MECH 539: Computational Aerodynamics
Department of Mechanical Engineering, McGill University

Project #5: Solve the Quasi One-Dimensional Euler Equations
Due 11th April, 2017

Solve the quasi-1D Euler equations for various finite-volume schemes. The nozzle geometry is given as,

$$S(x) = 1 - h\left[\sin\left(\pi x^{t_1}\right)\right]^{t_2}, \quad \text{for } 0 \le x \le 1$$

where $h = 0.15$ is the bump height, $t_1 = 0.8$ locates the maximum location of the bump and $t_2 = 3$ controls the width of the bump in the channel. The following are the flow conditions,

Specific heat ratio, $\gamma = 1.4$
Inlet Total Temperature, $T_t = 531.2$ R
Inlet Total Pressure, $p_t = 2117.0$ lb/ft^2
Gas Constant, $R = 1716$ ft·lb/slug·R

Provide the following in a written report:

1. Solve the quasi one-dimensional Euler equations using a scalar dissipation scheme for the spatial discretization, and a simple Euler explicit scheme for the temporal discretization if the exit static pressure, $p_{\text{exit}} = 0.8p_t$. Discretize the nozzle with 50 points. Show plots of the convergence of the density residual, pressure distribution across the channel, and Mach number distribution.

2. **Exit pressure study.** Solve the quasi one-dimensional Euler equations using the Scalar dissipation scheme for four exit static pressure to inlet total pressure ratios of 0.76, 0.72, 0.68, and 0.60 using 50 grid points. Tweek the value of ϵ for the scalar dissipation scheme until at least only two points are observed in the shock for the exit static pressure to inlet total pressure ratio of 0.76. Use the same value of ϵ value for the 0.72, 0.68, and 0.60 cases. Is the code stable at this value ϵ? If not, then what are the values for the other cases? For each exit pressure ratio, show the convergence of the density residual, pressure distribution across the channel, and Mach number distribution.

3. **Grid study.** Solve the equation for an exit pressure ratio of 0.72 for 25, 50, 100, and 200 grid points. Comment on the value of ϵ. Is there a normal shock in the channel? Is the location of the shock the same for all four grid sizes. Explain your answer. For each grid, show the convergence of the density residual as a function of the number of iterations, pressure distribution across the channel, and Mach number distribution on the same plot. Discuss your findings by comparing the solutions. Does it require the same number of iterations to converge the answer for each grid? Explain your answer. Evaluate the total pressure loss by comparing the values at the inlet and the outlet. Compare the the total pressure loss as you refine the grid.

4. **Temporal discretization scheme study.** Solve the equation for an exit pressure ratio of 0.72 and 100 grid points for two different time stepping schemes: Euler explicit and your choice of the following: Crank-Nicolson, or Jameson's Runge-Kutta. For each time-stepping scheme, show the convergence of the density residual as a function of the number of iterations and the convergence of the density residual as a function CPU time. Discuss your findings by comparing the solutions.

[**Note**:]

- When required to compare different schemes, solutions on a different number of grid points, please plot the results on the same graph, and make sure that that axis are labeled and legend provided for the reader.
- When pressure plots are required, please plot, the local static to the inlet total pressure ratio.

Please see the next page for additional notes——————————>

Additional Notes

Pseudo Code.

1. **Initialize.** Setup the grid, and initialize the state vector (density, momentum, and energy as well as static pressure) flow using the specified flow conditions. Impose the exit static pressure.

2. **Iteration Loop**

   - Compute the time step, Δt_i for each control volume based on the chosen temporal discretization scheme.
   - Compute the flux, $F_{i+1/2}$ across each edge based on the chosen scheme.
   - Compute the residual, $R_i^n = F^n_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - F^n_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - Q_i^n$, for all control volumes, $i = 2, ..., i_{\max} - 1$.
   - Update the state vector based on the chosen temporal discretization. For the explicit Euler, the equation would be $w_i^{n+1} = w_i^n - \frac{\Delta t_i}{V_i} R_i^n$, for all control volumes, $i = 2, ..., i_{\max} - 1$.
   - Update the static pressure, speed of sound, and Mach number for each control volume, $i = 2, ..., i_{\max} - 1$.
   - **Update Inlet and Exit Boundary Conditions using Characteristic Boundary Conditions**
     * Update the inlet boundary condition, by solving for the $u - c$ characteristic if inlet is subsonic. If inlet is supersonic, then do not update $i = 1$. (see page 12 of *NavierStokes-BoundaryConditions.pdf*)
     * Update the exit boundary condition, by solving all three characteristics for both subsonic and supersonic exit boundary conditions. Place a conditional statement on how the change in the static pressure, δp, is computed. (see pages 13 and 14 of *NavierStokes-BoundaryConditions.pdf*)
   - **Check for convergence** by monitoring, R_i, which should converge to machine zero.
   - Repeat the **iteration loop** until convergence.

Inlet and Exit Boundary Conditions.

1. **Inlet Boundary Conditions**

   - *Supersonic Inlet.* Specify the total pressure, p_t, total temperature, T_t, and Mach number, M at cell $i = 1$. Static pressure, p, static temperature, T, speed of sound, c, velocity, u, and energy, e can be initialized from these three values using the isentropic relations as stated in page 11 of *NavierStokes-BoundaryConditions.pdf*. As the solution iterates, then you do not update $i = 1$ since all three characteristics are running right.
   - *Subsonic Inlet.* Solve for the $u - c$ characteristic as follows.

