $k = 0.02$, population will double roughly ever 35 years. For any increase in population by some factor, the geneal formula is $t = \frac{\ln f}{k}$ where f is the factor to change by. This formula *should* work for both positive and negative values of k.

7.2 Heat Loss

Heat loss is when an object of some temperature T has a higher temperature than its surroundings' temperature T_{out}, so it loses heat and cools. This can be modeled using differential equations, because of course it can.

Call the rate of temperature change $\frac{dT}{dt}$. Rate of change should be proportional to the temperature difference, so $\frac{dT}{dt} = T - T_{out}$. There also should be a constant attached onto it, so $\frac{dT}{dt} = k(T - T_{out})$. Assume that all of these variables are positive. Since temperature should be decreasing and all variables are positive, we need a final negative sign on the outside to ensure a negative derivative: $\frac{T}{t} = -k(T - T_{out})$. Separation of variables can be applied here in much the same way as it could be in the prior model, so it won't be shown here.

This model has a name: Newton's Cooling Law.

7.3 Spring-Mass Model

The spring-mass model concerns the position of a mass on the end of a spring that is oscillating over time. The resting position of the mass is called equilibrium, and is defined as 0. y is the position of the object, and w is the weight (N) or mass (kg) of the object. Motion is governed by Newton's equation for force $F = ma = my''$; in this context, force is the force exerted by the spring. The spring force is also proportionate to the displacement, so $F = -ky$ (Hooke's law!). Putting everything together, $my'' = -ky$. Damping is needed for real-world scenarios due to air resistance. Unfortunately, air resistance is related to speed, so $F = -k_{air}y'$, for an equation of $my'' = -ky - k_{air}y'$. The nonhomogenous term is is the force added to keep the system running $f(t)$, so the rewritten final version of the model is $my''+k_{air}y'+ky = f(t)$. The constants have to be obtained from some data: air resistance k_{air}, the spring constant k, mass m, and initial conditions.

An example of solving a model using the eigen method is below:

$$\begin{aligned} x'' + 196x &= 0 \\ r^2 + 196 &= 0 \\ r^2 &= -196 \\ r &= \pm 14i \\ x &= c_1 e^{14it} + c_2 e^{-14it} \end{aligned}$$

This model ends in imaginary numbers, which is fine in mathematics but not fine as a physical meaning. As it turns out, there's a useful identity $e^{ix} = \cos x + i \sin x$ that can help with getting a real answer. Using a little guessing, the answer is $x = \sin 14t$... somehow.

A general solution for the simplified version $mX'' + kX = 0$ is $X = c_1 \cos\sqrt{\frac{k}{m}}t + c_2 \sin\sqrt{\frac{k}{m}}t$. The constant $\sqrt{\frac{k}{m}}$ is the angular frequency ω; a higher spring constant will yield quicker vibrations, and a lower mass will similarly yield quicker vibrations. Quadrupling the spring constant will double the frequency, and quatering the mass will also double the frequency (?). The amplitude is determined by the starting conditions given in the initial value problem: $x(0) = x_i, x'(0) = v_i$. With a bit of work, $x_i = c_1, c_2 = \frac{v_i}{\omega}$. The final solved equation is as follows:

$$x(t) = x_i \cos\omega t + \frac{v_i}{\omega}\sin\omega t$$

The amplitude is defined by the coefficients in front of sine and cosine; using a nifty identity of $\sin(a+b) = \sin a + \cos b + \cos a \sin b$ we can solve amplitude for $\sqrt{x_i^2 + (\frac{v_i}{\omega})^2}$. This lets us express the above formula as $A\sin(\omega t + \phi)$ where ϕ is the starting angle $\tan\phi = \frac{\omega x_i}{v_i}$.

7.4 Pendulum Model

Pendulums can be modelled as oscillators too, with a stable steady state at the bottom of their arc and an unstable steady state at the top (?). The angle of the pendulum is θ, the length of the rod is L, and the weight is W. Gravity is the only force, but only in a direction perpendicular to the bar connected to the weight: $\sin\theta = \frac{F}{-W}$, so $F = -W\sin\theta$. Force can be calculated using Newton's law again $F = ma = mS''$, where S is the arc length $S = L\theta$, for $mL\theta''$. Putting everything together and simplifying it all yields an equation of $L\theta'' + g\theta = 0$.

