PH 1121 Principles of Physics: Electricity and Magnetism

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1 Electric Charge

Objects can carry electrical charge, a signed quantity. Like charges repel, different charges attract, with strength proportional to the absolute magnitude of the charges. The magnitude of an observable charge is an integer number of e, $Q = Ne$. The unit of charge is the Coulomb, where $e = 1.6 \times 10^{-19}$C. Electrons have a charge of $-e$ and protons have a charge of e; this attraction between the two is what keeps atoms together. Protons, however, are much heaver, having a mass of 1.673×10^{-27}kg compared to electron's 9.109×10^{-31}kg. For any isolated system, the algebraic sum of all electric charges must remain constant.

1.1 Coulomb's Law

Coulomb's law is an inverse-square law. The law states that the magnitude of the electric force between two point charges is inversely proportional to the square of the distance between the two charges, and directly proportional to the product of the magnitude of the charges. This law was derived primarily through experimentation, and is expressed mathematically as:

$$\vec{F} = k\frac{q_1 q_2}{d^2}(\hat{r}_{12}) \tag{1}$$

The value k is simply a constant: $k = 9 \times 10^9\ \mathrm{m^2N/C^2}$. This constant is derived from $\frac{1}{4\pi\varepsilon_0}$, so long as the environment is a vacuum. In other materials, $k = \frac{1}{4\pi\varsigma_0\varepsilon}$. (note that ε_0 and ε appear to be magic numbers). $\hat{r}_{12}$ is simply a unit vector corresponding to the direction of the force experienced by charge 1 due to charge 2's influence.

These forces are subject to the superposition principle; to find the net force from one point on any other point, you need only sum the force from each other individual charge. For example, if you have three charges q_1, q_2, q_3, finding the total force on q_1 means finding $\vec{F}_{12} + \vec{F}_{13}$.

1.2 Electric field

Electric fields are created by charged entities. Like other fields (e.g. gravitational fields) they exist in space independent of whether there's matter there or not. Electric fields are mathematically represented as vector fields, in the form $\vec{E} = \frac{\vec{F}}{q}$. The electric field is always there regardless of the presence of a test charge.

To calculate the electric field of a point charge, use Coulomb's law. Since there is no test charge here, just remove the term for the second charge:

$$\vec{E} = k\frac{q_1}{d^2}\hat{r} \tag{2}$$

This creates a vector field of vectors all pointing away from or towards the charge (away from if positive, towards if negative); vectors that have the same distance from the charge have the same magnitude, and their magnitude falls off with the square of the distance to the charge.

Electric field due to a charged line Suppose that you have a continuous distribution of charges. Along the X axis let a line lie between $-a$ and a, with total charge Q that is distributed uniformly across the line. The linear charge density λ is then $\lambda = \frac{Q}{L} = \frac{Q}{2a}$. The challenge, now, is to find the electric field for an arbitrary point P located at coordinates $(0, R)$. The trick here involves "cutting" the line into many smaller pieces, each one identical and infinitely small — specifically, of length dx. The charge at a point is $dq = \lambda dx = \frac{Q}{2a}dx$. Then $d\vec{E} = \frac{kdq}{r^2}\hat{4}$, and $\vec{r} = -x\hat{\imath} + R\hat{J}$. Here, have an align*:

$$
\begin{aligned}
&= \frac{k\lambda dx}{r^3}\vec{r} \\
&= \frac{k\lambda dx}{(x^2+R^2)^{3/2}} \\
dE_x &= \frac{-k\lambda x dx}{(x^2+R^2)^{3/2}} \\
dE_y &= \frac{k\lambda R dx}{(x^2+R^2)^{3/2}} \\
E_x &= \int_{-a}^{a} dE_x \\
&= \int_{-a}^{a} \frac{-k\lambda x}{(x^2+R^2)^{3/2}} dx \\
E_y &= \int_{-a}^{a} \frac{k\lambda R dx}{(x^2+R^2)^{3/2}} dx \\
E_x &= 0 \\
E_y &= \frac{k\lambda 2a}{R\sqrt{a^2+R^2}}
\end{aligned}
$$

Note the interesting result that if the line is infinitely long, the result is then $\frac{2k\lambda}{R}\hat{j}$.

Electric field due to a charged disk Consider a charged disk with radius R and total charge Q. The area of the disk is obviously defined as $A = \pi R^2$. To find the surface charge density, just divide the total charge by the area to get $\sigma = \frac{Q}{A}$.

This is where things get calc-y. Define an infinitely thin ring on this disk to be r' in radius and of dr' thickness. If the ring is centered on the origin and facing a direction parallel to the Z axis, then the distance to a given point along the Z axis from the disk is defined by the angle θ and a distance r. By that logic, $dA = dr'r'd\theta$, and $d\vec{E} = \frac{kdq}{r^2}\hat{r}$.

