# MA1023 (Calc III) Notes

Christopher Myers

June 13, 2020

## 1 Improper Integral Convergence Tests

### 1.1 Comparison Test

The comparison test for improper integrals describes that if a function g(x) is less than a function f(x) and the improper integral of f(x) converges, the improper integral of g(x) converges as well:

$$\int_a^\infty f(x)dx \neq \infty \rightarrow \int_a^\infty g(x)dx \neq \infty \tag{1}$$

### 1.2 Limit Test

If f(x) and g(x) are both positive, take the limit (going to (infinity) of their quotient. If the resultant number is finite and nonzero, their integrals both behave the same in terms of convergent (both diverge or both converge).

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L \tag{2}$$

## 2 Sequences and Series

A *sequence* is a sequence $a_1 \cdots a_n$, where $n$ could be infinite. Notation examples follow:

$$\{a_1, a_2, a_3 \cdots a_n\}$$
$$\{a_n\}_1^\infty$$

A *series* is a summation of a sequence, like so:

$$\sum_{n=1}^{i} a_n \tag{3}$$

Series can be infinite, but they aren't necessarily finite:

$$\sum_{n=1}^{\infty} a_n \tag{4}$$

## 2.1 Convergence Tests

### 2.1.1 $\mathbf{N^{th}}$ term test

If the limit of a sequence doesn't go to zero, the summation of that series is divergent:

$$\lim_{n\to\infty} a_n \neq 0 \mapsto \delta$$

This doesn't work the other way around - a sequence might have a limit at zero, but that doesn't mean it converges. For example, 1/n goes to zero, but the summation $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

### 2.1.2 Squeeze Theorem

Take functions f(x), g(x), and h(x). If f(x) is less than g(x) and g(x) is less than h(x), and f(x) & h(x) have limits at zero, so does g(x), even if it oscillates.

$$f(x) < g(x) < h(x)$$
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = 0 \mapsto \lim_{x\to\infty} g(x) = 0$$

This applies especially to oscillating functions:

$$f(x) = \frac{1}{x^2}$$
$$g(x) = \frac{\sin x}{n^2} h(x) = \frac{1}{n^2}$$

## 2.2 Ratio Test

Divide the function(n+1) by the function. If the result is less than one, the function absolute converges. If one, inclusive, else, divergent.

$$\lim_{n\to\infty} |\frac{a_n + 1}{a_n}| =? \tag{5}$$

# 3 Power Series

For x and n where n goes to infinity and x is a constant, find the values for which a given sequence converges. Example:

$$\sum_{n=1}^{\infty} \frac{n^2(2x-4)^{2n}}{4^n(n^4+1)} \tag{6}$$

## 3.1 Computing Power Series

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \frac{x^2}{1-x}$$

Radius of converge is still for x < 1 ?
TODO: Fix fracs

$$f(x) = \frac{1}{1-x^2} \tag{7}$$

Power series is

$$\sum_{n=0}^{\infty} x^2(x^n) = \sum_{n=0}^{\infty} x^{2n} \tag{8}$$

Originates from $(x^2)^n$

Root test yields the same results, the power series converges for abs(x) < 1. Example 3:

$$f(x) = \frac{1}{9+x^2} \to \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{3^{2n+2}} \tag{9}$$

To compute, factor out 9 to get 1 + x over some number. Continue factoring until the above magic happens. (TODO: Review)

**Example 4**

$$f(x) = \frac{x^3}{9+x^2} \to \sum_{n=0}^{\infty} \frac{x^{2n+3}}{3^{2n+2}} \tag{10}$$

**Example 5**

$$f(x) = \frac{2}{(1-2x)^2} \tag{11}$$

To solve, look at series for $C_n(x-a)^n$:

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a) + \cdots \tag{12}$$

Otherwise known as applying the power rule: $\sum_{n=1}^{\infty} nC_n(x-a)^{n-1}$

Apply to the above 11 equation:

$$\frac{d}{dx}(\frac{1}{1-2x}) = \frac{2}{(1-2x)^2} \to \sum_{n=1}^{\infty} 2^n n x^{n-1} \tag{13}$$

. . . there's some calcy stuff going on above that I couldn't understand/copy in time.

**Example 6** Original function:

$$f(x) = ln(5-x) \tag{14}$$

To get the power series, start by getting the integral. Use logarithm rules to break it up first, then do term-by-term for the power series thingy:

$$\int \sum_{n=0}^{\infty} C_n(x-a)^n = \sum_{n=0}^{\infty} \frac{C_n(x-a)^{n+1}}{n+!} + C \tag{15}$$

Integrate: $-\int \frac{1}{5-x}dx = -\int \frac{1}{5}[\frac{1}{1-x/5}]dx = -\int \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}dx = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^{n+1}} + C.$

The series (before integration) is convergent for x < 5. C (constant from integration) must be found. When x=0, $\ln 5 = C$ (magic!). Therefore $f(x) = \ln(5-x) = \ln 5 - \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)5^{n+1}}$

# 4 Taylor Series

Taylor series are ways to approximate given functions using polynomials. They are similar to power series but with their own specific form.