Using the information from the spring-mass model, we can solve the pendulum model to yield a general formula:

$$\theta(t) = c_1\sqrt{\frac{g}{L}}t + c_2 \sin\sqrt{\frac{g}{L}}t$$

The constants c_1 and c_2 can be derived in the same way as the ones for the spring-mass model were. To find the period, use $t = \frac{2\pi}{\sqrt{\frac{g}{L}}}$

8 Systems

Systems are models that involve two or more models. The scalar form for a a system is $y' = ay+bz+f(x)$ and $x' = cy+dz+g(x)$, while the matrix (or vector) form puts constants in one matrix, variables in another, and multiplies them together. More generally, $u' = f(t, u, v)$ and $v' = g(t, u, v)$. Homogenous systems discard the free term ($f(x)$ or $g(x)$) on the end.

Solutions to these problems involves finding a vector function for as many components (y, z, etc.) as there in the initial problem. Better still, there are (as usual) initial conditions to take into account. Many of these systems do not have general methods yet, and are instead processed numerically.

Suppose there are two populations, one of wolves and the other of sheep. The variable u is the population of wolves, and the variable v is the population of sheep. The wolf population follows a population model, but since the wolves rely on sheep for food, their population also relies on sheep: $\frac{du}{dt} = aU + bV$. Sheep are also grow faster with increased population, but their population decreases with wolf population, so $\frac{dv}{dt} = cV - dU$.

The matrix form of the system is as follows:

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} a & b \\ -d & c \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

A solution to this system will look something like this:

$$\begin{bmatrix} y \\ x \end{bmatrix} = c_1 \begin{bmatrix} y_{h1} \\ z_{h1} \end{bmatrix} + c_2 \begin{bmatrix} y_{h2} \\ z_{h2} \end{bmatrix} + \begin{bmatrix} y_p \\ z_p \end{bmatrix}$$

Linear independence for systems is defined similarly to linear indpendence for simpler models: $\{ \begin{smallmatrix} u_1 & u_2 \\ v_1 & v_2 \end{smallmatrix} \}$ are linearly independent if $| \begin{smallmatrix} u_1 & u_2 \\ v_1 & v_2 \end{smallmatrix} | \neq 0$ (i.e. if the Wronskian is not zero. The same theorem from earlier about how the Wronskian is either always zero or never zero holds here too.

Example 1 Solve the system $y' = -y + 32, z' = 3y - z$.

Start by guessing what y and z look like; in this case, $y = e^{rx}$ and $z = e^{rx}$. There cannot be different values for r for complicated reasons involving cancellation, but y and z can be different, if we allow constants p and q in front of the two guesses. So far, we have $y = pe^{rx}$ and $z = qe^{rx}$, so do some substitution work:

$$\begin{aligned} y' &= pre^{rx} \\ z' &= qre^{rx} \\ pre^{rx} &= -pe^{rx} + 3qe^{rx} \\ qre^{rx} &= 3pe^{rx} - qe^{rx} \end{aligned}$$

It is intended that there is no term involving x left. This has been done using magic. Seriously, I don't know how it was actually done.

$$\begin{aligned} pr &= -p + 3q \\ qr &= 3p - q \\ (-1 - r)p + 3q = 0 & \\ 3p + (-1 - r)q = 0 & \end{aligned}$$

In order for this to have nontrivial solutions, the determinate of the matrix formed by the system's coefficients must equal 0. This can also be expressed as $|A - rI| = 0$ where A is the matrix of coefficients and I is a 2x2 identity matrix.

$$\begin{aligned}
\begin{bmatrix} -1-r & 3 \\ 3 & -1-r \end{bmatrix} &= 0 \\
(-1-r)^2 - q &= 0 \\
-1-r &= \pm 3 \\
-r &= 1 \pm 3 \\
-r &= 4, -2 \\
r &= 2, -4 \\
r_1 &= 1 \\
r_2 &= -4
\end{aligned}$$

There are two values for r, so there are two possible solutions. For $r_1 = 2$:

$$\begin{aligned}
(-1-r)p + 3q &= 0 \\
-3p + 3q &= 0 \\
p &= q
\end{aligned}$$

This means that you can use any number for p and q as long as $p = q$. The value isn't important, just their ratio. For simplicity, choose 1. The final solution is $y_1 = e^{2x}$ and $z_1 = e^{2x}$.

