Calc IV (MA1024) Notes

Christopher Myers

December 13, 2016

Contents

Part I
Differentiation

1 Functions of Several Variables

In this course, derivatives of functions of multiple variables will be calculated, as opposed to derivatives of functions of one variable. For example, $f(x, y)$ might be used instead of just $f(x)$. An example of such a function is below:

$$z = \frac{x^2}{9} + \frac{y^2}{16} = f(x, y)$$

The above's domain is the XY plane, and has a range of all real numbers. For example, $f(1, 4) = \frac{10}{9}$.

1.1 Graphing

In a function of one variable, the resultant graph is two dimensional. In a function of multiple variables, different dimensions are added on. In the previous example, Z would be given as height above the XY plane, forming a 3D graph that looks like a sort of dome (in reality a *parabaloid*). All functions of multiple variables create surfaces in some form or another.

A useful technique is using a *slice*. To take a slice, set one variable to be constant, in effect creating a 2D graph. That graph is just a cross section of some part of the graph.

Level curves are produced by taking a slice along the XY plane of some 3D function. For example, the level curves produced by a parboloid are a series of concentric ellipses.

1.2 Contours

Contours can be created by taking slices along planes at various places by setting a single variable (e.g. Z) equal to a constant. For example, the contours of a parabolid produced by slicing along the XY plane (i.e. setting Z to a constant) are concentric ellipses. Note that other kinds of contours can be created by setting other variables as constant, producing vertical slices instead. (In the above parabaloid example, paraboloid slices create parabolas.)

2 Partial Derivatives

Partial derivatives can be found by treating the variable not being differentiated with respect to as fixed/constant.

The notation is given by showing the derivative of the function with respect to the variable being varied. For example, the partial derivative of f with respect to X is $\frac{\delta f}{\delta y}$. If you're taking the derivative of f at $(3, 16)$, then you get $\frac{\delta f}{\delta y}(3, 16) = 2$.

Note that there is an alternative, more recent notation where the variable differentiated with respect to is put in a subscript, like so: $f_y(3, 16) = 2$.

Example 1 Find the partial derivatives for $x^3y^3 + 2x^3 + 3y^2 + \sin(xy)$.

$$\frac{\delta f}{\delta y} = 4x^3y^3 + 6y + \cos(xy)x$$
$$\frac{\delta f}{\delta x} = 3x^2y^4 + 6x^2 + \cos(xy)y$$

Note that the fixed variable *must* be brought out front of the cosine due to the chain rule.

2.1 Second Derivatives

It's easy enough to find second derivatives, just repeat the same process. The notation is similar: $\frac{\delta^2 f}{\delta y^2}$ for deriving a function f with respect to y. In the alternative notation style, make the subscript a double subscript (e.g. f_{yy}).

2.2 Mixed Partial Derivatives

It is possible to differentiate with respect to x and *then* y, creating a "mixed partial derivative", given by the notation f_{yx}. This is also easy to compute, just switch out the variable being differentiated. Note that this is commutative: the order shown in the notation doesn't matter, doing x then y produces the same results as y then x. Or, $f_{xy} = f_{yx}$.

2.3 Interpretation of Partial Derivatives

In partial derivatives, only one variable varies at a time, no matter how many variables there are. This means that it's possible to create interpretations of partial derivatives that yield useful results.

Example 1 - Temperature Suppose there is a function $T(x, y, t)$ that returns temperature (F) at location (x, y) at time t (minutes), and that $\frac{\delta T}{\delta t}(1, 7, 2) = 0.5$. In this context, the value 0.5 means that the rate of temperature change at $t = 2$ at point $(1, 7)$ is 0.5 (half a degree per minute).

Now suppose that $\frac{\delta T}{\delta x}(1, 7, 2) = -0.7$. Since the variable being differentiated is now x, it means that at point $(1, 7)$ and at time $t = 2$, temperature is decreasing by 0.7 degrees.

This same thought process can be applied to any other variable in a function that can be differentiated. Partial derivatives will give you rate of variance *with respect to a single variable.*

Example 2 - Vibrations Suppose there is a function $u(x, t)$ which yields the displacement in inches at time t for some vibration. If you were to take $\frac{\delta u}{\delta t}$, the result would be the rate of change of displacement with respect to change…otherwise known as velocity. In the context of a vibration, it's how fast the vibrating string is moving in a given location. Since the derivative of speed is acceleration, taking the second derivative will yield acceleration.

3 Directional Derivation

Partial derivatives are useful but they are not general enough to account for all cases where a slope might be needed. The slope needs to be computed in any direction, not just x or y.

The total change in height is defined as follows:

$$\Delta f = \frac{\delta f}{\delta x}\Delta x + \frac{\delta f}{\delta y}\Delta y$$

The change in height is not sufficient, it needs to be divided by distance to be effective:

$$\begin{aligned}\frac{\Delta f}{\Delta s} &= \frac{\Delta f}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \\ &= \frac{\delta f}{\delta x}\frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \frac{\delta f}{\delta y}\frac{\delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}\end{aligned}$$

Now give this a vector interpretation. Define a vector u as $u = \frac{\Delta x}{\sqrt{(\Delta x)^2+(\Delta y)^2}} i + \frac{\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} j$. [1] Here, u is a unit vector pointing from one point to the other, but it's only one unit long.

Also define a new vector function f^Δ as $\frac{\delta f}{\delta x} i + \frac{\delta f}{\delta y} j$ This is the gradient vector of the original scalar function f [2]. The final result of all of this is as follows:

[1] Note that u is a unit vector with length 1.

[2] Usually this would be represented as a sort of $f\Delta$ where the Δ is upside down, but I don't know how to type that out.

$$\frac{\delta f}{\Delta s} = f^{\Delta} \cdot \hat{u}$$

So, to calculate the rate of change, take the dot product of the gradient vector and the unit vector $\hat{u}$.

Example 1 $f(x, y) = x^2 y$. P_1 is at (2, 1) and $\hat{u} = \frac{\hat{i}}{\sqrt{5}} + \frac{2\hat{j}}{\sqrt{5}}$. What is the rate of change in the $\hat{u}$ direction?

To solve, get the gradient vector, $2xy\hat{i} + x^2\hat{j}$, by taking partial derivatives and adding them. The gradient vector at (2, 1) would thus be $4i + 4j$. Now take the dot product...

$$\begin{aligned}\frac{\Delta f}{\Delta s} &= 4\hat{i} + 4\hat{j} \cdot \frac{\hat{i} + 2\hat{j}}{\sqrt{5}} \\ &= \frac{12}{\sqrt{5}}\end{aligned}$$

This means that if you were to go one unit in the direction of $\hat{u}$, the height will change by about $\frac{12}{\sqrt{5}}$.

