Euler’s formula and topology of surfaces

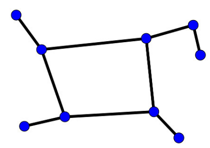
5.0.1 Introduction In this chapter we will be considering surfaces, which is necessary background to understanding orbifolds and where the “magic theorems” come from.

We will be doing *topology* rather than *geometry*. In topology we study features of things which are unchanged by a continuous deformation and so to a topologist the top half of a sphere (including the equator), the closed unit disc in the plane, a (closed) square in the plane and half of the closed square rolled in to a cone are all the same thing. To a geometer, the first is a (positively) curved surface whereas the others are flat and the last two feature singular points such as corners on a boundary or a “cone point”.

So for this Chapter, a disc is the same as a triangle and a football the same as a rugby ball.

5.1 A quick guide to Euler characteristics

5.1.1 Definition By a *map* on a surface we mean a connected graph such that the regions into which the surface is divided are connected and simply connected. (The final condition means that any loop in a region can be continuously shrunk down to a point.)



5.1.2 Euler’s theorem Euler’s famous theorem says that for a map on a sphere that divides it into F regions and has V vertices and E edges we have

$$V - E + F = 2.$$

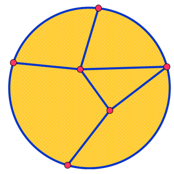
The “2” on the right-hand side is the *Euler characteristic of the sphere* and we will denote Euler characteristics generally by the Greek letter χ (“chi”).

This is proved in Year 1 PPS - see Liebeck’s book if you have forgotten. We will also sketch a proof below.

5.1.3 Example Imagine the little map (above-right) drawn on a sphere. We have $V = 9$, $E = 9$ and $F = 2$, where the two regions are the inside of the quadrilateral and the rest of the sphere outside of it.

It is convenient to allow "dangling" edges and vertices as in this picture, which are not part of a border of any region. Observe that if one picks the dangling "cherries", including their stalk, then one removes exactly one vertex and one edge each time leaving $V - E + F$ unchanged. So dangling pieces can always be removed.

5.1.4 Exercise (BM) Find a map on a sphere which violates the condition of its regions being simply connected and such that $V - E + F \neq 2$.



5.1.5 Definition The Euler characteristic for a map on a surface with boundary is defined in the same way as for one without, except there is an extra requirement that one has edges joined by vertices running round every boundary.

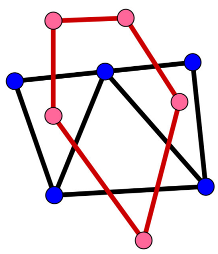
5.1.6 Exercise (BS) Compute χ using the map in the closed disc on the right. (Note that here $F = 5$, the disk including boundary is our whole surface, there is no "region outside".)

5.1.7 Theorem For the closed disk, $\chi = 1$.

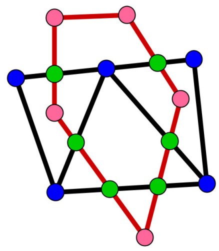
Proof. Imagine the disc as drawn on the surface of the sphere. Then calculating χ for the sphere is the same as calculating it for the disc, except that one counts one extra region (the exterior of the disc). □

5.1.8 Vague Definition For us, a *surface* is a 2-dimensional object, closed (so that if, like the disk, it has a boundary, then the boundary is included) and "compact" (which in the case of surfaces in $\mathbb{R}^n$ means that they do not go off to infinity).

5.1.9 Theorem Given a compact surface, the Euler characteristic $\chi = V - E + F$ is independent of the choice of map used to calculate it.



A (non-examinable) sketch proof for the case without boundary goes as follows. Suppose you have two maps M and N on a surface such as the black and red ones to the right. Deform them a little if necessary so that the vertices of each do not lie on edges or vertices of the other, and so that where edges of one meet edges of the other they cross transversely (meaning that the cross one another cleanly rather than, say, meeting tangentially and veering off again).



Now, wherever an edge of G meets an edge of G' add that point as a vertex to both graphs. These extra vertices are the green ones in the lower picture. Note that adding such a vertex divides an existing edge of G in two and so leaves χ unchanged. And the same applies for χ'. Now consider the combined graph H (all 16 vertices 10 regions and 24 edges in the lower picture). We claim that the Euler characteristic of H is equal to that of G and equal to that of G' and so $\chi = \chi'$.

