PH 2550 Atmospheric and Space Environments

Christopher Myers

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Contents

1 Basic Space Environment

There are several layers to the region around the Earth:

- Neutral atmosphere — the atmosphere that we see and are used to, up to about 100 km.
- Ionosphere — from about 100 km up is a region of mostly ionized particles.
- Plasmasphere — part of the ionosphere. Consists of a layer of plasma.
- Van Allen radiation belts — charged particles exhibit circular motion due to the Earth's magnetic field. Particles also drift between the north pole and the south pole. This results in a series of radiation belts.
- Interstellar space — outside Earth's sphere of influence entirely is interstellar space, which is still more dangerous due to lack of protection from Earth's magnetic field. In this region can be found strong UV rays and soft X-rays (?), and various solar phenomena.

Also of note are different physical objects that can be found, notably small meteoroids and debris from spacecraft. Small meteoroids are usually *very* small, on the order of grains of sand to the size of marbles (with the latter being exceedingly rare), and can be found anywhere, but space debris is man-made and found primarily in low earth orbit. This debris field consists of everything from pieces of metal to bolts and flecks of paint, and due to the very high speeds involved in orbit they can cause significant damage all on their own. Cosmic rays are also a threat (usually in the form of protons of energy around 1 GeV), although they come from interstellar space.

Spacecraft reliability is measured using N, the number of parts tested, and t, the time elapsed. Here, ΔN is the number of parts that fail, so $\frac{\Delta N}{\Delta t} = -kN(t)$. This can be integrated and solved to get $N(t) = N_0 e^{-kt}$ where k is the proportionality constant, t is time, $N(t)$ is the number of parts active at a given time, and N_0 is the original starting number of parts. Alternatively, use $\eta = \frac{1}{k}$ (in units of seconds) to get $N(t) = N_0 e^{-t/\eta}$. The number of parts that have failed is simply $F(t) = N_0 - N(t)$; simplifying yields $F(t) = N_0(1 - e^{-t/\eta})$. Taking the derivative of *that* ultimately gets you your failure rate.

The reliability function is defined as $R(t) = \frac{N(t)}{N_0}$; it's really just the fraction of parts that have survived to a given time. It can also be interpreted as the probability of survival of an individual part. If instead you take $\frac{F(t)}{N_0}$ you'll get the fraction of parts that failed instead, and thus get the probability of a given part failing $P(t)$. If you take the derivative of failure probability with respect to time, you get $p(t) = \frac{1}{\eta} e^{-t/\eta}$. This turns out to be a probability density function.

1.1 Measurements of distances

One AU is the average distance between the Earth and the Sun. Distances to stars are measured using triangulation, a method called parallax. If you define a distance between two points in the Earth's orbit B (for baseline) and the distance to the star as D, the distance is equal to the baseline divided by the angle formed in radians, $D = \frac{B}{\Phi}$. Note that normal trig functions are skipped here because, for very small angles, they aren't needed (this is the small angle approximation). The angle is formed by the triangle between the star and the two points in Earth's orbit. Using the same type of math, we can define a parsec as the distance where the angle formed by the triangle is 1 arcsecond (or 1 degree divided by 60 minutes, divided by 60 seconds). This value turns out to be about 3.6 lightyears.

The irradiance of a star is measured by its energy output in watts; divide that by $4\pi D^2$ ($\frac{\Phi}{4\pi D^2}$) to get irradiance, measured in W/m^2. So, if you multiply by m^2 again, you can get watts again, and calculate the amount of energy received by a solar panel, for instance. Stars that are 5 magnitudes of brightness different, the ratio of their irradiances is about 100. The magnitude of a star is defined as $m - m_0 = -2.5 \log(\frac{E}{E_0})$ where E is the irradiance. However, apparent brightness decreases with distance, so some adjustment is needed to the magnitude scale. The absolute magnitude of a star is defined as what its magnitude would appear to be were the star exactly 10 parsecs away, and is denoted as M. The absolute magnitude is then calculated as $m - M = -2.5 \log(\frac{E(D)}{E(10)})$.

Certain stars call Cepheids have a brightness that varies with time, usually over a period of days. If you know the period P, you can get the absolute magnitude $M = (-2.43 \pm 0.12)([\log_{10} P] - 1) - (4.05 \pm 0.02)$. Each number can also be represented as k_1, k_g, and k_2, respectively, and P is the period of the Cepheid in days. Since you know the apparent magnitude of the star and now the absolute magnitude, you can plug these values into the formula for absolute magnitude to compute how far away the Cepheid is. You can also use $D_\Phi = 10^{0.2[(m-M)+1]}$. The useful thing about Cepheids is that you can use them to approximate distances to incredibly distant objects (like other galaxies) just by finding a Cepheid near it.

This allows for the calibration of yet another scale, the Hubble expansion scale. Turns out, distant objects show a strong redshift, indicating that everything is expanding away from everything else. The speed at which objects are moving away can be calculated using this redshift and the relativistic doppler equation, usually by looking at redshifted emission lines from hydrogen. If you can use the Cepheid scale to get the distance to a galaxy, you can calibrate the Hubble expansion scale, which says $D = \frac{1}{H_0} v$, where H_0 is the Hubble constant and v is the velocity of the galaxy, as determined by its redshift. This can be used to measure the distances of galaxies billions of lightyears away.

1.2 Formation of stars

Stars are formed from protoplanetary clouds, sometimes called proplyds (?). The cloud is characterized by the values T, p, n, m, where m is the average mass of particles in the cloud, T is the temperature, and p is the density (?). When put into the Jeans equation, you can get the radius needed for it to collapse as:

$$R_J = \sqrt{\frac{15 k_B T}{4 \pi p G m}}$$

...where R_J is the Jeans radius and k_B is the Boltzmann constant. This equation tells you that if a cloud has a radius bigger than the Jeans radius, it's going to collapse. You can also find the Jeans mass:

$$M_J = (\frac{3}{4\pi p})^{1/2} (\frac{5 k_B T}{G m})^{3/2}$$

Predictably, this means that if the mass is greater than M_J, the cloud is unstable and will collapse (and hopefully form a star!)

