Tiling the plane

2.1 Polygons

A *polygon* is a chain of line segments (*sides* or sometimes *edges* joined at *vertices* or *corners*) which form a closed loop. We assume this is in the plane unless we state otherwise. A polygon is *simple* if it does not intersect itself, in which case it has an interior which is a region bounded by line segments. We will use the word "polygon" for both the chain of segments and for its interior, according to context. A simple polygon is *convex* if any line that meets its interior does so in a single connected segment.

We refer to a polygon with n sides as an n-gon or by a familiar name such as "triangle", "quadrilateral", "pentagon", etc.

We are most interested in polygons with some symmetry. For example, we know that a quadrilateral has D_4 symmetry if it is a square and D_2 symmetry if it is an oblong. On the other hand, a generic quadrilateral has trivial symmetry group.

2.1.1 Definition A *flag* of a polygon is a triple (v, e, f) where v is a vertex, e is the centre of an edge attached to that vertex and f is the polygon's centre.

In general, to geometers a "flag" is a 0-dimensional object, a 1-dimensional object, and so on up to a k-dimensional object with each object contained in the next.

2.1.2 Definition A *regular* polygon is one where the symmetry group acts transitively on its flags. (Thus, given vertices v_1, v_2 attached to edges e_1, e_2 respectively, there is a symmetry of the figure taking v_1 to v_2 and e_1 to e_2.)

The convex, regular polygons are exactly all the *regular n-gons* (the equilateral triangle, square, regular pentagon, etc). Regular polygons and polyhedra are often denoted by their Schläfli symbol. The regular, convex n-gon has Schläfli symbol $\{n\}$.

2.1.3 For interest There are also non-convex regular *star polygons* (see `http://mathworld.wolfram.com/StarPolygon.html`). These are denoted $\{\frac{n}{m}\}$ where $n \geq 5$ and $2 \leq m < n/2$. If the gcd of m and n is not 1 then $\{\frac{n}{m}\}$ is not a star polygon but a *compound*.

2.1.4 Definition A polygon is *vertex transitive* ("v-transitive" for short) if its symmetry group acts transitively on its vertices. It is *edge transitive* ("e-transitive" for short) if its symmetry group acts transitively on its edges.

2.1.5 Problem (NL) A triangle is v-regular if and only if it is equilateral. Similarly for e-regular.

Consider now convex quadrilaterals. A generic quadrilateral has no non-trivial symmetries. Special, more symmetric quadrilaterals include rectangles, rhombuses, parallelograms, isosceles trapeziums, kites and squares. In each case what is the symmetry group and which cases are e-transitive and which are v-transitive.

2.2 Honeycombs

In this chapter we will consider only tilings that are "edge to edge", meaning that where two polygons meet, they do so only at a vertex or they share a complete common edge. Unless otherwise stated, tilings are by regular polygons.

2.2.1 Definition A *regular tiling* or *honeycomb* in the plane is a tiling of the plane by identical regular polygons. It is easy to see that there are exactly three possibilities:

- The *quadrille*: a tiling by squares with four meeting at each vertex. Symmetry group $*442$. Schäfli symbol $\{4,4\}$.
- The *deltille*: a tiling by equilateral triangles with six meeting at each vertex. Symmetry group $*632$. Schäfli symbol $\{3,6\}$.
- The *hextille*: a tiling by regular hexagons with three meeting at each vertex. Symmetry group $*632$. Schäfli symbol $\{6,3\}$.

In each case, flags in each polygon in the tiling are fundamental domains for the symmetry group and so given two flags in two polygons in the tiling, there is a unique symmetry taking one to the other.

The *Schäfli symbol* $\{p, q\}$ indicates that the tiling is by regular p-gons and q are meeting at each vertex.