Example 2 $f(x, y) = x^2y^2 + 4xy + 1$. What is the rate of of f at (3,1) in the direction of $\sqrt{3}\hat{i} + \hat{j}$ direction?

$$\begin{aligned}\frac{\delta f}{\delta x} &= 2xy^2 + 4y \\ \frac{\delta f}{\delta y} &= 2x^2y + 4x \\ \frac{\Delta f}{\Delta s} &= (2xy^2 + 4y)\hat{i} + (2x^2y + 4x)\hat{j} \cdot \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} \\ &= 10\hat{i} + 30\hat{j} \cdot \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} \\ &= 5\sqrt{3} + 15 \\ &\approx 23.7\end{aligned}$$

Don't forget to normalize u to $\hat{u}$!

3.1 Notation

The notation for the directional derivative of f in the direction of $\hat{u}$ is $D\hat{u}f$.

3.2 Highest Rate of Increase

According to vector basics, $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$. This can be applied to directional derivatives:

$$\begin{aligned}D\hat{u}f &= f^{\Delta} \cdot \hat{u} \\ &= |\vec{f}| \cos\theta\end{aligned}$$

For the greatest increase in the function, θ should equal zero. In other words, to increase the function the most, move in the f^{Δ} direction – gradients give you the direction of maximum increase.

You can also determine that highest rate of increase, which will simply be the magnitude of the gradient vector ($|f^{\Delta}|$) For example, in the previous Example 2, the highest gradient would be $10\sqrt{10}$, or about 31.6. There is no other direction that gives any higher gradient.

If you were to graph such a directional vector alongside a series of contour lines, you would find that the gradient vector would be orthogonal (perpendicular) to the level curves. So, the f^{Δ} vector will always be perpendular to the level curve through the associated point P on the curve.

Example 3 $T(x, y) = x^2 + 4y^2 + 20 = 0$ at (x, y). If you start at (4, 1) and wish get warmer fastest, what direction should you move in?

$$f^{\Delta} = 2x\hat{\imath} + 8y\hat{\jmath}$$
$$f^{\Delta}(4, 1) = 8\hat{\imath} + 8\hat{\jmath}$$

So head 45^o. The gradient in this direction is thus $8\sqrt{2} \approx 11.3$-

Example 4 $f(3, 6) = 10, f^{\Delta}(3, 6) = 5\hat{\imath} + 12\hat{\jmath}$. 1) What information can be gotten from this, 2), what is $D\hat{u}f(3, 6)$ if $\hat{u} = 4\hat{\imath} - 3\hat{\jmath}$?, and 3) what can be said about the level curve through (3, 6)?

1. You can determine the slope in any given direction as well as the direction to head in for the maximum slope.
2. $D\hat{u}f(3, 6) = 5\hat{\imath} + 12\hat{\jmath} \cdot \frac{4}{5}\hat{\imath} - \frac{3}{5}\hat{\jmath} = -3.2$
3. The unit vector through (3,6) must be somehow perpendicular to the gradient vector.

4 Tangents

4.1 Tangent Lines

There are two types of tangent *lines*, geometrical and algebraic. Geometrical tangent lines are simple enough - choose a point on a curve and draw a line tangent to the curve through it. Algebraic is where, instead, you take the derivative of the function and then graph a line through that point using $y = f(x_0) - f'(x_0)(x - x_0)$.

4.2 Tangent Planes

If a line is tangent to a 2D curve, a plane must thus be the tangent to a 3D curve. Hence, a tangent *plane* is a plane that is tangent to a given point on the 3D curve.

Planes can be defined by three points just as lines can be defined by two, by following the formula $\vec{N} \cdot (x - P_0) = 0$ for a normal vector $\vec{N}$ and points x and P_0. Planes are essentially just functions of two variables, like $z = f(x, y)$.

To get a tangent line, start by fixing variables and getting parametric equations...

$$\vec{r}(y) = (x, y, f(x, y))$$
$$\vec{r}'(y) = (0, 1, \frac{\delta f}{\delta y}$$
$$\vec{s}(x) = (x, y, f(x, y))$$
$$\vec{s}'(x) = (1, 0, \frac{\delta f}{\delta x})$$

Now there should be two tangent vectors. To get a normal vector, take the cross product of the two:

$$\vec{N} = \vec{s}' \times \vec{r}' = -\frac{\delta f}{\delta x}\hat{\imath} - \frac{\delta f}{\delta y}\hat{\jmath} + \hat{k}$$

Example 1 Referring to the last "Example 4", get the tangent plane at (3, 6, 10) with $f^{\Delta} = 5\hat{\imath} + 12\hat{\jmath}$.

$$\begin{aligned}
\vec{N} &= -5\hat{\imath} - 12\hat{\jmath} + \hat{k} \\
0 &= (-5\hat{\imath} - 12\hat{\jmath} + \hat{k}) \cdot ((x-3)\hat{\imath} + (y-6)\hat{\jmath} + (z-10)\hat{k}) \\
&= -5(x-3) - 12(y-6) + (z-10) \\
z &= 10 + 5(x-3) + 12(y-6)
\end{aligned}$$

Note that the last equation is the 3D analog of a tangent *line* in elementary calculus.

5 Surfaces

Functions of three variables must necessarily be four dimensional, making these extremely difficult if not impossible to graph. In such functions of three variables, the analog of a level curve is a level *surface*, a surface where the value at a given point of the function remains constant. These are formed by setting one of the three variables to a constant.

5.1 Gradient

The gradient is still computable using $f^{\Delta} = \frac{\delta f}{\delta x}\hat{\imath} + \frac{\delta f}{\delta y}\hat{\jmath} + \frac{\delta f}{\delta z}\hat{k}$. The resultant vector is now orthogonal to the level surface.

Example 1 An ellipsoid can be created by the formula $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = d^2$.

Example 2 What does the following function look like as a level surface?

$$f(x, y, z) = \frac{1}{\sqrt{(x-2)^2 + (y-4)^2 + z^2}} = 10$$

Answer: it looks like a sphere centered at (2, 4, 0) with a radius of $\frac{1}{10}$ This means that for the function $f(x, y, z)$, every point on that sphere has a value of 10.

Example 3 What does the shape produced by $x^2 + y^2 - z^2$ look like?

If you set Z to zero, circles are created on the XY plane If x is set to zero, hyperbolas are created on the ZY plane. If you set Y to zero, more hyperbolas are created. The result is a shape with circular cross sections looking down on it and hyperbolas from the other perspectives - a "hyperboloid of one sheet" (think of a reactor cooling tower).

Example 4 What does the shape produced by $-x^2 - y^2 + z^2 = 1$ look like?

If you take horizontal slices along the YZ plane, you get circles provided that the absolute value of constant k (in $x^2 + y^2 = k^2 - 1$) is greater than 1. If you set X to zero and take slices along the YZ plane, the result is $-y^2 + z^2 = 1$, producing hyberbolas centered on the Z axis (axis determined by figuring out intercepts). If you set y to zero, more hyperbolas are created.