For $r_2 = -4$:

$$\begin{aligned}
(-1+4)p + 3q &= 0 \\
3p + 3q &= 0 \\
p + q &= 0
\end{aligned}$$

This solution also allows for all real numbers, so long as $p = -q$.

8.1 General Steps to Solve

Systems generally have this form:

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

...at least the ones we're studying, anyways.

Step 1 Get the eigen value. Use $(a-r)(d-r) - bc = 0$

Step 2 Get the eigen vector $\frac{p}{q}$. Use $(a-r)p + bq = 0$. Use r_1 or r_2 to get different solutions as needed.

Step 3 The solution will follow this form:

$$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} e^{r_1 x} + c_2 \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} e^{r_2 x}$$

This is the vector form; the scalar form uses two equations instead:

$$\begin{aligned}
y &= c_1 p_1 e^{r_1 x} + c_2 p_2 e^{r_2 x} \\
z &= c_2 q_1 e^{r_1 x} + c_2 q_2 e^{r_2 x} \ :
\end{aligned}$$

Example 2 Solve $u' = u + 3v, v' = -2v$

The matrix used to find the eigen value is $\begin{smallmatrix} 1-r & 3 \\ 0-r & -2 \end{smallmatrix}$. This results in $(1-r)(-2-r) = 0$ yielding $r = 1, -2$.

For step 2, substitute in values of r into the relevant equation to get $(1-1)p + 3q = 0$ for r_1, yielding $q = 0$, while p can be any number; for simplicity, choose 1. Substituting in r_2 yields $(1+2)p + 3q = 0$ for $p + q = 0$. Choose $p = 1$ and $q = -1$ for simplicity (again). The values don't matter, just the ratio.

Since all required values have been calculated, the final solution can be formulated as:

$$\begin{bmatrix} u \\ v \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{x} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2x}$$

Example 3 Solve $y' = z$, $z' = -9y + 6z$.

The matrix formed to find the eigen value is $\begin{smallmatrix} 0-r & 1 \\ -9 & 6-r \end{smallmatrix}$ for $(-r)(6-r) + 9 = 0$, $(r-3)^2$, and thus $r = 3$. This is a repeated root problem, so the row of the matrix used to find p and q doesn't matter. Use $-3p = q$ for $p = 1$ and $q = 3$. Since there's a repeated root, the solution constains something in the form of

$$x \begin{bmatrix} p_2 x + d_1 \\ q_2 x + d_2 \end{bmatrix} e^{rx}$$

(Note: this is the equivalent to putting an x in front of a term for a traditional eigen value solution). Remember the formula $(A - rI) \begin{smallmatrix} p_2 \\ q_2 \end{smallmatrix} = \begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}$ from earlier? This is where it comes in. Keep in mind that $r = 3$

$$\begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ q_2 \end{bmatrix}$$
$$-3p_2 + q_2 = 1$$
$$-9p_2 + 3q_2 = 3$$

Only one equation needs to be satisfied, so choose $-3p_2 + q_2 = 1$. This time we get to choose $p_2 = 0$, leading to $q_2 = 1$, forming the final solution:

$$\begin{bmatrix} y_z \\ z_z \end{bmatrix} = e^{3x}(x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}) = e^{3x} \begin{bmatrix} x \\ 3x + 1 \end{bmatrix}$$

Where did d_1 and d_2 go? Oh well, I guess that that "final solution will contain something like..." statement from earlier was mistaken!

Example 4 Solve the system $y' = -z, z' = 16y$. This solution will involve complex roots.

