STOPPING TIMES

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Definition 1. A *binary string of length* n is a tuple $(x_1, x_2, \ldots, x_n)$ where each x_k is 0 or 1. Such a string is said to be *stopped* at index k if every index of the tuple in $(k/2, k]$ is zero. The *stopping time* of a binary string is the smallest k such that the string is stopped at index k, or ∞ if no such k exists. Note that, except 1 and ∞, every stopping time is even.

We are primarily interested in binary strings of length n which have stopping time n. Indeed, let $g(n)$ be the number of such strings. Our first goal is to show that $g(n)$ satifies the recurrence

$$\begin{aligned} g(1) = g(2) &= g(4) = 1 \\ g(4n) &= 2g(4n-2) \\ g(4n+2) &= 2g(4n) - g(2n). \end{aligned}$$

This will identify the sequence $a(n) = g(2n)$ as the *Narayana–Zidek–Capell numbers*, A2083 in the OEIS. This recurrence was discovered thanks to the OEIS.

Our first definition discusses *finite* binary strings, but it is more convenient to work with *infinite* binary strings which are nonzero in only finitely many places. This is equivalent, and has the benefit that every such infinite string has a finite stopping time when considered as a sufficiently long, finite binary string.

Definition 2. Let

$$V = \bigoplus_{k=1}^{\infty} (\mathbb{Z}/2\mathbb{Z})$$

be the direct sum of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$. That is, the set of all infinite tuples $(x_1, x_2, x_3, \ldots)$ where each x_k is 0 or 1, and only finitely many are 1. For each positive integer k, let $\octagon_k$ be the set of elements of V which are zero beyond position k and, when the first k entries are regarded as a finite binary string, they have stopping time k. That is, $\octagon_1 = \{0\} \subset V$,

$$\octagon_{2k} = \{v \in V : \forall j > k, v_j = 0, \text{ and } v \text{ has stopping time } 2k\}$$

and $\octagon_{2k+1} = \emptyset$ for every integer $k \geq 1$.

It is clear that $g(n) = |\bigcirc_n|$ for every positive integer n.

Theorem 1. *The sequence* $g(n)$ *satisfies the recurrence*

$$\begin{aligned} g(1) = g(2) &= g(4) = 1 \\ g(4n) &= 2g(4n-2) \\ g(4n+2) &= 2g(4n) - g(2n). \end{aligned}$$

Proof. Define $e_0, e_1 : V \to V$ by

$$e_0(b) = \begin{cases} 0 \text{ if } b = 0 \\ (b_1, ..., b_j, 0, 1, 0, 0, ...) \text{ if } b = (b_1, ..., b_j, 1, 0, 0, ...) \end{cases}$$

and

$$e_1(b) = \begin{cases} (1, 0, 0, ...) \text{ if } b = 0 \\ (b_1, ..., b_j, 1, 1, 0, 0, ...) \text{ if } b = (b_1, ..., b_j, 1, 0, 0, ...). \end{cases}$$

Note that e_0 and e_1 are injective, $e_0(V) \cap e_1(V) = \emptyset$, and $e_0(V) \cup e_1(V) = V$. Let $f : V \to V$ be given by

$$f(b) = \begin{cases} (b_1, ..., b_{j-1}, 1, 0, 0, ...) \text{ if } \exists j > 0 \text{ s.t. } b = (b_1, ..., b_j, 1, 0, 0, ...) \\ 0 \text{ otherwise,} \end{cases}$$

so that $f \circ e_0 = f \circ e_1 = \mathrm{id}_V$. The map e_1 takes the element of $\bigcirc_1$ to $\bigcirc_2$, the element of $\bigcirc_2$ to $\bigcirc_4$, and in general for $k \geq 1$,

$$e_1 : \bigcirc_{2k} \to \bigcirc_{2k+2}.$$

Similarly, for $k \geq 2$,

$$e_0 : \bigcirc_{4k-2} \to \bigcirc_{4k}.$$

Let $\bigcirc^1_{2k} = \{(b_1, ..., b_{2k}, 1, 0, 0, ...) \in V \text{ s.t. } (b_1, ..., b_{2k}, 0, 0, ...) \in \bigcirc_{2k}\}$. Then

$$e_0 : \bigcirc_{4k} \to \bigcirc_{4k+2} \cup \bigcirc^1_{2k}$$

for all $k \geq 1$. Now $e_0(\bigcirc_{4k}) \supseteq \bigcirc^1_{2k}$, since if $b = (b_1, ..., b_{k-1}, 1, 0, 0, ...) \in \bigcirc_{2k}$ then $(b_1, ..., b_{k-1}, 1, 0, 0, ..., 0, 1, 0, 0, ...) \in \bigcirc_{4k}$, where the final '1' is preceded by $k-1$ '0's. Finally, for all $k \geq 1$,

$$e_0(\bigcirc_{2k}) \cup e_1(\bigcirc_{2k}) \supseteq \bigcirc_{2k+2}.$$

$\square$