## 4.1 Derivation

$$\sum_{n=0}^{\infty} c_n(x-a)^2 = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots + c_n(x-a)^n$$

Power series are differentiable by term... At x=a, the power series is convergent (leaves only $C_0$, a finite number). So...

$$f(a) = \sum_{n=0}^{\infty} c_n(x-a)^n = 0f(a) \qquad = C_0$$

Differentiate:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n+1}$$

At x = a, $f'(a) = c_1$. Continue with the process; take the second derivative:

$$f''(x) = 2C_2 + 2*3C_3(x-a) + 3*4(x-a)^2 + \cdots + (n-1)nC_n(x-a)^{n-2}$$

At x=a, $f''(a) = 2C_2, C_2 = \frac{f''(a)}{2}$. The process can continue indefinitely:

$$C_3 = \frac{f'''(a)}{2*3}$$
$$C_4 = \frac{f^{iv}(a)}{2*3*4}$$

So the final formula for $C_n$ is $C_n = \frac{f^{(n)}(a)}{n!}$

### 4.1.1 Form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{16}$$

## 4.2 Maclaurin Series

The Maclaurin series is simply the Taylor series when a=0:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{17}$$

**Concept** $f(x) = f(a) + f'(a)(x-a)$: The equation of the tangent line, or the linear approximation of a given function. Taylor polynomials approximate given functions with polynomials. The higher the degree, the better the approximation (or as n goes to $\infty$)

### 4.2.1 Maclaurin Series Examples

**Example 1** Start with the Maclaurin series.

$$
\begin{aligned}
f(x) &= (1-x)^{-1} \\
f(a) &= f(0) \to 1 \\
f'(x) &= \frac{1}{(1-x)^2}, f'(0) = 1 \\
f''(x) &= \frac{2}{(1-x)^3}, f''(0) = 2 \\
f'''(x) &= \frac{2*3}{(1-x)^4}, f'''(0) = 2*3 \\
f^{iv}(x) &= \frac{2*3*4}{(1-x)^5}, f^{iv}(0) = 2*3*4 \\
f^{v}(x) &= \frac{n!}{(1-x)^{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n
\end{aligned}
$$

(TODO: Fix align*) Don't simplify, just look for a pattern. To find the radius of convergence, use the ratio test (or *maybe* the root test).

The Taylor/Maclaurin series are more useful as they work for an arbitrary function f(x).

**Example 2** "Something more interesting!" (still Maclaurin series)

$$
f(x) = e^x f(0) \qquad = 1 f'(x) = e^x \mapsto f'(0) = 1 f^n(x) \qquad = e^x \mapsto f'(0) = 1 e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
$$

Note that this implies that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ ... which it does.

**Example 3** "Next obvious choice"

$$
\begin{aligned}
f(x) &= \sin x, f(0) = 0 \\
f'(x) &= \cos x, f'(0) = 1 \\
f''(x) &= -\sin x, f''(0) = 0 \\
f'''(x) &= -\cos x, f'''(0) = -1 \\
f^{iv}(x) &= \sin x, f^{iv}(0) = 0
\end{aligned}
$$

All odd derivatives are abs(1), all evens are 0.

$$
f^{2n}(0) = 0 f^{2n+1}(0) \qquad = (-1)^n
$$

All even terms disappear due to being zero.

$$= \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Home exercise: calculate the Maclaurin series of the cosine of x. Actually, doing this will just result in $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

## 4.3 Alternative Methods

Consider $f(x) = \cos(x^2)$. Using the definition of the Maclaurin series would be difficult, so instead use the power series…something:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$$

The three basic series to know are $e^x$, $\cos x$, and $\sin x$. They can be used freely on the exam without showing how they were derived.

**Example 4**

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$
$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

In the above, parts of the $e^x$ and $e^{-x}$ series cancel out due to $e^{-x}$ having some negative terms. Specifically, all odd terms cancel out (series expansion is needed for this).