2.2.2 Duality / reciprocation A trick with tilings by regular polygons is to take the the centres of the tiles to be vertices in a new "dual" tiling. Two vertices are joined by an edge in the new tiling if and only if the corresponding polygons in the original tiling had a common edge. This

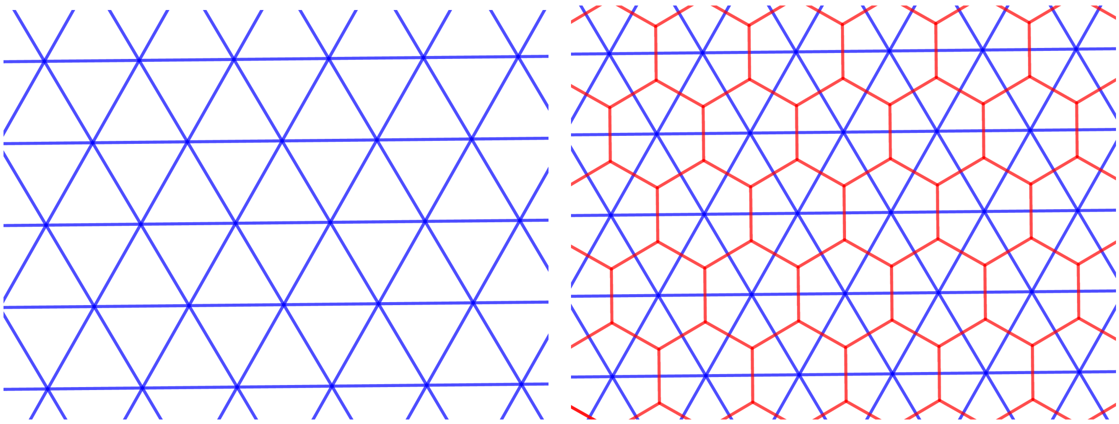


Figure 2.1: Deltille $\{3,6\}$ and with the dual hexatille $\{6,3\}$.

duality always reverses the numbers in the Schäfli symbol: as we see above, the detille and hextille are dual and the dual of the quadrille $\{4, 4\}$ is itself. Draw the picture and check!

2.3 Archimedean tilings

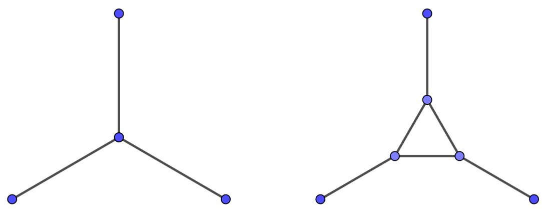


Figure 2.2: Truncating a vertex in a hextille.

2.3.1 Truncation If one takes a vertex in the $\{p, q\}$ honeycomb, one can replace the vertices by little regular q-gons, centred at the original vertex as illustrated for $q = 3$ in Figure 2.2. The resulting tiling consists in the hextille case of equilateral triangles and 12-gons.

If you slowly increase the size of the equilateral triangles, at some point the 12-gons become regular and we have a new tiling of the plane by regular polygons, called the *truncated hextille*. See Figure 2.3 below.

2.3.2 Exercise (NS) What do you get by truncating the quadrille and by truncating the deltille?

2.3.3 Vertex transitive tilings The truncated hextille has exactly the same symmetries as the original hextille: it's $*632$. The tiling now has two sorts of tiles and also two sorts of edges (those separating two 12-gons and those separating a 12-gon from a triangle). But it has only one type of vertex. More precisely, given two vertices there is a symmetry of the pattern taking one to the other. We say that the tiling is *vertex transitive* of *v-transitive* for short. The word *uniform* is aalso often used.

Tilings that are v-transitive but not regular are called *Archimedean or semiregular*.

2.3.1 The Wythoff construction

2.3.1 The Wythoff construction constructs most of the v-transitive tilings of the plane by regular polygons; it constructs all those with kaleidoscopic symmetry group of the form $G = *2pq$.

To be more precise, for each such group we find all tilings by regular polygons on which the group acts by symmetries v-transitively. We are most interested in the case where G is the full symmetry group of the tiling but sometimes we get cases where G is a proper subgroup of the full symmetry group in which case we say the tiling is *relative* rather than absolute. We will see examples soon.

