STOPPED BINARY STRINGS

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Abstract. A binary string $(x_1, \dots, x_n)$ is *stopped at index* k if all indices in $(k/2, k]$ are zero. The *stopping time* of a binary string is the smallest such k, if any exist. We show that the number of binary strings of length $2n$ with stopping time $2n$ is the nth Narayana–Zidek–Cappel number. By considering binary expansions, we produce two new integer sequences (the "stopped" integers) and a measurable subset of the unit interval with unknown measure. Generalizing these results to arbitrary bases produces infinite families of integer sequences not in the OEIS and infinitely many new constants denoting the measures of certain sets of the unit interval.

A new, experimental machine is placed into a factory. Despite its inventor's extolments, management is skeptical. They subject the machine to a rigorous *testing* process: Once a week, an inspector will visit the machine and mark down a 1 if a problem is detected, and a 0 otherwise. The first time the machine *consecutively* passes half of the total number of inspections, inspections will cease. This process creates a *binary string*, and the question at hand is whether it ends in sufficiently many 0's to denote a successful testing regimen. We call such successful binary string *stopped*. Some questions are immediate: Given a fixed number of inspections, how many binary strings are stopped? If we inspect *forever*, how many inspections will eventually result in a stopped string? And so on. This paper explores some of these questions. Along the way we will produce an infinite family of sequences which do not appear in the OEIS, an infinite family of measurable sets on the real line whose measures are unknown, and two particular integer sequences which do not appear in the OEIS.

A binary string $(x_1, x_2, \dots x_n)$ is said to be *stopped at index* $k \leq n$ if $x_j = 0$ for all $j \in (k/2, k]$. The *stopping time* of a binary string is the smallest k such that the string is stopped at index k, or ∞ if no such k exists. Note that, except 1 and ∞, every stopping time is even. We call a string of length n with stopping time n "stopped" without reference to an index.

The motivating result for this paper is that the number of binary sequences of length $2n$ with stopping time $2n$ equals the nth Narayana–Zidek–Cappel number T_n, defined by the recurrence

$$
\begin{aligned}
T_1 &= 1 \\
T_2 &= 1 \\
T_{2n} &= 2T_{2n-1} && (n \geq 2) \\
T_{2n+1} &= 2T_{2n} - T_n && (n \geq 1).
\end{aligned}
$$

This sequence is A2083 in the OEIS [**?**]. It originates from [**?**], where the authors show that T_n enumerates the number of random knock-out tournaments with n players.

Our definition of a stopped string, and indeed any condition on binary sequences, suggests various integer sequences and sets of reals by passing to the relevant binary expansions. The definition also generalizes to bases other than 2, the only change being that any occurrence of "1" should be replaced with "any nonzero digit." For simplicity we will sometimes prove the binary results and merely quote the generalizations.

The rest of this paper is organized as follows. Section **??** establishes the identity between the number of stopped binary strings and the Narayana–Zidek–Capell numbers, generalizes this result to arbitrary bases, and gives some elementary results about the enumerating sequences. Section **??** defines a set of reals in the unit interval whose binary expansions are "stopped," and considers the measure of this set. Section **??** gives two new integer sequences and discusses their natural densities.

1. Counting stopped binary strings

Let $g(n)$ be the number of binary strings of length n with stopping time n. Since stopping times are even except for 1, we know that $g(n)$ is zero for all odd n except $n = 1$, where $g(1) = 1$. Thus the interesting sequence is $g(2n)$, which we will show (in two ways!) equals the Narayana–Zidek–Capell sequence T_n.

It is convenient to work with *infinite* binary strings which are nonzero in only finitely many places. This is equivalent to studying finite binary strings, but has the benefit that every such infinite string has a finite stopping time when considered as a sufficiently long, finite binary string.