## 4.4 Taylor Series Examples

**Example 1** The idea of a Taylor series is the same as Maclaurin series, except that $a$ is not 0. For example, $f(x) = \frac{1}{\sqrt{x}}, a = 9$. In general, taylor series involve some work.

$$f(x) = x^{-1/2}$$
$$f(9) = \frac{1}{3}$$
$$f'(x) = -0.5x^{-3/2} \mapsto \frac{1}{3^3}$$
$$f''(x) = (-1/2)(-3/2)x^{-5/2} \mapsto$$
$$frac{-}1 * 32^2 * 3^5$$
$$f'''(x) = (-1/2)(-3/2)(-5/2)x^{-7/2} \mapsto -\frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{1}{3^7} = \frac{1*3*5}{2^3 * 3^7}$$
$$f^{(n)}(9) = (-1)^n \frac{1*3*5*7*\cdots*(2n-1)}{2^n 3^{2n+1}}$$

To deal with the above, pragma notation (?) is needed:

$$\frac{1}{\sqrt{x}} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{n=1}^{\infty}(2n-1)}{2^n 3^{2n+1} n!}(x-a)^n$$

The radius of convergence must be computed – use the ratio test.

$$\lim_{n\to\infty} | \frac{(-1)^{n+1}(1*3*5\cdots*(2(n+1)-1))(x-9)^{n+1}}{2^{n+1}3^{2n+3}(n+1)!} * \frac{2^n 3^{2n+1} n!}{1*3*5*\cdots*(2n-1)(x-n)^n)(-1)^n} |$$

I need another align* for this.

$$= \lim_{n\to\infty} |x-9| \frac{2n+1}{(2*9)(n+1)}$$
$$= \lim_{n\to\infty} \frac{|x-9|}{9} < 1$$

# 5 Parametric Curves

Parametric curves allow for curves and loops and such to be graphed. Such lines cannot be drawn as a function of x $f(x) = \ldots$ as they do not count as functions.

Under parametric curves, you can find $(x, y)$ coordinates by equation.

## 5.1 Definition

Parametric curves are formed by two functions for the two variables in an $(x, y)$ coordinate:

$$x = f(t)$$
$$y = g(t)$$

## 5.2 Basic Examples

**Example 1**

$$
\begin{aligned}
x &= t^2 - 2t \\
y &= t + 1
\end{aligned}
$$

A simple way of graphing these is to draw a table with inputs for $t$ and outputs for x and y. Those x and y pairs make coordinates which can in turn be graphed. The above definition yields a sideways parabola.

**Solving for f(x)**

$$
\begin{aligned}
t &= y - 1 \\
x &= (y-1)^2 - 2(y-1) \\
&= y^2 - 4y + 3
\end{aligned}
$$

**Example 2**

$$
\begin{aligned}
x &= \cos t \\
y &= \sin t
\end{aligned}
$$

The above forms a circle. It does not matter what coefficient is on $t$, as the $x^2+y^2=1$ for all cases of the coefficient. Putting a coefficient $r$ on the trig functions, however, *will* change the resultant circle, changing it to being of radius $r$. The same rules for circle construction follow from here on out as well.

**Example 3**

$$
\begin{aligned}
x &= \sin t \\
y &= \sin^2 t \\
\rightarrow y &= (\sin t)^2 = x^2
\end{aligned}
$$

This results in a parabola, where the "point" formed by the equations repeatedly traverses the parabola.

## 5.3 Cycloid

The best way to describe this is that if you were to trace the path followed by a single point on a rolling circle, a cycloid would be formed. This ends up looking like a "hopping" path. This curve cannot be represented without parametrics.

### 5.3.1 Derivation

Assume a circle with center $c$, radius $r$, edge point $P$, bottom point $T$ and central angle $t$. The distance the circle rolls is then $|OT|$, or the arc $PT\ldots$ which is just $rt$.

Let center be $C(rt, r)$. P as $P(x, y)$, and point Q as a point on CT level with point P.

$$
\begin{aligned}
x &= |OT| - |PQ| = rt - r\sin t \\
&= r(t - \sin t) \\
y &= |TC| - ||CQ = r - r\cos t \\
&= r(1 - \cos t)
\end{aligned}
$$

## 5.4 Calculating Derivatives of Parametric Curves

A parametric curve is called *smooth* if f'(t) and g'(t) are continuous and are never zero at the same time.

$$\begin{aligned} y &= F(x) \\ g(t) &= F(f(t)) \end{aligned}$$

Differentiate both sides, applying the chain rule...

$$\begin{aligned} g'(t) &= F'(f(t))f'(t) \\ &= F'(x)f'(t) \\ F'(x) &= \frac{g'(t)}{f'(t)} \end{aligned}$$

This is often shown as $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. Never try and cancel out the math spaghetti above, it is only notation.

When $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$, a horizontal tangent line is yielded. When in the opposite siutation ($\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0$), a vertical tangent line is yielded.