For tilings in the plane, the only possibilities are $*632$ and $*442$ but we proceed with a general p and q because it turns out we can use other values in the spherical and hyperbolic cases.

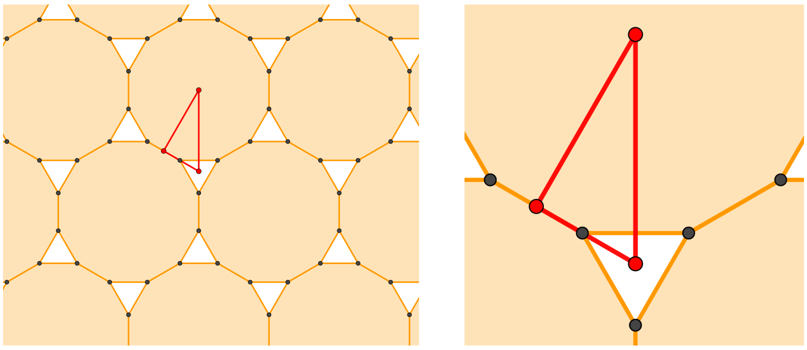
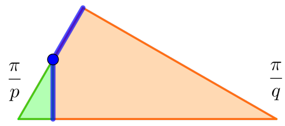


Figure 2.3: Truncated hextille with fundamental domain.

2.3.2 Wythoff's idea In figure 2.3 we see a fundamental domain in the truncated hextille and an expanded picture. On the right we see that picture in a different orientation. The base angles of the triangle are π/p and π/q where in this example we have $p = 3$ and $q = 6$. Note the following facts.



- There is one and only one vertex of the tiling in the orbifold. It is the blue point. (This follows from the vertex transitivity.)
- The blue half-edges meet the reflecting boundary of the orbifold orthogonally. (If they did not then the edge would change direction at the boundary.)
- The blue vertex lies on the angle bisector of the opposite angle and consequently the two half edges are the same length. (Thus the two half-edges generate a regular orange 12-gon when kaleidoscoped about the right-hand vertex.)
- The lower half-edge generates a green equilateral triangle when kaleidoscoped about the left-hand vertex.

2.3.3 Face code We define the *face code* of a vertex-transitive tiling to be the list of the number of sides of each n-gon surrounding a vertex. It is defined only up to cyclic permutations and reversal. For the truncated hextille the face code is $(12)^2(3)$ because at each vertex we encounter a 12-gon then a 12-gon then a 3-gon as we go round a vertex. The face code can be seen from the orbifold picture by considering the result of reflecting the orbifold in its shortest side.

2.3.4 Wythoff's construction Wythoff observed that in general if one is to have $*2pq$ acting vertex transitively on a tiling by regular polygons, that for essentially the same reasons sketched above for the truncated hextille there are exactly seven possibilities.

- The blue tiling vertex can be at one of the three corners of the orbifold with an edge joining it perpendicularly to the opposite side. (Note that in the case of the lower two vertices, the edge is one of the edges of the original orbifold.)
- It can be at the point on one of the three sides where the angle bisector of the opposite angle meets it. There are two blue edges joining the vertex orthogonally to the two other sides.

- Finally, the blue vertex can be at the meeting of the three angle bisectors (the "incentre" of the triangle), with a blue edge joining it orthogonally to each of the sides.

2.3.5 The $*632$ regular and Archimedean tilings Table 2.1 shows the seven results of the Wythoff construction for $*632$. It produces two honeycombs (regular tilings) that we know about and four v-regular "Archimedean" tilings. A case of note is the "truncated deltille". A little thought reveals that truncating a deltille leads to a hextille and so not to a new v-regular tiling. But since it comes from the Wythoff construction we see more: the resulting hextille will be two coloured (orange and green) and the $*632$ symmetry group of the coloured pattern is a subgroup of the symmetry group of the whole pattern (a "larger" $*632$) which still acts transitively on the vertices.

