Topics in Analysis, Examples Sheet 1. W. T. G. Michaelmas 2004

1. Let X be a non-compact subset of $\mathbb{R}^2$. Prove that there is a continuous unbounded function from X to $\mathbb{R}$.

2. Prove that a function f from a metric space X to a metric space Y is continuous if and only if $f^{-1}(A)$ is closed in X for every closed set $A \subset Y$.

3. Let K be a compact subset of $\mathbb{R}^2$ and let F be a closed subset of $\mathbb{R}^2$ such that $K \cap F = \emptyset$. Give two proofs that there exists $\delta > 0$ such that $d(x, y) \geqslant \delta$ for every $x \in K$ and every $y \in F$, one proof directly from the compactness of K and the other using sequential compactness instead.

4. Let I, J and K be three arcs, each making an angle of $2\pi/3$, whose union is the whole of the unit circle (as in lectures). Let A, B and C be *open* subsets of the closed unit disc D with $I \subset A$, $J \subset B$ and $K \subset C$. Suppose that $A \cup B \cup C$ is the whole of D. Deduce that $A \cap B \cap C$ is non-empty from the corresponding result about closed sets. [Hint: One way to do it is to start by defining $F \subset A$ to be the set of all x such that $d(x, A^c) \geqslant \max\{d(x, B^c), d(x, C^c)\}$, and similarly for sets $G \subset B$ and $H \subset C$.]

5. Let f be a continuous function from the *open* unit disc to itself. Must it have a fixed point? Does your answer change if f is a surjection?

6. Let X and Y be metric spaces, let K be a compact subset of X and let $f : X \to Y$ be continuous. Prove that $f(K)$ is closed.

7. Let f be a continuous function from a compact metric space X to itself, and suppose that $d(f(x), f(y)) = d(x, y)$ for every $x, y \in X$. Prove that f is a surjection. [Hint: if not, pick $x \notin f(X)$, show that there is a ball about x that misses $f(X)$ and consider the sequence $x, f(x), f(f(x)), \ldots$.]

8. Let x be a point in the closed unit disc D, let K be a compact subset of D not containing x and define a map $f_x : K \to \partial D$ as follows. Given $y \in K$, take the line from x to y and extend it (in the x-to-y direction) until it first hits the boundary. Call this point $f_x(y)$. Prove that f_x is a continuous function.

9. Let $g : D \to D$ be a continuous function and let $x \in D$ be a point such that $x \neq g(x)$. Let $h(x) = f_x(g(x))$, where f_x is defined as in question 8 (on some open set containing x). Prove that h is continuous at x.

10. Let $C[0,1]$ be the metric space consisting of all continuous functions $f : [0,1] \to \mathbb{R}$, with the distance $d(f,g)$ between two functions f and g defined to be the supremum of $|f(x) - g(x)|$ over all $x \in [0,1]$.

(i) Explain why this supremum is in fact a maximum.

(ii) Let X consist of all functions f in $C[0,1]$ that take values in $[0,1]$. Prove that X is not a sequentially compact subset of $C[0,1]$.

(iii) This shows that X is not compact. Prove the same result by exhibiting an open cover of X that has no finite subcover.

11. Prove that every compact metric space is complete. The *diameter* of a metric space X is the supremum of $d(x,y)$ over all points $x, y \in X$. Give an example of a metric space that is complete and has diameter 1 but is not compact.

12. Let A be a 3×3-matrix with positive entries. Use Brouwer's fixed-point theorem to prove that A has an eigenvector with positive entries. [Hint: use A to define a map from T to itself, where T is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.]

Topics in Analysis, Examples Sheet 2. W. T. G. Michaelmas 2004

General remark. Unless it is clearly inappropriate, you may quote from IA and IB.