The final result is another kind of hyperboloid, but this time with a gap in the center. These are called "hyperboloids of two sheets".

Example 5 What does the shape produced by $z = x^2 - y^2$ look like?

Taking horizontal slices produces $k = x^2 - y^2$. If sliced above the XY plane (i.e. k > 0) a hyperbola will be produced on X axis. If k < 0, hyperbolas will now be centered on the y axis. Stranger, if $k = 0$, $x = y$ *and* $x = -y$.

Setting X to zero produces $z = -y^2$, or a simple parabola opening down. If y is set to zero instead, $z = x^2$, producing a parabola pointing *up* instead.

The final result is a "saddle surface" that cannot be easily graphed by a human. It is of significance in calculus because from one side, the center is a maximum, but from the other, it is a minimum. This center point (at $(0, 0, 0)$) is called a *saddle point*.

All of these surfaces are called quadric surfaces.

6 Local Extrema of $f(x, y)$

6.1 Single-Variable Review

For $y = f(x)$, take the derivative and find its zeros to yield the local mins/maxes – i.e. solve for x in $f'(x) = 0$. Doing this doesn't guarantee actual mins/maxes (it instead generates candidates), so you need to use a test on each candidate:

1. First derivative test - test if the derivative's sign changes across the candidate. Positve to negative means a maximum, negative to positive means a minimum, and no change means that the candidate is eliminated.

2. Second derivative test - verify that the second derivative at the candidate point is non-zero. If the second derivative is positive, the candidate is a minimum, and vice versa. Otherwise, no information is gained.

6.2 Multivariable Local Extrema

In 3D, critical points are where the gradient is zero - i.e. $f^{\Delta} = \vec{0} = 0i + 0j$. The leaves two equations and two unknowns (for each partial derivative). Unfortunately, there is no first derivative test as there are many possible directions to approach any given point from. The second derivative test, however, is more interesting – local minimums and maximums *can* be generated, but so can saddle points.

Saddle points are analagous to inflection point (?) in single-variable equations such as $y = x^3$ at $x = 0$. Saddle points may look like a minimum or a maximum depending on how you look at them, but they are *not* maximums or minimums.

6.2.1 Second Derivative Test

Assume there is a critical point $P(x, y)$ that is already known. To conduct the test, find all the second derivatives: f_{xx}, f_{yy}, and f_{xy}. Put the results of these in a matrix with the pure partial derivatives on the top-left:bottom-right diagonal and the mixed derivatives elsewhere, then take the determinate of that matrix:

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2$$

The result of this should be a single number which can be interpreted using the following:

1. $D > 0$ – if $f_{xx} > 0$, the point is a minimum. If $f_{xx} < 0$, the point is a minimum. The case where $f_{xx} = 0$ is not possible as it is inconsistent with $D > 0$.

2. $D < 0$ – The point is a saddle point.

3. $D = 0$ – Test inconclusive.

Example 1 $f(x, y) = x^3 + y^3 - 12xy$

$$f_x = 0 = 3x^2 - 12y$$
$$f_y = 0 = 3y^2 - 12x$$

To find critical points, solve for where those equations intersect. The more interesting intersection is at $(4,4)$, so that point shall be examined.

$$\begin{aligned} f_{xx} &= 6x \\ f_{yy} &= 6y \\ f_{xy} &= -12 \\ D &= 36xy - 144 \\ D(4,4) &= 576 - 144 \\ &= 432 \end{aligned}$$

Since D is positive, something good of this will come. To further examine this, look at f_{xx}, which here is 24. This means that $(4,4)$ is a minimum.

Now repeat the process for the other critical point at $(0,0)$. Here, $D = -144$. Since $D < 0$, a saddle point has been found.

Example 2 $f(x,y) = 3xy^2 - x^3$

$$\begin{aligned} f_x &= 3y^2 - 3x^2 = 0 \\ f_y &= 6xy \end{aligned}$$

The above two equations intersect at only the origin, so the origin must be the only critical point.

$$\begin{aligned} f_{xx} &= -6x \\ f_{yy} &= 6x \\ f_{xy} &= 6y \\ D &= 0 \end{aligned}$$

Since $D = 0$, the second derivative test has failed and no useful information has been discovered.

7 Regression Analysis

7.1 Background

Assume data of two values (organized by points x, y) with n data points. To average them, add all points and divide by n:

$$\bar{x} = \frac{\sum\limits_{i=1}^{n} x_i}{n}$$

To get the variance, do the following:

$$\sigma^2 = \frac{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

These may not be ideal forms of analysis for all needs, so linear regression analysis can be used. Linear regression analysis tries to get a straight line that best fits a given set of data. Since the line will almost never go through all lines, there are many possibilities. To solve for the best possible line, multivariable calculus can be used.

7.2 Linear Regression

Define the total error as E for any straight line for a given set of data. The goal is to then minimize error. At a conceptual level, this works by drawing a line and trying to minimize the vertical distance between the line and each point. Each distance is squared and summed to yield the total error for the line and associated data.

$$E = \sum_{i=4}^{n} (ax_i + b - y_1)^2$$

The goal of the regression analysis is now to find variables a and b that minimize E – or find the minimum of $E(a, b)$. Start by finding critical points:

$$\frac{\delta E}{\delta a} = \sum_{i=1}^{n} 2(ax_i + b - y_i)(x_i) = 0$$
$$\frac{\delta E}{\delta b} = \sum_{i=1}^{n} 2(ax_i + b - y_i)(1) = 0$$

There are two equations and two unknowns that must be solved for.

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$
$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} 1 = \sum_{i=1}^{n} y_i$$

This is where some data needs to be supplied. This data has been supplied from elsewhere and results in equations $54a + 14b = 76$ and $14a + 4b = 18$. Solving those equations yields $a = 2.6$ and $b = -4.6$, for a final linear regression of $y = 2.6x - 4.6$. Any other values for a and b will yield more error.

7.3 R^2 Values and Data

R^2 is not a measure of how well the line fits the data but is instead a maeasure of how linear the data is; higher values ($0 \leq R^2 \leq 1$) mean the data is more linear. For example, values from 0.8 to 0.9+ are considered good in the physical sciences.

