L-functions for 1 1 1 1 - motives

General theory

L-functions

X: Calabi-Yau-threefold defined over $\mathbb{Z}$, smooth over $\mathbb{Q}$ with Hodge numbers $h^{30}, h^{12}, h^{21}, h^{03} = 1$;
Then $H^3X := H^3_{et}(\overline{X}, \mathbb{Q}_l)$ is a *four-dimensional symplectic $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ representation*.

- General Euler factors:
$$Q_p(T) := \det\left(1 - T \cdot \mathrm{Frob}_p \mid H^3X^{I_p}\right)$$
L–function:
$$L(H^3X, s) = \prod_p Q_p(p^{-s})^{-1}.$$
- Special form in our case with α_p and β_p:
$$Q_p(T) = 1 + \alpha_p T + \beta_p p T^2 + \alpha_p p^3 T^3 + p^6 T^4.$$
- Conductor:
Integer N such that $L(H^3X, s)$ satisfies the functional equation.
- Functional equation:
The completed L–function
$$\Lambda(s) = \left(\frac{N}{\pi^4}\right)^{s/2} \Gamma\left(\frac{s-1}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L(H^3X, s)$$
admits analytic continuation and satisfies
$$\Lambda(s) = \pm\Lambda(4 - s).$$

Differential operators

Calabi-Yau manifolds with $h^{12} = 1$ vary in a one-dimensional moduli space, and usually occur as members of a pencil $\mathcal{X} \longrightarrow \mathbf{P}^1$ defined over $\mathbb{Q}$. In such a situation on can find a Picard-Fuchs operator $L \in \mathbb{Q}\langle t, \frac{d}{dt}\rangle$ describing the variations of cohomology.

- Calabi-Yau operator:
Picard-Fuchs Operator of order 4 from the AESZ-list written in the general form
$$L : P_0(t)\theta^4 + P_1(t)\theta^3 + P_2(t)\theta^2 + P_3(t)\theta + P_4(t)$$
where $\theta = t\frac{d}{dt}$ and the $P_i(t)$ are polynomials. Let $\Delta(t) = P_0(t)$.
- Frobenius base:
The operators from AESZ have a MUM-point at 0 and we have a Frobenius base of solutions around 0:
$$\begin{aligned} f_0 &= A(t), \\ f_1 &= f_0(t)\log(t) + B(t), \\ f_2 &= \frac{1}{2}f_0(t)\log(t)^2 + 2f_1(t)\log(t) + C(t), \\ f_3 &= \frac{1}{6}f_0(t)\log(t)^3 + \frac{1}{2}f_1(t)\log(t)^2 + f_2(t)\log(t) + D(t), \end{aligned}$$
where $A(t) \in \mathbb{Q}[|t|]$, $B(t), C(t), D(t) \in t\mathbb{Q}[|t|]$.
- Fundamental solution matrix:
$$F(t) = \begin{pmatrix} f_0 & \theta f_0 & \theta^2 f_0 & \theta^3 f_0 \\ f_1 & \theta f_1 & \theta^2 f_1 & \theta^3 f_1 \\ f_2 & \theta f_2 & \theta^2 f_2 & \theta^3 f_2 \\ f_3 & \theta f_3 & \theta^2 f_3 & \theta^3 f_3 \end{pmatrix}$$
and modified fundamental solution matrix with log-terms removed:
$$E(t) := \begin{pmatrix} A & \theta(A) & \theta^2(A) & \theta^3(A) \\ B & A + \theta(B) & 2\theta(A) + \theta^2(B) & 3\theta^2(A) + \theta^3(B) \\ C & B + \theta(C) & A + 2\theta(B) + \theta^2(C) & 3\theta(A) + 3\theta^2(B) + \theta^3(C) \\ D & C + \theta(D) & B + 2\theta(C) + \theta^2(D) & 3\theta(B) + 3\theta^2(C) + \theta^3(D) \end{pmatrix}$$