**Example 1** Find the equation of the tangent line to $x = r(t - \sin t), y = r(1 - \cos t)$ at $t = \frac{\pi}{3}$. Start by taking the derivative...

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{r \sin t}{r(1 - \cos t} \\ &= \frac{\sin t}{1 - \cos t} \end{aligned}$$

Then just find the slope at point $\frac{\pi}{3}$ and find the equation of the tangent line.

$$\begin{aligned} \frac{\sin \frac{\pi}{3}}{1 - \cos \frac{\pi}{3}} &= \frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{2}} = \sqrt{3} \\ y &= y_0 + m(x - x_0) \\ &= \frac{r}{2} + \sqrt{3}(x - r(\frac{\pi}{3} - \frac{\sqrt{3}}{2})) \end{aligned}$$

Now take the second derivative:

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}(\frac{dy}{dx}) \\
&= \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} \\
\frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}(\frac{\sin t}{1-\cos t}}{r(1-\cos t} \\
&= \frac{\cos t(1-\cos t)-\sin^2 t}{r(1-\cos t)^3} \\
&= \frac{-(1-\cos t}{r(1-\cos t)^3} \\
&= \frac{-1}{r(1-cos)^2}
\end{aligned}
$$

Regardless of the values of t, the trajectory (sp?) of the cycloid is always concave down, since the second derivative is always less than zero.

## 5.5 Integrating Parametric Curves

Proof. . . or something:

$$
\begin{aligned}
y &= F(x) \\
A &= \int\limits_a^b F(x)dx \\
&= \int\limits_a^b ydx \\
x &= f(t) \\
y &= g(t) \\
\ldots \\
Area &= \int\limits_a^b ydx \\
&= \int\limits_a^b g(t)f'(t)dt
\end{aligned}
$$

**Example 1** Integrate the cycloid from 0 to $2\pi$

$$
\begin{aligned}
A &= \int_0^{2\pi} r(1-\cos t)r(1-\cos t)dt \\
&= r^2 \int_0^{2\pi} (1-\cos t)^2 \\
&= r^2 \int_0^{2\pi} (1-2\cos t+\cos^2 t)dt \\
&= r^2 (2\pi - \int_0^{2\pi} 2\cos t + \int_0^{2\pi} \cos^2 t dt)
\end{aligned}
$$

Don't think about the integral as the antiderivative but as the area under the curve. So $\int_0^{2\pi} 2\cos t$ must be zero. Some trig identities will be needed for the below.

$$
\begin{aligned}
&= r^2(2\pi + \int_0^{2\pi} \cos^2 t dt) \\
\cos^2 x &= \frac{1}{2}(1+\cos 2x) \\
\sin^2 x &= \frac{1}{2}(1-\cos 2x) \\
\cos 2x &= \cos^2 x - \sin^2 x \\
\sin 2x &= 2\sin x \cos x \\
&= r^2(2\pi \int_0^{2\pi} \frac{1}{2}(1+2\cos 2t)dt) \\
&= r^2(2\pi + \pi + \frac{1}{2}\int_0^{2\pi} \cos 2t dt) \\
&= 3\pi r^2
\end{aligned}
$$

### 5.5.1 Revolutions

The volume of a revolution around the x axis is defined by $\int_a^b \pi y^2 dx$, and the volume of a revolution around the y axis is defined by $\int_a^b 2\pi xy dx$. Or, volume is $\int_a^b A(x)dx$.

Quote of the day: "It's not my fault that you didn't express yourself ..."

**Example 1 - Volume of a Sphere** Find the volume of a sphere by revolving a circle around the x axis. Doing this with a traditional circle is very difficult, but not as much if parametrics are used. The definition of a parametric circle is below:

$$x = r\cos t$$
$$y = r\sin t$$

This must be integrated from 0 to $\pi$.

$$V_x = \int_0^{\pi} \pi(r^2\sin^2 t)(-r\sin t)dt$$
$$= -r^3\pi \int_0^{\pi} \sin^3 t dt$$

Integrating $\sin^3$ is painful, so write it as $\sin^2 t \sin t$ Then use u-substitution:

$$= -r^3\pi \int_0^{\pi} (1-\cos^2 t)\sin t dt$$

$u = \cos t$, $du = -\sin t dt\ldots$

$$= \frac{4}{3}\pi r^3$$

And that is the equation for the volume of a circle.

