ECE 2010 Introduction to Electrical & Computer Engineering

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1 Basic Electric Circuits

Simple electric circuits contain a voltage source — to provide electromotive force to make the circuit run — and a load. In electrical notation, $\vec{E}$ references an electric field and $\vec{H}$ references a magnetic field. In a fluid mechanics analogy, this is equivalent to a pump that pumps liquid through pipes and some arbitrary load, like a sandbox. In electrical systems, charges interact by electrical fields, not by mechanical contact forces. Since electrical fields propagate at the speed of light, there is near-instantaneous transmission of signals.

Voltage itself is just a measure of the strength of an electrical field; it measures potential difference. In SI terms, $1V = \frac{1J}{1C}$. If you have a uniform electric field $E = 10$V/m, a point A (observation point) and a reference point A^0, then the voltage is $V_{AA^0} = \int\limits_{A}^{A^0} \vec{E} \cdot d\vec{e}$. Since the dot products will something magical involving canceling out here, and $\cos 0 = 1$ (seriously, a diagram would help here!) $V_{AA^0} = \int\limits_{A}^{A^0} Edx \cos 0 = Ex$. In other words: multiply voltage is the product of the electrical field and the distance between the two points. All that physics stuff doesn't matter quite as much.

Electric current, then, is just flow of charges, as measured in Amps, a derived unit of the form C/s. Using this information combined with voltage information, it is possible to calculate power of a circuit by multiplying current by voltage. For instance, the power delivered by a voltage source of 10V to a load given that the demand is 0.5 A is 5 watts.

1.1 Ohm's Law

Ohm's law is a law that relates voltage to current and resistance: $V = IR$. To derive this, consider a conductive cylinder with an n-concentration of free charges. Current flows in one side and out the other, and the voltage from one end to the other can be measured as V. The current passing through a cross section over one second is $I = Aqnv$. The Coulomb force F acting on each charge is $F = qE$, all scalar since we are only considering movement along the X axis. The velocity of a charge is $v = \mu E$ where μ is the mobility of charges, an important constant for conductors. From above, the field $E = \frac{V}{e}$.

Rewriting $I = AqnV$ using the above information yields $V = \frac{e}{Aqn\mu}I$. This looks complicated until you realize that $\frac{e}{Aqn\mu}$ is just resistance, measured in Ω (ohms). The quantity Ωm is then the resistivity of a given material. The inverse of resistance is $S = 1/\Omega$, so S/m is then the conductivity of a material.

For resistor power draw, you can use $P = VI$, $P = RI^2$, or $P = \frac{V^2}{R}$.

1.2 Power sources

Independent voltage source An ideal independent voltage source is one with a positive and negative terminal. Instead of producing some amount of current, voltage sources produce a voltage between their two terminals. Current flows out of negative terminal and out of the positive terminal, but the amount of current will vary with the resistance along the circuit. When the V-I characteristic graph is shown, it appears as a vertical line that intersects the voltage value on the X axis; that is to say, the current can be anything as long as voltage remains constant. Obviously this is impossible for real voltage sources, which tend to bend off towards the Y axis as current grows.

When applied to a hydraulic analogy, this is roughly the same as a pump that provides constant torque but not constant speed. In diagrams they are represented by a circle with two leads, one labeled positive and the other labeled negative.

Independent current source An ideal independent current source is similar to a voltage source, but instead of providing constant voltage, it provides constant power. Once again, current flows out of the negative terminal and flows into the positive terminal. On a V-I characteristic graph, current sources show a flat line that intersects the Y axis at the appropriate current value. Again similar to voltage sources, real current sources have a tendency to bend towards the X axis the farther they go out along it.

When applied to a hydraulic analogy, this is roughly the same as a pump that provides constant speed but not torque. In diagrams they are represented by a circle with two leads and an arrow pointing in the direction of current flow.

Dependent sources Dependent sources provide power in a way that depends on some other factor. In diagrams they appear as their independent counterparts, but with a diamond shaped body instead of a circle. Usually these have four terminals — a pair for input and a pair for output.

A voltage-controlled voltage source is one example of a dependent source. These sources may relate their output voltage to their input voltage: $V_{out} = AV_{in}$. If $A = 10$, that would make this a voltage amplifier with a factor of 10. A similar concept exists for current sources.

Another variant of this is the voltage-controlled current source, called a transconductance amplifier. They read the input voltage and scale the output current according to $I_{out} = GV_{in}$. The flipped version of this is a current-controlled voltage source, which reads the input and scales the output voltage according to $V_{out} = RI_{in}$. These are called transresistance amplifiers, and are much more common than transconductance amplifiers.

1.3 Measuring devices

An ideal voltmeter provides no conductive path of its own, and as such can be wired in parallel. When wired into a circuit, ideal voltmeters will show the voltage between two points.

Ammeters, however, do provide a conductive path, but there in an ideal ammeter there should be no resistance along this path. Ammeters should be wired in series into the circuits they measure, but care should be taken to protect against short circuits.

1.4 Circuit ground

There are different kinds of circuit grounds, but they all provide approximately the same service. The first type is the earth ground, which should involve a connection to the ground at 0 volts. These earth grounds are often embedded in the actual earth. In diagrams these look like a long line with two shorter, parallel lines ordered below it.

The second type of circuit ground is a chassis ground, used in situations where earth grounds are not available (e.g. aircraft). In diagrams these have a shape something like a pitchfork. Finally, the third type of ground is a simple common (neutral) ground, and on diagrams it looks like a triangle.

If a circuit ground is connected to the wrong side of a circuit — after current has already gone through a resistor of some kind — then no current will flow out to ground. For some reason this means putting circuit ground near the negative terminal.

1.5 Wall plug supply

A wall plug supply has three prongs — one round, one long, and one short. The short prong is connected the positive terminal of a 120 VAC voltage source, the long prong is connected to neutral ground (???), and the round prong is connected to earth ground. It is important to note that somewhere along the line there is a connection between the neutral ground and earth ground. This means that *technically* it's safe to connect the long prong to the round one, but not the short one to neutral.

1.6 Circuit networking

Suppose you have a circuit with resistors A and B wired in series, connected to resistors C and D that are together wired in parallel. In circuit terminology, a branch is a two terminal circuit element, so there are four branches in this circuit. Nodes are then connections of two or more branches, so this circuit has 6 nodes (a diagram would be useful here, but I can't draw one). Nodes that are connected to each other with no other components in between them have equal voltages. Loops in the circuit are then closed paths, and meshes are loops that do not contain any other loops.

This is where networking theory (for planar circuits) becomes relevant:

$$b = n + m - 1 \tag{1}$$

...where b is the number of branches, n is the number of simple nodes, and m is the number of meshes. This is also where Kirchhoff's current law becomes relevant: the net current across any node is zero; current cannot disappear. This the electrical engineering version of the mass conservation law. For the purpose of simplifying circuits, nodes that are interconnected without intermediate components can be treated as single nodes (they share current and voltage).

A similar law for voltages exists too (Kirchhoff's voltage law): the sum of voltages for any closed loop is zero. This is the electrical engineering version of the energy conservation law. Since this works across loops and not just the whole circuit, you can break down a circuit into its meshes and use Kirchhoff's voltage law on each one individually, greatly simplifying the process.