Get the eigen equation by getting the determinant of $\begin{bmatrix} -r & -1 \\ 16 & -r \end{bmatrix}$. This yields the equation $r^2 + 16 = 0$ and thus $r = \pm 4i$. We need eigen vectors, so proceed to step 2.

Use $r = 4i$ to get $(-4i)p - q = 0$. Quick complex numbers review: If you take $(a+bi)(c+di)$, the result is $(ac - bd) + (bc + ad)i$. Dividing complex numbers $\frac{a+bi}{c+di}$ yields $\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$. Anyways, the origianl equation will yield $4ip + q = 0$, so choose $p = 1$, so therefore $q = -4i$, giving us a solution:

$$\begin{bmatrix} z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -4i \end{bmatrix} e^{4ix}$$

We'll need a second solution, and the entire thing needs to be made real. To do this, go back to the Euler formula and get $\begin{smallmatrix} 1 \\ -4i \end{smallmatrix}(\cos 4x + i \sin 4x)$. ...

$$= \begin{bmatrix} \cos 4x + i \sin 4x \\ 4 \sin 4x - 4i \cos 4x \end{bmatrix}$$
$$= \begin{bmatrix} \cos 4x \\ 4 \sin 4x \end{bmatrix} + \begin{bmatrix} \sin 4x \\ -4 \cos 4x \end{bmatrix} i$$

It turns out that both halves of the above are valid solutions when the i is discarded. The final solution is:

$$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 \begin{bmatrix} \cos 4x \\ 4\sin 4x \end{bmatrix} + c_2 \begin{bmatrix} \sin 4x \\ -4\cos 4x \end{bmatrix}$$

Example 5 Find a solution for the double-root system $y' = 3y - z, z' = 3z$.

Step 1: Get the eigen value using $\begin{bmatrix} 3-r & 1 \\ 0 & 3-r \end{bmatrix} = 0$ for $r = 3$.

Step 2: Get the eigen vector using $(3-3)p - q = 0$. This yields $-q = 0$. Since p can be anything, choose $p = 1$.

Step 3: Get another solution:

$$\begin{aligned}
\begin{bmatrix} y_2 \\ z_2 \end{bmatrix} &= \begin{bmatrix} p_1 t + p_2 \\ q_1 t + q_2 \end{bmatrix} e^{rt} \\
\begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix} \begin{bmatrix} p_2 \\ q_2 \end{bmatrix} &= \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} \\
(3-3)p_2 - q_2 &= p_1 \\
q_2 &= -1 \\
p_2 &= 0 \\
\begin{bmatrix} y_2 \\ z_2 \end{bmatrix} &= \begin{bmatrix} t \\ -1 \end{bmatrix} e^{3t} \\
\begin{bmatrix} y \\ z \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} t \\ -1 \end{bmatrix} e^{et}
\end{aligned}$$

8.2 Nonhomogenous Systems

Nonhomogenous systems take the form of $y' = ay + bz + f(t)$, $z' = cy + dz + g(t)$. Their solutions look like this:

$$\begin{bmatrix} y_g \\ z_g \end{bmatrix} = c_1 \begin{bmatrix} y_{h1} \\ z_{h1} \end{bmatrix} + c_2 \begin{bmatrix} y_{h2} \\ z_{h2} \end{bmatrix} + \begin{bmatrix} y_p \\ z_p \end{bmatrix}.$$

Example 1 Solve the system $y' = -4y + 2z + 10, z' = -3y + 3z + 5t$ for its particular solutions.