## 5.6 Arc Length

The arc length is defined as follows:

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

...or in other words, obtain the arc length from intervals, then sum them. For completely accurate answers, make the intervals infinitely small. The tricky part is just calculating the derivative of a parametric curve. Further refinement is shown below.

$$= \int_a^b \sqrt{\frac{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}{\frac{dx}{dt}^2}} \frac{dx}{dt} dt$$
$$= \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

**Example - find the length of one arc of the cycloid**

$$x = r(t - \sin t)$$
$$y = r(1 - \cos t)$$

…for $0 \leq t \leq 2\pi$

$$L = \int_0^{2\pi} \sqrt{(r(1-\cos t))^2 + (r\sin t)^2} dt$$
$$= r \int_0^{2\pi} \sqrt{2 - 2\cos t} dt$$

This must now be integrated. It will not be fun. The trig identity $\sin^2 t = \frac{1}{2}(1 - \cos(2t)$ (a half angle formula) will be needed, albeit slightly reordered: $\sin^2(\frac{t}{2}) = \frac{1}{2}(1 - \cos t)$

$$= r \int_0^{2\pi} \sqrt{4\sin^2(\frac{t}{2})} dt$$
$$= 2r \int_0^{2\pi} |\sin(\frac{t}{2})| dt$$

Because of $\frac{t}{2}$, the domain must also be halfed, becoming $0 \leq \frac{t}{2} \leq \pi$. Anyways, the antiderivative of the above is…

(Actually, just forget about the domain thing)

$$2r \int_0^{2\pi} \sin\frac{t}{2} = 2r(-2\cos(\frac{t}{2})|_0^{2\pi}$$
$$= 8r$$

## 5.7 Surface Area

To obtain the surface area, revolve a function around the x axis:

$$S_x = \int_a^b 2\pi y dS$$
$$S_y = \int_a^b 2\pi x dS$$
$$dS = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

**"Simple" Example** Show that the surface area of a sphere of some radius $r$ is $4\pi r^2$. A circle will be revolved around the x axis. ($x = r\cos t, y = r\sin t, 0 \leq t \leq \pi$

$$\begin{aligned}
S &= \int_0^{\pi} 2\pi r \sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2} dt \\
&= \int_0^{\pi} 2\pi r^2 \sin t dt \\
&= 2\pi r^2 (-\cos t|_0^{\pi}) \\
&= 4\pi r^2
\end{aligned}$$

# 6 Polar Coordinates

Polar coordinates are defined by a "pole"/ray of length r and of angle $\theta$. The standard form is $P(r, \theta)$. Or in other words, they are a rate or an angle. Be warned: the distance $r$ **c**an be negative ($P(-r, \theta)$), which evaluates to $P(r, \theta + \pi)$.

It is often (read: always) desired to convert coordinates from the polar system to traditional Cartesian coordinates. Doing this (among other useful calculations) involves simple right-angle trigonometry:

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
x^2 + y^2 &= r^2 \\
\tan\theta &= \frac{y}{x}
\end{aligned}$$

**Example 1** Convert $P(2, \frac{\pi}{3}$ to cartesian coordinates.

$$\begin{aligned}
x &= 2\cos\frac{\pi}{3} \\
y &= 2\sin\frac{\pi}{3} \\
&= (1, \sqrt{3})
\end{aligned}$$

**Example 2** Convert $(1, -1)$ to polar coordinates.

$$\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
-1 &= \tan\theta \\
&= P(\sqrt{2}, \frac{7\pi}{4})
\end{aligned}$$

Note that $\frac{7\pi}{4}$ can also be represented as $-\frac{\pi}{4}$ ...and there are technically many other representations of this, accommodating $r$ being positive or negative.

**Example 3** A circle can be created if r is some constant and $y = f(\theta)$. Note that this creates a circle centered at the origin.

The resulting formula results in $x^2 + y^2 = 4$, or the exact (cartesian) formula for a circle centered at the origin.

**Example 4** If $\theta = \frac{\pi}{3}$ is graphed, a line straight out from the origin is produced. This is akin to saying something like $x = 3$ in cartesian coordinates.

**Example 5** $r = 2\cos\theta$

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x \\ x^2 - 2x + y^2 &= 0 \\ x^2 - 2x + 1 - 1 + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

Above, completing the square is used. This produces a circle of radius 1 centered at $(1, 0)$

This principle can be applied for equations $r = 2a\sin\theta$ and $r = 2a\cos\theta$. The former produces a circle with radius $a$ centered at $(0, a)$ while the latter produces a circle with radius $a$ centered at $(a, 0)$.

**Example 6** Graphing $r = 1 + \sin\theta$ from 0 to $2\pi$ results in a sine wave in regular cartesian coordinates, but in polar coordinates creates a graph with a sort of hump shape on top and a dimpled structure on the bottom. This can be easily determined by examining the values of the function in each of the four quadrants.

In reality, these graphs are called cardiods. They are defined by the functions $a(1 + / - \sin\theta$ and $a(1 + / - \cos\theta)$.