If batteries are connected in series — say, a pair of 9V/1A batteries — the total voltage is the sum of the voltages (18V) while the current is the same as current across just one battery (1A). If instead the batteries are connected in parallel, the voltage will be the same as the voltage across one battery (9V) while the current will be the sum of the independent battery's voltages (2A).

1.7 Voltage divider

Suppose you have a circuit with a single power source connected to two resistors in series. Using Kirchhoff's voltage and current laws, it can be derived that $V_1 = \frac{R_1}{R_1+R_2}V_s$ and $V_2 = \frac{R_2}{R_1+R_2}V_s$. This is called a *voltage divider*, probably because it divides the voltage across the resistors based on their resistance.

If you put terminals across the second resistor in the series, you can measure the voltage across them as the resistance changes (e.g. if one of the resistors is a variable resistor like a thermsistor or a potentiometer). About 90% of all sensor circuits use voltage dividers in one way or another for this purpose. Note that the resistor that the measurement terminals span can be either the sensor component itself or the fixed resistor. When combined with a MOSFET, simple sensor circuits like this can also be used to control output sources such as lights and motors.

1.8 Current Divider

Current dividers are similar to voltage dividers, but they work in exactly the opposite way: $V = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}I_s$. Other equations:

$$I_1 = \frac{V}{R_1} = \frac{R_2}{R_1+R_2}I_s$$
$$I_2 = \frac{V}{R_2} = \frac{R_1}{R_1+R_2}I_s$$

Unlike voltage dividers, where most of the voltage goes through the load with the highest resistance, current dividers send most of their current through to the lowest resistance resistor.

1.9 Current limited circuit

In a circuit, the current will always be limited by the largest resistance. It might be more appropriate to say that $I = \frac{V}{R_1+R_2}$, but if R_1 is much greater than R_2, it's appropriate to say that the circuit is *limited* by resistor R_1, since even if R_2 goes to zero (say, if it were a variable resistor), the current can be at most $\frac{V}{R_1}$.

1.10 MOSFETs — Voltage Controlled Switches

MOSFETs are switches in that they control the amount of current that can pass by voltage. They have three terminals: a drain terminal (current input), a source (?) terminal (current output), and a gate terminal (voltage control signal). Also unlike physics switches, MOSFETs are completely solid state; instead of relying on mechanical principles and physical contact between conductors, MOSFETs have a transistor at their core.

1.11 Source transformation theory

Consider an ideal voltage source in series with a resistor. This could be called a practical voltage source, since this is how units like batteries actually behave. Now consider the same circuit, but with the resistor wired in parallel instead. This is also a practical power source, and is actually very close to the behavior of solar panels. In both cases, current is limited by the resistor.

Anyways, two linear networks are equivalent if their open-circuit voltage and closed-circuit current coincide. The current of the closed circuit is $I_{sc} = \frac{V_T}{R_T} = I_n$, while the voltage for the open circuit is $V_{oc} = V_T = R_n I_n$ (or the voltage across the resistor in parallel). Doing a little math yields $R_N = R_T$ and $I_N = \frac{V_T}{R_T}$. If these are satisfied, the two power sources are equivalent from the perspective of the actual load applied.

1.12 Thevenin/Norton's Theorems

These theorems state that any linear network of sources or resistances with two terminals A and B for the whole network, can be replaced by a single voltage source V_T and a single resistance R_T and be completely equivalent to the first network. If the resistor is wired in series, it's a Thevenin-equivalent circuit, but if the resistor is wired in parallel, it's a Norton-equivalent circuit.

Example Consider a 5V voltage source wired in series with a 5kΩ resistor and in parallel with another 5kΩ resistor. What is the resistance of the resistor in the Thevenin-equivalent circuit? What is the voltage of the source in the same circuit?

First, turn off the voltage source; replace it with a wire. Shifting the series resistor around a little bit will demonstrate that the two resistors are actually wired in parallel, so their resistance is $\frac{1}{5\text{k}\Omega} + \frac{1}{5\text{k}\Omega} = \frac{1}{R_T} = 2.5\text{k}\Omega$.

So what about the voltage source? I don't actually know. Maybe it's the same?

Example 2 Consider a 1 mA current source wired in parallel with a 1 kΩ load. In a thevenin-equivalent circuit, the resistor will be the same, so what would the voltage be in such a circuit that had a voltage source instead?

All you have to do is find the voltage across the resistor: 1 V. That means that the voltage of your voltage source is 1 V.

Example 3 Consider a 3 V voltage source wired in series with two 1 kΩ resistors and in parallel with a third (forming a ring with two terminals that includes the voltage source). What are V_T and R_T in the Thevenin-equivalent circuit?

First, find the resistance. The two series resistors can be added together, and using the principle for the above on the parallel resistor will yield $\frac{2}{3}$kΩ. Now look at the voltage across each resistor: since the resistors are all equal, divide the voltage 3 to get 1 V for each resistor. Somehow the end voltage ends up being 1 V, but I'm not sure where that came from.

1.13 Maximum power transfer theorem

For a practical voltage source (regardless of which type), the load power is maximized when the resistance load R_L is equal to the resistor wired into the power source R_T. This is the result of some clever math and calculus work that is not shown here.

2 Operational Amplifiers

Opamps are drawn as a triangle with two inputs on one base (one for positive, one for negative) and one output on the vertex opposite. The positive input is called the non-inverting output, the negative input is called the inverting input, and the output is just the output voltage. There can be two extra terminals connected to the sides, one called V_{cc} (positive rail), and the other called $-V_{cc}$ (negative rail).

During operation, the output of an opamp is characterized by $V_{out} = A(v^+ - v^-)$, where A is an open load gain, a large value on the order of 10^5 to 10^8, or even approaching ∞. This formula is always valid. For the rails, the voltage out will always be greater than the voltage input... unless it's negative. Something like that, anyways. This behavior is called *railing*.

The equivalent circuit for an opamp looks a little strange. Between the two inputs there is a resistor R_{in}, called the input resistance. Next to the resistor is a dependent voltage source, the positive end of which is connected to a resistor R_{out} and then V_{out}. The negative terminal of this dependent voltage source is then connected to $-V_{cc}$. Because the voltage controlled voltage source should be idea, the formula $\frac{R_{in}}{R_{out}+R_L}\frac{R_{in}}{R_{in}+R_S}V_S A = V_{out}$ should hold. That means that the two fractions up there should evaluate to 1, requiring that $R_{in} \to \infty$ and $A \to \infty$.

2.1 Negative Feedback

Suppose you wire up an opamp with the positive terminal wired up to a positive voltage source, but you wire the negative terminal into the voltage output, with any load in between. This will create negative feedback where a portion of the output voltage is returned to the negative/inverting input.

Here's an example. The output voltage is defined by $V_{out} = A(v^+ - v^-)$ — call the quantity in the parentheses V_x. Assume that 50% of V_x is returned back in 1 µs. Let $V^+ = 10V$, and $V^- = 0V$ at $t = 0$.

At $t = 1$, 50% of 10V is returned back to V^- (as 5V), and the differential voltage V_x then becomes 5V as well. At $t = 2$, another 50% is sent over, so $V^- = 7.5$ and $V_x = 2.5$. This process continues with V^- approaching but never quite reaching 10V, just as V_x approaches but never exactly reaches 0V. In reality, this all happens very fast: it takes only 4 µs for V_x to reach 0.625.