That means that the electric field of a charge ring q and radius r' is $E_z = \frac{kqz}{(r'^2+z^2)^{3/2}}$. Anyways, align*..

$$
\begin{aligned}
dE_z &= \frac{kz\sigma 2\pi r' dr'}{(r'^2+z^2)^{3/2}} \\
&= 2\pi k\sigma z \int_{0}^{R} \frac{r'dr'}{(r'^2+z^2)^{3/2}}
\end{aligned}
$$

After running through that calculation (which turns out to be reasonably easy), the final formula is:

$$
E_z = 2\pi k\sigma(1 - \frac{z}{\sqrt{R^2+z^2}}) \tag{3}
$$

Electric dipole in uniform electric field If you have a dipole with charges of $+q$ and $-q$ separated by distance $\vec{r}$, the dipole moment is $\vec{=}q\vec{r}$. If the dipole is in a uniform electric field, it makes an angle θ with the field. The torque on the dipole is then given by:

$$
\vec{\tau} = \vec{r} \times (q\vec{E}) \tag{4}
$$

This is also equivalent to $\vec{p} \times \vec{E}$. The magnitude of the torque is then $|\vec{\tau}| = |\vec{p}||\vec{E}|\sin\theta$. When the angle is 0, there is no torque. The work done by changing angles should then be $U(\theta) = -\vec{p}\vec{E} = -pE\cos\theta$.

1.3 Field lines

Field lines are used to show the magnitude and direction of electric fields. They have a few properties:

- The tangent direction at each point on the line is the direction of the electric field.
- Electric field lines never intersect
- Electric field lines terminate on charges; the originate from positive charges and end on negative ones.
- The density of electric field lines is proportional to the intensity of the electric field at that location.

1.4 Electric Flux

Electric flux is the amount of electric field through a defined surface. Using the concept of electric field lines as a guide, flux is equivalent to the amount of field lines passing through a point. Mathematically, flux is given by:

$$\phi_E = \vec{E} \cdot \vec{A} \cos\theta \tag{5}$$

...where $\vec{E}$ is the electric field, $\vec{A}$ is the surface, and θ is the angle to that surface. Since this is the dot product between two vectors, electric flux is actually a scalar quantity. If $\vec{A}$ defines a flat plane, the angle θ should be 90^o when the plane is perpendicular to the field.

Also, Guass's law is relevant here. Gauss's law states that the total electric flux through a closed surface is equal to the total (net) charge enclosed by the surface and divided by ε_0:

$$\phi_E * = \int \vec{E} \cdot \vec{A} = \frac{q_{enc}}{\varepsilon_0} \tag{6}$$

For example, consider a point charge located in the center of a cube. You can define $\phi_E = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6$, or just $6\phi_1$ where $\phi_1 = \frac{q}{6\varepsilon_0}$. For any closed conductor, the electric field close to the surface is $\frac{\sigma}{\varepsilon_0}$, while the inside will have no electric field. This principle is responsible for the effect of Faraday cages, which prevent electric fields from penetrating their walls.

Example Consider an infinite sized plane with surface charge density σ and a cylindrical Gaussian surface sticking through it. There are three ϕ values: one for each end and one for the main cylinder side surface. The charge enclosed by the surface is just $q_{encl} = \sigma A$. Based on Guass's law, $\phi_E = \frac{q_{encl}}{\varepsilon_0}$. Aaaand...example over. Sorry.

Example 2 Consider an infinitely long positively charged line with linear charge density λ. $\phi_E = E_A \cdot 2\pi r L$, and $q_{encl} = \lambda L$. Since $\phi_E = \frac{q_{encl}}{\varepsilon_0}$, putting it all together yields $E_A = \frac{\lambda}{2\pi r \varepsilon_0}$.

Example 3 A charged sphere contains a charge within it (...duh...) with charge density p_0 and radius R. $\phi_E = \int \vec{E} \cdot d\vec{A} dr$ (?) and $q_{encl} = P_0 \cdot \frac{4}{3}\pi r^2$. Also, end example. Again.

2 Electric Potential Energy

Warning: this section might be a little incoherent

Suppose you have two points A and B within an electric field. A path between them can be found that yields a value $W_e = \int_A^B \vec{F} \cdot d\vec{l}$. The potential difference between these points is δV, or $-(V_B - V_A)$. The voltage can be found as $V = \frac{U}{q}$ where U is the potential energy and q is the charge in question.. More math stuff: For the same two charges in the same field, you can use $\frac{1}{2}mV_B^2 + qV_B = \frac{1}{2}mV_A^2 + qV_A$. This is effectively an energy conservation law.

It's possible to calculate electric potential using an electric field by using $V_A - V_B = \int_A^B \vec{E} \cdot d\vec{l}$, where $\vec{l}$ is the path between the two charges. Because electric force is a conservative force, the actual path taken does not matter; a straight line works just as well as a *very* loopy curve.

Example Consider a positive charge centered at the origin and two points A and B that are distances r_A and r_B, respectively. Find the potential difference.

Integrating this using a straight line would be tricky, so draw two circles centered around the origin, one that contains A and the other that contains B. Assuming $r_B > r_A$, draw a straight line radially inward from B to the circle containing A, and call it C. Now you can integrate like so:

$$
\begin{aligned}
V &= \int_A^C \vec{E} \cdot d\vec{l} + \int_C^B \vec{E} \cdot d\vec{l} \\
&= \int_C^B E dl \\
&= \frac{q}{4\pi\varepsilon_0}(-\frac{1}{r})|_{r_C=r_A}^{r_B} \\
V_A - V_B &= \frac{q}{4\pi\varepsilon_0}(\frac{1}{r_A} + \frac{1}{r_B})
\end{aligned}
$$

For one point and something something infinity, $V = \frac{q}{4\pi\varepsilon_0 r}$. Most of the complex integration work is skipped above, but it shouldn't be too difficult to figure out.