1.3 Gravity

Escape velocity can be calculated using:

$$v_f = \sqrt{\frac{2GM}{R_E}}$$

...where G is the gravitational constant, M is the mass of the body, and R_E is distance from the center of mass of the body to the object (or in this case, the radius of the Earth). This can be used both to calculate how much speed a spacecraft has to have leaving the surface of the earth and the speed we can expect an incoming particle to have upon hitting the surface. For Earth, this ends up being about 11.2 km/s. For the Sun, it's about 600 km/s, so solar particles that leave the solar system have to be doing at *least* that speed in order to make it out into interstellar space.

Suppose you have a system of the Sun, the Earth, and a meteor. The distance from the sun to the earth is R_E, the distance from the Earth to the meteor is r_E, and the distance from the Sun to the meteor is r_s. The final formula for getting the velocity of the meteor uses two calculations for potential energy:

$$v_f = \sqrt{2G(\frac{M_E}{r_E f} + \frac{M_s}{r_s f})}$$

The sun obviously helps bring the speed up a *lot*. For the Earth and Sun, that speed ends up being about 40 km/s.

2 The Sun

The sun is a normal G2 star, one of about 100 billion stars in the galaxy. It has a diameter of 1,390,000 km and a mass of $1.989 \cdot 10^{30}$ kg, making up about 99.8% of the mass of the solar system. Inside, energy is created in the core when hydrogen is fused into helium (ions, no electrons yet). Energy fflows out from the core, through the radiative zone and out into the corona layer. Energy from the Sun passes through and imaginary disc that has a diameter equal to Earth's diameter; the flux of energy through the disc is about 1370 watts per square meter. The amount of energy that hits a square meter on the Earth's surface is greatest when the patch of land is perpendicular to the incoming energy. Unlike the Earth, the Sun does not rotate rigidly (since it's not a solid body). At the equator, it takes about 25 days to rotate around, but at the pole it takes 35 days to rotate; this is called differential rotation. There's also internal rotation within the sun, where different regions rotate faster than others; this may give rise to the Sun's magnetic field.

The Sun's corona is threaded with a network of magnetic fields; solar storms and flares result from changes in the structure and connections of these fields. Every 11 years, the Sun moves through a period of decreased sunspot activity (?). Sunspots are dark spots on the surface of the sun, where temperatures drop to about 3700 K (compared to 5700 K elsewhere). They typically last for several days. Sunspots have a strong magnetic fields, about 1000 times greater than the average field strength of the sun itself. Sunspots usually appear in pairs, with different polarities: one is magnetic north, the other is magnetic south. Sunspots have reached a maximum every 11 years, ever since 1700, but there is evidence of some sort of cycle on a 55 to 57 year scale. Because the pre-1700 decrease in sunspot activity coincide with the end of a "little ice age", sunspots are hypothesized to be involved in climate... somehow.

Prominences are dense clouds of material suspended above the surface of the Sun by loops of magnetic field. When these break down, solar flares — huge explosions from the surface of the sun — can arise. Coronal Mass Ejections (CMEs) can also happen, where huge bubbles of gas threaded with magnetic fields are ejected from the Sun over the course of several hours. CMEs disrupt the flow of solar wind and can disrupt electrical systems on Earth. If a large one were to hit Earth, most of the US would likely lose power just due to the influence of those magnetic fields, not to mention the damage to all of our satellites. Due to convection from inside the sun, the photosphere is subdivided into 1000–2000km cells called granules; fluid rises in the middle and sinks at darker edges, with the whole cycle taking about 20 minutes. Supregranules are composed of multiple granules and last for 1–2 days.

The heliosphere is the region centered at the Sun that contains the interplanetary magnetic field and the solar wind — it's effectively the extent of the influence of the sun. Until the 1990's it was thought to be limited to just the Sun itself, but according to data from the Ulysses spacecraft it extends a little more than that. The outer boundary is not entirely understood yes.

3 Radiation

3.1 Light pressure

Irradiance is defined as the radiant power of electromagnetic radiation incident on a unit area, in units of watts per square meter. S_e is defined as the solar constant (the irradiance of the Sun at the distance of the Earth), or $13366.1 W/m^2$. The total power of the sun can follow from $4\pi S_e a^2$ where a is 1 AU, giving a very large number.

For N photons, $\Delta\varepsilon = N\varepsilon$ and $\Delta p = Np$, so $\Delta\varepsilon = \Delta pc$, where ε is energy and momentum of a photon is p. These are related: $\varepsilon = pc$. Incident photos absorbed by a surface impart momentum to the surface:

$$\Delta p = \frac{1}{c}\Delta\varepsilon$$

If photos are absorbed in time Δt, then by the above the rate at which momentum is imparted is:

$$\frac{\Delta p}{\Delta t} = \frac{1}{c}\frac{\Delta\varepsilon}{\Delta t}$$

Force is recognized as:

$$F = \frac{\Delta p}{\Delta t}$$

And absorbed power P_{abs} is:

$$P_{abs} = \frac{\Delta \varepsilon}{\Delta t}$$

The flux of sunlight at distance r from the Sun is:

$$S = \frac{P}{4\pi r^2}$$

A surface of area A that faces the Sun absorbs radiant energy at a rate:

$$\frac{\Delta \varepsilon}{\Delta t} = SA$$

Another expression of absorbed power, equating two different equations here, is: [nope]

3.2 Black bodies

A black body is a theoretical object that is a perfect radiator; the Sun's spectrum is nearly identical to one. The energy flux from black body irradiators at different temperatures changes the wavelength of emitted light. The radiation peak moves to shorter wavelengths as temperature increases; the area under any one curve is the total flux of energy emitted by a radiator at a given temperature. The higher the temperature, the greater the flux.

