STOPPING TIMES

ROBERT DOUGHERTY-BLISS AND CHARLES KENNEY

Definition 1. A *binary string of length* n is a tuple $(x_1, x_2, \dots, x_n)$ where each x_k is 0 or 1. Such a string is said to be *stopped* at index k if every index of the tuple in $(k/2, k]$ is zero. The *stopping time* of a binary string is the smallest k such that the string is stopped at index k, or ∞ if no such k exists. Note that, except 1 and ∞, every stopping time is even.

We are primarily interested in binary strings of length n which have stopping time n. Indeed, let $g(n)$ be the number of such strings. Our first goal is to show that $g(n)$ satifies the recurrence

$$\begin{aligned} g(1) = g(2) &= g(4) = 1 \\ g(4n) &= 2g(4n-2) \\ g(4n+2) &= 2g(4n) - g(2n). \end{aligned}$$

This will identify the sequence $a(n) = g(2n)$ as the *Narayana–Zidek–Capell numbers*, A2083 in the OEIS, which begin

$$1, 1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, 654, \dots$$

This recurrence was discovered thanks to the OEIS.

Our first definition discusses *finite* binary strings, but it is more convenient to work with *infinite* binary strings which are nonzero in only finitely many places. This is equivalent, and has the benefit that every such infinite string has a finite stopping time when considered as a sufficiently long, finite binary string.

Definition 2. Let

$$V = \bigoplus_{k=1}^{\infty} (\mathbb{Z}/2\mathbb{Z})$$

be the direct sum of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$. That is, the set of all infinite tuples $(x_1, x_2, x_3, \dots)$ where each x_k is 0 or 1, and only finitely many are 1. For each positive integer k, let $\bigcirc_k$ be the set of elements of V which are zero beyond position k and, when the first k entries are regarded as a finite binary string, they have stopping time k. That is, $\bigcirc_1 = \{0\} \subset V$,

$$\bigcirc_{2k} = \{v \in V : \forall j > k, v_j = 0, \text{ and } v \text{ has stopping time } 2k\}$$

and $\bigcirc_{2k+1} = \emptyset$ for every integer $k \geq 1$.

It is clear that $g(n) = |\bigcirc_n|$ for every positive integer n.

Theorem 1. *The sequence $g(n)$ satisfies the recurrence*

$$\begin{aligned} g(1) = g(2) = g(4) &= 1 \\ g(4n) &= 2g(4n-2) \\ g(4n+2) &= 2g(4n) - g(2n). \end{aligned}$$

Proof. (1) Each nonzero element in V has a final nonzero entry (a one). Let $e_0 \colon V \to V$ be a map which inserts a 0 in the position of this final entry, shifting the previous final entry to the right one space. Similarly, let e_1 insert a 1. Symbolically,

$$e_0(b) = \begin{cases} 0 \text{ if } b = 0 \\ (b_1, ..., b_j, 0, 1, 0, 0, ...) \text{ if } b = (b_1, ..., b_j, 1, 0, 0, ...) \end{cases}$$

and

$$e_1(b) = \begin{cases} (1, 0, 0, \dots) \text{ if } b = 0 \\ (b_1, ..., b_j, 1, 1, 0, 0, ...) \text{ if } b = (b_1, ..., b_j, 1, 0, 0, ...). \end{cases}$$

Note that e_0 and e_1 are injective, $e_0(V) \cap e_1(V) = \emptyset$, and $e_0(V) \cup e_1(V) = V$.

For any positive integer k,

$$\bigcirc_{2k+2} \subseteq e_0(\bigcirc_{2k}) \cup e_1(\bigcirc_{2k}).$$

Indeed, the final nonzero entry of every string in $\bigcirc_{2k+2}$ occurs at position $k+1$. Removing the immediately preceding entry produces a string in $\bigcirc_{2k}$—e_0 and e_1 simply add the entry back.