Definition 1. Let

$$V = \bigoplus_{k=1}^{\infty} (\mathbb{Z}/2\mathbb{Z})$$

be the direct sum of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$. That is, the set of all infinite tuples $(x_1, x_2, x_3, \dots)$ where each x_k is 0 or 1, and only finitely many are 1. For each positive integer k, let $\bigcirc_k$ be the set of elements of V which are zero beyond position k and, when the first k entries are regarded as a finite binary string, they have stopping time k.[1] That is, $\octagon_1 = \{0\} \subset V$,

$$\octagon_{2k} = \{v \in V : \forall j > k, v_j = 0, \text{ and } v \text{ has stopping time } 2k\}$$

and $\octagon_{2k+1} = \emptyset$ for every integer $k \geq 1$.

Note that $g(n) = |\octagon_n|$ for every positive integer n.

Theorem 1. *The sequence $g(n)$ satisfies the recurrence*

$$\begin{aligned} g(1) = g(2) = g(4) &= 1 \\ g(4n) &= 2g(4n-2) \\ g(4n+2) &= 2g(4n) - g(2n). \end{aligned}$$

First, we will prove this directly from our definitions.

Proof. Each nonzero element in V has a final nonzero entry (equal to 1). Let $e_0 \colon V \to V$ be a map which inserts a 0 in the position of this final entry, shifting the previous final entry to the right one space. Similarly, let e_1 insert a 1. Symbolically,

$$e_0(b) = \begin{cases} 0 \text{ if } b = 0 \\ (b_1, ..., b_j, 0, 1, 0, 0, ...) \text{ if } b = (b_1, ..., b_j, 1, 0, 0, ...) \end{cases}$$

and

$$e_1(b) = \begin{cases} (1, 0, 0, \dots) \text{ if } b = 0 \\ (b_1, ..., b_j, 1, 1, 0, 0, ...) \text{ if } b = (b_1, ..., b_j, 1, 0, 0, ...). \end{cases}$$

Note that e_0 and e_1 are injective, $e_0(V) \cap e_1(V) = \emptyset$, and $e_0(V) \cup e_1(V) = V$.

For any positive integer k,

$$\octagon_{2k+2} \subseteq e_0(\octagon_{2k}) \cup e_1(\octagon_{2k}).$$

Indeed, the final nonzero entry of every string in $\octagon_{2k+2}$ occurs at position $k+1$. Removing the immediately preceding entry produces a string in $\octagon_{2k}$—e_0 and e_1 simply add the entry back.

In the other direction, for $k \geq 2$ we have

$$e_1(\octagon_{2k}) \subseteq \octagon_{2k+2},$$

because inserting a 1 into position k will never decrease the stopping time. For *odd* $k \geq 3$ we have

$$e_0(\octagon_{2k}) \subseteq \octagon_{2k+2},$$

because inserting a 0 at position k will only decrease the stopping time if the new stopping time *is* k; this is impossible if k is odd. This shows that

$$\octagon_{2k+2} = e_0(\octagon_{2k}) \cup e_1(\octagon_{2k}),$$

[1] As a result, all entries after $k/2$ are 0

and thus

$$g(2k+2) = 2g(2k)$$

for $k \geq 3$ odd.

To handle the even case, let

$$\hexagon_{2k}^{1} = \{(b_1, ..., b_{2k}, 1, 0, 0, ...) \in V \text{ s.t. } (b_1, ..., b_{2k}, 0, 0, ...) \in \hexagon_{2k}\}.$$

(Note that $|\hexagon_{2k}^{1}| = |\hexagon_{2k}| = g(2k)$.) Then

$$e_0(\hexagon_{4k}) \subseteq \hexagon_{4k+2} \cup \hexagon_{2k}^{1}.$$

This implies

$$\hexagon_{4k+2} \cup \hexagon_{2k}^{1} = e_0(\hexagon_{4k}) \cup e_1(\hexagon_{4k}).$$