**Example 7 - Flower** Graph $r = \cos 2\theta$. This produces a four-leaved "flower" shape that was described as a rose in-class, but is better described as a daisy.

It is generally advised to first graph any given polar function in cartesian coordinates first, then transfer over the observed behavior to polar coordinates.

## 6.1 Derivatives of Polar Functions

The equation of a tangent line *can* be found in polar coordinates by thinking about polar coordinates as being parametric curves with formulas $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Since the derivative of parametric curves can be found, the same technique can be applied to the above two formulas: $\frac{dx}{dy} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

**Example 1** Find the equation of the tangent line at $\theta = \frac{\pi}{3}$ of $1 + \sin\theta$. Remember trig identities.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta} \\ &= \frac{\cos\theta + \sin 2\theta}{\cos 2\theta - \sin\theta} \\ &= \frac{\cos\frac{\pi}{3} + \sin\frac{2\pi}{3}}{\cos\frac{2\pi}{3} - \sin\frac{\pi}{3}} \\ &= -1 \end{aligned}$$

Then find the equation of the tangent line!

## 6.2 Areas in Polar Coordinates

In a graph, assume $r = f(\theta)$ and you have two angles: a larger angle $\alpha$ and a smaller angle $\beta$, where $\theta$ is between $\alpha$ and $\beta$. For these conditions, find the area of the sector formed by these two angles.

To proceed, divide the region between them into many subregions, calculute the area of one, and add them to get an approximation of the area. The accuracy of the approximation is improved as the width of each subsector goes to zero…i.e. this involes integration.

$$\alpha = \theta_1 \theta_2 \ldots \theta_n = \beta$$
$$\Delta\theta = \theta_i - \theta_{i-1}$$

Take a center point (angle) $\theta{*}_i$ and get $r{*}_i$ from it. The area of any type of sector of a circle $= \frac{1}{2}\theta r^2$.

$$\begin{aligned}
\Delta A_i &= \frac{1}{2}(r{*}_i)^2 \Delta\theta \\
A(R) &= \sum_{i=1}^{n} \Delta A_i \\
&= \sum_{i=1}^{n} \frac{1}{2}(r{*}_i)^2 \Delta\theta \\
A(R) &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}(r{*}_i)^2 \Delta\theta \\
&= \int\limits_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta
\end{aligned}$$

Or, $\int\limits_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 d\theta$

**Example 1** Find the area enclosed by one loop of a four-leaved rose (given by $r = \cos 2\theta$. Range: $\frac{-\pi}{4} \leq \theta \leq \frac{\pi}{4}$.

$$\begin{aligned}
A &= \int\limits_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta d\theta \\
&= \int\limits_{0}^{\pi/4} \cos^2 2\theta d\theta \\
&= \int\limits_{0}^{\pi/4} \frac{1}{2}(1 + \cos 4\theta) d\theta \\
&= (\frac{\theta}{2} + \frac{\sin 4\theta}{8})|_0^{\pi/4} \\
A &= \frac{\pi}{8}
\end{aligned}$$

**Example 2** Find the area of the region that lies under $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$. The first one creates a circle, the second one creates a cardiod. First, however, the intersection of the two graphs must be found. Start by setting them equal to each other:

$$3\sin\theta = 1 + \sin\theta$$
$$\sin\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Now take the area under the circle and and subtract the area under the cariod: $A = \int\limits_{\pi/6}^{5\pi/6} \frac{1}{2}(\sin\theta)^2 d\theta - \int\limits_{\pi/6}^{5\pi/6} \frac{1}{2}(1+\sin\theta)^2 d\theta$.

$$= \frac{1}{2}\int\limits_{\pi/6}^{5\pi/6} ((3\sin\theta)^2 - (1+\sin\theta)^2))d\theta$$
$$= \int\limits_{\pi/6}^{pi/2} (8\sin^2\theta - 1 - 2\sin\theta)d\theta$$
$$= \pi$$

For $\sin^2$, use the double angle (?) identity. All of that above is omitted.

### 6.3 Calculating Arc Length of Polar Curves

$$r = f(\theta)$$
$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta$$
$$L = \int\limits_{\alpha}^{\beta} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$$

Now simplify the square root:

$$= \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}$$
$$= \sqrt{f'^2(\theta)\cos^2\theta - 2f'(\theta)\cos\theta f(\theta)\sin\theta + f^2(\theta)\sin^2\theta) + f'^2(\theta)\sin^2\theta + 2f'(\theta)\sin\theta)f(\theta)\cos\theta + f^2(\theta)\cos^2\theta} \quad = CARPAL_T U$$
$$= \sqrt{f'^2(\theta)(\cos^2\theta + \sin^2\theta) + f^2(\theta)(\sin^2\theta + \cos^2\theta)}$$
$$= \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}$$

So $L = \int\limits_{\alpha}^{\beta} \sqrt{[f'(\theta)]^2 + [f(\theta)]^2} d\theta$, or just $\int\limits_{\alpha}^{\beta} \sqrt{[\frac{dr}{d\theta}]^2 + r^2} d\theta$.