If you consider stars as black body irradiators, the colors and temperatures of stars can be easily explained. Yellowish stars peak in the visible range; reddish, white, and bluish-white have radiation peaks outside the visible range. Reddish stars must then be cooler, and white/blue stars are then hotter. To calculate the emission:

$$L_v(T_b, v) = \frac{2hv^3}{c^2 e^{INCOMPLETE}}$$

According to the Stefan-Boltzmann Law, if a substance is an ideal emitter, the total amount of radiation given off is proportional to the fourth power of its absolute temperature. This is important because it means that the amount of radiation given off is a very sensitive function of temperature; small changes in temperature can result in huge changes in radiation. The Stefan-Boltzmann law is:

$$M_b = \varsigma\varsigma AeT^4$$

...where sigma is magical, e is emissivity (1 for a black body), a T is absolute temperature.

The Planck function is:

$$E = hv = \frac{hc}{\lambda}$$

Where E is energy, h is Planck's constant in J/s or eV/s, v is frequency in Hertz, and λ is wavelength.

Wien's law can be represented as $\lambda_m = a/T$ where a is a constant. This is important because it shows that hotter bodies have shorter wavelengths.

According to the inverse square law, the amount of radiation passing through an area is inversely proportional to the square of the distance. That is, if you double the distance to the source, the radiation received will decrease by a factor of 4. This means that relatively small changes in the distance of a planet to the Sun can have big changes on the amount of energy that planet receives.

Not all incoming sunlight is absorbed by the Earth system; an albedo is involved. Albedo is just the fraction of light that that a surface reflects, where 0 is no reflection and 1 is complete reflection. The albedo of the earth varies greatly by location; the equator, for instance, has a very low albedo, whereas higher regions (closer towards the north and south poles) have much higher albedos. A bond albedo A_b is the ratio of reflected flux F_{ref} to the incident flux F_{inc}: $A_b = \frac{F_{ref}}{F_{inc}}$. Since absorbed flux equals incident flux minus reflected flux, the fluxes are related by:

$$F_{abs} = F_{inc} - F_{ref} = F_{inc} - A_b F_{inc} = (1 - A_b)F_{inc}$$

Planetary radiant exitance $M(T)$ variese with planet blackbody temperature. Incident power P_{inc} is the product of planet cross-sectional area A_c and solar irradiance S_m at Mars, for instance: $P_{inc} = A_c S_m$. The power absorbed is then…didn't have time to copy it down.

Example Determine the black body temperature of the sun given that the total power radiated (luminosity) is $3.846 \cdot 10^{26}$ W, and use a second method where the only data is that radiation wavelength at maximum intensity is 518.8 nm.

For the first method, use the Stefan-Boltzmann law to determine temperature. For the second, use Wien's displacement law with the given wavelength to derive a temperature (5778 K).

Example 2 Determine the black body temperature of a spacecraft in the orbit of Jupiter that has a bond albedo of 0.3 and no internal heat dissipation. Ignore the heat input from Jupiter.

To solve, you'll need albedo, the ratio for areas, the area of the spacecraft, the power output of the sun, and another important constant (?). With a little work, you can derive a formula to solve for temperature using the Stefan-Boltzmann law, among others, and find that the temperature is 111.7 K.

3.3 Antireflection coatings & ultraviolet degradation

Antireflection coatings are designed to reduce the amount of incident electromagnetic power lost by spacecraft. They're used on solar cells and other optical devices, and often consist of one or more thin layers of transparent material whose indices of refraction differ from the underlying substrate, the goal being to reduce certain refraction angles. According to Brewster's law, $\theta_B = \arctan(\frac{n_t}{n_i})$.

The cross sectional area of all the particles on a surface (?) combined is $dA = \varsigma nAdx$. The fraction of the beam scattered is $\frac{dE}{E} = -\frac{dA}{A} = -\varsigma ndx$ — dE is negative. Integrate as a function f energy to obtain $\ln\frac{E(x)}{E(0)}$, and finally $E(x) = E_0 e^{-\varsigma nx} = E_0 e^{-x/X}$, where $X = \frac{1}{\varsigma n}$.

4 Orbits around Earth

Newton's law of gravity states that $F = \frac{GMm}{r^2}$. If you take the derivative (gradient) of this with respect to r, you get $\frac{dF}{dr} = -\frac{2GMm}{r^3}$ — this tracks infinitesimally small changes of F in relation to infinitesimally small changes in r. Now suppose we define a $\Delta r < r$ but still bigger than dr. You can actually pull a $\frac{-2}{r}$ out of the gradient function to get $\frac{-2F}{r}$; multiply both sides by Δr to get a simple expression $\Delta F = \frac{-2F}{r}\Delta r$. Morphing this into an acceleration form will yield $\Delta g = -2g(r)\frac{\Delta r}{r}$.