* Compute $\frac{\partial p}{\partial u}$ from taking the derivative of p with respect to u in the isentropic relations.

$$\frac{\partial p}{\partial u} = p_t \left(\frac{\gamma}{\gamma - 1}\right) \left[1 - \frac{\gamma - 1}{\gamma + 1} \frac{(u_1^n)^2}{a_*^2}\right]^{1/(\gamma-1)} \cdot \left(-2\frac{\gamma - 1}{\gamma + 1}\frac{u_1^n}{a_*^2}\right)$$

where $a_*^2 = 2\gamma\left(\frac{\gamma-1}{\gamma+1}\right) c_v T_t$, and $c_v = R/(\gamma - 1)$.

* Compute δu

$$\lambda = \left(\frac{u_2^n + u_1^n}{2} - \frac{c_2^n + c_1^n}{2}\right)\frac{(\Delta t)_1}{\Delta x} \quad \text{where,}\ \ (\Delta t)_1 = \frac{\text{CFL}\Delta x}{u_1^n + c_1^n}$$

$$\delta u = \frac{-\lambda\left[p_2^n - p_1^n - \rho_1^n c_1^n (u_2^n - u_1^n)\right]}{\frac{\partial p}{\partial u} - \rho_1^n c_1^n}$$

* Update flow properties.

$$\begin{aligned}
u_1^{n+1} &= u_1^n + \delta u \\
T_1^{n+1} &= T_t \left[1 - \frac{\gamma - 1}{\gamma + 1}\frac{(u_1^n)^2}{a_*^2}\right] \\
p_1^{n+1} &= p_t \left[\frac{T_1^{n+1}}{T_t}\right]^{\gamma/(\gamma-1)} \\
\rho_1^{n+1} &= p_1^{n+1}/(RT_1^{n+1}) \\
e_1^{n+1} &= \rho_1^{n+1}\left[c_v T_1^{n+1} + \frac{1}{2}(u_1^{n+1})^2\right] \\
c_1^{n+1} &= \sqrt{\frac{\gamma p_1^{n+1}}{\rho_1^{n+1}}} \\
\text{Mach}_1^{n+1} &= u_1^{n+1}/c_1^{n+1}
\end{aligned}$$

2. **Exit Boundary Conditions**

– *Supersonic and Subsonic Exit.*

∗ Compute eigenvalues.

$$\lambda_1 = \left(\frac{u^n_{imax} + u^n_{imax-1}}{2} \right) \frac{(\Delta t)_{imax}}{\Delta x}$$
$$\lambda_2 = \left(\frac{u^n_{imax} + u^n_{imax-1}}{2} + \frac{c^n_{imax} + c^n_{imax-1}}{2} \right) \frac{(\Delta t)_{imax}}{\Delta x}$$
$$\lambda_3 = \left(\frac{u^n_{imax} + u^n_{imax-1}}{2} - \frac{c^n_{imax} + c^n_{imax-1}}{2} \right) \frac{(\Delta t)_{imax}}{\Delta x}$$
$$\text{where, } (\Delta t)_{imax} = \frac{\text{CFL}\Delta x}{u^n_{imax} + c^n_{imax}}$$

∗ Compute characteristic relations.

$$R_1 = -\lambda_1 \left[\rho^n_{imax} - \rho^n_{imax-1} - \frac{1}{(c^n_{imax})^2} (p^n_{imax} - p^n_{imax-1}) \right]$$
$$R_2 = -\lambda_2 \left[p^n_{imax} - p^n_{imax-1} + \rho^n_{imax} c^n_{imax} (u^n_{imax} - u^n_{imax-1}) \right]$$
$$R_3 = -\lambda_3 \left[p^n_{imax} - p^n_{imax-1} - \rho^n_{imax} c^n_{imax} (u^n_{imax} - u^n_{imax-1}) \right]$$

∗ Compute exit Mach number

$$\text{Mach}^n_{imax} = \frac{(u^n_{imax} + u^n_{imax-1})/2}{(c^n_{imax} + c^n_{imax-1})/2}$$

∗ Compute δp based on either a subsonic or supersonic exit.

if $\text{Mach}^n_{imax} > 1$ **then**
  $\delta p = \frac{1}{2}(R_2 + R_3)$
else
  $\delta p = 0$
end if

∗ Update $\delta \rho$ and δu

$$\delta \rho = R_1 + \frac{\delta p}{(c^n_{imax})^2}$$
$$\delta u = \frac{R_2 - \delta p}{\rho^n_{imax} c^n_{imax}}$$

* Update flow properties.

$$
\begin{aligned}
\rho_{imax}^{n+1} &= \rho_{imax}^{n} + \delta\rho \\
u_{imax}^{n+1} &= u_{imax}^{n} + \delta u \\
p_{imax}^{n+1} &= p_{imax}^{n} + \delta p \\
T_{imax}^{n+1} &= \frac{p_{imax}^{n+1}}{\rho_{imax}^{n+1} R} \\
e_{imax}^{n+1} &= \rho_{imax}^{n+1}\left[c_v T_{imax}^{n+1} + \frac{1}{2}(u_{imax}^{n+1})^2\right] \\
c_{imax}^{n+1} &= \sqrt{\frac{\gamma p_{imax}^{n+1}}{\rho_{imax}^{n+1}}} \\
\mathrm{Mach}_{imax}^{n+1} &= u_{imax}^{n+1}/c_{imax}^{n+1}
\end{aligned}
$$