**Example 1** Find the length of $r = 1 + \sin\theta$.

$$L = \int\limits_{0}^{2\pi} \sqrt{\cos^2\theta + (1+\sin\theta)^2} d\theta$$
$$= \sqrt{2 + 2\sin\theta} d\theta$$

Multiply by the conjugate...

$$= \int_0^{2\pi} \sqrt{2+2\sin\theta} * \frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{(2+2\sin\theta)(2-2\sin\theta)}}{\sqrt{2-2\sin\theta}} \quad \mathrm{d}\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{4-4\sin^2\theta}}{\sqrt{2-2\sin\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{2\cos\theta}{\sqrt{2-2\sin\theta}} d\theta$$

This was the entire point of multiplying by the conjugate. Now u-substitution can be used: $u = 2-2\sin\theta$, $du = -2\cos\theta d\theta$.

After working all of this out, the length is 8.

# 7 Vectors

## 7.1 3D Coordiniate System

In two dimensions, a point can be represented with two coordinates: $x$ and $y$. This is limiting and less realistic, so a 3D coordinate sytem is used, adding $z$ as a coordinate: $(x, y, z)$. X is faces out of the paper, Z faces up, and Y faces right.

## 7.2 Formulas

### 7.2.1 Distance Between Two Points

$$|P_1P_2| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

### 7.2.2 Formula for a Sphere

The formula for a sphere is given by taking any point $P$ on the sphere with coordinates $(x, y, z)$. Thus, the definition a sphere is:

$$r^2 = x^2 + y^2 + z^2$$

This gives a sphere centered at $(0, 0, 0)$. A more general formula is:

$$r^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$$

...which gives a sphere centered at $(x_0, y_0, z_0)$ with radius $r$. A demonstration for simplifying down this is shown below.

$$\begin{aligned} x^2 + y^2 + z^2 + 2x - 4z + 1 &= 0 \\ x^2 + 2x + 1 - 1 + y^2 + z^2 - 4z + 4 - 4 + 1 &= 0 \\ (x+1)^2 + y^2 + (z-2)^2 &= 4. \end{aligned}$$

This gives a sphere centered at $(-1, 0, 2)$ with radius 2.

## 7.3 Vector Definitions

Vectors are rays that have direction and magnitude. If you have two points A and B, you can always define a direction from A to B. In this context, A is the initial point and B is the terminating point. The different values for a vector – one for each dimension – are called *components*. If the initial point for a vector is the origin, it is called a position vector.

fsubsectionMagnitude of a Vector

$$|V| = \sqrt{x^2 + y^2 + z^2}$$

**Example** For $P(1,2,3)$ and $Q(-2,0,4)$, find $|PQ|$

$$\begin{aligned} PQ &= (-3,-2,1) \\ |PQ| &= \sqrt{3^2 + 2^2 + 1^2} \\ &= \sqrt{14} \end{aligned}$$

Note: a zero vector is where all components of the vector are zeros.

## 7.4 Unit Vectors

Unit vectors are vectors where the magnitude is 1. The three basic unit vectors go along the axes $x, y, z$ and are vectors $i, j, k$ respectively. To find the unit vector in the direction of any given vector, normalize it by dividing each of its components by its magnitude:

$$u = \frac{v}{|v|}$$

**Example** For the vector from $P(1,1,1)$ to $Q(3,0,4)$, find the unit vector.

$$\begin{aligned} PQ &= (2,-1,3) \\ u &= \frac{PQ}{|PQ|} \\ &= (\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}) \end{aligned}$$

Note that the above can also be written as $(i\frac{2}{\sqrt{14}} - j\frac{-1}{\sqrt{14}} + k\frac{3}{\sqrt{14}})$.

## 7.5 Dot Product

Also known as the scalar project. Defined as multiplying two vectors' respective components by each other and summing them:

$$u \cdot v = u_x v_x + u_y v_y + u_z + v_z$$

### 7.5.1 Angle Between Vectors

For two vectors $u$ and $v$ and an angle $\theta$ between them,

$$u \cdot v = |u||v| \cos\theta$$

From this it can be easily determined if vectors are orthogonal (perpendicular) or parallel. If the dot product between two vectors is 0, the two vectors are orthogonal if the dot product is zero. If the dot product is 1, the two vectors are instead parallel.