Consider an Earth-Moon system with polar coordinates centered on the center of the Earth; the distance from the origin to a point mass m on the Earth's surface is a, and the angle to that location is ψ. The distance to the center of the earth is R and the distance from the moon's center to the point mass m is ρ. Potential energy is $V = -\frac{GMm}{r}$; divide both sides by m to get potential $U = -\frac{GM}{r}$, in units of J/kg (note that M and m are not related to point mass m for the exclusive purpose of presenting this formula). Thus, $-\frac{dU}{dR} = F$. The tidal potential of the moon $V^+(a,\psi)$ can be written as $U^+(a,\psi) = -\frac{GM}{\rho}$…but we're not using ρ, we're using ψ, so the potential field is actually:

$$U^+(\alpha,\psi) = -\frac{GM}{\sqrt{R^2 + a^2 - 2Ra\cos\psi}}$$

This is an ugly formula, so do some math. You can extract an R out from underneath the radical, and you can assume that a is much smaller than R (consider that a is effectively the radius of the Earth while R is the distance from the Earth to the moon). That means that you can perform a binomial series expansion:

$$U^+ = \sum_{n=0}^{\infty} (\frac{a}{R})^n P_n(\cos\psi)$$

$$U_0 = -\frac{GM}{r}$$

$$U_1 = -\frac{GM}{r}(\frac{a}{R}\cos\psi)$$

$$U_2 = -\frac{GM}{r}(\frac{a^2}{2R^2}[3\cos^2\psi - 1])$$

If you take the partial derivative (?!!) of U_2 you get the first term for the differential gravity of the moon on the Earth. This is painful and there wasn't enough time to show it here. This whole series represents the differential force experienced by the Earth; higher terms correspond to different things.

If, on a new diagram, you represent an angle from the Y axis θ as colatitude, a distance from the origin (again at the center of the Earth) as r, and a second angle λ as latitude, you get a coordinate set (r, θ, λ). Now the challenge is to find potential. Represent an infinitesimally small piece of mass as dm and the distance from it to the point (r, θ, λ) as ρ. This lets you set up an integral for potential:

$$dU = -\frac{Gdm}{\rho}$$

$$U^* = \frac{GM}{r}[\sum_{n=0}^{\infty} (\frac{a}{r})^n \sum_{m=0}^{n} (\bar{C}_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda) P_{n,m}(\cos\theta)]$$

Note that the negative is gone; this is a matter of notation.

Anyways, there's a few things to calculate involving orbits. Mean anomaly is the average orbital angular speed M in radians per second; it's a measure of the perturbation of the orbit due to a disturbing potential. The second is the argument of the periapsis ω, which is the angular position of the periapsis. In the plane of the orbit, if you draw an angle from the reference plane's intersection (or orbited body) to the location of the periapsis, you have your argument of the periapsis. The right ascending node Ω is the location where the orbital plane intersects the reference plane such that the satellite is ascending, and the right descending node is the same thing, just with the satellite descending instead. In simple systems the orbital plane and all three of these values are actually constants, but in reality they change over time; calculating that rate of change is then of some importance. Since orbital period is $T = \frac{2\pi}{\omega}$, if ω is changing, that means that orbital period must also be changing over time!

4.1 Multi-body spacecraft

Suppose a spacecraft consisting of two differently-massed pieces is in orbit around the Earth. The bigger mass is m_i (inner) and the smaller mass is m_o, since the bigger mass is on the inside orbit. The centripetal acceleration is $ma_c = m\omega^2 r$; some math yields $\omega^2 = \frac{GM}{r^3}$, which is in fact Kepler's third law. This basically means that the inner mass will travel ahead of the second mass simply because it's in a higher orbit — the two masses have different angular speeds.

Now suppose that you've linked the two masses together by some sort of tether, such that their angular speeds are synchronized. Suppose that the orbital radius of the inner mass is r_0 (note that it's *not* r_o). The force on the inner mass is $f_i = m_i\omega_i^2 r_0 = \frac{GMm_i}{r^2}$. As for the outer piece, the distance between the inner and outer pieces is Δr, so $f_o = m_o\omega_o^2(r_0 + \Delta r) = \frac{GMm_o}{r_0+\Delta r}$. Now you have the two forces, but you need the inner mass tugging on the outer mass. So, the total force $f_o' = f_o + f_i$. Doing this math gets amusingly complicated and involves yet another series expansion; working through that and retaining terms only to first order yields:

$$f = 3m_o w_i^2 \Delta r$$

That's the force that the larger piece needs to exert on the smaller piece in order to tug it along. The difference in gravitational acceleration between the two can be represented as $\Delta g = 2g(r_0)\frac{\Delta r}{r} = 2\omega_i^2 \Delta F$.

Imagine attaching the two spacecraft pieces by a bar of negligible mass; the distance from the Earth to the center of mass of the spacecraft system is r. For the piece closer to the Earth there's a slightly greater force acting on it, and a slightly smaller force acting on the more distance piece; this creates a torque. If the distance to the center of mass is ρ and the distance from the center of mass to the different pieces (extending radially outward from the Earth) is Δr. Anyways, the moment of inertia of this system (assuming equal masses of both pieces) is $I = 2m\rho^2$. Torque is $\tau = 2\Delta r \Delta F$, and with a little work you can get another formula $\tau = 3\frac{GM}{r^3} I \cos\beta \sin\beta$; the simplified version is $\tau = 3\frac{GM}{r^3} I\beta$.

5 Charged Particles & the Earth's Magnetic Field

For an electric field E and a charge q, force is $F = qE$. Change in kinetic energy is $\Delta K = -qE\Delta r$, where Δr is the distance over which the particle moved. A more specific version is $\Delta K = -q\vec{E} \cdot \Delta\vec{r}$, which effectively picks out the parts of the field that are parallel (?) to the direction of movement. Furthermore, for charged magnetic fields, $\vec{F} = q\vec{v} \times \vec{B}$ — force is perpendicular to both velocity and the direction of the magnetic field (the cross product picks out the perpendicular component). A simplified version is then $F = qvB$, which assumes velocity and magnetic field are perpendicular. These properties allow for uniform circular motion of particles in a magnetic field, called cyclotron motion. If you set magnetic force equal to the required centripetal force in $qvB = m\omega^2 r$ (where $\omega = 2\pi f$ and f is frequency), you can calculate the values of different variables (like the strength of a magnetic field) using the others. Objects with a magnetic moment will experience a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$. Also, $\vec{\tau} = \frac{d\vec{L}}{dt}$ (???). I don't know where/when this becomes relevant.