Frobenius method by Dwork, Candelas, de la Ossa, van Straten

- $U_p(0)$:
Limit Frobenius matrix:
$$U_p(0) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ p^3 \cdot x_p & 0 & 0 & p^3 \end{pmatrix},$$
x_p conjecturally equal to $r \cdot \zeta_p(3)$, $r \in \mathbb{Q}$ some rational.
- $U_p(t)$:
Dwork's Frobenius matrix:
$$U_p(t) := E(t^p)^{-1} \cdot U_p(0) \cdot E(t) \in \mathrm{Mat}(4, 4, \mathbb{Q}[[t]]).$$
Conjecturally one has
$$U_p(\mathrm{teich}(t)) \overset{?}{=} \text{Matrix of } \mathrm{Frob}_p : H^3X_t \to H^3X_t$$
and
$$Q_{p,t}(T) = \det(1 - U(\mathrm{teich}(t))T),$$
$\mathrm{teich}(t) \in \mathbb{Z}_p$ Teichmüller lift of $t \in \mathbb{F}_p$.
- p-adic expansion:
The matrix $U(t)$ has the p-adic expansion
$$U_p(t) = \frac{V_0(t)}{\Delta(t)^{p\cdot\delta_0}} + p \cdot \frac{V_1(t)}{\Delta(t)^{p\cdot\delta_1}} + p^2 \frac{V_2(t)}{\Delta(t)^{p\cdot\delta_2}} + p^3 \cdot \frac{V_3(t)}{\Delta(t)^{p\cdot\delta_3}} + p^4 \cdot \frac{V_4(t)}{\Delta(t)^{p\cdot\delta_4}} + p^5 \cdot \frac{V_5(t)}{\Delta(t)^{p\cdot\delta_5}} + \dots,$$
with $V_i(t) \in \mathrm{Mat}(4, 4, \mathbb{Q}[t])$.
- Weil bounds:
For the zeros $z_1, \dots, z_4$ of $Q_p(T)$ we have:
$$|z_i| = p^{-3/2}.$$
So for α_p and β_p in $Q_p(T)$ it is enough to compute up to $\mod p^4$ and adjust the result such that the zeros fulfill the Weil bounds:
$$U_p(t) \mod p^4 \equiv \frac{V_0(t)}{\Delta(t)^{p\cdot\delta_0}} + p \cdot \frac{V_1(t)}{\Delta(t)^{p\cdot\delta_1}} + p^2 \frac{V_2(t)}{\Delta(t)^{p\cdot\delta_2}} + p^3 \cdot \frac{V_3(t)}{\Delta(t)^{p\cdot\delta_3}} \mod p^4.$$

Implemented functions

Differential operators

`aap` stored in the file `order4`	CY-operator as array of the form $[cX^4, (a_4X^4 + \cdots + a_0), \ldots, (b_4X^4 + \cdots + b_0)]$ where X stands for θ.
`delta` Input: operator `aap`	Output: The discriminant $P_0(t)$ of the operator *aap*
`degreeCY` Input: operator `aap`	Output: The degree r of the operator *aap* which is the degree of `delta(aap)`
`coeffT` Input: operator `aap`, integer n	Output: A polynomial of the form $a_n + Tb_n + T^2c_n + T^3d_n$ where a_n, b_n, c_n, d_n are the n-th coefficients of $A(t), B(t), C(t), D(t)$.
`an_vec` Input: operator `aap`, integers N, i	Output: vector of length N of coefficients of the Frobenius base a_n if $i = 1$, b_n if $i = 2$, c_n if $i = 3$, d_n if $i = 4$.

Computing Euler factors with Frobenius method by Dwork, Candelas, de la Ossa, van Straten

`Esols` Input: operator `aap`, integer N	Output: The matrix $E(t) \in Q[\vert t\vert]$ with power series entries truncated at N
`UU` Input: matrix $E(t)$, prime p, integer N	Output: The matrix $U_p(t) \in Q[\vert t\vert]$ with power series entries truncated at N
`Umake` Input: operator `aap`, prime p	Output: The matrix $U_p(t) \mod p^4 \in Q[t]$ which has polynomial entries
`Triple` Input: operator `aap`, prime p, rational t	Output: The coefficients of $Q_p(T)$ as triple $(p, \alpha_p(t), \beta_p(t))$
`ListOfTriples` Input: operator `aap`, lower and upper bound *pmine* and *pmax* for list, rational t	Output: List of triples $[(pmin, \alpha_{pmax}(t), \beta_{pmin}(t)),$ $\ldots, (pmax, \alpha_{pmax}(t), \beta_{pmax}(t)]$