(That the left contains the right is clear given the preceding inclusion. For the other direction, note $\hexagon_{2k}^{1} \subseteq e_0(\hexagon_{4k})$, since if $(b_1, ..., b_{2k}, 0, 0, ...) \in \hexagon_{2k}$ then $(b_1, ..., b_{2k-1}, 1, 0, 0, ...)$ has stopping time precisely $4k$.) Therefore

$$g(4k+2) + |\hexagon_{2k}^{1}| = 2g(4k),$$

or

$$g(4k+2) = 2g(4k) - g(2k).$$

This establishes the recurrence. $\square$

Now we will prove our theorem combinatorially by relating our sequence to a certain class of compositions. Recall that a *composition* of an integer n is a vector in $(\mathbb{Z}_{>0})^m$, for some $m \in \{1, 2, ..., n\}$, whose components add to n. The nth Narayana-Zidek-Capell number, T_n, is the cardinality of the set

$$R_n = \left\{ \text{Compositions } p \text{ of } n \text{ satisfying } p_1 = 1 \text{ and } p_k \leq \sum_{j=1}^{k-1} p_j \right\}.$$

Proof. For $j \in \mathbb{Z}_{>0}$, let $f(j) = (0, 0, ..., 0, 1)$, with $j-1$ zeroes preceding the one. Then for $p \in (\mathbb{Z}_{>0})^m$, let

$$F(p) = (f(p_1), f(p_2), ..., f(p_m), 0, 0, ...),$$

concatenated in the natural way, with a tail of zeroes appended at the end. Then $F : R_n \xrightarrow{\sim} \hexagon_{2n}$ is a bijection, so $|\hexagon_{2n}| = g(2n) = T_n$. Indeed, if $p \in R_n$, then since $\sum_{j=1}^{m} p_j = n$, the final nonzero entry in $F(p)$ is in spot n. Thus $F(p)$ is stopped at time $2n$, and is not stopped at time t for any $t \in [n, 2n-1]$. And if the zeroes from $f(p_r)$ in $F(p)$ caused $F(p)$ to be stopped at some time $t \leq n-1$,

$$F(p) = (f(p_1), ..., f(p_{r-1}), 0, 0, ..., \underset{\substack{\uparrow \\ \text{index } t}}{0}, ...)$$

then $p_r > \lceil t/2 \rceil \geq \sum_{j=1}^{r-1} p_j$. But this would contradict the fact that $p \in R_n$. And, if $b \in \bigcirc_{2n}$ has 1s at indices $a_1 = 1, a_2, ..., a_\ell = n$, then

$$\begin{aligned} F^{-1} &: \bigcirc_{2n} \to R_n \\ F^{-1} &: b \mapsto (a_1, a_2 - a_1, a_3 - a_2, ..., a_\ell - a_{\ell-1}) \end{aligned}$$

provides an inverse to F, since the index a_r of the rth one in b is at most twice the index a_{r-1} of the $(r-1)$st one. $\square$

A slightly different recurrence holds for arbitrary bases. Let $g_b(n)$ be the number of b-ary strings of length n with stopping time n. Again, stopping time n means that all entries with indices in $(n/2, n]$ are 0, but for $1 \leq m < n$, some entry in $(m/2, m]$ is nonzero. In the experimental machine example, this corresponds to the case where there are $b-1$ distinct errors the machine could throw in any given week. Then

$$\begin{aligned} g_b(1) &= 1 \\ g_b(2) &= b - 1 \\ g_b(4) &= (b-1)^2 \\ g_b(4n) &= b g_b(4n-2) && (n \geq 2) \\ g_b(4n+2) &= b g_b(4n) - (b-1) g_b(2n) && (n \geq 1). \end{aligned}$$

For convenience, let $r_b(n) = g_b(2n)$, so that the recurrences read

$$\begin{aligned} r_b(1) &= b - 1 \\ r_b(2) &= (b-1)^2 \\ r_b(2n) &= b r_b(2n-1) \\ r_b(2n+1) &= b r_b(2n) - (b-1) r_b(n). \end{aligned}$$

The sequences $r_b(n)$ seem new for $b \geq 3$ in the sense that they do not appear in the OEIS. All $r_b(n)$ satisfy the following scaled-meta-Fibonacci property.