7.4 Quadratic Regression

The same process from above can be used to fit parabolas to data:

$$E = \sum_{i=1}^{n} (ax_i^2 + bx_i + c - y_i)^2$$

Now try to find the minimum of E...

$$\frac{\delta E}{\delta a} = \sum_{i=1}^{n} 2(ax_i^2 + bx_i - y_i)(x_i^2) = 0$$
$$\frac{\delta E}{\delta b} = \sum_{i=1}^{n} 2(ax_i^2 + bx_i - y_i)(x_i) = 0$$
$$\frac{\delta E}{\delta b} = \sum_{i=1}^{n} 2(ax_i^2 + bx_i - y_i)(1) = 0$$

Now rearrange and simplify:

$$a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$$
$$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$
$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$$

This is extremely painful to solve out, but the process is expandable to polynomials of any degree in theory. Pain caused increases with the factorial of the degree of the polynomial being regressed.

8 Chain Rule

Suppose there is a scalar function $f(x, y, z)$ for all real numbers that represents temperature T. There is also a parametric function $\vec{r}(t)$ for $t \geq 0$ that has component functions $x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. The problem is: as you move around the curve, how fast is temperature changing? In mathematical notation, what is $\frac{df}{dt}$ at any t?[3]

$$\frac{f(\vec{r}(t + \Delta t)) - f(\vec{r}(t))}{\Delta t}$$

This is problematic because temperature does not depend directly on time, but instead on a position in space *given by* time.

$$= \frac{\Delta f}{\Delta x}\frac{\Delta x}{\Delta t} + \frac{\Delta f}{\Delta y}\frac{\Delta y}{\Delta t} + \frac{\Delta f}{\Delta z}\frac{\Delta z}{\Delta t}$$

Now let Δt go to zero. Each respective component (x,y,z) also goes to zero, resuling in:

$$\frac{df}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt} + \frac{\delta f}{\delta z}\frac{dz}{dt}$$

This is essentially just a three-dimensional version of the chain rule as a scalar function. It may be easier to represent this using vector notation, so a dot product version is simply $f^{\Delta} \cdot \vec{r}'(t)$. This will come out to be zero if f^{Δ} and $\vec{r}'$ are orthoganal, because in such case $\vec{r}'$ is in the direction of a level curve.

[3] d is used instead of δ intentionally

Example 1 $f(x, y, z) = x^2 + y^2 + z^2$, $\vec{r}(t) = \cos t\hat{\imath} + \sin t\hat{\jmath} + 3t\hat{k}$ Function f produces spheres centered at the origin and its gradient should produce a normal line that is essentially just a radiant line moved about the origin. $\vec{r}(t)$ creates a helix with a z difference between consecutive coils of 3π (due to the $3t$ term on the end). Temperature (given by f) will thus vary with position on the helix...

$$
\begin{aligned}
f^{\Delta} &= 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{z} \\
\vec{r}' &= -\sin t\hat{+}\cos t\hat{\jmath} + 3k \\
\frac{df}{dt} &= f^{\Delta} \cdot \vec{r}'(t) = -2x\sin t + 2y\cos t + 6z \\
&= -2\cos t\sin t + 2\sin t\cos t + 6(3t) \\
&= 18t
\end{aligned}
$$

The z component of the function is all that really mattered. If the derivative was done correctly, *everything* should be in terms of t (which it is here).

Part II
Integration

9 Double Integrals

Single/elementary integrals rely on one variable and are "easy". Previous notation was along the lines of $\int_a^b f(x)dx$, but this doesn't work for multivariable integrals. Instead, notation is now $\int_R f(x,y)dA$ where R is a shape on the xy plane and A is some very small part of A that is being differentiated with respect to.[4] So the result of such integration is *volume under a surface above R*.

Example 1 Integrate $f(x,y) = 100 - x^2 - y^2$ for $R = 1 \le x \le 5$ and $2 \le y \le 6$. Since regions are no longer defined as a region from a to b, but as 2D shapes, simple squares will be used for now.

Fix x between 1 and 5 and have a vertical slice parallel to the yz plane that is Δx thick. Solving for the volume of the slice will yield the area times its thickness, so the goal is to find the area under one infinitely thin section – a single 2D graph. This can be done by regular integration in a manner similar to partial derivation. Treat variables *not* being integrated as constant. In this case, start by integrating with respect to x:

$$\begin{aligned} A &= \int_2^6 f(x,y)dy \\ &= \int_2^6 100 - x^2 - y^2 dy \\ &= 100y - x^2y - \frac{y^3}{3}|_2^6 \\ &= \frac{992}{3} - 4x^2 \end{aligned}$$

This is the integration equivalent of a partial derivative - it's a partial antiderivative.

Now that the area of one slice has been computed, add up the volumes of all slices to get the total volume. This just takes another integral.

$$\begin{aligned} V &= \int_1^5 \frac{992}{3} - 4x^2 dx \\ &= 1157.\bar{3} \end{aligned}$$

Note that the result will come out the same no matter the order of the integration; the slices summed will still add up to the same volume. In effect, this means that you can integrate x first and y second *or* y first and x second.

Double integration can be written with two integral signs (which technically should have parentheses, but these are often omitted):

$$\int\int f(x,y)dxdy$$
$$\int(\int f(x,y)dx)dy$$

[4] A is the equivalent to the very very tiny "bar" that could be drawn under a single variable graph when doing simple summations.

Example 2 For a surface $z = 100 - x^2 - y^2$, integrate for a region R described by $y = -3x + 4$ where $x \geq 0$ and $y \geq 0$ (a triangle).

Start by fixing y and letting x vary. This will get tricky since R is not rectangular; since we are integrating with respect to x, make sure the limits on the integral are in terms of x. Remember to use the fundamental theorem of calculus when working with the limits of integration.

$$\begin{aligned}
A &= \int\limits_0^{-\frac{y}{2}+2} 100 - x^2 - y^2 dx \\
&= 100x - \frac{x^3}{3} - xy^2|_0^{-\frac{y}{2}+2} \\
&= 100(-\frac{y^2}{2} + 2) - \frac{(-\frac{y^2}{2} + 2)^3}{3} - (-\frac{y}{2} + 2)y^2 \\
V &= \int\limits_0^4 100(-\frac{y^2}{2} + 2) - \frac{(-\frac{y^2}{2} + 2)^3}{3} - (-\frac{y}{2} + 2)y^2 dy \\
&= \ldots
\end{aligned}$$

The end answer should be a number, but that isn't shown here.

Example 3 Set up an interal for the entire volume under the paraboloid from earlier $(100 - x^2 - y^2)$ above the XY plane.

As a cheat way to minimize work, find the volume under the first octant and multiply by four. This means finding the area under the paraboloid for one quarter of a circle with radius 10.

$$\begin{aligned}
b_y &= \sqrt{100 - x^2} \\
V &= \int\limits_0^{10} \int\limits_{y=0}^{\sqrt{100-x^2}} 100 - x^2 - y^2 dy dx
\end{aligned}$$

Example 4 Consider the integral $\int\limits_0^4 \int\limits_{y/2}^2 e^{x^2} dx dy$. This integral cannot actually be computed in its current order as e^{x^2} doesn't have an antiderivative.