Example 2 Place a solid cylinder with charge per unit length λ inside a second, larger cylinder. The first cylinder has a radius of r_a and the second has radius of r_b (the cylinders are A and B, respectively). Find the potential difference between the inner and outer cylinders.

$$
\begin{aligned}
V_A - V_B &= \int_A^B \vec{E} \cdot d\vec{l} \\
&= \int_A^C \vec{E} \cdot d\vec{l} + \int_C^B \vec{E} \cdot d\vec{l} \\
&= \int_{r_A}^{r_B} \frac{\lambda}{2\pi\varepsilon_0 r} dr \\
V_A - V_B &= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r_B}{r_A}
\end{aligned}
$$

This formula can be applied for any arbitrary set of cylinders.

Example 3 Find the electric field in vector notation when $V_{(x,y,z)} = 3x^2 + 2y + 100$ V.

Use $\vec{E} = -\frac{\delta V}{\delta x}\hat{\imath} - \frac{\delta V}{\delta x}\hat{\jmath} - \frac{\delta V}{\delta z}\hat{k}$. Doing this yields $-6x\hat{\imath} - 2æ$ (there are partial derivatives involved).

Example A charge insulator bar of length L and total charge Q is located on the x axis. Find the electric potential at point P, which is located at $y = d$.

Chop the whole thing into small segments dx. Then $dq = \lambda dx = \frac{Q}{L} dx$. That means that $dV = \frac{kdq}{\sqrt{x^2+d^2}}$. Anyways, doing some integration yields $\frac{kQ}{L} \ln x + \sqrt{x^2 + d^2}|_{-L/2}^{L/2}$. Solving that yields:

$$V(y) = \frac{kQ}{L} \ln \frac{\frac{l}{2} + \sqrt{\frac{L^2}{4} + y^2}}{\sqrt{\frac{L^2}{4} + y^2} - \frac{L}{2}}$$

3 Capacitance

Capacitors are devices that function sort of like batteries in that they provide voltage, but instead of storing power in chemical form, they store it in electrical fields. In circuit diagrams, capacitors usually appear as two parallel lines of equal length, with terminals coming out from the middle of each.

Capacitors are measured in C/V, or the Faraday (F). Capacitance does not depend on how much charge is stored in the capacitor. Increase or decreasing the charge on the capacitor leads to increase or decrease in the potential difference between the plates. Capacitance itself depends on the physical properties of the capacitor, such as shape, size, and arrangement of the conductors. Physically, capacitors are usually constructed with two sheets of a conductor with a dielectric material between them.

Parallel plate capacitor Two conducting plates of area A are separated by distance d. For now, suppose that there is only a vacuum between the plates. The electric field is a uniform electric field from the positive plate to the negative plate. The constant σ is $\frac{Q}{A}$ and the field is then $\frac{Q}{A\varepsilon_0}$. To find potential difference, just find $E \cdot d$. The capacitance is:

$$C = \frac{Q}{\Delta V} =$$

Alternatively, $C = \frac{\varepsilon_0 A}{d}$.

Spherical capacitor Spherical capacitors have a sphere of radius a inside another sphere of radius b. The field is defined as $E = \frac{Q}{4\pi\varepsilon_0 r^2}$, so the potential difference (after some integration) evaluates to $\frac{Q}{4\pi\varepsilon_0}(\frac{1}{a} - \frac{1}{b})$, and the capacitance $\frac{Q}{\Delta V}$ evaluates to $\frac{4\pi\varepsilon_0}{\frac{1}{a} - \frac{1}{b}}$.

Example A given capacitor is made of two large metal plates with area of $5000cm^2$ each. The disks are separated by a distance of $d = 1.0mm$. The two disks are connected to the two ends of a 12 volt battery. What is the capacitance of the capacitor? What is the charge stored in the capacitor? What is the electric field between the plates?

For the first part, $C = \frac{\varepsilon_0 A}{d}$. For the second, $Q = CV$. For the third, $dE = V$, and $E = \frac{V}{d}$. Fill in the numbers as you see fit.

Now, while keeping the plates connected to the battery, if the plates were pulled away from each other such that their separation doubles to 2 mm, how will the above answers change?

The capacitance will half. Since Q is dependent on C, Q wil also be halved.

3.1 Serial and parallel connection of capacitors

Different types of capacitor connections result in different characteristics for the resulting circuit. If you connect capacitors in series (a serial connection), each capacitor will have the same charge Q. The total capacitance will unfortunately be lower than the sum:

$$C_{sum}^{-1} = C_1^{-1} + C_2^{-1} + C_3^{-1} + \ldots$$

Parallel connections, of course, have different properties. The voltage will be the same across all of them, and the total charge will be simply the sumo of all their charges. Unlike series wiring, parallel wired capacitors will have a total capacitance that is the sum of their individual capacitances.

3.2 Stored energy

Capacitors work by keeping two plates of different charge apart from each other. One plate will accrue a negative charge, the other will accrue a positive charge of the same magnitude. This results in an electric field between the plates and thus a potential difference. Ultimately it takes work to create that potential difference, so there is some measure of energy involved. More specifically, $dW = \Delta V dq$. This evaluates to $dW = \frac{q}{c}dq$, so the value for work done is $W = \int dW = \int_0^Q \frac{q}{c}dq$. The result of that should be obvious: $V_E = W = \frac{Q^2}{2C}$.