The Earth's magnetic field is $\mu = 7.9 \times 10^{22}$ Am^2. This unit becomes painfully obvious when you consider that $\mu = IA$, where I is current through a closed loop of area A. On a diagram, consider r as a distance from the center, and θ as the angle from magnetic north. The magnetic field can be defined as $\vec{B} = \frac{\mu_0}{4\pi}\frac{\mu_E}{r^3}(2\cos\hat{r} + \sin\theta\hat{\theta})$. This is not an inverse square law, it's an inverse *cube* law. For example, at a distance of about 10 Earth radii, the magnetic field is only about 30 nT in strength. Note that μ_0 is a universal magnetic constant and μ_E is the strength of the Earth's magnetic field. The radial component of a magnetic field, which points radially inward/outward, is B_r. The unit vector that points in the direction of increasing θ is $\hat{\theta}$, and the unit vector that points in the direction of increasing r is $\hat{r}$; these are the unit vectors seen in the formula. Here are some formulas about them:

$$\begin{aligned} V(r,\theta) &= ? \\ B_r &= \frac{\Delta V}{\Delta r} \\ B_\phi &= \frac{1}{r}\frac{\Delta V}{\Delta\phi} \\ B_\theta &= \frac{1}{r\sin\theta}\frac{\Delta V}{\delta\theta} \end{aligned}$$

Moving on! Electric and magnetic fields have energy density. N/m^2 is pressure and energy density is J/m^3, but since a joule is just an Nm, $\frac{N}{m^2} \cdot \frac{m}{m} = \frac{J}{m^3}$. This is a useful explanation for why magnetic fields can exert forces on particles from the incoming solar wind. If a planet doesn't have a magnetic field to break the wind, the solar wind then impacts upon the atmosphere instead. This results in much deeper penetration of solar wind into the atmosphere, but it means that particles in the atmosphere can sometimes achieve escape velocity due to these impacts with solar particles, stripping away atmospheres over long periods of time. For at least mars, the closest these particles can get to the surface is called the ionosphere standoff.

Anyways, for a shell defined by $drA = 4\pi r^2 dr$ (a volume) and incoming proton count/density (not sure which) n_i, the number of protons inside the shell is $N = n4\pi r^2 dr$. If you expand the shell to be r_s, you get $N = n_f 4\pi r_f^2 dr$, the density decreases. If you set these equal to each other, you get $n_f = n_i(\frac{r_i}{r_f})^2$; if the two shells don't have equal proton flow rates, you don't have a steady state, and you just get protons piling up somewhere. So, $\frac{dN}{dt} = 4\pi r^2 n \frac{dr}{dt}$, and $N = 4\pi r_i^2 nv$ (with a corresponding formula for final values). Final velocity is given by $v_f = \sqrt{v_i^2 + 2GM_s(\frac{1}{r_f - r_i})}$. Doing more magic math yields $n_f = n_i \frac{v_i}{v_f}(\frac{r_i^2}{r_f})$; just make sure to do the velocity calculation first. For one final refinement, solar wind is gas, with a temperature! This isn't covered yet, but taking temperature into account leads to a somewhat more complicated formula.

Suppose you have a material of depth x; the rate at which a beam passing through it loses energy is $S = \frac{-dE}{dx}$, which is also approximately equal to the rate at which the beam deposits energy, $LET = \frac{dE_d}{dx}$ (linear energy transfer). The rate R at which energy is lost varies with an exponential decay curve. The average penetration distance is then $R = S\frac{dE}{LET}$ (?!!! this makes no sense). The units of LET_ρ (LET per unit density, kg/m^3) are then energy per length density, which after a little rearranging will end up being something like Jcm^2/g (or any other set of length and mass units).

5.1 Bethe-Bloch

If you take the derivative of $E(x) = e^{-x/X}$ (where $x = \frac{1}{n\sigma}$), you get $\frac{dE}{dx} = -nrE(x)$. Stopping power is $S = \frac{-dE}{dx} = n\sigma E_{e,o}$ where $E_{e,o} = m_e c^2$ (rest mass of an electron); this is the Bethe-Bloch formula. It has about the same form as the other formulas referenced here.

5.2 Organic radiation absorption

When radiation impacts an atom, it usually ionizes it. If this atom is part of a molecule, the atom will usually lose its position in that molecule and the molecule will be broken apart. In an organic context, this causes tissue damage. Radiation absorption is measured in J/kg. Examples of radiation include X-rays, gamma rays, alpha particles, beta particles, electrons, protons, neutrons, and muons. There's a dose threshold for different organs before they stop functioning (this is why our bodies don't immediately shutdown despite that background radiation is a part of life on Earth). However, stochastic effects can still occur, wherein your chance of cancer later in life increases; just because the radiation doesn't kill in the short term, it doesn't mean that it can't kill years later. Ultimately for these kinds of effects, there *is* no dose threshold.

There are a *lot* of units for radiation doses. The Roentgen R is in units of coulombs per kilogram (ionization of the air due to radiation). One Roentgen is equal to $2.6 \cdot 10^{-4} C/kg$, because numbers and units are stupid. The Gray Gy is 1 J/kg, and is referred to as absorbed dose. This is different from the Sievert Sv, which is also 1 J/kg, which is "equivalent dose", which has to do with how different types of radiation are more or less effective — in formulas it appears as H_t, because *REASONS!*. Effective dose is written in Sieverts (1 Sv), and is (again) in units of J/kg. Yet another unit is the rad, which is 0.01 Gy. The rem (radiation equivalent man…for some reason) is 0.01 Sv. Relevant quote: "Protons have a mind of their own!" One Roentgen is $9.3 \cdot 10^-3$ Gy; 1 Gy = 10 rad, and 1 Sievert is 100 rem.