To compensate, change the order of integration. This requires some knowledge of the region of integration, but it can be done. Here, the region of integration is a triangle defined by lines $y = 2x$ and $x = 2$.

$$\begin{aligned}
V &= \int\limits_0^2 \int\limits_0^{2x} e^{x^2} dy dx \\
&= \int\limits_0^2 e^{x^2} y|_0^{2x} dx \\
&= \int\limits_0^2 e^{x^2} 2x dx \\
&= e^{x^2}|_0^2 \\
&= e^4 - 1
\end{aligned}$$

Example 5 Compute $\int\limits_0^\pi \int\limits_x^\pi \frac{\sin y}{y} dydx$. This will need to be reordered to make this integral possible. Do *not* try to cancel out y and end with sin, it won't work.

To reorder this double integral, start by determining the area it's integrating over - here, that area is defined by a triangle with a hypotenuse defined by $y = x$ - the area is between the line and the y axis from 0 to π, and the top right corner is just (π, π).

$$\begin{aligned} V &= \int\limits_0^\pi \int\limits_0^y \frac{\sin y}{y} dxdy \\ &= \int\limits_0^\pi \frac{\sin y}{y} x|_0^y dy \\ &= \int\limits_0^\pi \sin y dy \\ &= 2 \end{aligned}$$

Example 6 Set up $\int\limits_R f(x, y)dA$ for an R bounded by $y = x^2$ and $y = x + 2$.

When graphed, the region is a curved section underneat the line and above the parabola, which intersect at $(-1, 1)$ and $(2, 4)$.

This will be integrated in $dxdy$ order. One way to figure out the limits of integration is to pick a random interior point and ask how far to the left/right (since this is with respect to x) it could go. This one is tricky - it depends on where y is, there are different areas this hypothetical point could bump into depending on its y value. This is where those intersections come in - two different integrals will be used, one for anything between the two intersections (with respect to y) and the other for anything below the bottom-most intersection.

Only the integral for the y area between the intersections is shown here:

$$\int\limits_{-1}^{2} \int\limits_{x^2}^{x+2} f(x, y)dydx$$

9.1 Polar Integration

In many cases it doesn't make sense to use cartesian coordinates, particularly where circles and circle-y curves are used. In these cases, it is preferable to use polar coordinates, the coordinate system based on circles.

This primarily involves changing the region of integration over to polar coordinates. Instead of integrating for miniscule boxes, now you are integrating in miniscule arcs. dA becomes $rdrd\theta$.

Example 7 Integrate $100 - x^2 - y^2$ for everything above the xy plane.

To start, convert the equation itself to polar. Since $x^2+y^2 = r^2$ in polar, the equation turns into $100-r^2$. The integral itself is now much simpler:

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{10} (100 - r^2) r dr d\theta \\
&= \int_0^{2\pi} (\frac{100r^2}{2} - \frac{r^4}{4}|_0^{10} d\theta
\end{aligned}
$$

Example 8 Integrate $f(x, y) = e^{\frac{x^2+y^2}{2}}$ for all values. Since this function has circular symmetry, polar integration can be used.

In cartesian coordinates, this looks very nasty: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}$. This isn't actually computable, so convert to polar using $x^2 + y^2 = r^2$.

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} r dr d\theta \\
&= \int_0^{2\pi} \lim_{b\to\infty} -e^{-\frac{r^2}{2}}|_0^b d\theta \\
&= \int_0^{2\pi} \lim_{b\to\infty} 1 - e^{-\frac{b^2}{2}} d\theta \\
&= \int_0^{2\pi} 1 d\theta \\
&= 2\pi
\end{aligned}
$$

Example 9 Of alternate interest is integrating $f(x) = e^{-\frac{x^2}{2}}$, as it is important in statistics. Ideally it should come out to 1 so the function can be scaled. Start by defining $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}$, then squaring it...

$$
\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} * \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \\
&= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} * \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}
\end{aligned}
$$

Now write it as an iterated integral:

$$
\begin{aligned}
V &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} dxdy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy \\
&= 2\pi
\end{aligned}
$$

Now we have the *square* of the integral originally wanted, so just take the square root of the answer obtained above, for a final result of $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} = \sqrt{2\pi}$.

This is useful because any bell curve function can now be normalized by dividing it by $\sqrt{2\pi}$. So, the formula for the standard normal distribution is thus $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

Example 10 Convert $\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \frac{2}{\sqrt{1+x^2+y^2}} \, \mathrm{d}y \, \mathrm{d}x$ to polar and evaluate it. This integral is not impossible to do in cartesian coordinates, but it is very lengthy.

First, to make the integral clearer, draw a sketch of the region of integration. Here, the region of integration is a semicircle above the x axis of radius 3. This is surprisingly easy to convert to polar coordinates, as r can be set to 0-3 and the angle is just π.

$$
\begin{aligned}
V &= \int_{0}^{\pi} \int_{0}^{3} \frac{2}{\sqrt{1+r^2}} r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_{0}^{\pi} (2\sqrt{1+r^2}|_0^3) \, \mathrm{d}\theta \\
&= \int_{0}^{\pi} 2\sqrt{10} - 2 \\
&= \pi(2\sqrt{2} - 2)
\end{aligned}
$$

Example 11 Convert to polar and find the volume under the paraboloid $z = 36 - x^2 + y^2$ and above the triangle bounded by the x axis, the y axis, and $2x + y = 3$.

On the XY axis a circle of radius 6 is formed, but the defined triangle cuts a piece of that out, making the limits of integration more difficult to find. Limits for θ are easy enough to find, as this integration happens in one octant only, for 0 to $\frac{\pi}{2}$. The formula for the given line is then converted into polar for $2r\cos\theta + r\sin\theta = 3$. Solving for r then yields $\frac{3}{2\cos\theta + \sin\theta}$

Turns out, this integral *is* easier to do in cartesian coordinates, but the problem called for polar…

$$
V = \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{3}{2\cos\theta+\sin\theta}} (36 - r^2) r \, \mathrm{d}r \, \mathrm{d}\theta
$$

…yeah, this one isn't going to be completed.

Example 12 Consider, int the first octant, 2 cylinders of radius 4 centered on the x and y axes. Find the volume of the region enclosed by them.

The region of integration can be broken down into two identical triangles; just find the area above one and double it. The equation of one of the cylinders is $x^2 + z^2 = 16$, so solving for z yields $\sqrt{16 - x^2}$, so this will be used as the function integrated. The region of integration looks like a triangle, so the limits of integration (in dydx order) are then 0 to x and 0 to 4. Plug in relevant values and evaluate.

$$\int_0^4 \int_0^x \sqrt{16 - x^2}\, \mathrm{d}y\, \mathrm{d}x$$

9.2 Finding Volumes of Shapes

This is just an application of double integrals. It's possible to use double integrals to find the volume of various shapes.