Proposition 1. *For all $b \geq 2$ and $n \geq 2$,*

$$r_b(n) = (b-1) \sum_{k=1}^{\lfloor n/2 \rfloor} r_b(n-k).$$

Proof. The result holds for $n = 2$ since

$$r_b(2) = (b-1) \sum_{k=1}^{1} r_b(k).$$

For $2n \geq 4$, we can use induction to write

$$\begin{aligned} r_b(2n) &= (b-1)r_b(2n-1) + r_b(2n-1) \\ &= (b-1)r_b(2n-1) + (b-1)\sum_{k=1}^{n-1} r_b(2n-1-k) \\ &= (b-1)\sum_{k=1}^{n} r_b(2n-k). \end{aligned}$$

A similar argument applies for $2n+1 \geq 3$. □

Meta-Fibonacci sequences are those whose terms are generated by summing some number of the previous terms. Specifically, a sequence $a(n)$ is $p(n)$-Fibonacci if

$$a(n) = \sum_{k=1}^{p(n)} a(n-k).$$

Our previous proposition shows that $T_n = r_2(n)$, the Narayana–Zidek–Capell numbers, is a meta-Fibonacci sequence. This fact is well-known and can be used to show that $r_2(n)/2^n$ converges to a positive constant (see [**?**]). However, sequences of the form

$$a(n) = \alpha \sum_{k=1}^{p(n)} a(n-k)$$

seem to be unstudied for $\alpha \neq 1$.

Proposition 2. *For every $b \geq 2$, $g_b(2n)/b^n$ is monotonically decreasing. In particular,*

$$\frac{g_b(2n)}{b^n} \leq \frac{b-1}{b}$$

for $n \geq 1$.

Proof. Since $g_b(2n)$ is positive by definition, the recurrence for $g_b(2n)$ implies

$$g_b(2n) \leq b g_b(2n-2)$$

for all $n \geq 2$, which is equivalent to our claim that $g_b(2n)/b^n$ is monotonically decreasing. In particular,

$$\frac{g_b(2n)}{b^n} \leq \frac{g_b(2)}{b^1} = \frac{b-1}{b},$$

as desired. □

Our proposition implies that $g_b(2n)/b^n$ converges to some nonnegative constant, and we suspect that it is a *positive* constant. As we have said, the case $b = 2$ generates the Narayana–Zidek–Capell numbers T_n, and it is well-known that $T_n/2^n$ converges to a positive constant. See, for example, Capell and Narayana's original paper [**?**], or Emerson's argument in [**?**]. (According to the OEIS, the constant is exactly twice the Atkinson-Negro-Santoro constant given in Section 2.28 of Finch's book of mathematical constants [**?**].)

2. Stopped reals in the unit interval

Suppose we choose a binary string $(0.b_1b_2b_3b_4...)_2$ uniformly at random; that is, each bit b_i has an equal probability of being 0 or 1, independently of all the others. What is the probability that the resulting sequence has finite stopping time? There is a great chance it will be stopped early, by picking $b_1 = 0, b_2 = 0$, or $b_3b_4 = 00$. On the other hand, the longer the string $(0.b_1b_2b_3...b_k...)_2$ becomes without being stopped, the less likely b_{k+1} is to make it stop. There are two salient differences between considering stopping times for binary (and later b-ary) sequences, and stopping times in $\oplus_{i=1}^{\infty}(\mathbb{Z}/2\mathbb{Z})$.

(1) Binary sequences most likely never terminate.
(2) A real number in $[0, 1]$ can have more than one binary expansion, one of which is stopped (by an infinite tail of 0s), the other of which may or may not be (with an infinite tail of 1s).

Since the definition of "stopped at time k" depends only on the first k bits in the expansion, (1) won't cause trouble, and in fact leads to our probabilistic questions. (2) is innocuous as well, at least for questions of probability: the set of *dyadic* rationals in $[0, 1]$, i.e. those which have two distinct binary expansions, has measure 0. However, to avoid confusion, all definitions that follow use the terminating binary expansion, when the two disagree.