Formula time. $H_{T,R}$ (t for type of tissue, R for radiation type) is $W_R D_{T,R}$ — W_R is a weight factor, $D_{T,R}$ is the dose. If you sum all of these together, you get $H_T = \sum\limits_R W_R D_{T,R}$ as the overall dose; a similar operation can be performed over tissue type instead. Total charge deposited is $Q = \frac{eE_{dep}}{<I_1>}$ where I_1 is average ionization energy (e is the charge of an electron); it's useful for characterizing radiation damage to electronics. Energy deposited is $E_{dep} = s\rho L$, where s is path length, ρ is material density (kg/m^3), and L is linear energy transfer (rate at which energy is deposited per meter of penetration, J/m).

5.3 Neutral Atmospheres

5.4 Atmospheric pressure with constant temperature

If you have a volume of air, a certain mass will be contained; if you take the total mass and divide it by the volume, you get the density ρ in kg/m^3. If you multiply the average mass of molecules in the air by the total number of molecules, you also get the mass. You can do the same thing with mass and density to get $dm = \rho A dh$.

Imagine taking an infinitely thin slice of this air column with area A and height dh; then, $V = Adh$. Pressure on the area is p in N/m^2 (pascals), so pA is just force in Newtons. The pressure on the top will be greater than the pressure on the bottom (?), so we'll call that $p + dp$; multiply that by A to get the pressure on the top. Finally, the force of gravity is dmg. Since the air ultimately isn't going anywhere, these forces should be in equilibrium: $-(p+dp)A - dmg + pA = 0$. Rearranging and simplifying (area ultimately cancels out) yields $dp = -\rho g dh$.

We know from thermodynamics that $p = \frac{N}{V}kT$; multiply by m/m to get $p = \frac{\rho}{m}kt$. Solving for ρ yields $\rho = \frac{pm}{kT}$, which can be substituted in for the ρ term in the earlier atmosphere pressure equation: $dp = -\frac{pm}{kT}gdh$ — just remember that p is a function of height. For now we'll treat g as a constant (suppose we use a range of elevations such that g doesn't change significantly) and T as a constant as well. Anyways, by separation of variables you get $\frac{dp}{p} = -\frac{mg}{kT}dh$. Integrate: $\int\limits_{p_0}^{p(h)} \frac{dp}{p} = \int\limits_{0}^{h} -\frac{mg}{kT}dh$. Solving results in $\log\frac{p}{p_0} = -\frac{mg}{kT}$, and rearranging yields the final formula $p(h) = p_0 e^{-\frac{mg}{kT}h}$. Note that the k in these formulas is *not* the R gas constant 8314 J/kmolK, it is in fact the Boltzmann constant.

You can also get height as $H = \frac{kT}{mg}$ or $H = \frac{RT}{Mg}$ (where M is the number of moles...?). At sea level, $\rho = 1.217\frac{kg}{m^3}$ (h=0). If you go up 26 km — 3H — or well into the ozone layer, you get $\rho = .05\rho_0$. Going up 78 km — 9H — brings you to the turbopause[1] and to about $1.2 \cdot 10^{-4}p_0$. Finally, at h= 12H = 105 km, you get to the Karman line, pressure is just $6 \cdot 10^{-6}p_0$.

5.5 Atmospheric pressure with variable temperature

Unfortunately, temperature in an atmosphere is not constant, it's $T(h) = T_0 + Lh$, where L is the temperature gradient (how much the temperature changes over a distance). At least, for now it is, since this is just a simple first-order linear variation. If you plug in this value for T in the original $dp = -\frac{pm}{kT}gdh$ and solve (derivation not shown here), you'll get $p = p_0(\frac{T(h)}{T_0})^{-\frac{mg}{kL}}$. In reality the atmosphere's temperature isn't at all linear, but this is a good way of approximating it.

5.6 Drag on orbits

Spacecraft have total mechanical energy $E = -\frac{\mu m}{2a}$ ($\mu = G(M+m)$), and the force on them is $F_d = \frac{1}{2}\rho AC_d n^2$ (I have no idea what any of these variables mean). If you take the derivative of E over time, you get $P = \frac{\mu}{2a^2}$, which is in units of watts since it's a measure of energy over time. You can also say that $P = \vec{F} \cdot \vec{v}$, but if the two are in the opposite direction you get $P = -Fv$. Sorry if these notes seem disjointed! If you work some magic, you can find the rate of change of the orbit semimajor axis a, which turns out to be $\dot{a} = -\rho(\frac{C_d A}{m})(\frac{a}{r})^{3/2}\sqrt{\mu a}$. If you use one of Kepler's law, which says that orbital period $\tau = 2\pi\sqrt{\frac{a^3}{m}}$ and take the derivative of that, you can find the time rate of change of the orbital period: $\dot{\tau} = -3\pi\rho a\frac{C_d A}{m}(\frac{a}{r})^{3/2}$. Also useful to know is that the reciprocal of one of these terms is $\beta = \frac{m}{C_d A}$ in units of kg/m^2; this is called the ballistic coefficient, and it's the measure of a spacecraft's ability to ram itself through an atmosphere.

5.7 Spacecraft Erosion

The very far upper reaches of the atmosphere — above the turbopause — are divided into strata, one of which contains oxygen. Due to UV bombardment, this oxygen is primarily atomic oxygen, which can and *will* happily bond to things it comes into contact with, including spacecraft. Over time, this causes erosion, quantified by erosion yield E.