This is useful for showing the energy density of an electric field:

$$\begin{aligned}
V_E &= \frac{1}{2}\frac{Q^2}{C} \\
Q &= \sigma A \\
E &= \frac{\sigma}{\varepsilon_0} \\
V_E &= \frac{1}{2}\varepsilon_0 A d E^2
\end{aligned}$$

Note that part of the last term Ad is actually the volume of the capacitor. The above doesn't actually involve energy density, so to find potential energy per unit volume, use $u_e = \frac{V_E}{Volume} = \frac{1}{2}\varepsilon_0 E^2$. This has units of $\mathrm{J/m^3}$.

Example Consider a sphere of radius a with a total charge of q on it. What is the energy density of the sphere?

To find energy density, use $V_E = u_E \cdot volume$, but that means finding the electric field first. This can be easily found to be $\frac{Q}{4\pi\varepsilon_0 r^2}$. Plugging that into a place yields $u_e(r) = \frac{Q^2}{32\pi^2\varepsilon_0 r^4}$. Now multiply by the surface area of a hypothetical "shell" (at infinite distance) around the sphere, getting $dV_E = u_E(r) \cdot 4\pi r^2 dr$. Integrate that from a to ∞ with respect to r and you'll get $\frac{Q^2}{8\pi\varepsilon_0 a}$. This is the final answer.

3.3 Dielectrics

Dielectric materials are materials that are sandwiched between the two conducting plates in a capacitor. They don't conduct electricity on their own, but if the electric field is too high they can experience dielectric breakdown and ionize, allowing current to flow.

When used (instead of a vacuum), dielectrics can increase the capacitance of a capacitor by a factor of k, the dielectric constant of the material. This capacitance increase is given by $C = kC_0$. For example, air has a value of about 1.00059, paper is about 3.7, glass is $4-6$, and water (pure) is 80. The dielectric strength has to be reasonably high though — the same materials have strengths of 3, 16, 9, and [something], respectively. That means that while glass has a higher dielectric constant than paper, paper can take much more load across it before it experiences dielectric breakdown.

An electric field across a dielectric material can polarize (?) that material, causing molecules/dipoles/somethings inside to line up in a direction that follows the field. The electric field within a dielectric between capacitor plates is $\vec{E} = \frac{E_v}{k}$, and $\Delta V = \vec{E} \cdot \vec{d} = Ed = \frac{E_0 d}{k}$, so solving a bit yields the earlier formula of $C = kC_0$.

Example A non-conducting slab of thickness t, area A, and dielectric constant κ_e is inserted into the space between the plates of a parallel-plate capacitor with spacing d, charge Q, and area A. The slab is not necessarily halfway between the capacitor plates. What is the capacitance of the system?

Doing some magic math — seriously, I have no idea where this comes from — yields:

$$\frac{1}{\frac{d-t}{\varepsilon_0 A} + \frac{t}{\varepsilon_0 A \kappa}}.$$

4 Voltage, Current, & Resistance

Current through a cross sectional area is just the net charge flowing through the area per unit time, given by $I = \frac{dQ}{dt}$. This unit of measurement is the Ampere, or just Amp (A), a derived unit given as $1\frac{C}{s}$. Here, we'll define the direction of current as the flow direction of positive charges. In reality, it's actually negative charges that flow (electrons), and they flow against the direction of the electric field. Either way, current is only a scalar quantity, so there is no unfortunate vector math to be done. Electric fields travel at the speed of light, but the actual charges inside the wire travel much slower, on the order of millimeters per second, at a speed called *drift velocity*, or v_d. Current itself can be given as $nq|v_d|A$, where n is the concentration of moving charged particles, q is the charge per particle, v_d is the drift speed, and A is the cross-sectional area. There's also value $\vec{J}$, for current density, given by $\vec{J} = nq\vec{v_d}$.

Note that although electric fields travel at the speed of light (at least in a vacuum. . . ?), the propagation speed through a wire isn't quite that fast. It's still extremely fast, on the order of 10^6m/s, but it's not the speed of light. A value τ is involved here somewhere, it's involved in conductance: $\sigma = \frac{nq^2\tau}{m}$; resistivity is just the inverse of that. For most metals, resistivity increases with increasing temperature, but semiconductors have the opposite relation. Semiconductors (like silicon) start with a fairly high (but still relatively low) resistivity, but it drops off sharply as temperature increases.

Voltage is just the difference in potential along a given path. It's given by $V = V_a - V_b = \int \vec{E} \cdot d\vec{l}$. If the electric field is uniform, finding voltage between points is as simple as multiplying the distance by the magnitude of the field. The ratio of voltage to current is called resistance, and is given by the formula $R = \frac{V}{I}$, or in the better-known formula of $V = IR$. This is known as Ohm's law, and predictably the unit of measurement for resistance is the Ohm (Ω).

Naturally, energy will be dissipated when current flows through a resistor. Suppose you have a conductor with two ends a and b, with voltages V_a and V_b respectively where $V_a > V_b$. There's also some value dq in here somewhere. Anyways, $E_a = dqV_a + \frac{1}{2}mV_d^2$ and $E_b = dqV_b + \frac{1}{2}mV_d^2$. Therefore, $\Delta E = E_b - E_a = dq(V_b - V_a)$, which gives the energy consumption of the resistor (also, $I = \frac{dq}{dt}$. yes, there is calculus involved).

Anyways, this all comes down to $W = \int_0^t IV dt = \int_0^t I^2 R dt$. This gives the energy used by the resistor, so dividing by time yields power in the forms of $P = IV$, $P = I^2R$, and $P = \frac{V^2}{R}$, all in the units of Watts, or joules per second.