There are two summing-point constraints (whatever those are...). One is the case where there is no current into the amplifier. The other is differential voltage $V_x = 0$, but only with negative feedback.

2.2 Non-inverting amplifier circuit

Once again, consider an opamp with V^+ wired up to some voltage source and V_{out} wired somewhere unimportant. If you were to connect a resistor in between V_{out} and V^-, you'd have another negative feedback system, so put another resistor on the V^- side of the first resistor, but have it lead away to ground. The current through the ground resistor $R1$ is called i_1, while current across the other resistor $R2$ is called i_2.

Now consider the previous summing constraints. If there is no current through the amplifier, $i_1 = i_2 = 0$. The second summing constraint applies here because there is negative feedback. With a little math $V_{out} = (1 + \frac{R_2}{R_1})V_{in}$ and $A_{closed-loop} = 1 + \frac{R_2}{R_1}$. Using this method, it is possible to build what is effectively a voltage amplifier.

2.3 Inverting amplifier circuit

This time, things get strange. Wire a voltage source in series with a resistor $R1$ into the negative input. Also wire the positive input into V_{out}, again with a resistor $R2$ in series, and with the connection formed between $R1$ and V^-. Wire the positive input into ground. Here, $V_{out} = -\frac{R2}{R1}V_{in}$ and $A_{cl} = -\frac{R2}{R1}$. This is a voltage amplifier that inverts the incoming voltage.

2.4 Buffer/voltage follower

Start by constructing a non-inverting amplifier circuit, but don't add any resistors. That means that $V_{out} = V_{in}$ and $A_{cl} = 1$. If there is no power in, there will be no power out. This forms a power amplifier. A little math yields that $V_{out} = \frac{A}{A+1}V_{in}$.

2.5 "Push/Pull" Mode

Push Mode Consider an opamp circuit wired up with the output connected to a resistor $R_2 = 49k\Omega$ and then to the negative input. Also connected to the input is a resistor $R_1 = 1k\Omega$, which is connected to ground. Connected to the output is a resistor $R_4 = 500\Omega$ that is connected to ground also. A pair of 8V batteries are connected to the power-in ports of the opamp, and the positive input is connected to a 0.1V source.

Finally, ground (common) is connected to the spot between the batteries, and all "ground" connections are interconnected. This circuit has negative feedback, and $A_{cl} = 1 + \frac{49}{1} = 50$. If $V_{out} = 5V$, solve for all the currents.

A small amount of current flows through $R1$, about 0.1 mA. The same amount of current then flows through the start of the common ground connection. Out of opamp's input flows 10 mA, so combining the two on common ground yields 10.1 mA. By KCL, that means that the power supply to which common ground is connected must be generating 10.1 mA as well, which it sends into the amplifier. That also means that 10.1 mA is flowing out the opamp, both from V_{out} and from the other power supply terminal. 10 mA of this goes into common ground, while the remaining 0.1 mA goes back to the feedback loop.

What this means is that there is a power loop running through the opamp, out V_{out}, through R_4 into common ground, back to the power supply, and through the opamp again. The second loop (the feedback loop) then controls the whole thing. Here, the amplifier sources the current.

Pull Mode This is basically the same thing as the circuit from "push mode", just with $V_{in} = -0.1$ instead. Similarly, that means that $V_{out} = -5V$ and $V* = -0.1V$. Current directions are reversed, so the feedback loop runs in the other direction, and current (0.1 mA) flows out of common ground and into the feedback loop, just as current flows into V_{out} instead. That makes the total current flowing from common ground 10.1 mA. The whole circuit shows the same power loop effect, except this time current flows out of the opamp's negative rail instead of the positive one.

Unlike the previous circuit, this one sinks the current. However, a small feedback current still controls the much larger main current.

2.6 Amplifier Circuit Design

Choosing proper resistor values Consider a noninverting amplifier with R_1 connected to ground, R_2 connected to feedback, and R_L connected to V_{out} and common ground. Also consider a similar setup for in inverting amplifier. The feedback loop doesn't actually do anything involving power delivery, it just has to do with regulating voltage gain (?). However, energy is still dissipated by the resistors in the loop, so care must be taken when choosing resistors.

To avoid high losses, R_1 and R_2 should be greater than 50 to 100Ω. They can't be too high either, as high resistors are close to being open circuits (sort of), so R_1 and R_2 should be less than or equal to 1 MΩ. Obviously, their resistance should be stable too. Finally, the output resistor R_L should be at least

Using multiple stages Putting too much gain on one amplifier is a bad idea. In general, don't use a gain higher than 100, or you're likely to run into problems with stability. If for some reason you really *do* need gain that absurdly high, use multiple stages of opamps for your amplification. As long as each individual stage doesn't have a gain higher than 100, you should be able to construct a multistage amplifier that gives the gain you want. Stage gains multiply, so you could construct a 10,000 gain amplifier using a pair of 100 gain amplifiers.

Input resistance Also known as impedance. If you connect a sensor to an amplifier, the sensor doesn't "see" the whole amplifier circuit, it sees the input resistance to the amplifier circuit R_{in}. In the case of a noninverting amplifier, no current can flow into the amplifier and $R_{in} = \frac{V_{in}}{I_{in}}$, so R_{in} must necessarily be ∞. Inverting amplifiers, however, *can* allow current to flow through their input. No current flows into the opamp itself, so the impedance is simply R_1.

Voltage amplifier vs. matched amplifier Once again, consider a simple sensor circuit connected to an amplifier and a resistance R_S on the sensor's side. For a matched amplifier, R_{in} should be equal to R_S. All wireless and RF work uses this, as it delivers maximum power from the sensor to the amplifier circuit. Also, there's some magic words in here about "no reflections", whatever that means (seriously, I was just told to write it down...).

In case of voltage amplifiers, choose $R_{in} = \infty$. This effectively creates an open circuit, and is used for all general purpose amplifiers.

2.7 Instrumentation Amplifier

Differential sensor (wheatstone bridge) A typical sensor uses a voltage divider in order to be able to output different voltages as a form of readout. Start by constructing a wheatstone bridge with $R_3 = 350\Omega$, $R_4 = 350 \pm 1\Omega$, a voltage at b of $4.5V \pm 6mV$, and $V_{cc} = 9$V. For the second voltage divider (connected in parallel to the first), use the exact same settings (including a voltage of 4.5V at a).

The differential voltage is given by $V_D = V_a - V_b = \pm 6mV$. The common mode voltage V_{cm} is given by $\frac{V_a+V_b}{2}$. The challenge, now, is to construct an amplifier circuit that can amplify this tiny voltage into something more useful, but we need an amplifier with two inputs; neither inverting nor non-inverting amplifiers have that, they each have only one input.