Suppose you have a spacecraft with cross-sectional area A and length L; then $V = LA$. The orbital velocity of the spacecraft is $v = \sqrt{\frac{GM_E}{R_e+h}}$. You can also write volume in terms of velocity (somehow), where $V = LA = v + A$. The number of atoms encountered are then $N = nV = nv + A$, where n is the number density of atoms. Define a new quantity F for fluence, the number of oxygen atoms per unit area: $f = \frac{N}{A}$. It can also be written as $f = nvt$, and $N = FA$.

Now, whenever an oxygen atoms impacts the spacecraft, some material is lost in the form of volume Vol, which is measured per oxygen atom. The volume eroded can be written as Ad where A is the area of a single erosion area and d is the erosion depth. Erosion yield can the be written as $E = \frac{Ad}{N}$. You can also

[1]The turbopause is area where the atmosphere is so thin that turbulent mixing of the air stops and air molecules start to stratify

find erosion depth by doing $d = Ef$. If you want to find erosion depth, first compute v, use $f = nvt$ (?!!), and look up erosion yield in a table (since it varies by material).

5.8 Light through the atmosphere

The refractive index of light is $n = \frac{c}{v}$, where v is the speed of light through the material. The distance travelled by light is $ds = cdt_c$, and $ds = vdt_v$. Solving both yields $dt_c = \frac{ds}{c}$ and $dt_v = \frac{ds}{v}$ (these t values are travel time at different velocities, given by the subscript). Doing more solving and calculus yields $(\frac{1}{v} - \frac{1}{c})ds = dt$. v is a function of s though (where s is the height (?) in the atmosphere), but for keeping things simple assume that v is a constant. You can write the path delay as $dr = cdt = (n-1)ds$. Anyways, $\Delta r = \int (n(s) - 1)ds$.

5.9 Ions

Ionization can be given as $q = q_{\max} e^{1-z-e^{-z}}$, where $z = \frac{h-h_m}{H}$ (height above the earth minus the height of max ionization, all over the height of the atmosphere). The potential of a spherical body with a charge is given by $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ (the first term is the Couluomb constant).

If you have a plasma composed of positively and negatively charged particles and an electromagnetic wave passes through them, the particles will experience a sinusoidal motion, defined by $a = \frac{F}{m} = -\frac{k}{m}x$, and frequency is $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$. The plasma frequency pretty much comes down to the disturbance of the electrons, and is given by $f_{pe} = \frac{1}{2\pi}\sqrt{\frac{n_e}{\varepsilon_0}\frac{e^2}{m_e}}$. You can also use $a = \frac{F}{m} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m}\frac{1}{r^2}$; these equations are both found by setting up a second order differential equation and solving, which supposedly is easy. Note that n_e is the number density of electrons.[2]

If you have a lot of waves travelling together at the same time, they form a wave packet, with amplitude high in the center and diminishing at the edges. The packet does not travel at the speed of light, instead at a group velocity defined by $v_g = c\sqrt{1 - \frac{f_{pe}^2}{f}}$. If you take group speed and multiply it by a tiny distance dt, you get ds, the distance travelled ($v_g dt = ds$). Since number density of electrons is a function of height s, f_{pe} is a function of s, and since v_g depends on that, v_g is also a function of s, so the integral ends up being $t = \int dt = \int \frac{ds}{v_g(s)}$. Doing a lot of calculus yields time delay due to the ionosphere as $\Delta t_g = \frac{e^2}{8\pi\varepsilon_0 m_e c f^2}\int n_e ds$ — the integral portion is TEC, or Total Electron Content in units of number of electrons per cross sectional area.

6 Spacecraft

6.1 Time in Earth's shadow

A spacecraft will experience time in Earth's shadow when it's in an equatorial orbit, and will spend progressively less time as its orbital height increases. The angular velocity of the spacecraft is $\omega = \sqrt{\frac{GM}{r^3}}$, and there's a relation to time in shadow given by $\frac{\theta}{\omega} = t$, where θ is the angle formed by the intersections of lines from each edge of the earth with the spacecraft's orbit. Things get more complicated when you have a very high, very slow orbit, because the motion of the Earth around the sun will have an effect on the amount of time the spacecraft spends in shadow. That particular effect won't be covered here, though.

6.2 Contamination

The interior and exterior of spacecraft can be contaminated, especially during the manufacturing process — this contamination rate is measured in units of kg/ms. Quartz crystals can also be affected by contamination, with material deposits affecting their frequency. If you have a sensitivity S_t, a quartz crystal area, and a time rate of change for frequency, you get $\dot{m}_c = S_t A\dot{f}$ — the rate at which stuff in your spacecraft accumulates.

[2] Relevant lecture quote: "If you pack more squirrels in the room, their tails twitch faster!"

6.3 BDRF

The directional reflectance distribution function defines how light reflects off the surface of a spacecraft. A mirror reflection is when lights is mirrored perfectly across the axis perpendicular to the impact location. For glossy materials, the reflection is not perfect, so reflected light goes in more or less the same direction, but it's a little scattered. For the purposes of this function, ϕ is the angle on a surface (the XY plane?), and θ_i and θ_r are the incident and reflectance angles as measured from a line perpendicular to the surface. ϕ_r is the angle on the surface of the reflection. The reflection can also be biased based on the wavelength of light. Anyways:

$$f_r(\theta_i, \phi_i, \theta_r, \phi_r, \lambda) = \frac{\Delta L}{\Delta E} = \frac{dL}{dE}$$

It's a measure of the ratio of reflected light to incident light, basically.

6.4 Debris

The amount of particles striking a spacecraft is measured in units of number of particles per square meter per year, per square radian: $\frac{\#}{m^2 yr(rad)^2}$ (square radian accounts for particles coming from some area of the sky…?). Flux is then measured in $\frac{\#}{m^2 s}$. In general, as the mass of the impacting particle increases, the rate at which particles that size decreases (exponentially?), by virtue of there being far fewer big particles than small particles.