Definition 2. Let S_k be the set of reals in $[0, 1]$ which have stopping time k when regarded as an infinite binary string. Membership in this set is determined by examining only the first k bits of a binary expansion, so S_k is measureable (in fact, it is a union of half-open dyadic intervals). Let $S = \cup_{k\geq 1}S_k$ be the set of all stopped reals in $[0, 1]$, which is measurable since the S_k are.

Let us get a feel for what the sets S_k "look like." First, observe that every element of S_k consists of a binary string which has stopping time k followed by *any* binary string at all. Thus, each string with stopping time k determines an interval of length 2^{-k} included in S_k, and the disjoint union of these intervals is *all* of S_k. In this way,

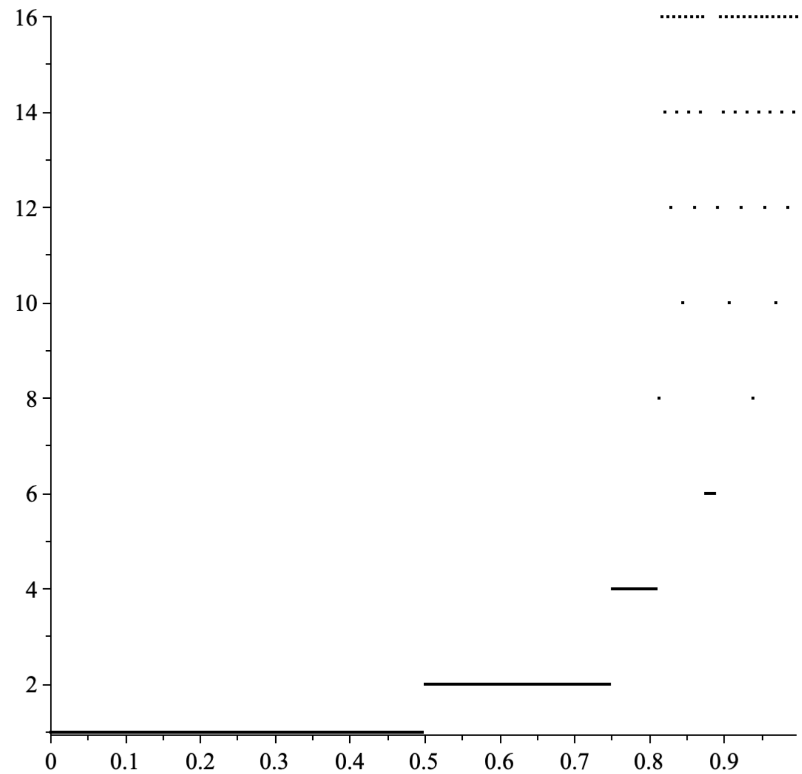

FIGURE 1. Plot of binary sequence stopping times. Stopping times take values in $[0, \infty]$.

we can compute the following sets:

$$\begin{aligned} S_1 &= [0, 1/2) \\ S_2 &= [1/2, 3/4) \\ S_4 &= [3/4, 13/16) \\ S_6 &= [7/8, 57/64) \\ S_8 &= [13/16, 209/256) \cup [15/16, 241/256). \end{aligned}$$

This simple description of S_k gives us a nice expression for the measure of S.

Theorem 2. *Let β_2 be the measure of S. Then*

$$\beta_2 = \sum_{k \geq 1} \frac{g(k)}{2^k} = \frac{1}{2} + \sum_{k \geq 1} \frac{g(2k)}{4^k} \approx 0.841657913173647.$$

Proof. The measure of S_k is $g(k)/2^{2k}$, where $g(k)$ is the number of binary strings of length k with stopping time k. Since S is the pairwise disjoint union of the S_k, the "exact" result follows immediately. $\square$