Differential amplifier This type of amplifier mixes inverting and non-inverting amplifiers. Start by wiring a resistor (R_1) into the negative input (V_b), then wiring a resistor after that one (R_2) that bridges the gap between V_{out} and the negative input. For the positive input (V_a), wire another resistor in (R_3), but immediately after that, wire in a fourth resistor (R_4) that connects to ground. The currents across each resistor are named similarly. Here's how you solve for everything, using summing constraints

$$
\begin{aligned}
I_1 = I_2 \rightarrow \frac{V_b - V^*}{R_1} &= \frac{V^* - V_{out}}{R_2} \\
I_3 = I_4 \rightarrow \frac{V_a - V^*}{R_3} &= \frac{V^* - 0}{R_1} \\
V_b &= V^* + \frac{R_1}{R_2}(V^* - V_{out}) \\
V_a &= V^* + \frac{R_3}{R_4}(V^* - 0)
\end{aligned}
$$

Now assume that $\frac{R_1}{R_2} = \frac{R_3}{R_4}$, which is usually the case in these types of circuits anyways. After doing some magic math:

$$V_{out} = \frac{R_2}{R_1}(V_a - V_b)$$

That means that the differential closed loop gain is just $A_{cl} = \frac{R_2}{R_1}$.

Buffer stage Also known as a unity common-mode gain stage. Suppose you take two non-inverting amplifiers and wire V_b into one and V_a into the other. Instead of a common ground connection to the negative inputs, wire the two ground connections together and get rid of ground altogether. This is a critical step in making the whole thing work. The current running along the loop in the V_b amplifier is called I_b, and vice versa for the other amplifier. Call the outputs for each V_b^* and V_a^*, respectively.

Solving for this thing's behavior once again involves the summing point constraints:

$$I = I_b \rightarrow \frac{V_a - V_b}{2R_3} = \frac{V_b - V_b^*}{R_4}$$
$$I = I_a \rightarrow \frac{V_a - V_b}{2R_3} = \frac{V_a^* - V_a}{R_4}$$
$$V_b^* = V_b - \frac{R_4}{2R_3}(V_a - V_b)$$
$$V_a^* = V_a + \frac{R_4}{2R_3}(V_a - V_b)$$
$$V_a^* - V_b^* = V_a - V_b + \frac{R_4}{R_3}(V_a - V_b)$$
$$V_a^* - V_b^* = (1 + \frac{R_4}{R_3})(V_a - V_b)$$

The above is the final formula for this thing's output. Putting one of these between the connection from your wheatstone bridge and your difference amplifier will create the fabled instrumentation amplifier. *That* output will given by $V_{out} = (1 + \frac{R_4}{R_3})\frac{R_2}{R_1}(V_a - V_b)$.

2.8 Various Amplifiers

Transresistance amplifier These are build just like inverting amplifiers, but with no input resistance (here, the only resistor is called R_f). The current in I is equal to $\frac{0V - V_{out}}{R_f}$, so $V_{out} = -R_f I_{in}$.

Transconductance amplifier Built similarly to non-inverting amplifiers, these have a resistor R_L attached to output, R_F leading from that to ground, and another wire from R_L into the negative input. V_{in} is wired into the positive input. Here, $I_{out} = R_f V_{in}$. That means the current only depends on the control voltage and the feedback resistance; the load resistance has nothing to do with anything.

Current amplifier These look like weird transresistance amplifiers. The feedback resistor is called R_2, a resistor from V_{out} to R_2 is R_L, and a resistor from R_L to ground is R_1. I_{in} is wired into the negative input, and positive input is wired into ground. The current I_{out} is given by $1 + \frac{R_2}{R_1} I_{in}$.

General feedback system For this we'll need something a little more abstract. There is a source block that emits a signal X_{in}. It could be anything, even a function of time. That signal is connected to a summing block that does something. Into the summing block also feeds a negative input x_f (for feedback). The summing block performs some kind of difference operation and outputs a signal x_e into an amplifier block, which outputs a signal Ax_e. Part of this signal goes into X_{out}, while the remaining part goes into the feedback gain block β. β's output $\beta x_{out} = x_f$ then gets routed back into the summing block's feedback input. To summarize:

- x — a signal, e.g. voltage, current, pressure, pH, etc.
- A — open-loop gain block. $x_{out} = Ax_e$.
- β — feedback gain block. $x_f = \beta x_{out}$.
- Σ — summing block. $x_e = x_{in} - x_f$.

Now, solve for the behavior of the system…

$$\begin{aligned}
x_{out} &= Ax_e \\
&= A(x_{in} - x_f) \\
&= A(x_{in} - \beta x_{out}) \\
x_{out} &= Ax_{in} - \beta A x_{out} \\
&= x_{out} + \beta A x_{out} \\
x_{out} &= \frac{A}{1 + \beta A} x_{in}
\end{aligned}$$

The last line above contains the expression for the closed loop gain of the circuit: $\frac{1}{1+\beta A}$. If you let $A \to \infty$, then $A_{cl} = \frac{1}{\beta}$, and thus does not depend on the value A. Also note that as a result of all of this, the error signal x_e tends toward 0, which is exactly the second summing point constraint of opamps.

3 Dynamic Circuit Elements

3.1 Capacitors

Capacitors are constructed as two parallel plates (usually) with some dielectric material in between (usually). Current flowing through will force charge to accumulate on both plates, where energy is stored in the resulting electric field. The more charged the plates are, the more difficult it becomes for current to flow through. In a waterworks analogy, capacitors are analogous to a sheet of rubber stretched over a pipe; for a time, they'll allow current to flow, but when the pressure is released they'll exert pressure of their own in the opposite direction, at least for a while.

In the passive reference configuration, current flows from positive to negative, and there is a corresponding voltage drop across it. Current is given by:

$$I_c = C\frac{dV_c}{dt}$$

…where C is the capacitance in Farads (usually given in pF or μF) and $\frac{dV_c}{t}$ is the derivative of the voltage across the capacitor over time. Actually, this yields something called the Maxwell displacement current. Interestingly, while DC current will eventually stop flowing through a capacitor, AC current will be able to pass unhindered. Furthermore, capacitors can create large currents for a very short period of time, as they can discharge extremely quickly.

3.1.1 Capacitors in Amplifiers

Here's a strange idea — take an inverting amplifier, but instead of an R_2 resistor, stick a capacitor in there instead. This is called a Miller Integrator, and it yields $V_{out} = -\frac{1}{RC}\int\limits_0^t V_{in}t'dt'$

3.1.2 Energy-release capacitor circuit

Consider a circuit with an open switch, a capacitor of strength 10 µF, and a resistor $R = 10\Omega$. The voltage across the capacitor is 10V at $t = 0$, but after $t = 0$, the switch is closed. As time approaches infinity, the voltage across the capacitor approaches 0. The problem now is to find the transient behavior of the circuit, i.e. how it behaves during $0 < t < \infty$.

By KVL, $V_R = V_C$. By KCL, $I_C = -I_R$. In the passive reference configuration, $I_C = C\frac{dV_C}{dt} = -\frac{V_C}{R}$ (after a little math). This is where things get into differential equations, because to solve this RC circuit's behavior, you need to do some magic math & calculus:

$$
\begin{aligned}
C\frac{dV_C}{dt} + \frac{F_C}{R} &= 0 \\
\frac{dV_C}{dt} + \frac{V_c}{RC} &= 0 \\
RC &= \tau \\
\frac{dV_c}{dt} + \frac{V_C}{\tau} = 0 \\
V_e &= ke^{-t/\tau}
\end{aligned}
$$

…where k is an unknown constant that can be solved for using the initial conditions of the circuit. Using this particular example where $V_C(t = 0) = 10V$, $k = 10V$, so $V_C(t) = 10e^{-t/\tau}$. Since we have the voltage, you can solve for the current and get $-\frac{10}{R}e^{-t/\tau}$.