Consider a simple model of a boulder of mass M made of N little particles of mass m, so $M = Nm$. If you divide by sides by volume V, you get $\frac{M}{V} = \frac{N}{V}m = nm$, where n is the number density of particles. Given that $m = \frac{4}{3}\pi(\frac{s}{2})^3\rho$, mass is roughly proportional to the cube of size. A little extra math yields that $m \propto \frac{1}{n}$, and $f \propto n$. From somewhere else comes the formula $3 \ln s \approx \ln f$ (somehow). This simple model roughly explains why more massive particles are far more rare than tiny dust-sized ones.

Another model of the same thing is the Grun model:

$$\begin{aligned} F_c(m,h) &= F(m)G_e(h)\Psi(h) \\ F(m) &= F_1(m) + F_2(m) + F_3(m) \end{aligned}$$

…as a function of particle mass and height above the Earth, where $\Psi(h)$ is the Earth's shielding factor, and $G_e(h)$ is the gravitational focusing factor. The final integrated function F_c (which is for masses *greater than* m) is in units of flux (Grun flux? Not sure).

You can also use these models to predict the collision rate of particles, using a Poisson distribution:

$$P(k, \lambda t) = \frac{1}{k!}(\lambda t)^k e^{-\lambda k}$$

This gives the probability that there will be k collisions in a time t under average rate of events λ per unit time. This could be any unit time, even years, since these collision/"strike" events aren't strictly common. For instance, the probability of zero events is $e^{-\lambda t}$. Subtracting this probability from 1 will get you the probability of being struck by any number of events in the same time period.

6.5 Impacts

For a target "as thick as you need it to be", penetration depth is $P_\infty = 5.24 d_p^{19/18} BHN^{-0.25}\sqrt{\frac{\rho_p}{\rho_t}}(v\cos\theta)^\gamma$, where $.67 < \gamma < .98$. Using this formula, you can solve for the diameter of the projectile d_p — note that ρ_p and ρ_t are the densities of the projectile and target, respectively. Critical diameter (on the threshold of perforation) is $d_c = \frac{?}{K_f K_1 \rho_p^\beta \rho_t^k V^\gamma (\cos\theta)^\eta}^{\frac{1}{\gamma}}$.

You can think about penetration as a projectile pushing out a plug the size of itself from the material it impacts. This creates shearing force, which is the main resistive force in this simplified model. Shearing force is $f_{sh}\frac{h\pi d}{2}$, where h is the height of the plug, d is the diameter, and all of it is divided by 2 because it's the average length of the plug — this formula is for *average* shear force. Anyways, using conservation of momentum you can work out that $-m\Delta v =< \Delta F > \Delta t = f_{sh}\frac{\pi dh}{2}\Delta t$ (?????). Average speed is $< v >= \frac{v}{2}$. Mass of the projectile is given by $\frac{4}{3}\pi(\frac{d}{2})^3\rho$.

When you work all of this out, you get $h = kd\sqrt{\rho}v$ (k is the shear force constant). This is an approximation of impact depth, but comparing against the original d_p formula from above, this model is reasonably accurate and much simpler to work with.

7 Thermal Control

Thermal control is critical to spacecraft. The range of astrophysical temperatures is extremely large, with 6000K as just the surface temperature of the sun. The range in which humans can live, however, is much smaller, ranging from 313k to a bit lower, while our equipment can survive at temperatures from 393K to 173K. The steady state constant temperature of a system is T. Energy in is $\dot{q}_{in} = (1 - A_b)S_e$ (where $S_e = \frac{\phi}{4\pi r_e^2}$, r_e is the radius of the Earth's orbit and ϕ is the luminosity (?) of the sun). Energy out is $\dot{q}_{out} = M(T) = \varepsilon\sigma_{SB}T^4$, where ε is emittance (1 for perfect black bodies) and σ_{SB} is the Stefan-Boltzman constant.

Consider a spacecraft wall with a temperature on one side (the outside) and a temperature on the other side (the inside). The difference between the two is ΔT, which is about proportional to $\dot{q}$ (energy flow?). Define a distance scale S, where proceeding from the outside in is an increase in S. $\dot{q} = \frac{dq}{dt}$ in units of W/m^2, and $\frac{dq}{dt} = -h_c\frac{dT}{ds}$ — h_c is the convective transfer coefficient, and it's different for different materials. Copper, for instance, has a very high convective transfer coefficient, while insulators have a very low one. This equation can be rewritten as $\Delta T = \int dT = \int \frac{\dot{q}}{h_c}ds$, which leads to $\dot{q} = -h_c\frac{\Delta T}{\Delta S}$.

There's a Kirkhoff law that says at a particular wavelength, absorbance α is $\alpha_\lambda = \varepsilon_\lambda$, i.e. absorbance is equal to emissivity at a certain wavelength. This is useful in calculating the thermodynamics of a Dewar flask (double walled container with vacuum in the middle): $\dot{q}A = \dot{Q} = \varepsilon_1\varepsilon_2\sigma_{SB}A_1F_{1\to2}(T_1^4 - T_2^4)$ where $F_{1\to2}$ is the view factor and anything with a subscript 1 is relates to the inner side of the outer wall, and vice versa for subscript 2. The view factor is the fraction of light that is captured by the surfaces involved. In a Dewar flask, one surface is radiating into another that is completely enclosed inside it, so the view factor is almost exactly 1.

In the same Dewar flask there's a temperature difference across each surface, with radiated power in between them — power through each wall must be equal and so is $\dot{q}_{cond}$. Each wall has its own ΔT; the model can be simplified by assuming that the two ΔT's are equal.

$\int_a^b \sum_0^{666} x^2 + \sqrt{\varepsilon}$