To see this, start with the black/blue graph G with the added green vertices. Choose a region in G and add in all the red/pink edges and vertices within it. Since a disk $\chi = 1$, this does not change χ for G. Repeat until you arrive at H.

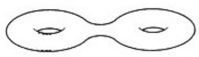
5.1.10 Exercise (BS) Recall that in Chapter 1 we constructed a torus by taking a rectangle and zipping the edges together. Think of the rectangle as being a single "face" for our map and think of the edges of the rectangle as being edges of our map and the corners being vertices. As one identifies the edges to obtain a torus, some of the edges and corners get identified so that the resulting map on the torus has fewer edges and vertices than we started with.

Decide how many vertices ad edges the map on the torus has and deduce that for a torus $\chi = 0$.

5.1.11 Exercise (NS) For a 1-dimensional object, a map divides the object into intervals (which are edges) and the Euler characteristic is just $V - E$. Draw a graph on the circle and deduce that $\chi = 0$ and draw a graph on a closed interval and deduce that $\chi = 1$.

5.1.12 Exercise (NM) Compute χ for (the surface of) a cylinder (including its two circular boundaries but without caps on the end) and for the Möbius band including its single circular boundary.

5.1.13 Definition A *connected sum* of two surfaces Σ_1, Σ_2 is the surface that results if one punches a hole in both surfaces and connects the boundaries of the resulting holes together with a tube. It is denoted $\Sigma_1 \# \Sigma_2$. A connected sum of two toruses is shown on the left. Taking a connected sum with a torus is called "adding a handle" to a surface. The picture on the right is a "torus with a handle" or a "sphere with two handles".



5.1.14 Theorem The Euler characteristic of a connected sum is given by

$$\chi_{\Sigma_1 \# \Sigma_2} = \chi_{\Sigma_1} + \chi_{\Sigma_2} - 2.$$

In particular, adding a handle to a surface reduces its Euler characteristic by two.

5.1.15 Exercise (NM) Prove the above theorem. (Try cutting a triangular face out of each surface and connecting them together with a triangular prism.)

5.2 The projective plane

5.2.1 The group $N\times$ has a glide g which is reflection in the equator followed by a rotation about the poles of π/N. Thus g^2 is a rotation about the poles by $2\pi/N$ and so generally there is a gyration point at the poles. (There is only one type of gyration because g swaps the North and South poles over.)

In the particular case $N = 1$ there is no gyration and $1\times = \times$ is the 2-element subgroup of $O(3)$ containing just $\pm\mathbb{I}$.

5.2.2 The orbifold of $\times$ can be thought of in several different ways. Firstly, a topologist would say it is just "the sphere with opposite points (i.e. $\mathbf{x}$ and $-\mathbf{x}$) identified".

Alternatively we could try and find a fundamental domain for $\times$. If we take the open Northern hemisphere, then each point is in a different orbit of the group. If we include the equator however, then opposite points on it have to be identified: you have to imagine that if you walk South over the equator you magically reappear at the diametrically opposite point heading North.

If we are just trying to get a grip on the topology, we could flatten the closed Northern hemisphere out into a disk, keeping the rule about walking off the edge and appearing at the opposite point.

Finally, if you are an algebraic geometer you would identify the orbifold with the set of all 1-dimensional subspaces of $\mathbb{R}^3$ and know it as *the projective plane* $\mathbb{R}P_2$.

5.2.3 Imagine that before identifying the opposite points on the sphere you cut two identically sized discs centred on the poles out of the sphere leaving just a strip around the equator. The identification will turn that strip into a Möbius band, as you can probably convince yourself (see exercise below).

Now the identification identifies the two discs you cut out, and so the Möbius band is the result of punching an open disc out of $\mathbb{R}P_2$ leaving a single boundary curve. Reversing this, we see that you can also construct $\mathbb{R}P_2$ by taking a Möbius band and a disk and "zipping together" their boundary curves. Of course, you cannot do this in $\mathbb{R}^3$ without some self-intersection unpleasantness, but that can be avoided by dodging into the fourth dimension where necessary.

5.2.4 Exercise (NS) Cut out a long thin strip of paper and imagine that it is wrapped around the equator of a sphere. Observe that "identifying opposite points" on the sphere can be realised by wrapping the strip twice round a Möbius band.