Since the constant τ is given in units of seconds, it can be used to help graph the discharge behavior of the capacitor. Here, τ is 0.1 ms, and the graph looks like an exponential decay from 10V to 0V and exponential decay of current from -1A to 0A. When $t = \tau$, the voltage will be proportional to the original voltage multiplied by $\frac{1}{e} = 0.368$, so about a third of the original voltage will remain.

3.1.3 Energy-accumulating capacitor circuit

This is basically the same circuit as before, but where the capacitor starts in a discharged state and a voltage source of 10V wired in. Once again, suppose that $R = 10\Omega$ and $C =$ µF. I'm going to skip all the math this time, as it's basically the same as before, but in reverse. The differential equation is $\frac{dV_C}{dt} + \frac{V_C}{\tau} = \frac{10}{\tau}$. Doing a little magic math yields $V_C = ke^{-t/\tau} + 10$. The constant k can be found using the initial conditions, so $k = -10$. The final result is $10(1 - e^{-t/\tau})$.

3.1.4 Non-ideal digital waveform

Assume an input V_s that is the input from a logic gate, and is wired in series with a capacitor and a resistor R. The voltage source doesn't produce smooth input, it produces something like a square wave, a peak at 10V for every '1' bit that is transmitted. Also suppose that R has V_{out} across it. If there is no capacitor, $V_{out} = V_s$, but the capacitor is always present. When the input source sends a '1', the capacitor charges, and when a '0' is transmitted, the capacitor discharges. This can be used to turn a digital waveform into a regular waveform. If the time constant is small, these generated peaks will be narrower, and vice versa for larger time constants.

3.2 Inductor

Similar to capacitors, inductors store energy, but they store it in a magnetic field instead. Once again, in the passive reference configuration there is a voltage drop across the inductor. Inductors are constructed using coiled wire, often with some sort of core in the middle, and behave exactly the opposite to capacitors. In a way, they give current flowing across them "inertia" — they'll keep the voltage going until they've discharged. In a waterworks analogy, inductors are like a water wheel in part of the pipe that takes time to spin up (and thus resists the flow), but once started will provide its own pressure to keep things moving until it slows down.

Voltage is given by:

$$V_L = L\frac{dI_L}{dt}$$

…where L is the inductance given in Henrys (weird unit, I know) and $\frac{dI_L}{dt}$ is the derivative of current with respect to time. This is known as Faraday's law of induction. Unlike capacitors, DC current can flow through a conductor easily, but AC current struggles more. In fact, as the frequency of the AC current increases, the inductor begins to behave more and more as a simple open circuit. Also, again in contrast to capacitors, inductors can briefly create very high voltages (instead of very high currents).

3.2.1 Energy-release inductor circuit with a current source

In plain English, this just refers to a circuit that discharges an inductor. Consider a circuit with a current source that provides current $I_s = 1A$. In parallel, wire a switch, a 1 kΩ resistor (through which current I_R flows), and an inductor of strength 1 mH with current I_L through it. Note that each of these should each individually be wired in series.

When $t = 0$ (switch open), there is a DC steady state with 1A flowing through the inductor. When the switch is closed, however, all the current will start flowing through the newly closed switch instead (it's a short circuit), effectively turning off the current source. This leaves a circuit with an inductor and a resistor wired in a loop together; through the inductor we'll say is a voltage V_L, and across the resistor is a voltage V_R.

By KCL, $I_R = -I_L$, and by KVL $V_L = V_R$. However, $V_L = \frac{dI_L}{dt} = RI_R = -RI_L$, so work a little magic math (read: differential equations):

$$\begin{aligned} L\frac{dI_L}{dt} + RI_L &= 0 \\ \frac{dI_L}{dt} + \frac{I_L}{L/R} &= 0 \\ \tau &= \frac{L}{R} \\ \frac{dI_L}{dt} + \frac{I_L}{\tau} &= 0 \\ I_L(t) &= ke^{-t/\tau} \end{aligned}$$

This is extremely close to the way capacitors discharge, complete with time constant τ in units of seconds. To solve for constant k, use conditions $t \leq 0 : I_L = 1A$ and $t \to \infty : I_L = 0$. So, $I_L(t = 0) = ke^{-0/\tau}$ and thus the final equations for this circuit are:

$$\begin{aligned} I_L(t) &= 1e^{-t/\tau} \\ V_L &= -1Re^{-t/\tau} \end{aligned}$$

Note that the value 1 comes from the starting conditions of the circuit, which involved a current of 1A. When graphed, this circuit discharges in a way identical to the discharge of a capacitor. Before the switch is closed, the current remains steady. When the switch is closed, the current drops off exponentially according to $I_L = 1e^{-t/\tau}$. Similarly, voltage follows an exponential decay curve, but this time leading from 1000V to 0V. When time equivalent to one time constant has passed (i.e. when $t = \tau$), 36.8% of the original value remains; that means 0.368A and -368V for this circuit.

The result of this circuit ends up being a very large voltage spike of 1000V, but it lasts only a brief moment, in this case only 1 μs. This makes these types of circuits useful for applications where massive voltage is required very quickly, like an ignition circuit (?).

3.2.2 Energy release inductor circuit with a voltage source

The above circuit is pretty impractical since it involves shorting out a current source. In reality, energy release inductor circuits are more often made with voltage sources. Wired in series with the voltage source is closed switch and a resistor $R_0 = 10\Omega$. Wired in parallel after those two components are the 1 kΩ resistor and the 1 mH inductor. Assume the same notation for currents, resistances, and voltages as from the previous example. At $t = 0$, the DC steady state is $I_L = 1A$. After $t = 0$ the switch is closed, so as $t \to \infty$, the DC steady state is $I_L = 0$.

We know from the previous circuit that $I_L = ke^{-t/\tau}$. From that, $I_L(t = 0) = k = 1A = \frac{10V}{R_0}$, so $I_L(t) = \frac{10V}{R_0}e^{-t/\tau}$. Using Ohm's law (?), $V_L(t) = -10V\frac{R}{R_0}e^{-t/\tau}$. This means that the ratio of the two resistors, $\frac{R}{R_0}$ actually defines the voltage spike seen when the circuit discharges.

3.2.3 Energy-accumulating inductor circuit

Create the same circuit as in the energy release inductor circuit with a current source, but with the switch being open at $t = 0$ and a current source instead. At $t = 0$, $I_L = 0$. As $t \to \infty$, $I_L = 1A$. For this circuit, $I_L = S + ke^{-t/\tau}$, and using the two known conditions we can find $S = 1A$ and $k = -1A$. So, $I_L(t) = 1A(1 - e^{-t/\tau})$, the exact same formula for the charging of a capacitor.