Example 10 Find the area above the XY plane, inside a circular cylinder with radius 10 centered on the z axis, and below the plane $6x + 15y + 3z = 48$. Since this has circular symmetry, we can automatically jump to using polar coordinates.

First find the function by looking at the plane. The plane must be solved for z, yielding $z = 16 - 2x - 5y$. This isn't very polar (there are four variables!) so continue converting to the final result of $16 - 2r\cos\theta - 5r\sin\theta$. The limits of integration are simply based on the radius and the angle being integrated (2π).

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{2} 16 - 2r\cos\theta - 5r\sin\theta r dr d\theta \\
&= \int_0^{2\pi} 16\frac{r^2}{2} - \frac{2}{3}r^3\cos\theta - \frac{5}{3}r^3\sin\theta|_0^{10} d\theta \\
&= \int_0^{2\pi} 800 - \frac{2000}{3}\cos\theta - \frac{5000}{3}\sin\theta \\
&= 800\theta - \frac{2000}{3}\sin\theta + \frac{5000}{3}\cos\theta|_0^{2\pi} \\
&= 1600\pi
\end{aligned}$$

This is an odd answer - it has *nothing* to do with the coefficients. That means that the plane capping this sloped cylinder off has nothing to do with the answer either- in effect, the whole thing is only dependent on the radius and height! This is because the region cut off by the "low" part of the plane is equal in volume to the region gained by the "high" part of the plane.

Example 11 Find the volume of a sphere with radius 10 that has a hole of radius 2 punched through the center of it.

This can be approached as a simple integration problem where the sphere is defined as $x^2 + y^2 + z^2 = 10$ or $sqrt100 - x^2 - y^2 = f(x, y)$. Since this is obviously a circley problem, polar coordinates can be used, resulting in $\sqrt{100 - r^2}$ instead. Now the only challenge is finding the limits of integration...

$$\int_0^{\frac{\pi}{2}} \int_2^{10} \sqrt{100 - r^2} r dr d\theta$$

This will find the volume of one octant. Use u substitution where $u = 100 - r^2$, $du = -2rdr$, and $\frac{du}{2} = rdr$ resulting in $\int u^{\frac{1}{2}}(-\frac{-du}{2}) = -\frac{1}{3}u^{\frac{3}{2}}$.

$$V = \int_0^{\frac{\pi}{2}} (-\frac{1}{3}(100 - r^2)^{\frac{3}{2}}|_0^{10})d\theta$$

Then just finish the integration (not shown, sorry!).

Example 12 Find the volume of a cone[5] with a height H and radius R. The answer – $\frac{1}{3}\pi HR^2$ – is already known, it just needs a proof.

To solve this, consider taking a vertical cross section of the cone. From this position, the cone looks like a triangle with the hypotenuse defined by $y = \frac{H}{R}y$. Thus, the height at any given point on the cone is $H - \frac{H}{R}r$. The limits are simple as the region is just a circle.

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^R (H - \frac{H}{R}r)rdrd\theta \\ &= \int_0^{2\pi} \frac{Hr^2}{2} - \frac{H}{3R}r^3 d\theta \\ &= 2\pi\frac{HR^2}{6} \\ &= \frac{\pi R^2 H}{3} \end{aligned}$$

10 Triple Integrals

Triple integrals are essentially the same as double integrals, except that instead of taking the form $\int f dv$, these take the form of $\int f dV$, as you are now integrating over a volume.

10.1 Cartesian Coordinates

Example 1 Set up a triple integral to determine the volume in the first octant bounded by the plane $5x + 2y + z = 10$, then evaluate it.

In three dimensions, this plane forms a bounded triangular prism-shaped area. This just means solving for the limits of integration and constructing the appropriate integrals using them. Intersections for the axes are at (0,0,10) for z and (0,5,0) for y.

$$\begin{aligned} V &= \int_0^2 \int_0^{5-5y/2} \int_0^{10-5x-2y} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{50}{3} \end{aligned}$$

For a shape in the first octant defined by general formula $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$, find a formula for the volume. This is just a more general version of the previous Example 1. The formula is quite simple despite that the integration itself is lengthy: $\frac{abc}{6}$.

[5]Technically this is a *right circular cone*. The most general form for a cone is just some closed region and an apex point that is connected to all the edges of that region. For such cones, volume is simply $\frac{1}{3}AH$ where H is the height of the apex and A is the area (?) of the region.

Example 2 For z > 0, find the volume of the elliptic paraboloid $z + x^2 + \frac{y^2}{4} = 100$.

The vertex is centered about the z axis 100 units up, Fortunately, this shape is symettrical about the x and y axes, so the volume in only one quadrant can be found and multiplied by four to find the desired total volume above the xy plane. All that is truly required is solving for the limits of integration.

$$\int_0^{10} \int_0^{2\sqrt{100-x^2}} \int_0^{100-x^2-y^2/4} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

This is an extremely painful integral to solve, so the actual solving process is not shown.

10.2 Cylindrical Coordinates

A coordinate system is just a set of directions that tell you how to get from some point (usually the origin) to another point. By default we use Cartesian coordinates, but more interesting coordinate systems – such as polar – exist.

Cylindrical coordinates are a 3D coordinate system that forms a hybrid between polar and cartesian. They take the form of (r, θ, z), where r is the radius, θ is the angle, and z is the height. All of the polar coordinate conversions and substitution rules still apply here.

To do integrals in cylindrical coordinateds, $dV = (rdrd\theta)dz = rdrd\theta dz$. This one involves integrating over 3D sectors of a circle. So in sum, integrating looks like this:

$$\int\int\int f(r, \theta, z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

Under this system, there are *six* different orders of integration, so pick the one that is most convenient.

Example 1 Find the volume of $z = 100 - r^2$ for $z > 0$. If sketched, this is an upside- down parabaloid with a max at (0, 0, 100).

The tricky part of this is finding the limits of integration. For θ, just use 0 to 2π. z is a function of r, and the height is defined by $100 - r^2$, so integrate from 0 to $100 - r^2$. r itself can vary from 0 to 10 only (as the equation $z = 100 - r^2$ cannot be less than 0).

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^{10} \int_0^{100-r^2} (100 - r^2) r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\pi \int_0^{10} (rz|_0^{100-r^2})\,\mathrm{d}r \\
&= 2\pi \int_0^{10} r(100 - r^2)\,\mathrm{d}r \\
&= 2\pi(50r^2 - \frac{r^4}{4}|_0^{10}) \\
&= 2\pi(5000 - 2500) \\
&= 5000\pi
\end{aligned}$$

10.3 Spherical Coordinates

Spherical coordinates are the 3D equivalent of polar coordinates. . . this time without the Cartesian component thrown in (like in cylindrical). The first number given is a distance rho (ρ), the second is an angle θ along the XY plane, and the third is the angle of elevation (ϕ). In sum, these coordinates are given as (ρ, θ, ϕ). Note that in some notation, ϕ is given as an angle from the z access and not from the XY plane.