In the previous section we established that the sequence $T_n = g(2n)$ is A2083 in the OEIS. It is known that $T_n = O(2^n)$, so the above series converges very quickly. Using the recurrence of the previous section to generate the first hundred terms of $g(2k)$ it is easy to approximate the sum and provide rigorous lower bounds. To get more rigorous upper bounds, first note $g(2n)/2^n$ is monotonically decreasing. This implies that the error of using

$$\frac{1}{2} + \sum_{1\le k<n} \frac{g(2k)}{4^k}$$

as an approximation to the measure of S is

$$\sum_{k\ge n} \frac{g(2k)}{4^k} \le \frac{g(2n)}{2^n} \sum_{k\ge n} \frac{1}{2^k} = \frac{2g(2n)}{4^n} = O(1/2^n).$$

For example, using the first 99 terms ($n = 100$) will give an error of no more than

$$\frac{22284668265087}{158456325028528675187087900672} \approx 1.406360286 \times 10^{-16}.$$

The constant β_2 seems new. Accordingly, we conjecture that it is irrational and even transcendental, but we really know nothing about it. The only information we have is that it is a value of the ordinary generating function of $g(n)$.

Definition 3. Let

$$G(z) = \sum_{k\ge 0} g(k)z^k = z + \sum_{k\ge 1} g(2k)z^{2k}$$

be the ordinary generating function of $g(n)$, and

$$A(z) = \sum_{k\ge 1} T_k z^k$$

the ordinary generating function of T_k. Note that $G(z) = z + A(z^2)$.

Proposition 3. *The generating function $A(z)$ is analytic on a disk of radius $1/2$ centered at the origin and satisfies*

$$A(z) = \frac{z(1 - A(z^2))}{1 - 2z}.$$

Proof. The equation is a routine computation using the recurrence for the Narayana-Zidek-Capell numbers proved in the previous section. We have:

$$\begin{aligned}\sum_{k\geq 1} T_k z^k &= \sum_{k\geq 1} T_{2k} z^{2k} + \sum_{k\geq 0} T_{2k+1} z^{2k+1} \\ &= \sum_{k\geq 1} 2T_{2k-1} z^{2k} + T_1 z + \sum_{k\geq 1} (2T_{2k} - T_k) z^{2k+1} \\ &= 2z \sum_{k\geq 0} T_{2k+1} z^{2k+1} + 2z \sum_{k\geq 1} T_{2k} z^{2k} - zA(z^2) + z \\ &= 2zA(z) + z(1 - A(z^2)).\end{aligned}$$

Solving this for $A(z)$ yields the result. Since $T_n = O(2^n)$, we see that $A(z)$ converges everywhere that $\sum_{k\geq 0} (2z)^k = (1-2z)^{-1}$ does, which is at least a disk of radius $1/2$. □

It is clear that

$$\beta_2 = G(1/2) = \frac{1}{2} + A(1/4).$$

But again, this is very little information. We suspect that neither $A(z)$ nor $G(z)$ are algebraic.

Expanding reals in $[0, 1]$ with different bases yields different "stopped sets" and different measures. The same arguments apply, except now the measure of the $S_{b,k}$ will be $g_b(k)/b^k$. Thus, the measure of the set of b-ary stopped reals is

$$\beta_b = \frac{1}{b} + \sum_{k\geq 1} \frac{g_b(2k)}{b^{2k}}.$$

These constants go as follows:

$$\begin{aligned}\beta_2 &\approx 0.84165791317364708989 \\ \beta_3 &\approx 0.62119074589923243760 \\ \beta_4 &\approx 0.48141057151328202149 \\ \beta_5 &\approx 0.39071175514852239829 \\ \beta_6 &\approx 0.32805820924751380527.\end{aligned}$$

These seem to be decreasing, and indeed they are, down to 0.

Theorem 3. *For $b \geq 2$,*

$$\frac{2}{b} - \frac{1}{b^2} \leq \beta_b \leq \frac{2}{b}.$$

Proof. The definition of g_b shows that it is positive, and the recurrence established in the previous section shows that $g_b(2k)/b^k$ is monotonically decreasing. In particular,

$$g_b(2k)/b^k \le g_b(2)/b = \frac{b-1}{b}.$$

for $k \ge 1$. Therefore

$$\beta_b \le \frac{1}{b} + \frac{b-1}{b} \sum_{k \ge 1} \frac{1}{b^k} = \frac{2}{b}.$$

The lower bound is from the definition of β_b. □

3. Stopped integers

In this section we define two integer sequences related to stopped binary strings. They are the *maximally stopped* and *prestopped* integers.