Example Consider a copper wire with diameter of 2.05 mm and resistivity of $\rho = 1.72 * 10^{-8}$. If the cable is 10 meters long, what is its resistance?

$$\begin{aligned} R &= 1.72 * 10^{-8} \frac{10}{\pi 0.001025^2} \\ &= 0.052\Omega \end{aligned}$$

In order to get charges flowing, a source of electromotive force (EMF) is used. EMF is measured in units of Volts too, and is usually provided by a dedicated power source like a laboratory power supply or a battery. When we model out circuits, we usually model them as "ideal" circuits; ideal batteries, for instance, have no internal resistance, but real ones do.

The voltage through any closed loop is exactly 0, so that means that voltage must drop across various components in a circuit.

5 Circuits

5.1 Circuit Rules

There are a number of useful rules about circuits that can help you solve for currents and voltages across different components:

1. All resistors in a serial connection have the same current.
2. The effective voltage across a series of resistors is equal to the sum of the voltages across each resistor.
3. The amount of voltage across a given resistor in a series of resistors is proportional to the fraction of the resistance that resistor makes up in the summed resistance of all the resistors in that series.
4. Resistors wired in parallel have current divided amongst them. For example, if you have a 3-way junction, the current will be split 3 ways.
5. The effective resistance across resistors wired in parallel is $\frac{1}{R_s} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_n}$. This means that wiring two resistors in parallel will result in a value *lower* than wiring them in series, or in most cases even if you had just used one resistor instead.
6. The sum of all currents flowing into and out of a given node is zero (Kirchoff's current law).
7. The sum of all voltages in any given closed loop is 0 (Kirchoff's voltage law).

5.2 RC Circuit

An RC circuit is a simple circuit containing a resistor and a capacitor wired in series. Often these will be wired up to a battery and used as a means of charging the capacitor; the resistor is needed as a current limiter. Using KCV, you can find:

$$\begin{aligned}
\varepsilon - iR - V_c &= 0 \\
\varepsilon - iR - \frac{q}{C} &= 0 \\
\varepsilon - \frac{dq}{dt}R - \frac{q}{C} &= 0 \\
C\varepsilon - \frac{dq}{dt}RC - q &= 0 \\
-\frac{dq}{dt}RC &= (q - C\varepsilon) \\
\frac{dq}{q - C\varepsilon} &= -\frac{dt}{RC}
\end{aligned}$$

Integrate on both sides:

$$\log(q - c\varepsilon) = -\frac{t}{RC} + c$$

Q can now be expressed as a function of time:

$$\begin{aligned}
q - C\varepsilon &= e^{-\frac{t}{RC}} e^{c} \\
q &= C\varepsilon + ce^{-\frac{t}{RC}}
\end{aligned}$$

The only thing left is to find the value of the constant c. When $t = 0$, $q = 0$, so solving for the constant gives $-C\varepsilon$:

$$q = C\varepsilon(1 - e^{-\frac{t}{RC}})$$

Note that as time elapses, the inner term will approach 1, so the charge will just be $q = C\varepsilon$. Also note that you can find the current $i = \frac{dq}{dt}$ to get $i = \frac{\varepsilon}{R}e^{-\frac{t}{RC}}$. This relation shows that for a brief moment, the current will be very high, but will experience exponential falloff as time progresses. This is similar to the charging behavior, just in reverse.

Suppose you have an RC circuit with a charged capacitor and a newly closed switch. We know that $-iR+\frac{q}{C}=0$, so current flowing through will lead to a decrease in charge. That means that $\frac{dq}{dt}R+\frac{q}{C}=0\ldots$ oh hell, here's an align*:

$$\begin{aligned}\frac{dq}{q} &= -\frac{dt}{RC}\\ \log q &= -\frac{t}{RC}+c\\ q &= e^{-\frac{t}{RC}}e^{c}\\ q &= ce^{-\frac{t}{RC}}\end{aligned}$$

Assuming that $q=Q_0$ at $t=0$, the final equations are:

$$\begin{aligned}q &= Q_0e^{-\frac{t}{RC}}\\ i &= \frac{Q_0}{RC}e^{-\frac{t}{RC}}\end{aligned}$$

This, again, shows that the charge on a capacitor decreases exponentially as it discharges; if you want high current, you must discharge it very quickly using a very low resistance value.

6 Magnetism

Magnets generate magnetic fields[1], allowing objects to interact with each other over distance. For a permanent magnet (dipole thingy), opposite ends attract, positive ends repel. A magnet (or ferrous material) under another's influence will line up with the field lines it's being influenced by. Magnetic field lines emanate from the north pole and terminate (sort of, they actually flow through) the south pole. Note that unlike electric charges, it is impossible to get a stable magnetic monopole; cutting a magnet in half will just result in two magnets, not isolated magnetic poles.

The unit for measuring magnetic field is the Tesla (T), measured as N/Am. This is a ridiculously strong electric field; for reference, the strongest magnet in world has a magnetic field 100 Teslas strong, and even then it only lasts for brief moments. The Earth is also a giant magnet, with the magnetic south pole and the geographic north pole, and vice versa for the geographic south pole. Despite its size, the Earth's magnetic field isn't actually that strong — it varies from 25 to 60 µT

6.1 Force on a moving charge

Force is given by:

$$\vec{F}=q(\vec{v}\times\vec{B})$$

In other words, the force is proportional to the magnitude of the charge and the cross product of the charge's velocity and the magnetic field at that point. If the charge is positive, the force will be in the same direction as $\vec{v}\times\vec{B}$; if negative, it will be in the opposite direction. This can be simplified down to $F=qvB\sin\theta$; the direction can be determined using the right hand rule.