5.2.5 There is a canonical map p from the sphere S^2 to $\mathbb{R}P_2$ which takes the point on the sphere to the point in $\mathbb{R}P_2$ that it defines. Each point in $\mathbb{R}P_2$ has two inverse images in S^2. Notice that this does *not* mean that S^2 is somehow to separate copies of $\mathbb{R}P_2$: it cannot be since S^2 is connected. Rather it means that S^2 is what is called a "bundle" over $\mathbb{R}P_2$. The situation is rather like that illustrated in the picture in 0.2.17 except that there there are an infinite number of inverse images whereas here if you go round twice, you are back where you started.

We discuss this example here for three reasons. Firstly it is additional motivation toward the fact that we need to understand surfaces better in order to understand orbifolds. Secondly $\mathbb{R}P_2$ will be an important to us in that discussion. Thirdly, it's fun.

5.2.6 Theorem The Euler characteristic of the projective plane $\mathbb{R}P_2$ is $\chi = 1$.

Proof. Consider a map on the sphere (such as that corresponding to a cube) which has the property that it is unchanged under $\mathbf{x} \mapsto -\mathbf{x}$. This defines a map on $\mathbb{R}P_2$ with half as many regions, edges and vertices. $\square$

5.2.7 Making surfaces by identifying edges of a rectangle We observed earlier how to identify the edges of a square (or rectangle) in pairs to make a torus.

If we take a rectangle and just identify one pair of opposite sides (in the same direction) we obtain a cylinder (without end caps). If we identify them in opposite directions we get a Möbius band.

The projective plane comes from identifying both pairs of opposite sides, taking opposite orientation in each case, and there is one final possibility: identify one pair of opposite edges in corresponding directions and one pair reverse: the result is a Klein bottle.

5.3 The classification of surfaces

5.3.1 Aim Our aim is to classify all possible compact surfaces (allowing boundaries). Our strategy will be to construct surfaces by performing operations on a sphere. We will then argue that every surface is equivalent to one of those we have constructed.

5.3.2 Definition A surface is *orientable* if it is possible to continuously choose over the whole of it which direction of rotation in the surface we regard as "positive".

A sphere is orientable: choose for example, rotations to be positive if they are anticlockwise as viewed from outside the sphere. A Möbius band is not orientable: a clock face transported once round the band returns to its starting point as a mirror image. (You may say that that the clock face has come back on the "other side" of the Möbius band. But when we think of surfaces we are imaging the surface to be the whole universe: it does not have "two sides" any more than our three-dimensional world has two sides.)

For a surface in $\mathbb{R}^3$ that does not cross itself, a continuous choice of unit normal over the surface is possible if and only if the surface is orientable.

5.3.3 Equivalent condition If a surface contains a Möbius band, then it clearly cannot be orientable. The converse is true also: if a surface is not orientable there is a path in the surface such that if you carry a clock along it, when you return to the start you have a mirror image clock. A thin band around this path will be a Möbius band. (Or it will if the path does not cross itself. We will no worry about that issue here.)

5.3.4 We now consider four operations that we can use to construct a surface from a sphere.

Punching holes

5.3.5 Definition By "punching a hole" in a surface we mean removing something that is topologically an open disc, leaving the surface with a new boundary curve.

5.3.6 Theorem Punching out a hole reduces χ by 1.

To see this, simply choose a map where the hole to be punched out is the interior of a region.

5.3.7 Notation Our notation for surfaces uses symbols to denote operations performed on a sphere to obtain the surface. We denote the sphere with k holes punched out by $*^k$. It has $\chi = 2 - k$. Note that for topological purposes it does not matter what the configuration or shape of the holes is. Provided they do not overlap, they can be resized and moved about without changing the surface topologically.

5.3.8 Exercise (NS) A closed disc is $*$ and a cylinder is $**$ or $*^2$. Check that this gives the same Euler characteristic as you found in 5.1.12.