3.3 Timer/clock circuit

Positive Feedback Consider an opamp circuit with the inverting input connected to ground and the noninverting input connected to an input resistor R_1 and a feedback loop from the output containing a resistor R_2 By the first summing point constraint, no current flows into the amplifier. The input voltage to the noninverting input is given by $V^+ = \frac{R_1}{R_1+R_2}V_{out}$. However, since the inverting input is at zero, the second summing point constraint (the voltage difference between the two inputs is exactly zero) is now violated.[1]

This requires a closer look at positive feedback. Suppose that 50% of V_{out} is returned back in 1μs, that $R_1 = R_2$, $A = 10^6$, and $V_{out} = AV^+$. As time elapses, V^+ will increase from 0 to 1/2 mV to 1/4 V, while V_{out} will go from 1 μV to 1/2 V to V_{cc}. This means that regardless of what you choose, a positive feedback circuit will always end up yielding either V_{cc} or $-V_{cc}$.

That means the above two conditions are stable states. When $V_{out} = V_{cc}$, $V^+ = \frac{R_1}{R_1+R_2}V_{cc}$, and when $V_{out} = -V_{cc}$, $V^+ = -\frac{R_1}{R_1+R_2}V_{cc}$. This creates something of a voltage hill, from which it is extremely easy for voltage to go to either V_{cc} or $-V_{cc}$ depending on what the input is.

Concept of a memory cell Positive feedback circuits like the one above can be used to create simple memory cells since they'll keep their value even after the input voltage returns to 0 (that is, if you provide 1V of input, for example, even when the input drops back down to 0V the circuit will continue providing V_{cc} volts of output). To accomplish this, call the inverting input the "write trigger". As long as V_{cc} is greater than the input voltage, it's acceptable to have noise on the write trigger; the circuit will maintain its '1' (turned ON) state. To clear the set bit, send a quick pulse (preferably of the same magnitude as V_{cc}) to the write trigger, and the output will drop to $-V_{cc}$, which can be interpreted as a '0'. Applying an input of $-V_{cc}$ will do the opposite, causing output to go to V_{cc} instead, which can be interpreted as a '1'.

In reality, this isn't how actual memory is stored, since that would be horribly expensive and impractical. However, it is still a good example of a usage for positive feedback loops, and does demonstrate the concept behind simple memory cells.

Concept of an oscillator Unfortunately, this is another opamp circuit, using the same construction as the positive feedback loop, but with some additions. There's a loop connecting V_{out} to a resistor R and then to the negative input. Also connected to the negative input is a capacitor C, which is then connected to ground.

What ends up happening is that the capacitor will slowly charge until it hits a point ($\frac{V_{cc}}{2}$ if $R_1 = R_2$), at which point the circuit will abruptly flip from $V_{out} = V_{cc}$ to $V_{out} = -V_{cc}$. The capacitor will discharge and begin charging in other direction (...you know what I mean), until once again it hits that critical point and the circuit flips back. This process continues forever, generating a square wave output from V_{out}. The frequency can be calculated using information about the RC circuit built into the thing, using $F = \frac{1}{\tau}$.

3.4 Summary

I don't know why this is here. Anyways,

- RC Circuit Discharge: Use $\frac{dV_C}{dt} + \frac{V_C}{\tau} = 0$ to solve and get $V = V_c e^{-t/\tau}$. Remember that $\tau = RC$. At $t = \tau$, $V = 0.368V_c$. To get current, divide by negative resistance.
- RC Circuit Charge: $V = V_C(1 - e^{-t/\tau})$.

[1]Why this summing point constraint applies here is a mystery to me. It's only supposed to apply in circuits that have a negative feedback loop, but this one is a positive feedback loop.

- LR Circuit Discharge: $I = I_L e^{-t/\tau}$, where $\tau = \frac{L}{R}$. This is based on the differential equation $\frac{dI_L}{dt} + \frac{I_L R}{L} = 0$. To get voltage, multiply by negative resistance.
- LR Circuit Charge; $I = I_S(1 - e^{-t/\tau})$.

4 AC Circuits

AC circuits are circuits in which the current follows a sine wave instead of a simple direct flow of current. They are named AC for **A**lternating **C**urrent, as opposed to the **D**irect **C**urrent (DC) circuits covered so far.

4.1 Basics

The basic harmonic signal of alternating current is given by $V(t) = V_m \cos(\omega t + \phi)$. Any AC circuit or anything involving radio can be characterized using this formula. The maximum voltage produced is given by the term V_m, the frequency is given using ω (in the form of angular frequency, radians per second), and the phase (also known as offset, at least to me anyways, because it shifts the wave in time) is given by ϕ. The amount of time between peaks or troughs is the period T. ω can be calculated using $\omega = 2\pi f$ where f is frequency in Hertz (hz), and can also be found by $f = \frac{1}{t}$.

4.2 Phase

Phase is a relative measure, given against a base of the cosine function. If $\phi > 0$, the signal is said to *lead* the base signal cos by ϕ. On a graph, that means that the phase shifted wave appears to be offset to the left from the original wave. If $\phi < 0$, the signal is said to *lag* the base signal by ϕ, and on a graph this looks as if the wave is shifted to the right.

4.3 Phasers

Unfortunately, this has nothing to do with phasers as a type of weapon. Any harmonic signal like $V(t) = V_m \cos(\omega t + \phi)$ is redundant; all that's needed is V_m and ϕ. You can replace it all with a complex number $V = V_m e^{j\phi}$. Using more magic math, $e^{j\phi} = \cos\phi + j\sin\phi$. When graphed out with the real component on the x axis and the imaginary component on the y axis, you'll get a line from the origin whose length is given by V_m and whose angle from the x axis is given by ϕ. In the end, $V_m \cos(\omega t + \phi) = Re(V_m e^{j\omega t + j\phi})$, which simplifies to $V_m e^{j\phi}$.

Example Suppose you have a signal $V(t) = 2\cos(\omega t + \frac{\pi}{4})$. Converting this to a phasor turns it into $2e^{j\frac{\pi}{4}}$.

Example 2 Converting the phasor $2e^{-\frac{\pi}{4}j}$ to a regular signal will yield $2\cos(\omega t - \frac{\pi}{4})$.

4.4 Impedance

If you have a source of resistance in a circuit, current $I(t)$ will flow across it and there will be a corresponding voltage drop. In the context of AC, current and voltage are given by cosine functions, so things are a little more complicated. The phasor of voltage and current are both complex numbers with units of volts and amps, respectively. If you divide these two phasor, you get $Z = \frac{V}{I}$, or impedance (measured in Ohms). When graphed, voltage and current in this simple system will be in phase and of the same frequency, with current as a smaller wave than voltage. With a little math, you can prove (using $\frac{V}{I} = R$) that $|Z| = R$.

Capacitors Capacitors work a little differently. If you have a capacitor in series instead, you get $V(t) = V_m \cos(\omega t + \phi)$ and $i(t) = C\frac{dV}{dt} = C(-\omega)V_m \sin(\omega t + \phi)$ (or, using cosine, $i(t) = \omega c V_m \cos(\omega t + \phi + \frac{\pi}{2})$). That means that current leads voltage by $\frac{\pi}{2}$. Doing some magic math involving $Z_C = \frac{V}{I}$ shows that the impedance of a capacitor is:

$$Z = \frac{1}{j\omega C}$$

...and the magnitude of it is $|Z_C| = \frac{1}{\omega C}$.