A special case occurs when a charged particle moves at right angles to a uniform field. Because velocity is always perpendicular to the resulting magnetic force, the particle will travel in uniform circular motion[2]. The radius of this circle can be found by $F=\frac{mv^2}{r}$. Since force is given by qvB (the particle is moving at an

[1] Do I get a Nobel Prize for this discovery?

[2] It's useful to know that a positive particle in this situation will rotate clockwise if the field is coming out of the plane it is orbiting on. It will rotate counterclockwise if the field goes the other way, or if the particle is negative instead

angle of 90 degrees relative to the magnetic field), $R = \frac{mv}{qB}$. The period of its "orbit" is given by $T = \frac{2\pi m}{qB}$. This effect can actually result in some useful effects. Particles can be trapped in magnetic fields, providing useful ways to contain them, or ways of containing and accelerating them in devices like cyclotrons.

6.2 Motion of a charged particle under both electric and magnetic fields

The two forces can be stacked on top of each other: $\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$. This will generally create very complicated behavior that is best modeled numerically using computer simulations. It's also possible for a uniform electric field and a uniform magnetic field to be perpendicular to each other (cross fields) such that the electric field cancels with the magnetic force: $\vec{E} = -\vec{v} \times \vec{B}$.

A useful application of cross fields is Thomson's experiment to measure the charge to mass ratio of electrons ($\frac{e}{m} = 1.758820099 \cdot 10^{11} C/kg$). The experiment worked by sending an electron beam through a cross field — magnetic field on the horizontal axis, electric field on the vertical axis — and having the beam hit upon a screen (like a CRT screen). The deflection of the electron beam could be measured by looking at the screen.

A second useful application is the mass spectrometer, which can determine the amount of an element in a sample. They work by first vaporizing the sample, then ionizing it so it can be affected by the electric and magnetic fields. The sample's particles are then accelerated through a cross field, where particles will be deflected in amounts that vary by their mass (lower mass means higher deflection). By looking at the distributions of deflections (as detected when the ions impact on a detector), you can determine what the sample was made of.

6.3 Magnetic force on a straight current-carrying conductor

Each charge in a charge carrier, such as a copper wire, experiences a magnetic force as described above. The total number of charged particles is given by $N = nAL$ where AL is the volume (area times length) and n is the density of free charges. The total force on the conductor is given by $\vec{F} = N\vec{F}_q = Nq(\vec{v}_d \times \vec{B})$. With a little bit of work to turn this into current, you get $\vec{I}(\vec{l} \times \vec{B})$, where $\vec{l}$ contains the direction and length of the conductor. Alternatively, $F = ILB \sin\theta$, if you ditch the vectors.

6.4 Magnetic force on a curved conductor

Now the conductor isn't straight anymore. In general, that makes things a headache, because now you have to cut the wire into infinitesimally thin (straight) rods and integrate for those over the whole conductor. This is best shown by example...

Example A wire bent into a semicircle of radius R forms a closed circuit and carries a current I. The wire lies in the xy plane, and a uniform magnetic field is directed along the positive y axis. Find the magnitude and direction of the magnetic force acting on the straight portion of the wire and on the curved portion.

With a little magic math, $\vec{F}_{straight} = 2IRB\hat{k}$. For the circular part, $d\vec{F} = Id\vec{l} \times \vec{B}$. Since the field's direction is known, $\vec{F}_{arc} = \int_0^\pi IRB \sin\theta d\theta(-\hat{k})$ to get $-2IRB\hat{k}$... Eh. This isn't finished. I think.

Anyways, what ends up happening with wire loops in a magnetic field is that you get a torque force on them; this is the principle on which motors work. If you have a coil of length a and width b, there are two forces keeping it rotating, $F = IaB\sin(\phi)$ and $F' = Ibb\sin(90-\phi)$ The torque ends up being $|\tau| = IabB\sin\phi$ where ϕ is the angle the coil makes with the magnetic field. The torque is also given as $\tau = (I\vec{A}) \times \vec{B}$.

The magnetic dipole moment is just $\vec{\mu} = I\vec{A}$, for one loop. If there are multiple turns of the loop, then $\vec{\mu} = NI\vec{A}$. As long as you can find out the area, you should be good to go — for instance, when a circle is on the XY plane, $\vec{\mu} = I\pi r^2$. To find torque on a magnetic dipole, just take the cross product of the dipole moment with the magnetic field:

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Potential energy is then given by:

$$U = -\vec{\mu} \cdot B = -\mu B \cos\phi$$

6.5 Magnetic field due to a moving charge

Moving charges generate a magnetic field. This can be seen plainly in electromagnets — passing a current through a coil of wire can generate a very strong magnetic field. It is given by:

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{r^2}$$

…where μ_0 is a constant ($\mu_0 = 4\pi \cdot 10^{-7} \frac{Tm}{A}$)[3] and r is a vector pointing from the charge to the place where you want to measure the magnetic field. This means the magnetic field is perpendicular to the direction of the charged particle, explaining how coil-based electromagnets work.