Take a guess at what particular solutions will look like – in this case, $y_p = At + B$ and $z_p = Ct + D$. Take the derivative of each and substitute them into the system to get the following:

$$\begin{aligned}
A &= -4(At + B) + 2(Ct + D) + 10 \\
&= -4B + 2D + 10 \\
0 &= -4A + 2C
\end{aligned}$$

$$\begin{aligned}
C &= -3(At + B) + 3(Ct + D) + 5t \\
&= -3B + 3D \\
0 &= -3A + 3C + 5
\end{aligned}$$

This is now a system for the unknowns A, B, C, and D. Solve:

$$
\begin{aligned}
A + 4B - 2D &= 10 \\
4A - 2C &= 0 \\
3B + C - 3D &= 0 \\
3A - 3C &= 5 \\
C &= 2A \\
3A - 6A &= 5 \\
A &= -\frac{5}{3} \\
C &= -\frac{10}{3} \\
4B - 2D &= \frac{35}{3} \\
3B - 3D &= \frac{10}{3}
\end{aligned}
$$

There are a variety of methods that can be used to solve the system just above (the last two equations), but this time we'll use a determinant technique.

$$
\begin{aligned}
B &= \frac{|\begin{bmatrix} \frac{35}{3} & -2 \\ \frac{10}{3} & -3 \end{bmatrix}|}{|\begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}|} \\
&= \frac{-35 + \frac{20}{3}}{-12 + 6} \\
&= \frac{\frac{85}{3}}{-6} \\
&= \frac{85}{18}
\end{aligned}
$$

The same matrix-y process can be repeated on B, but substituting the right column of the numerator instead. Why would you want to do this? No idea, now that we have $B = \frac{85}{18}$, $D = \frac{65}{18}$. The useful part about this technique is that it's valid for any system that follows a similar form.

The final two particular solutions are $y_p = \frac{85}{18} - \frac{5}{3}t$ and $z_p = \frac{65}{18} - \frac{10}{3}t$.

Example 2 Solve the system from example 1.

Get the eigen value:

$$
\begin{aligned}
(-4 - r)(3 - r) + 6 &= 0 \\
r^2 + r - 6 &= 0 \\
(r + 3)(r - 2) &= 0 \\
r_1 &= 2 \\
r_2 &= -3
\end{aligned}
$$

Now get the eigen vectors:

$$
\begin{aligned}
(-4-2)p + 2q &= 0 \\
-6p + 2q &= 0 \\
-3p + q &= 0 \\
p &= 1 \\
q &= 3
\end{aligned}
$$

Now for the other value of r...

$$
\begin{aligned}
(-4+3)p + 2q &= 0 \\
-p + 2q &= 0 \\
q_2 &= 1 \\
p_2 &= 2
\end{aligned}
$$

The general solution is now just the homogenous solution added to the particular solution:

$$
\begin{bmatrix} y_g \\ z_g \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} \frac{85}{18} - \frac{5}{3}t \\ \frac{65}{18} - \frac{10}{3}t \end{bmatrix}
$$

Example 3 For the above solution, find the constants using initial conditions $y(0) = 1, z(0) = 0$.

$$
\begin{aligned}
\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} c_1 + 2c_2 + \frac{85}{18} \\ 3c_1 + c_2 + \frac{65}{18} \end{bmatrix} \\
c_1 + 2c_2 &= -\frac{67}{18} \\
3c_1 + c_2 &= -\frac{65}{18}
\end{aligned}
$$

Wolfram Alpha says that $c_1 = -\frac{7}{10}, c_2 = -\frac{68}{45}$, so the general solution to this system for the given initial values is:

$$
\begin{bmatrix} y_g \\ z_g \end{bmatrix} = -\frac{7}{10} \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} - \frac{68}{45} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} \frac{85}{18} - \frac{5}{3}t \\ \frac{65}{18} - \frac{10}{3}t \end{bmatrix}
$$

9 Laplace Transform

Laplace transforms change differential equations from one form to another, easier form...somehow. Somehow, functions that get transformed into another space.

Here are some terms. $f(t)$ is the function to be transformed, and $F(s)$ is the transformation:

$$
F(s) = L(f) = \int_0^\infty f(t)e^{-st}dt
$$

Try doing it on a few examples.

Example 1 Find $L(1)$.

$$\begin{aligned} L(1) &= \int_0^\infty e^{-st}dt \\ &= \frac{e^{-st}}{-s}|_0^\infty \\ \lim_{t\to\infty} e^{-t} &= 0 \\ 0 + \frac{1}{s} &= 0 \\ &= \frac{1}{s} \end{aligned}$$

Example 2 Find $L(e^{at})$

Using integration by parts (read: magic)...

$$\begin{aligned} L(t) &= \frac{e^{-(s-a)t}}{-(s-a)}|_0^\infty \\ &= \frac{1}{s-a} \end{aligned}$$

Example 3 Find $L(\sin at)$. This one should be fun...