Inductors Inductors work differently than capacitors, but since they're still first order dynamic circuit elements, the impedance should at least have a similar form. Here, let's do this math-style this time:

$$\begin{aligned}
i(t) &= I_m \cos(\omega t + \phi) = \mathbf{I} \\
V(t) &= L\frac{dI}{dt} = -I_m \omega \sin(\omega t + \phi) \\
&= \omega L I_m \cos(\omega t + \phi + \frac{\pi}{2}) \\
\mathbf{V} &= \omega L I_m e^{j\phi} e^{j\frac{\pi}{2}}
\end{aligned}$$

This shows that current lags voltage by $\frac{\pi}{2}$. Impedance is given by:

$$\begin{aligned}
\mathbf{Z} &= \omega L e^{j\frac{\pi}{2}} \\
\mathbf{Z}_L &= j\omega L \\
|\mathbf{Z}_L| &= \omega L
\end{aligned}$$

Application: Bypass capacitor Suppose you have a pair of power supplies wired in series, one AC, one DC, and in series a load is wired to them. To filter out the AC current, it is possible to wire in a capacitor in parallel with the load — the AC current will go through the capacitor instead of the resistor, while the DC current will go through the load. The challenge, then, is to figure out an appropriate impedance for the resistor. Set the capacitance to $1000\mu F$ and the frequency to 300 Hz...

$$\begin{aligned}
\mathbf{Z}_C &= \frac{1}{j\omega C} \\
\omega &= 2\pi f \\
\mathbf{Z}_C &= -j\frac{1}{\omega C} \\
&= -j0.53\omega
\end{aligned}$$

The resistor should be set to 0.53 Ω in order for the capactitor to effectively filter out the AC current.

Application: Blocking capacitor The opposite effect is possible if the capacitor is wired in series with the load instead of in parallel. This time, suppose you have a load of 8 Ω instead, and you must find the capacitance. Using the same process as above but in reverse, it is possible to find an appropriate capacitance value.

4.5 AC RC circuit

The circuit setup here involves an AC voltage source, a resistor, and a capacitor. The AC voltage source has $V_m = 5V$ and $\omega = 5000$, the resistor is a 1kΩ resistor, and the capacitor has a capacitance of 1 μF. You could spend a lot of time trying to solve this problem using differential equations, but if instead you change to the frequency domain and use phasor, things can be easier.

Step 1: Convert the circuit to phasor form. An imaginary current $\mathbf{I}$ will flow through the circuit, and the voltages are then $\mathbf{V}_S$, $\mathbf{V}_R$, and $\mathbf{V}_C$. Now that it's in the frequency domain, this is effectively a DC circuit.

Step 2: Solve the resulting DC circuit. This means finding all the impedances, which should be easy. $\mathbf{Z}_R = 1000\Omega$ and $\mathbf{Z}_C = \frac{1}{j\omega C} = -1000j$.[2] The equivalent impedance is:

$$\mathbf{Z}_{eq} = \mathbf{Z}_R + \mathbf{Z}_C = 1000 - 1000j$$

This could also be simplified to $1000(1-j)$.

Step 3: Find current. Do $\mathbf{I} = \frac{\mathbf{V}_S}{\mathbf{Z}_{eq}} = \frac{5}{\mathbf{Z}_{eq}}$. However, you need $\mathbf{Z}_{eq}$ in polar form to do this division, so get it in polar form as $1000\sqrt{2}e^{-j\frac{\pi}{4}}$. That makes the current $\frac{5}{1000\sqrt{2}e^{-j\frac{\pi}{4}}}$.[3]

Step 4: Find voltages across components. Once again treating it like a DC circuit, the voltage across the resistor is just $\mathbf{V}_R = \mathbf{Z}_R\mathbf{I}$, which evaluates to $\frac{5}{\sqrt{2}}e^{j\frac{\pi}{4}}$. Again using the polar form of $\mathbf{Z}_C$, the voltage across the capacitor is $\mathbf{V}_C = \frac{5}{\sqrt{2}}e^{-j\frac{\pi}{4}}$.

Step 5: Convert the phasors back to the time domain. It's important to have these in the form they've been solved as $(ce^{c_2 j})$ or else conversion will be more difficult. The current is $I(t) = \frac{5}{1000\sqrt{2}}\cos(\omega t + \frac{\pi}{4})$, the voltage across the resistor is $V_R(t) = \frac{5}{\sqrt{2}}\cos(\omega t + \frac{\pi}{4})$, and the voltage across the capacitor is $V_C(t) = \frac{5}{\sqrt{2}}\cos(\omega t - \frac{\pi}{4})$. This shows that in an AC circuit, capacitors will cause phase shifts. If the capacitor in this circuit was switched out to another resistor, ϕ would be 0.

4.6 AC RL Circuit

Construct the same circuit as the AC RC circuit, but with a 1H inductor instead. The resistor's impedance is already known, so the only impedance left to solve is $\mathbf{Z}_L = j\omega L = 1000j$. The equivalent resistance is then $\mathbf{Z}_{eq} = 1000 + 1000j$. Again, $\mathbf{Z}_{eq}$ needs to be converted to polar form, so it gets changed to $1000\sqrt{2}e^{j\frac{\pi}{4}}$. That makes the current across the circuit equal to $\frac{5}{1000\sqrt{2}}e^{-j\frac{\pi}{4}}$. Voltage across the resistor is then $\frac{5}{\sqrt{2}}e^{-j\frac{\pi}{4}}$, and voltage across the inductor is $\frac{5}{\sqrt{2}}e^{j\frac{\pi}{4}}$.

Converting back to the time domain, current is $I(t) = \frac{5}{1000\sqrt{2}}\cos(\omega t - \frac{\pi}{4})$, voltage across the resistor is $V_R(t) = \frac{5}{\sqrt{2}}\cos(\omega t - \frac{\pi}{4})$, and voltage across the inductor is $V_R(t) = \frac{5}{\sqrt{2}}\cos(\omega t + \frac{\pi}{4})$. This the same phase shift as the capacitor, just in the opposite direction.

4.7 RC filters

A low pass filter may be constructed with an AC source connected to a resistor and a capacitor after the resistor that bridges positive input and ground. Wired in parallel is the voltage output. An alternate output involves a single terminal AC voltage source, the resistor, and the capacitor afterward simply wired into ground.

This circuit can be treated as if it were a simple RC circuit in AC. The output voltage will be equal to the voltage across the capacitor. Fortunately, we already know how to solve this circuit, but this time we need to solve it for *all possible frequencies.* After some magic math:

$$\mathbf{V}_C = \mathbf{Z}_C\mathbf{I} = \frac{\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C}V_m$$

This is based on the principles behind a simple voltage divider. Expanding this yields:

$$\mathbf{V}_C = \frac{1}{1 + j\omega\tau}V_m$$

In polar form, this becomes:

$$\mathbf{V}_R = \frac{\omega\tau}{\sqrt{1 + (\omega\tau)^2}}\left(\frac{e^{j\frac{\pi}{2}}}{e^{j\arctan(\omega\tau)}}\right)$$

$$\mathbf{V}_C = \frac{1}{\sqrt{1 + (\omega\tau)^2}}\left(\frac{1}{e^{j\arctan(\omega\tau)}}\right)$$

[2] When solving this, it is useful to know that $\frac{1}{j} = -j$.