6.6 Biot-Savart Law

The Biot-Savart law defines the magnetic field generated by a current element. Start with the current element $Id\vec{l}$, in which you multiply the current by infinitesimally small sections of the wire $\vec{l}$. The magnetic field $d\vec{B}$ is given by:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{Id\vec{l} \times \hat{r}}{r^2}$$

For a moving charge, magic math yields a value of:

$$\vec{B} = \frac{\mu_0 q\vec{v} \times \vec{r}}{4\pi r^3}$$

<++>

Example — circle Consider a circular wire on the xy plane. An infinitesimally small section of the circle is $d\vec{l}$ and the current element is then given by $Id\vec{l}$. Using polar integration for this problem will make things easier, so instead use $Id\vec{l} = Idl\hat{\theta}$. By the Biot-Savart law…

$$\begin{aligned}
d\vec{B} &= \frac{\mu_0}{4\pi} \frac{Idl\theta \times \hat{4}}{r^2} \\
dB_z &= dB \cos\phi \\
\cos\phi &= \frac{R}{r} \\
dB_z &= \frac{dB \cdot R}{r} \\
dB_z &= \frac{\mu_0 I dl}{4\pi r^2} \\
&= \frac{\mu_0 I R^2 d\theta}{4\pi r^3} \\
B_z &= \int_0^{2\pi} \frac{\mu_0 I R^2 d\theta}{4\pi (R^2 + z^2)^{3/2}} \\
&= \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}
\end{aligned}$$

The vector $\hat{r}$ is vector pointing from a point P on the z axis to a point on the circle. Only the z component of the vector is being calculated above — the angle ϕ is the angle between the line from the z axis to the

[3] Notably, $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ — the speed of light changes depending on the environment!

circle and the z axis itself. Anyways, when you simplify to involve the magnetic dipole moment, you get this:

$$B_z = \frac{\mu_0 \mu}{2\pi(R^2+z^2)^{3/2}}$$

When you place two magnetic coils parallel to each other and along the same axis, the magnetic field strength will vary as normal, but the field between the rings will be very uniform.

Example 2 — straight wire A straight wire lies on the y axis from $y = a$ to $y = -a$. Find the magnetic field at a point P on the x axis.

The current element is given by $Id\vec{l} = Idy\hat{j}$, so $d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{l}\times\hat{r}}{r^2}$ where r is the line pointing from a point on on the wire to the point P. Below, θ is the angle between that line and the y axis.

$$\begin{aligned}
Id\vec{l}\times r &= Idy(-\hat{k}) \\
dB &= \frac{\mu_0}{4\pi}\frac{Idy\sin\theta}{r^2}(-\hat{k}) \\
&= \frac{\mu_0}{4\pi}I\frac{x}{r}\frac{dy}{r^2}(-\hat{k}) \\
&= \frac{\mu_0}{4\pi}\frac{Ixdy}{r^3}(-\hat{k}) \\
\vec{B} &= \int_{-a}^{a}\frac{\mu_0}{4\pi}\frac{Ixdy}{(x^2+y^2)^{3/2}}(-\hat{k}) \\
&= \frac{\mu_0 Ix}{4\pi}(-\hat{k})\int_{-a}^{a}\frac{dy}{(x^2+y^2)^{3/2}} \\
&= \frac{\mu_0 I}{2\pi x}\frac{a}{(x^2+a^2)^{1/2}}
\end{aligned}$$

Note that in the event of limits $x < a$ or the wire is infinitely long, $B = \frac{\mu_0 I}{2\pi x}$.

6.7 Ampere's Law

Guass's law for electric field states that $\int \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\varepsilon_0}$. There's a similar law for magnetic fields, but it works a little differently:

$$\int \vec{B}\cdot d\vec{A} = 0$$

This effectively bars magnetic monopoles from existing. For electric potential, another law exists for $V_A - V_B = \int_A^B \vec{E}\cdot d\vec{l}$. Electric force is conservative, so a point moving around a path and back to its starting point will not have any work done on it in total.

Anyways, Ampere's law gives a line integral for magnetic field along a closed path. If a current flows through (let's say into the plane of the drawing), a circle can be used to represent the magnetic field; the magnetic field will be tangent to the circle at every point along it. If the radius is r, then $B = \frac{\mu_0 I}{2\pi r}$. This circle is a closed path, so let's integrate over it. Cut it into infinitesimally small pieces of length $d\vec{l}$; at every point along the circuit, $\vec{B}$ and $d\vec{l}$ point in the same direction. So, instead of $\int \vec{B}\cdot d\vec{l}$, you can do $\int Bdl$ along the path. Since B is just a constant here, all you need to do is integrate along the path for dl. Since this is a circle, you end up getting $2\pi r$, which when multiplied by B yields μI_0.

Ampere's law states that, for any closed path, if you know the current passing through:

$$\int \vec{B} \cdot d\vec{l} = \mu_0 I_{encl}$$

…where I_{encl} is the sum of all the currents flowing through the path. The sign of the current is defined by the right hand rule: curl the fingers of your right hand around the integration path, and your thumb should point in the direction of the positive current.

Ampere's law applies to any closed path, but is most useful when calculating the strength of highly symmetric magnetic fields. Here's one strategy:

1. Identify the symmetry (distribution) of the field
2. Select the appropriate integration path, called an Amperian loop. It should pass through the point where you want to find the magnetic field, and should follow the symmetry of the magnetic field (making path integration easier)
3. Find the currents encircled by the path
4. Perform the rest of your calculations.

6.7.1 Magnetic field due to a long cylindrical conductor

Consider a conductor with current I, magnetic field $\vec{B}$, radius R, and a circle with radius r enclosing a cross section of it.