Use more integration by parts: $\int u'v = uv - \int uv'$

$$\begin{aligned} L(\sin at) &= \int_0^\infty \sin ate^{-st}dt \\ u' &= e^{-st} \\ u &= \frac{e^{-st}}{-s} \\ v &= \sin at \\ \frac{e^{-st}}{-s} \sin at|_0^\infty \end{aligned}$$

(This example is incomplete)

9.1 Formulas

$$
\begin{aligned}
L(1) &= \frac{1}{s} \\
L(t^n) &= \frac{n!}{s^{n+1}} \\
L(e^{at}) &= \frac{1}{s-a} \\
L(\sin at) &= \frac{a}{s^2+a^2} \\
L(\cos at) &= \frac{s}{s^2+a^2} \\
L(f') &= sF(s) - f(0) \\
L^{-1}(\frac{1}{s}) &= 1 \\
L^{-1}(\frac{1}{s^{n+1}}) &= \frac{t^n}{n!} \\
L^{-1}(\frac{1}{s-a}) &= e^{at}
\end{aligned}
$$

Note that Laplace transformations are linear. For example, you can turn $L(t+\sin at)$ into $L(t)+L(\sin at)$. This is formalized in the theorem $L(af+bg) = aL(f)+bL(g)$. The solution to this transform would be $\frac{1}{s^2}+\frac{a}{s^2+a^2}$.

Example 4 Take $L(f')$.

$$
\begin{aligned}
&= \int_0^\infty f'(t)e^{-st}dt \\
&= f(t)e^{-st}|_0^\infty - \int_0^\infty f(t)(-se^{-st})dt \\
\lim_{t\to\infty} f(t)e^{-st} &= 0 \\
&= sF(s) - f(0)
\end{aligned}
$$

This is a restatement of one of the formulas from the above subsection. Note that the function cannot be “terribly big”, and that $s > a$.

Example 5 Use Laplace transformations to solve the differential equation $y' + y = e^{-3t}$ with initial value $y(0) = 4$.

$$
\begin{aligned}
y' + y &= e^{-3t} \\
L(y'+y) &= L(e^{-3t}) \\
L(y') + L(y) &= \frac{1}{s+3} \\
sY - y(0) + Y &= \frac{1}{s+3} \\
(s+1)Y &= \frac{1}{s+3} + 4
\end{aligned}
$$

Note that setting the $s+1$ term above equal to zero will yield the eigen value!

$$Y = \frac{1}{(s+3)(s+1)} + \frac{4}{s+1}$$

The above *is* a solution, but it's in the wrong space. Convert it back to normal space using the inverse Laplace transform. If $L(f) = F$, then $L^{-1}(F) = f$, a rule that isn't at all helpful. The inverse is also linear. Anyways...

$$\begin{aligned} y &= L^{-1}(Y) \\ &= L^{-1}\left(\frac{1}{(s+3)(s+4)} + \frac{4}{s+1}\right) \end{aligned}$$

The last term can be solved easily, but the term on the left must be solved using partial fractions.

$$\begin{aligned} \frac{1}{(s+3)(s+1)} &= \frac{A}{s+3} + \frac{B}{s+1} \\ B &= \frac{1}{2} \\ A &= -\frac{1}{2} \\ L^{-1}\left(\frac{\frac{-1}{2}}{s+3} + \frac{\frac{1}{2}}{s+1} + \frac{4}{s+1}\right) &= -\frac{1}{2}e^{-3t} + \frac{9}{2}e^{-t} \end{aligned}$$

In order, the steps to solve are:

1. Take the Laplace transform of the equation
2. Solve for Y in the transformation. Use initial conditions where needed.
3. Take the inverse transform of the equation solved for. This is usually the hardest part.