[3] To convert to polar form, do $\mathbf{Z} = \sqrt{x^2 + y^2}e^{j\arctan\frac{y}{x}}$

And so in the time domain, the formulas are:

$$V_R(t) = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}} V_m \cos(\omega t + \frac{\pi}{2} - \phi)$$
$$V_C(t) = \frac{1}{\sqrt{1+(\omega\tau)^2}} V_m \cos(\omega t - \phi)$$

These functions front of the cosine are called *amplitude transfer functions*, usually denoted as $H_m(f)$. For a low pass filter (as shown here), the output is the voltage across the capacitor, while for a high pass filter the output is the voltage across the resistor, since the capacitor and the resistor are switched. The transfer function for the low pass filter is $H_m(f) = \frac{1}{\sqrt{1+(2\pi\tau f)^2}}$, and the "break frequency" is $f_B = \frac{1}{2\pi\tau f}$ (units of Hertz). That means the transfer function can be expressed as:

$$\frac{1}{\sqrt{1+(f/f_B)^2}}$$

As $f \to \infty$, this function will approach zero. In high pass filters, $H_m(f) = \frac{2\pi\tau f}{\sqrt{1+(2\pi\tau f)^2}}$. Using break frequency, the transfer function is:

$$\frac{f/f_B}{\sqrt{1+(f/f_B)^2}}$$

As $f \to \infty$, this function will approach 1.

4.8 LR Filters

Filters can also be constructed using inductors, but the position of the inductor and the resistor must be swapped relative to the corresponding RC filter circuit. That is, in a low pass filter the resistor comes first and the capacitor comes second, but if an inductor is used the inductor comes first and the resistor comes second, and vice versa for high pass filters. For the two filters to be equal, the time constant τ must be equal across both circuits, i.e. $\frac{L}{R} = RC$. Regardless, nobody *actually* uses inductors for filtering purposes, since they're too expensive and bulky in comparison to small and cheap capacitors.

4.9 Decibels

Warning: this section is very disjointed!

In the above two sections, transfer functions were described; the challenge now is to plot them. This involves the break frequency f_B, also known as a "corner" frequency. Decibels are a convenient way to do this on a logarithmic scale. This is because the values produced by transfer functions vary significantly, so logarithmic plots are most suitable.

Decibels, measured in dB, are dimensionless; they have no proper physical representation. Keeping in mind that $H_m(f)$ is the transfer function, the decibel form is $H_m(f)_{dB} = 20\log_{10} H_m(f)$. However, we want to plot power...

Anyways, let's start with some values for a low pass filter $f_B = 100$Hz (and using shorthand for $H_m(f)$ to become Hm and HdB). When frequency is $10^0, Hm = 1$; $10^1, Hm = 1$; $10^2, Hm = 0.71$. $10^3, Hm = 0.1$; $10^4, Hm = 0.01$, and so on. When frequency is $10^0, HdB = 0$; $10^1, HdB = 0$; $10^2, HdB = -3$. This number is the magic number where half of the *power* is lost. Anyways, when frequency is $10^3, HdB = -20$; $10^4, HdB = -40$, and so on. This is all based on plugging values into $H_m(f)_{dB} = 10\log_{10}(H_m(f))^2$. I have no idea where that comes from.

Bode plots are used to graph these sorts of relationships, where $x = f$ and $y = H_m(f)_{dB}$, with the x axis set to be a log scale. For a low-pass filter of $f_B = 100$Hz, the graph starts with 0 at the top and -40 dB on the bottom; the graph itself starts out flat at the top left, then curves into linear decay; the unit of time in which the frequency increases by an order of magnitude is called a *decade*. In this case, we say that the graph has 20 dB per decade. When the high pass filter of the same f_B is graphed instead, the result is exactly symmetrical across the magic line of -3 dB at $f = 10^2$, still with 20 dB per decade. In both cases, the region of values close to 0 on one side of the break frequency is called the *pass band*; for low pass filters, this is to the left of the break frequency, while for the high pass filters it's to the right.

4.10 Cascaded filters

Just like amplifiers, filters can be cascaded to improve filtering performance. Note that this is not the *only* way to improve filter performance, just the first go-to solution. Suppose you chain two low pass filters of $f_B = 100$ Hz together. On the Bode graph, f_B yields a value of -6 dB instead and the graph afterwards has 40 dB per decade, so cascading amplifiers ends up adding the amount of dB per decade together.

4.10.1 LC Circuit

An LC circuit can be constructed by wiring an inductor and a capacitor together in a loop. Across the inductor is voltage V_L and across the capacitor is voltage V_C; around the circuit, coming from the negative side of the capacitor and into the positive side of the inductor, is current $i(t)$.

As usual, start by converting this circuit into phasor form gets you $\mathbf{V}_L + \mathbf{V}_C = 0$ and (eh screw it, let's drop the mathbf notation) thus $(Z_L + Z_C)I = 0$. This can't happen for normal numbers, but it turns out that it *can* for imaginary numbers. All you have to do is add the two formulas together:

$$\begin{aligned} j\omega L + \frac{1}{j\omega C} &= 0 \\ \omega L - \frac{1}{\omega C} &= 0 \\ \omega_0 &= \frac{1}{\sqrt{LC}} \end{aligned}$$

This frequency is the undamped resonant frequency — converting this into frequency in Hz yields:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Now suppose you add an AC voltage source and a resistor of resistance R and voltage V_R across it, creating a series resonant circuit. The equivalent impedance across the circuit is $Z_{eq} = R + j\omega L + \frac{1}{j\omega C}$. When the circuit starts to resonate, however, the two impedances from the capacitor and the inductor cancel each other out, resulting in the equivalent resistance simply being R. Therefore, the circuit current is $\frac{\mathbf{V}_S}{R} = \frac{V_m}{R}$. The voltage across the resistor is $Z_R I = V_m$. The voltages across the capacitor and the inductor are:

$$\begin{aligned} V_L &= \frac{V_m}{R}\omega_0 L e^{j\pi/2} \\ V_C &= \frac{V_m}{R}\frac{1}{\omega_0 C}e^{-j\pi/2} \end{aligned}$$

A dimensionless quantity Q is involved that is the quality factor of the resonator: $Q = \frac{\sqrt{L/C}}{R}$. Using this value, you can rewrite the two voltages as:

$$\begin{aligned} V_L &= QV_m e^{j\pi/2} \\ V_C &= QV_m e^{-j\pi/2} \end{aligned}$$

On a phasor diagram, this looks like a vector for V_L pointing straight up and a vector V_C pointing straight down, both with magnitude QV_m. In the time domain, everything looks like this:

$$
\begin{aligned}
i(t) &= \frac{V_m}{R}\cos(\omega t) \\
V_R(t) &= V_m \cos(\omega t) \\
V_L(t) &= QV_m \cos(\omega t + \pi/2) \\
V_C(t) &= QV_m \cos(\omega t - \pi/2)
\end{aligned}
$$

Now you have to figure out how to choose the factor Q. Suppose you're working with a transmitter circuit with $L = 50\mu H$, C = 2nF, $f_0 = 500$Hz, and $w_0 = 2\pi f_0$. Using these numbers you can calculate that $Q = 3.2$. On the receiver end, take L=0.85 mH, C=120 pF, and R=10 to get $Q = 266$.