1. Let $f : [0,1] \to \mathbb{R}$ be a continuous function, let n be a positive integer and for $0 \leqslant i \leqslant n$ let $x_i = i/n$. Define a function $g_n : [0,1] \to \mathbb{R}$ by setting $g(x_i) = f(x_i)$ for each i, and making g linear on all the intervals $[x_{i-1}, x_i]$. Show that the functions g_n converge uniformly to f.

2. Let $p(z)$ be the quadratic polynomial $z^2 - 4z + 3$ and let $f : [0,1] \to \mathbb{C}$ be the closed path $f(t) = p(2e^{2\pi it})$. (Thus, the image of f is the image of the restriction of p to the circle of radius 2 and centre 0.)

(i) Calculate the winding number of f about 0 directly from the definition of winding number.

(ii) Can you give another proof that uses some of the results of the course?

3. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(t) = 0$ if $t \leqslant 1/2$ and $f(t) = 2t - 1$ if $1/2 \leqslant t \leqslant 1$. Find a polynomial p such that $|p(t) - f(t)| \leqslant 1/5$ for every $t \in [0,1]$.

4. It is clear that a function with a jump-discontinuity cannot be uniformly approximated by polynomials. However, that is not the only kind of discontinuity. True or false: no discontinuous function on $[0,1]$ can be uniformly approximated by polynomials?

5. Imitate the proof of the Weierstrass approximation theorem to prove that a continuous function $f : [0,1]^2 \to \mathbb{R}$ can be uniformly approximated by polynomials. [In the final version of this question I shall give an outline of how the proof should go, but it would be a good exercise to think about it without those hints first.]

6. Let $f : [0,1]^2 \to \mathbb{R}$ such that $f(x,y)$ is continuous in x for each fixed y and continuous in y for each fixed x. Does it follow that f is continuous?

7. Without looking at a book or at your old notes, prove that a continuous function from a compact metric space to $\mathbb{R}$ is uniformly continuous. [Write out the definition of uniform continuity, negate it, use the negated definition to generate a pair of sequences (x_n) and (y_n) and apply sequential compactness.]

8. Calculate the first five Chebyshev polynomials.

9. Calculate the first four Legendre polynomials. Do it both from the formula and by using orthogonality and check that your answers agree.

10. The Chebyshev polynomials form an orthogonal system with respect to a certain positive weight function w. That is, $\int_{-1}^{1} T_m(x)T_n(x)w(x)\,dx = 0$ whenever $m \neq n$. Work out what the weight function should be, and prove the orthogonality. [Hint: use an appropriate substitution for x!]

Topics in Analysis Examples Sheet 3 W. T. G.

1. For each n, let $f_n : [0,1] \to \mathbb{R}$ be a function and suppose that the functions f_n converge uniformly to a function f. Suppose also that f is bounded (above and below). Prove that for any positive integer m the functions $g_n(t) = f_n(t)^m$ converge uniformly to $g(t) = f(t)^m$.

2. Prove that there is no sequence of analytic functions f_n that converges uniformly on the unit circle to the function $1/z$ (which on the circle is the same as $\bar{z}$). Why does this not contradict Runge's theorem?

3. Construct a sequence of polynomials that converges uniformly to $1/z$ on the semicircle consisting of all points of the unit circle that have real part greater than or equal to 0.

4. Work out continued-fraction expansions for $71/49$ and $\sqrt{3}$.

5. Define a sequence (x_n) by $x_1 = 1$, $x_2 = 3$ and $x_n = x_{n-1} + x_{n-2}$ for $n \geqslant 3$. Prove that x_n/x_{n-1} converges to the golden ratio $(1+\sqrt{5})/2$. Describe the continued-fraction expansion of x_n/x_{n-1}.

6. Let x have continued-fraction expansion

$$1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}$$

with the 1s and 2s continuing to alternate. Calculate the value of x.