Setting each variable to a constant will yield different things. Making ρ a constant will yield a sphere with the radius of the constant. Making ϕ constant will generate a cone, and setting θ constant will yield a plane.

The general form for such problems takes the form of $\int\int\int \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Example 1 Find the volume of a sphere of radius R centered at the origin.

$$
\begin{aligned}
V &= \int_0^{\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3} \sin\phi \, d\phi \, d\theta \\
&= \int_0^{2\pi} \frac{2R^3}{3} \\
&= \frac{4}{3}\pi R^3
\end{aligned}
$$

Example 2 Find the volume of a section of a sphere of radius six with a center angle of 30ô.

$$
\int_0^{2\pi}\int_0^{\pi/6}\int_0^{6} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta
$$

Example 3 Find the volume of the same shape from before, but with a flat top. This is just a right circular cone with radius R and height capital H.

$$
\begin{aligned}
V &= \int_0^{2\pi}\int_0^{\arctan\frac{R}{H}}\int_0^{\frac{H}{\cos\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \frac{2\pi H^3}{3} \int_0^{\arctan R/H} \cos^{-3} \sin\phi \, d\phi
\end{aligned}
$$

This last part involves finding the integral of $\cos^2 \arctan\frac{R}{H}$. With a little bit of magic and a whole lot of geometry, this turns into $\frac{H^2}{R^2+H^2}$. Moral of this story: if there's a messy, triggy integral draw it out as a triangle as a means to help simplify it.

The result of the original integral from above (where this mess originated from) is then just $\frac{1}{3}\pi R^2 H$. Note that this is after a fair amount of simplification.

10.4 Parametric Surfaces

Surfaces can be parameratize by creating a vector function of two variables that goes from 2D to 3D. The vector function takes variables in some domain and creates a surface where every point on that surface is the function of the variables input. If you take a function of two variables and fix one, a curve is generated instead of a surface. To actually do these computations, parameterization is needed.

The goal of parameterization is to find a vector function $\vec{r}(s,t)$ and a region R in the st domain such that $\vec{r}$ produces the desired surface as you graph it. This means that X, Y, and Z are all given in terms of s and t[6]

To start, suppose that we are parameratizing simple cartesian planes in the form of $z = f(x,y)$. To parameterize this, set X equal to S, Y equal to T, and Z equal to f(s,t). R is then the region in the XY plane below the surface. For example, a parameterization of $5z + 3x + 13y = 30$ yields a $x = s$, $y = t$, and $z = 6 - \frac{3}{5}s - 2t$. Note that this is *not* the only parameterization of this function, there are plenty of other possibilities.

Example 1 - Parameterization Find the surface area of $(z = (x^2 + y^2)^2$ below the 26 plane. This function has circular symmetry about the z axis.

To parameratize this, turn use polar coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= r^4 1 \end{aligned}$$

The final parameratized function is $\vec{r}(r,\theta) = r\cos\theta\hat{i} + r\sin\theta\hat{j} + (r^4+1)\hat{k}$

Example 2 - Sphere Find the surface area of a sphere of radius R. First define a parametric equation for the sphere...

$$\vec{r}(\theta,\phi) = R\sin\phi\cos\theta\hat{i} + R\sin\phi\sin\theta\hat{j} + R\cos\theta\hat{k}$$

This should be integrated for $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. Theta and phi are really just parameters that give us points on the sphere. If theta is fixed and phi varies, the result is a vertical great circle; if phi is fixed and theta varies, a cicrle that goes around the Z axis is produced.

Now try to find the surface area of the sphere by integrating with respect to phi and theta. To do this, take a small rectangle in the $\theta - \phi$ plane and see what happens to it. If you can find the surface area of that, you can repeat the process many times to find the total surface area of the shape requested.

$\frac{\delta\vec{r}}{\delta\theta}$ and $\frac{\delta\vec{r}}{\delta\phi}$ are tangent vectors to the surface. They determine a parallelogram on the surface of the sphere, so this will serve as our "small rectangle" of integration. To find its area, take the magnitude of the cross product of the two vectors:

$$\begin{aligned} A &= |\vec{r}_\theta\Delta\theta \times \vec{r}_\phi\Delta\phi| \\ \vec{r}_\theta &= -R\sin\phi\sin\theta\hat{i} + R\sin\phi\cos\theta\hat{j} \\ \vec{r}_\phi &= R\cos\phi\cos\theta\hat{i} + R\cos\phi\sin\theta\hat{j} - R\sin\phi\hat{k} \\ A &= |-R^2\sin^2\phi\cos\theta\Delta\theta\Delta\phi\hat{i} - R^2\sin^2\phi\cos\theta\Delta\theta\Delta\phi\hat{k} - R^2\sin\phi\cos\phi\Delta\theta\Delta\phi\hat{k}| \\ &= R^2\sin\theta\Delta\theta\Delta\phi \end{aligned}$$

That last line is the area of one tiny patch on the sphere defined earlier with respect to theta and phi. Now that the area of one incredibly tiny patch has been determined, add all of them up! This is, of course, going to involve integration.

[6]These letters are not unique or special in any way; any set of letters can be substituted in for them.

$$
\begin{aligned}
SA &= \int_0^{\pi}\int_0^{2\pi} R^2 \sin\phi \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= 2\pi R \int_0^{\pi} \sin\phi \, \mathrm{d}\phi \\
&= 4\pi R^2
\end{aligned}
$$

Note that since this technically gives you the volume of a thin shell, you can integrate this with respect to R to get the volume of a sphere – this yields the classic $\frac{4}{3}\pi R^3$

Example 3 - Paraboloid Find the surface area of a paraboloid given by $z = x^2 + y^2$ and with height H.

This problem allows us to pick the parameterization used. For this problem, polar coordinates are ideal, so the parameterization used will be $\vec{r}(r,\theta) = r\cos\theta\hat{i} + r\sin\theta\hat{j} + r^2\hat{k}$. Finding bounds on theta is easy enough ($0 \leq \theta \leq 2\pi$), and r can be defined as a function of H – $0 \leq r \leq \sqrt{H}$. The process of finding the cross product and its magnitude is not shown here.

$$
\begin{aligned}
sA &= \sqrt{4r^4 + r^2}\Delta r \Delta\theta \\
&= \sqrt{4r^2 + 1} r \Delta r \Delta\theta \\
SA &= \int_0^{2\pi}\int_0^{\sqrt{H}} \sqrt{1 + 4r^2} r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \frac{\pi}{6}((1 + 4H)^{3/2} - 1)
\end{aligned}
$$

Example 4 - Cone Find the surface are of a right angle cone with radius R and height H. Assume that the apex of the cone is at the origin and the base lies somewhere above it.

