STOPPING TIMES

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Let

$$V = \bigoplus_{k=1}^{\infty} (\mathbb{Z}/2\mathbb{Z})$$

and

$$\octagon_1 = \{0\} \subset V,$$

while for $k \geq 1$,

$$\octagon_{2k} = \{v \in V : \forall j > k, v_j = 0, \text{ and } v \text{ has stopping time } 2k\}.$$

Let $g(n) = |\octagon_n|$. Then

$$g(4n) = 2g(4n-2)$$

and

$$g(4n+2) = 2g(4n) - g(2n).$$

Define $e_0, e_1 : V \to V$ by

$$e_0(b) = \begin{cases} 0 \text{ if } b = 0 \\ (b_1, ..., b_j, 0, 1, 0, 0, ...) \text{ if } b = (b_1, ..., b_j, 1, 0, 0, ...) \end{cases}$$

and

$$e_1(b) = \begin{cases} (1, 0, 0, ...) \text{ if } b = 0 \\ (b_1, ..., b_j, 1, 1, 0, 0, ...) \text{ if } b = (b_1, ..., b_j, 1, 0, 0, ...). \end{cases}$$

Note that e_0 and e_1 are injective, $e_0(V) \cap e_1(V) = \emptyset$, and $e_0(V) \cup e_1(V) = V$. Let $f : V \to V$ be given by

$$f(b) = \begin{cases} (b_1, ..., b_{j-1}, 1, 0, 0, ...) \text{ if } \exists j > 0 \text{ s.t. } b = (b_1, ..., b_j, 1, 0, 0, ...) \\ 0 \text{ otherwise,} \end{cases}$$

so that $f \circ e_0 = f \circ e_1 = \mathrm{id}_V$. The map e_1 takes the element of $\octagon_1$ to $\octagon_2$, the element of $\octagon_2$ to $\octagon_4$, and in general for $k \geq 1$,

$$e_1 : \octagon_{2k} \to \octagon_{2k+2}.$$

Similarly, for $k \geq 2$,

$$e_0 : \octagon_{4k-2} \to \octagon_{4k}.$$

Let $\octagon^1_{2k} = \{(b_1, ..., b_{2k}, 1, 0, 0, ...) \in V \text{ s.t. } (b_1, ..., b_{2k}, 0, 0, ...) \in \octagon_{2k}\}$. Then

$$e_0 : \octagon_{4k} \to \octagon_{4k+2} \cup \octagon^1_{2k}$$

for all $k \geq 1$. Now $e_0(\octagon_{4k}) \supseteq \octagon^1_{2k}$, since if $b = (b_1, ..., b_{k-1}, 1, 0, 0, ...) \in \octagon_{2k}$ then $(b_1, ..., b_{k-1}, 1, 0, 0, ..., 0, 1, 0, 0, ...) \in \octagon_{4k}$, where the final '1' is preceded by $k-1$ '0's. Finally, for all $k \geq 1$,

$$e_0(\octagon_{2k}) \cup e_1(\octagon_{2k}) \supseteq \octagon_{2k+2}.$$