Choose the path to be a counterclockwise circle of radius r. At every point along the path, the magnetic field will be in the same direction and will have the same strength. The line integral is then $\int Bdl = B \cdot 2\pi r$. The current enclosed can be found as $I_{encl} = J\pi r^2$ where J is the current density…

$$\begin{aligned} I_{encl} &= J\pi r^2 \\ &= \frac{I\pi r^2}{\pi r^2} \\ &= \frac{Ir^2}{R^2} \\ B \cdot 2\pi r &= \frac{\mu_0 I r^2}{R^2} \; B & &= \frac{\mu_0 I r}{2\pi R^2} \end{aligned}$$

This applies for when the point lies inside conductor. For a point outside the conductor, magnetic field strength will decrease with an inverse-square relationship to distance.

6.7.2 Magnetic field of a solenoid

Solenoids are long coils of wire used to generate magnetic fields. The magnetic field inside is uniform and points in the same direction as the solenoid. Finding the magnetic field is trickier here, though, since you can no longer use a circle (we're looking at a cross section of the solenoid from the side). So, use a rectangle instead! The path integral then becomes $\int \vec{B} \cdot d\vec{l} = BL$ where L is the length of the solenoid. The current enclosed can be found as $I_{encl} = NI$ where I is the solenoid current and N is the number of coils. So, $B = \frac{\mu_0 NI}{L}$, which simplifies to $\mu_0 nI$. In other words, magnetic field is proportional only to the current and the density of the coils.

6.7.3 Magnetic field through a torus

Imagine a circular solonoid, with a material core bent in a similar loop. That's what we're dealing with here.

6.8 Induction, Magnetic Flux, and Lentz's Law

Electromagnetic induction is the opposite of current in a wire generating a field: a changing magnetic field will induce a current in a wire. Note that the field *must* be a changing field, a simple static field won't do it.

Magnetic flux is a value similar to electric flux in that it's effectively the amount of "field lines" passing through some given surface. It can be found by taking the dot product of the magnetic field $\vec{B}$ and the area $\vec{A}$. If the surface is flat and the field is uniform, it can be repented simply as $BA\cos\phi$ where ϕ is the angle between the field and the surface. On the calc side of things, you cut the surface into many infinitesimally small surface dA, then integrate over them: $\int \vec{B} \cdot dA$.

Faraday's law is used on magnetic flux:

$$\varepsilon = -\frac{d\phi}{dt}$$

The faster flux changes, more emf will be generated. For example, consider a coil of wire with a constant field passing through it. As the angle changes, the flux will change: $\phi_B = BA\cos\theta$. That means EMF is then $|\varepsilon| BA\frac{d\cos\omega t}{dt}$ to get.

According to Lentz's law, the induced EMF (or current) will generate a magnetic field that exactly counteracts the change of magnetic flux.

EVERYTHING FROM HERE ON OUT WILL NOT BE COVERED ON THE EXAM

6.9 Applications of Induction

When an electromagnetic field induces a voltage in material, it does so in a way such that the magnetic field generated by the resulting current opposes the original field. This is mutual induction. Suppose you have a coil #1 with N_1 turns and current I_1, and a second coil #2 through which the magnetic flux is $\phi_{B2} = M_{21}I_1$ (what?). The emf in coil 2 is then $\varepsilon_2 = -\frac{d\phi_{B2}}{dt} = -M_{21}\frac{dI_1}{dt}$. The flux through coil 1 and coil 2 should be the same, as should the emf: $M_{12} = M_{21} = M$, where M is in units of Henrys, for induction (specifically it's 1 Tm^2/A). This has many useful applications. Paired coils can be used to charge devices without touching them, for instance, or by varying the amount of current, AC transformers can be built.

Self inductance can be used to create inductors, first-order circuit components with properties useful for things like filtering. These usually take the form of solenoids, which generate a magnetic field in the form of $\vec{B} = \frac{\mu_0 NI}{L}$. Here, $\phi_B = N \cdot BA = \frac{\mu_0 N^2 I(t) A}{l} = \frac{\mu_0 N^2 A}{l} I$. Note that l is the inductance of the solenoid, *not* its length. The emf is $\varepsilon = -\frac{d\phi_B}{dt}$, and the flux can be simplified down to $\phi_B = lI$, so $\varepsilon = L\frac{dI}{dt}$.

In practice, this means that inductors give circuits some "inertia" — you can't suddenly force current through, it has to be ramped up first because the inductor will resist it. Inductors obey laws similar to capacitors when charging in an LR circuit:

$$i = \frac{\varepsilon}{R}(1 - e^{-t/\tau})$$

... where ε is the voltage and τ is the time constant $\frac{L}{R}$. When discharging, inductors again work similarly to capacitors:

$$i = \frac{\varepsilon}{R}e^{-t/\tau}$$

The work done can be found by integrating the power over all time:

$$\begin{aligned} U &= \int_0^\infty P(t)dt \\ &= \int_0^\infty \frac{\varepsilon^2}{R}e^{-2t/\tau}dt \\ &= \frac{1}{2}LI^2 \end{aligned}$$

Again, this is very similar to the way capacitors work. It may have something to do with the fact that they're both first-order circuit elements... Anyways, to be especially evil, it's possible to conductor capacitors and inductors together to create an LC circuit. The voltage across the capacitor should always be equal to the voltage across the inductor, so using this knowledge and a little magic math you can derive the following:

$$\omega = \frac{1}{\sqrt{LC}}$$
$$q = Q_0 \cos(\omega t)$$
$$I = Q_0 \omega \sin(\omega t)$$