7. What rational with denominator less than 10 best approximates the number $71/49$?

8. Let C be a circle with centre 0 in the complex plane, traversed anticlockwise, and let K be a compact set disjoint from C. Define a function on K by

$$f(z) = \frac{1}{2\pi i}\int_C \frac{dw}{w-z}\ .$$

By splitting up the path integral into small pieces and approximating the contribution from each piece, prove that f can be uniformly approximated on K by functions of the form $f_n(z) = \sum_{i=1}^N a_i/(w_i - z)$, where the a_i and w_i are complex numbers with the w_i lying in C.

9. Prove that $\sqrt{3}+\sqrt{5}$ and e^2 are irrational.

Topics in Analysis Examples Sheet 4 W. T. G.

1. (i) For each n let F_n be the nth Fibonacci number, with $F_1 = F_2 = 1$. Prove the identities $F_{2n+1} = F_n^2 + F_{n+1}^2$ and $F_{2n} = F_n(F_{n-1} + F_{n+1})$. Let x_n stand for the ratio F_{n+1}/F_n. Use the above identities to express x_{2n} in terms of x_n.

(ii) If we set $y_k = x_{2^k}$, then we have expressed each y_k as a simple rational function of y_{k-1}. Explain why the sequence (y_k) converges very rapidly to the golden ratio. How does this relate to another method that produces rapid convergence?

2. What are the eigenvalues of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$? What are the connections between this, Fibonacci numbers and continued fractions?

3. Assuming the continued-fraction expansion of $\tan(x)$ given in lectures, derive an expansion for $\tanh(x)$. Deduce that

$$\frac{e+1}{e-1} = 2 + \cfrac{1}{6 + \frac{1}{10 + \frac{1}{14 + \dots}}}$$

4. Let X and Y be topological spaces, let $x \in X$ and let $f : X \to Y$. Say that f is *continuous at* x if for every neighbourhood N of $f(x)$ there exists a neighbourhood M of x such that $f(M) \subset N$. Prove that f is continuous if and only if it is continuous at every $x \in X$.

5. Let $X = \{0\} \cup \{n^{-1} : n \in \mathbb{N}\}$ and let X have the subspace topology inherited from $\mathbb{R}$. Which subsets of X are open? Which are closed? Which are compact?

6. The open sets in the *cofinite topology* on an infinite set X are the empty set and all subsets Y of X such that $X \setminus Y$ is finite. Write down and prove necessary and sufficient conditions for a function $f : X \to X$ to be continuous under this topology.

7. Let X be a Hausdorff topological space and let $x_1, \dots, x_n$ be n distinct points in X. Prove that there exist pairwise disjoint open sets $U_1, \dots, U_n$ with $x_i \in U_i$ for every i.

8. Let X be a complete metric space and let Y be another metric space that is homeomorphic to X. Must Y be complete?

9. Let $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ with the obvious topology coming from $\mathbb{C}$. Define an equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ iff $x - y$ is an integer. Prove that $\mathbb{T}$ is homeomorphic to $\mathbb{R}/\sim$ with the quotient topology.

10. Prove that every metric space is normal. Find an example of a Hausdorff topological space that is not normal.

11. Let X be a subset of $\mathbb{R}$. A point x in X is called *isolated* if there exists some $\delta > 0$ such that $|x - y| \geqslant \delta$ for every other $y \in X$. X is called *perfect* if it is closed and has no isolated points. Prove that a non-empty perfect set must be uncountable. (Hint: consider X as a metric space in its own right, and don't forget the Baire category theorem.)

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that for every $x > 0$ we have $f(nx) \to 0$ as $n \to \infty$. Prove that $f(x) \to 0$ as $x \to \infty$.

Topics in Analysis - Extra Examples Sheet. W. T. G.

Note. These problems are meant to help you to refamiliarize yourself with the content of the course, and to provide material for revision supervisions. They should not be taken as an indication of what the actual exam questions will be like. In particular, there are no bookwork questions on this sheet, and some of the problems, while not intended to be particularly difficult, are designed to make you think a bit harder than it would be reasonable to expect in an exam.