**Example** Determine the dot product between two vectors $u(1, -2, -1)$ and $v(-6, 2, -3)$.

$$\begin{aligned} u \cdot v &= -6 - 4 + 3 \\ &= -7 \end{aligned}$$

Since the dot product is neither zero nor 1, these vectors are neither parallel nor perpendicular. Now find the angle between them:

$$\begin{aligned} \cos\theta &= \frac{u \cdot v}{|u||v|} \\ &= \frac{-7}{\sqrt{6}\sqrt{49}} \\ \theta &= \arccos -\frac{1}{\sqrt{6}} \end{aligned}$$

**Example 2 - Physics** This isn't fully worked through, but dot products have physics application: Work can be represented as the dot product between the vectors for force and distance.

## 7.6 Cross Product

The cross product is an operation that creates a vector out of two vectors that is orthogonal to its two components. The formula for its calculation is shown below, omitting the matrix stuff:

$$u \times v = (u_2v_3 - u_3v_2)i - (u_1v_3 - u_3v_1)j + (u_1v_2 - u_2v_1)k$$

**Example 1** For two vectors $u(1, 2, 3)$ and $v(-2, -3, -4)$, calculate $u \times v$.

$$\begin{aligned} u \times v &= (u_2v_3 - u_3v_2)i - (u_1v_3 - u_3v_1)j + (u_1v_2 - u_2v_1)k \\ &= (-8 + 9)i - (-4 + 6)j + (-3 + 4)k \\ &= i - 2j + k \\ &= (1, -2, 1) \end{aligned}$$

### 7.6.1 Normal Unit Vector

For vectors $u$ and $v$ , $n$ (the unit normal vector to the plane determined by $u$ and $v$) is defined by:

$$u \times v = (|u||v| \sin\theta)n$$

### 7.6.2 Parallel Vectors

The length of the cross product is the area of the parallelogram defined by two vectors:

$$|u \times v| = |v||u| \sin\theta$$

If two vectors are parallel, the cross product will be zero.

**Example 1** Does $u \times v = v \times u$?
Answer: no. $u \times v = -(v \times u)$.

**Example 2** Find the area of the triangle with vertices $P(1,-1,0), Q(2,1,-1), R(-1,1,2)$. To solve, find the mangnitude of the cross product and divide by two (triangle area is defined by $\frac{1}{2}bh$).

$$
\begin{aligned}
A(\Delta PQR) &= \frac{1}{2}|PQ \times PR| \\
PQ &= (1,2,1) \\
PR &= (-2,2,2) \\
PQ \times PR &= 6i + 6k \\
|6i + 6k| &= 6\sqrt{2} \\
A(\Delta PQR) &= 3\sqrt{2}
\end{aligned}
$$

## 7.7 Lines and Planes

### 7.7.1 3D Lines

The equation of lines in 3D does not make sense to write in terms of the original coordinates (e.g. $z = \dots$). To solve this problem, use parametrics instead.

Start with a directional vector that points in the direction of the line. The easiest way to get such a vector is to construct one using two points on the line. Then multiply by some time value and solve for (x, y, z)

$$
\begin{aligned}
x &= x_0 + tv_1 \\
y &= y_0 + tv_2 \\
z &= z_0 + tv_3
\end{aligned}
$$

These are the parametric equations for 3D lines. Note that T can be any number, similar to $x$ in the classic $y = ax + b$ equation... sort of.

**Example** Find the parametric equation for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$. The directional vector of the line can be defined as vector $PQ$.

$$
\begin{aligned}
PQ &= (4,-3,7) \\
x &= 1 + 4t \\
y &= -1 - 3t \\
z &= 4 + 7t
\end{aligned}
$$

## 7.8 Planes

For any given plane, the normal vector is perpendicular to that plane. That means that that the normal vector is also perpendicular to any given vector on the plane. Below is the general equation for a plane along the vector $P_0P$ and perpendicular to normal vector $n(a,b,c)$.

$$
\begin{aligned}
(x - x_0, y - y_0, z - z_0) \cdot (a,b,c) &= 0 \\
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0
\end{aligned}
$$

**Example** Find the equation of the plane through $A(0,0,1), B(2,0,0), C(0,3,0)$. To do this, find vectors AC and BC, then take their cross product.

$$\begin{aligned}
AB &= (2,0,-1)\\
AC &= (0,3,-1)\\
n &= AB \times AC\\
&= 3i + 2j + 6k\\
&= (3,2,6)\\
P(ABC) &= 3(x-0) + 2(y-0) + 6(z-1)\\
6 &= 3x + 2y + 6z
\end{aligned}$$