Adding crosscaps

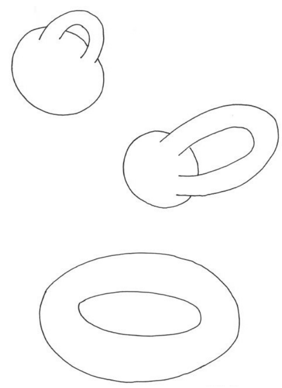
5.3.9 Definition By *adding a crosscap* to a surface we mean the process of cutting out a disc and then "filling the hole" by zipping the boundary of a Möbius band to it. We observed in §5.2 that cutting a disc out of a projective plane leaves a Möbius band, and so one could describe "adding a crosscap" as being taking the connected sum with a projective plane. We will write $\times$ for a crosscap. Thus $\times^m$ denotes the sphere with *m* crosscaps.

5.3.10 Theorem Adding a crosscap reduces χ by 1. The resulting surface is not orientable, irrespective of whether the original surface was.

Proof. The Euler characteristic follows from the formula for a connected sum and our knowledge that $\chi = 1$ for a projective plane. ☐

5.3.11 Exercise (NM) Explain why a Möbius band is the surface $*\times$. (The notation means a sphere with a hole punched out and also a crosscap added.)

Adding handles (standard and cross)

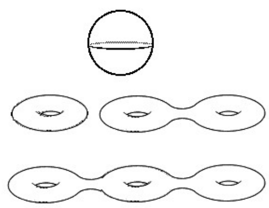


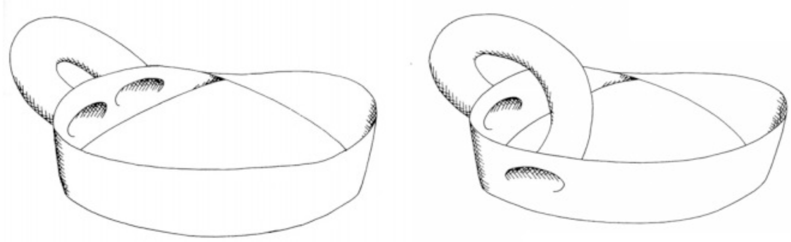
On the right at the top, a handle has been added to a sphere. Expanding the handle and shrinking the sphere it becomes clear we have a torus.

Adding more handles to a sphere, we obtain the sequence of surfaces in the bottom picture on the right. We will write $\mathbf{o}$ for a handle and so describe those surfaces as $\mathbf{o}^g$ where $g = 0, 1, 2, 3$ in the picture but can be any natural number.

Since adding a handle decreases χ by 2, the surface we are calling $\mathbf{o}^g$ has $\chi = 2 - 2g$.

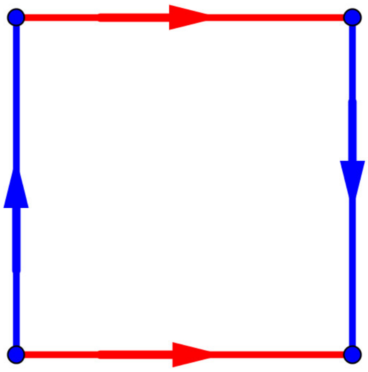
Adding handles as we have on the right maintains orientability. The surfaces pictured have an inside and an outside and the outward facing normal can be used to orient them.

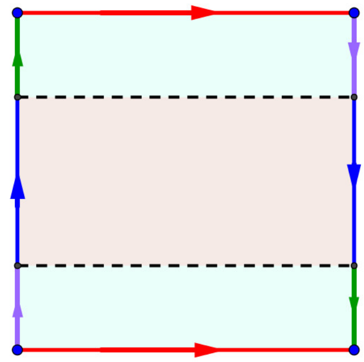




5.3.12 The pictures above show the addition of a handle (left) and "cross handle" (right) to a Möbius band. In the cross handle the other end of the connecting tube is connected to the "other side" of the surface.

5.3.13 The Klein bottle If we attach a cross handle to a sphere the result is a "Klein bottle": the example on the left was unfortunately manufactured at a 3-dimensional factory and so the handle has to pass through the sphere. In four dimensions one could avoid that. In general, adding a cross handle to a surface is the same as taking a connected sum with a Klein bottle.





5.3.14 Exercise (NS) We calculated $\chi = 0$ for the Klein bottle earlier because our argument about attaching handles a surface did not depend on whether we were talking about a proper handle or a cross handle. Check that by checking how many vertices and edges there are after the identification in the left-hand picture above.

5.3.15 Proposition A Klein bottle can be cut by a single closed curve into two Möbius bands.
Proof. To prove this, consider the picture on the right above. The two dotted lines together constitute a single closed curve on the Klein bottle that divides it into two and does not intersect itself.