1. Using the Brouwer fixed-point theorem directly, prove that there is a complex number z such that

$$z^8 + 3z^6 - 2z^5 + z^3 - z^2 + 10z = 1 \ .$$

Using other results from the course (which you should state), but not the fundamental theorem of algebra, prove that there is a complex number w such that

$$w^8 + 3w^6 - 2w^5 + 10w^3 - w^2 + w = 1 \ .$$

2. Take an $n \times n$ grid of points and colour all its vertices red, blue, green or yellow. Do this in such a way that all the points along the bottom are red, all the ones on the right are blue, all the ones along the top are green and all the ones on the left are yellow. (At the four corners of the grid there will be a conflict: choose one of the two possible colours in such a way that the corners are all of different colours.) Prove that for at least one of the $(n-1)^2$ little squares in the grid, at least three colours are used. Find an example where no such square uses all four colours.

3. Construct a sequence of polynomials (P_n) such that $P_n(x)$ converges uniformly to $|x|$ in the interval $[-1, 1]$.

4. (i) Let $\mathcal{F}$ be the set of all functions defined on $[-1, 1]$ that can be uniformly approximated by polynomials. Using the result of the previous question, prove that if f and g belong to $\mathcal{F}$, then so do $f \vee g$ and $f \wedge g$. (These are defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$.)

(ii) Use (i) to prove that any continuous piecewise linear function defined on $[-1, 1]$ belongs to $\mathcal{F}$. (A piecewise linear function is one that is allowed to change direction at finitely many points and is linear between them.)

(iii) Prove that any continuous function on $[-1,1]$ can be uniformly approximated by piecewise linear functions and deduce that $\mathcal{F} = C[-1,1]$. (If you get to this point, then you have given another proof of Weierstrass's approximation theorem.)

5. Without using the formula for the nth Legendre polynomial P_n, prove that it is an even function if n is even and an odd function if n is odd.

6. Construct a sequence of polynomials P_n such that, for every positive real number a and every positive real number $b < a$, $P_n(z)$ converges uniformly to z^{-1} on the disc $\{z \in \mathbb{C} : |z-a| < b\}$. (If you don't want to give an exact formula for P_n, then at least explain in principle how it can be done.)

7. (i) Suppose that x is a real number and it has a simple continued fraction expansion of the form

$$1 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Prove that there is a sequence of fractions (p_n/q_n) such that

$$|x - \frac{p_n}{q_n}| \leqslant \frac{1}{a_n q_n^2} .$$

(ii) If the coefficients a_n are unbounded, deduce that x cannot be a quadratic irrational. (It is possible to prove this by showing that the continued fraction for a quadratic irrational is eventually periodic. That is not the proof that is being suggested here. Note that if you know the continued fraction expansion of e then you can deduce that e is not the root of a quadratic polynomial with integer coefficients.)

8. (i) Let X be a non-compact topological space, and let x be a point not belonging to X. Define a topology on $X \cup \{x\}$ by taking all open sets $U \subset X$ together with all sets of the form $U \cup \{x\}$ such that U is open in X and $X \setminus U$ is compact. Prove that $X \cup \{x\}$ is compact under this topology. (It is called the *one-point compactification* of X.)

(ii) Let X be the complex plane $\mathbb{C}$. Prove that the one-point compactification of X is homeomorphic to the sphere $\{v \in \mathbb{R}^3 : \|v\|^2 = 1\}$.

9. Let X be an infinite-dimensional normed space. Prove that the linear span of any finite collection of vectors is nowhere dense (with respect to the metric $d(x,y) = \|x-y\|$. Deduce that if X is complete, then it is not the linear span of a countable collection of vectors. (The linear span of a subset $Y \subset X$ is the set of all *finite* linear combinations $\sum_{i=1}^n a_i y_i$ such that each y_i belongs to Y.)