Definition 4. A positive integer n is *maximally stopped* provided that its binary expansion has stopping time equal to its length. A positive integer n is *prestopped* if its binary expansion has length k and is the prefix of a binary string with stopping time $2k$.

The maximally stopped integers begin

$$2, 12, 56, 208, 240, 864, 928, 992, 3392, 3520, 3648, \ldots,$$

and the prestopped integers begin

$$2, 4, 5, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 32, 33, \ldots$$

Neither sequence appears in the OEIS.

Definition 5. Let $M(x)$ and $P(x)$ be the number of maximally stopped and prestopped integers $\le x$, respectively.

Proposition 4. *The counting functions $M(x)$ and $P(x)$ satisfy*

$$P(x) = \Theta(M(x^2)),$$

i.e., $cM(x^2) \le P(x) \le CM(x^2)$ for some constants c and C.

Proof. □

Proposition 5. *The counting functions $M(x)$ and $P(x)$ satisfy*

$$M(x) = \Theta(\sqrt{x})$$
$$P(x) = \Theta(x).$$

Proof. Since $M(x) = M(\lfloor x \rfloor)$, suppose that x is an integer and write $x = 2^m + k$ for an integer $m \geq 0$ and $0 \leq k < 2^m$. Every maximally stopped integer not exceeding x has a binary expansion with length not exceeding $m + 1$. Conversely, every such binary string corresponds to a unique maximally stopped integer. Every such string with length less than $m + 1$ corresponds to an integer $< x$, so

$$M(x) \geq \sum_{k<m+1} g(k).$$

On the other hand, every integer counted by $M(x)$ has length $\leq m + 1$, so

$$M(x) \leq \sum_{k \leq m+1} g(k).$$

Since $g(2k) = \Theta(2^k)$, the upper bound and lower bounds are both $\Theta(2^{m/2}) = \Theta(\sqrt{x})$. For example,

$$\sum_{k \leq m+1} g(k) = \Theta(1) + \Theta(\sum_{k \leq (m+1)/2} 2^k) = \Theta(2^{m/2}) = \Theta(\sqrt{x}).$$

It follows that $M(x) = \Theta(\sqrt{x})$. This implies $M(x)/x = O(x^{-1/2})$, so $M(x)/x \to 0$ as $x \to \infty$. □

An easy corollary of this result is that the maximally stopped integers grow like squares. That is, the nth maximally stopped integer is $\Theta(n^2)$. To establish, say, the lower bound, let m be the nth maximally stopped integer, and note that

$$n = M(m) \leq c\sqrt{m}$$

for some constant c. Then $m \geq n^2/c^2$. The upper bound is the same except for the constant.

Corollary 1. *The maximally stopped integers have natural density* 0*, i.e.,* $\lim_{x\to\infty} M(x)/x = 0$*. The prestopped integers do not have natural density* 0*.*

Proof. Since $M(x) = \Theta(\sqrt{x})$, we have $M(x)/x = \Theta(x^{-1/2})$, so $M(x)/x \to 0$ as $x \to \infty$. On the other hand, $P(x) = \Theta(x)$ implies that

$$cx \leq P(x) \leq Cx$$

for some positive constants c and C. Thus $P(x)/x$ does *not* go to 0 as $x \to \infty$. □

4. Conclusions and open questions

We have produced an infinite family of integer sequences, $g_b(n)$, an infinite family of real constants, β_b, and two new integer sequences, the maximally stopped and prestopped integers. These objects have left some open questions.

We can very quickly approximate the measures β_b using their series representation, but we do not know much about them. Are they expressible in terms of well-known constants such as e, π, and γ? Are they irrational or transcendental?

The prestopped integers introduced in Section **??** do not have natural density zero. Do they have *any* natural density? If not, what are their upper and lower natural densities?