Before identifying, cut along the dotted lines. The central beige area can now be identified by the blue arrows to form a Möbius band. Now identify the two pale blue strips along the red edge. Identifying green with green and lilac with lilac creates another Möbius band. □

Drawing these things together we have the following two results.

5.3.16 Proposition A Klein bottle is the connected sum of two projective planes. (So a Klein bottle is the surface ××.)

5.3.17 Proposition Adding a cross handle to a surface is the same as adding two crosscaps.

5.3.18 Exercise (NM) Explain how the two propositions just above follow from the previous discussions.

Crosscaps and handles

5.3.19 Proposition In the presence of a crosscap, a cross handle is the same thing as a handle. Thus two crosscaps are the same thing as a handle, provided there is another crosscap somewhere.

Proof. Look at the "handle on a Möbius band" picture above. If you fix one point of attachment and then take the other point of attachment of the handle and move it all the way round the band until it becomes close to the fixed one again, the handle becomes a cross handle! But the spare crosscap contains a Möbius band, so you can walk around that. □

5.3.20 Definition We say a surface is *tidy* if it is equivalent to one obtained by punching out discs, and adding crosscaps, handles and cross handles to a sphere.

5.3.21 Theorem Every tidy surface is equivalent to one that is of the form

$$*^a \mathbf{o}^b \text{ (where } a, b \geq 0) \quad \text{or} \quad *^a \times^c \text{ (where } a \geq 0, c \geq 1).$$

The Euler characteristics are $2 - a - 2b$ and $2 - a - c$ respectively. All these forms are distinct surfaces except for the obvious case where $b = c = 0$.

Proof. The first part follows from the above. The uniqueness follows because the number of boundary curves is equal to a. Then the Euler characteristic determines the surface except where we need to decide whether we have n handles or $2n$ crosscaps. But the former is orientable and the latter is not. □

5.3.22 Exercise (NL) Expand the above proof by adding some details.

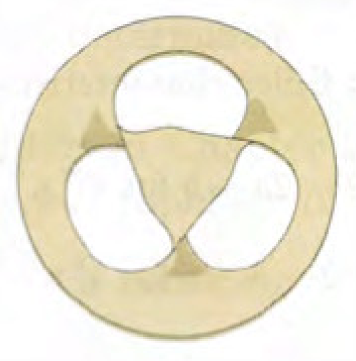
5.3.23 The classification of surfaces This fundamental result in geometry and topology states that the previous theorem is in fact a classification of all (connected, compact) surfaces. To see that, one needs to show that every surface is equivalent to a tidy one.

The key definition is of a "trianglarisable surface" meaning one can cut it into "triangles" that "zip together" along edges. it is non-trivial, but one can show that any "reasonable" surface is triangularisable.

The idea then is to imagine your surface arriving as a self-assembly kit from IKEA. You tip all the triangles with their zips onto the floor. Pick up a triangle. In your hand you now have a tidy surface which is a sphere with one punched hole. Now find another triangle that fastens to that one and add it. You still have a tidy surface – in fact its topology has not changed. Now keep adding triangles: the key is to show that each addition you make keeps the surface tidy. (After you finish you probably find some spare pieces on the floor of course.)

This argument is Conway's ZIP ("zero-irrelevancy proof"). You can find a more complete account of it on Learn under "Resources".

5.3.24 Exercise (NL) Consider the surface on the right (taken from the Conway book). What is its signature? To calculate this you need to count how many boundary components there are, find its Euler characteristic (perhaps by making a paper model) and check if it is orientable.



5.3.25 Surfaces and geometric groups Of course, the point of studying surfaces is to understand orbifolds and you will not be surprised to discover that all the wallpaper and spherical groups we have studied that have no numbers in their signature, the orbifold is the surface given by the same notation. The orbifold is made out of Euclidean space (flat) material for the wallpaper groups and out of "surface of a sphere" (constant positive curvature) material for the spherical groups.

For miracles, if one has a curve joining a point to its mirror image in the pattern which does not cross a mirror, then thickening that to a little strip gives us a Möbius band in the orbifold and so a crosscap.

5.3.26 Exercise (CM) For the four wallpaper groups and two spherical groups with no numbers in their signature, understand how the orbifold can be considered to be the corresponding surface.