In the other direction, for $k \geq 2$ we have

$$e_1(\bigcirc_{2k}) \subseteq \bigcirc_{2k+2},$$

because inserting a 1 into position k will never decrease the stopping time. For *odd* $k \geq 3$ we have

$$e_0(\bigcirc_{2k}) \subseteq \bigcirc_{2k+2},$$

because inserting a 0 at position k will only decrease the stopping time if the new stopping time *is* k; this is impossible if k is odd. This shows that

$$\bigcirc_{2k+2} = e_0(\bigcirc_{2k}) \cup e_1(\bigcirc_{2k}),$$

and thus

$$g(2k+2) = 2g(2k)$$

for k odd.

To handle the even case, let

$$\bigcirc^1_{2k} = \{(b_1, ..., b_{2k}, 1, 0, 0, ...) \in V \text{ s.t. } (b_1, ..., b_{2k}, 0, 0, ...) \in \bigcirc_{2k}\}.$$

(Note that $|\bigcirc^1_{2k}| = |\bigcirc_{2k}| = g(2k)$.) Then

$$e_0(\bigcirc_{4k}) \subseteq \bigcirc_{4k+2} \cup \bigcirc^1_{2k}.$$

This implies

$$\bigcirc_{4k+2} \cup \bigcirc^1_{2k} = e_0(\bigcirc_{4k}) \cup e_1(\bigcirc_{4k}).$$

(That the left contains the right is clear given the preceding inclusion. For the other direction, note $\bigcirc^1_{2k} \subseteq e_0(\bigcirc_{4k})$, since if $(b_1, ..., b_{2k}, 0, 0, ...) \in \bigcirc_{2k}$ then $(b_1, ..., b_{2k-1}, 1, 0, 0, ...)$ has stopping time precisely $4k$.) Therefore

$$g(4k+2) + |\bigcirc^1_{2k}| = 2g(4k),$$

or

$$g(4k+2) = 2g(4k) - g(2k).$$

This establishes the recurrence. □

Proof. (2)

Definition 3. A *composition* of n is a vector in $(\mathbb{Z}_{>0})^m$, for some $m \in \{1, 2, ..., n\}$, whose components add to n.

The nth Narayana-Zidek-Capell number, $a(n)$, is the cardinality of the set

$$R_n = \left\{ \text{Compositions } p \text{ of } n \text{ satisfying } p_1 = 1 \text{ and } p_k \leq \sum_{j=1}^{k-1} p_j \right\}.$$

For $j \in \mathbb{Z}_{>0}$, let $f(j) = (0, 0, ..., 0, 1)$, with $j-1$ zeroes preceding the one. Then for $p \in (\mathbb{Z}_{>0})^m$, let

$$F(p) = (f(p_1), f(p_2), ..., f(p_m), 0, 0, ...),$$

concatenated in the natural way, with a tail of zeroes appended at the end. Then $F : R_n \xrightarrow{\sim} \bigcirc_{2n}$ is a bijection, so $|\bigcirc_{2n}| = g(2n) = a(n)$. Indeed, if $p \in R_n$, then since $\sum_{j=1}^m p_j = n$, the final nonzero entry in $F(p)$ is in spot n. Thus $F(p)$ is stopped at time $2n$, and is not stopped at time t for any $t \in [n, 2n-1]$. And if the zeroes from $f(p_r)$ in $F(p)$ caused $F(p)$ to be stopped at some time $t \leq n-1$,

$$F(p) = (f(p_1), ..., f(p_{r-1}), 0, 0, ..., \underset{\substack{\uparrow \\ \text{index } t}}{0}, ...)$$

then $p_r > \lceil t/2 \rceil \geq \sum_{j=1}^{r-1} p_j$. But this would contradict the fact that $p \in R_n$. And, if $b \in \octagon_{2n}$ has 1s at indices $a_1 = 1, a_2, ..., a_\ell = n$, then

$$\begin{aligned} F^{-1} &: \octagon_{2n} \to R_n \\ F^{-1} &: b \mapsto (a_1, a_2 - a_1, a_3 - a_2, ..., a_\ell - a_{\ell-1}) \end{aligned}$$

provides an inverse to F, since the index a_r of the rth one in b is at most twice the index a_{r-1} of the $(r-1)$st one. $\square$