First, parameterize the cone: $\vec{r}(r,\theta) = r\cos\theta\hat{i} + r\sin\theta\hat{j} + \frac{H}{R}r\hat{k}$. This function should have a domain of $0 \leq \theta < 2\pi$ and $0 \leq r \leq R$.

Now find the magnitude of the normal vector of the surface by taking the cross product of the two tangents (?):

$$
\begin{aligned}
|\vec{N}| &= |\vec{r}_r \times \vec{r}_\theta| \\
\vec{N} &= \hat{i}(-\frac{H}{R} r \cos\theta 0 - \hat{j}(\frac{H}{R}\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) \\
&= \hat{i}(-\frac{H}{R} r \cos\theta 0 - \hat{j}(\frac{H}{R}\sin\theta) + \hat{k}r \\
|\vec{N}| &= \sqrt{\frac{H^2}{R^2} r^2 \cos^2\theta + \frac{H^2}{R^2} r^2 \sin^2\theta + r^2} \\
&= \sqrt{\frac{H^2}{R^2} r^2 + r^2} \\
&= r\sqrt{\frac{H^2}{R^2} + 1} \\
SA &= \int_0^{2\pi}\int_0^{R} r\sqrt{\frac{H^2}{R^2} + 1} \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \pi R\sqrt{\frac{H^2}{R^2} + 1}
\end{aligned}
$$

This isn't quite the "standard" formula for the surface area of a cone, but it yields the correct area nevertheless. To find a cleaner function for the surface area, add the area of the base to the area of a circle with the relevant area cut out in order to make a cone. This yields an answer of $\pi R^2 + (\frac{1}{2}\theta)R^2$. Something like that, anyways.

Example 5 - Circles in R^3 Suppose there is a circle $R\cos\theta\hat{\imath} + R\sin\theta\hat{\jmath}$ in the XY plane of radius R. Now let $\hat{u} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$, which looks like a vector pointing radially outward from the origin for a fixed θ. Consider a new vector $\cos\alpha\hat{u} + \sin\alpha\hat{k}$, a unit vector in the $\hat{u} - \hat{k}$ plane.

Now there is a circle in the XY plane with a pair of vectors on its edge, one pointing radially outward and the other pointing from the same point, but at an angle. As θ varies from 0 to 2π and α varies from 0 to π, a torus is formed. The final formula for this is then $(R\cos\theta\hat{\imath} + R\sin\theta\hat{\jmath}) + (S\cos\theta\hat{u} + S\sin\theta\hat{k})$ where R is the main radius of the torus and S is the radius of the circles that form the "ring" of the torus. Now, find the surface area!

Some of the process of solving this is omitted as it takes an extreme amount of time and space to write out. Fortunately, after this is over, things go relatively smoothly.

$$\begin{aligned}
|\vec{N}| &= |\vec{r}_\theta \times r_\alpha| \\
&= SR + S^2\cos\alpha \\
SA &= \int_0^\pi \int_0^{2\pi} SR + S^2\cos\alpha \,\mathrm{d}\theta \,\mathrm{d}\alpha \\
&= 2\pi \int_0^{2\pi} SR + S^2\cos\alpha \,\mathrm{d}\alpha \\
&= 4\pi^2 SR
\end{aligned}$$

Turns out, this could've been reasoned out without calculus. Imagine taking a pair of scissors and cutting it radially, allowing it to be stretched out into a cylinder. That cylinder is then $2\pi R$ long. If you cut it laterally then, you get a rectangle of surfcae area $4\pi^2 RS$.

10.5 Flux Through a Surface

10.5.1 Flux Through a Rectangle

<++>

To find flux through a surface, we first need vector fields. A vector function is a function that produces a vector for every relevant point. For example, consider a stream - every point in that stream is flowing in some direction, so you could call that stream a vector field.

Example 1 For water flowing right to left at 20 ft/s^2, how much water per second flows through a 1 ft^2 cross section?

The answer is actually very straightforward: 20 ft^3/s. To better visualize this, imagine the cross section is moving instead, and that it sweeps out a rectangular prism shaped area.

Example 2 Suppose that the square from the previous example is tilted at 45^o. How much water is flowing through the square now?

To solve this, a little trigonometry is needed. The apparent width has changed, but you need to multiply the height by $\cos 45$ to yield an apparent area of $\frac{1}{\sqrt{2}}$, so the new flow is now $10\sqrt{2}$ ft^3/s

Example 4 Suppose the square is now tilted at an angle θ to the vertical. What is the flow through the square now?

Using the same trigonometric trick from above, the flow is now $20\cos\theta\,{}^{\mathrm{ft}^3}\!/_{\mathrm{s}}$

A more general way of representing this is $\vec{V}\cdot\hat{N}$, or the dot product of the velocity to the normal vector of the surface.

10.5.2 Flux Through Any Surface

First parameterize the surface S, then set up an integral over the region for $\vec{V}\cdot\hat{N}$. This takes the form of $\int\int \vec{V}\cdot\hat{N}\,\mathrm{d}s\,\mathrm{d}t$

Example 1 For vector field $\vec{V}(x,y,z) = -y^2\hat{j} - 2z\hat{k}$ find the flux through surface $4x + 6y + z = 24$. This vector field generally points downward, so it should yield a good flux. Integration is needed here because the vector field changes over distance.

Start by parameterizing the surface to yield $\vec{r}(s,t) = s\hat{i} + \hat{j} + (24 - 4s - 6t)\hat{k}$, for (s,t) from (0,4) to (6, 0). To set up the integral, a unit normal vector is needed. This could be obtained by a cross product, but since this is a planel, cross products are overkill. The final unit normal vector is then $\frac{4\hat{i}+6\hat{j}+\hat{k}}{\sqrt{53}}$.

Now find $\vec{V}(\vec{r}(s,t))$ or the velocity on the surface. This should yield an answer in terms of the variables used, i.e. s and t here. In this case, the answer is $-t^2\hat{j} - 2(24 - 4s - 6t)\hat{k}$, calculated by evaluating the vector field for the given plane (???)

Next find the dot product: $\vec{V}\cdot\hat{N} = \frac{-6t^2 - 2(24-4s-6t)}{\sqrt{53}}$ This is now just a scalar in terms of t and s, but it needs to be integrated. Make sure to use the proper region of integration as define by the plane.

$$\int_0^4 \int_0^{\frac{3}{2}t+6} \frac{-6t^2 - 48 + 8s + 12t}{\sqrt{53}}\,\mathrm{d}s\,\mathrm{d}t$$

This is a "relatively harmless" (read: terrifying) double integration that is fairly trivial, so its solution is not shown.