2.3.6 The $*442$ regular and Archimedean tilings The case $*442$ is less productive: one obtains only the quadrille and the truncated quadrille (squares and octagons with face code $(8)^2(4)$). This is because the symmetry of the fundamental domain leads to two pairs of possibilities becoming identical and the others lead to relative cases.

2.3.7 The non-Wythoff v-regular tilings There are three non-Wythoff Archimedean tilings which we will discuss later: the snub quadrille, the snub hextille and the isosnub quadrille. See Table 2.2. In all then we have 11 v-regular tilings, 3 of which are regular.

2.3.8 Catalan tilings One can take the dual of the 8 Archimedean tilings and obtain 8 rather attractive *Catalan tilings*. These are not tilings by regular polygons. The same symmetry group acts transitively on the faces of the Catalan tiling because they correspond to vertices in the original tiling. The Catalan tiling will have different types of vertex, but because the faces of the original tiling were regular polygons, the vertices of the Catalan tiling are regular in the sense that the emerging edges are equally spaced around a vertex. Each edge of the Catalan tiling perpendicularly bisects and edge of the original tiling. (Why?)

You can find pictures of the 8 Catalan tilings at `https://en.wikipedia.org/wiki/List_of_convex_uniform_tilings`. The "Cairo tiling" in particular seems particularly attractive.

2.3.9 Exercise () Consider the Wythoff construction of the "truncated deltille". What colouring of the hextille does that give you?

2.3.10 Exercise () Consider the Wythoff construction for $*442$. There are three "relative cases" which give rise to colourings of tilings. Identify the colourings.

2.3.11 Exercise () What are the symmetry groups of the three non-Wythoffian Archimedean tilings?

2.3.12 Exercise () One of the Archimedean tilings is "chiral" meaning that its mirror image is not equivalent to the original under direct isometry. Which one?

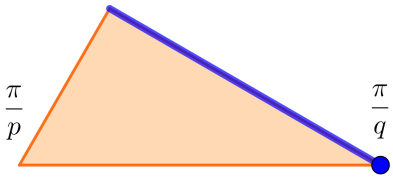
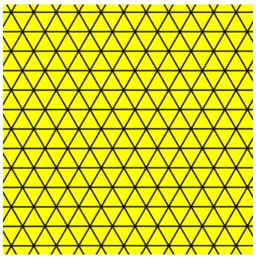
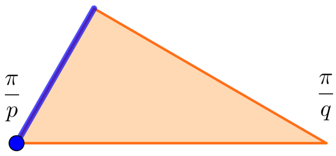
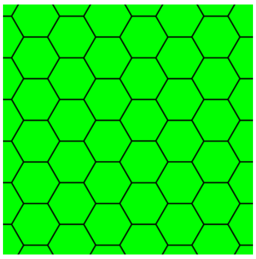
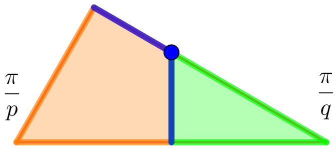
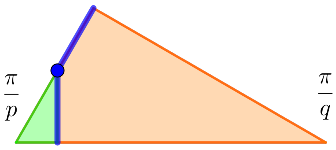
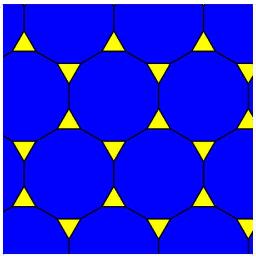
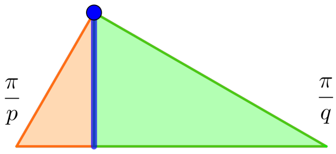
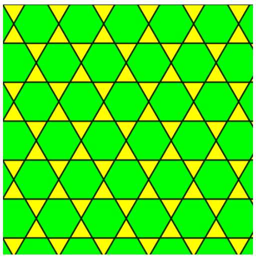
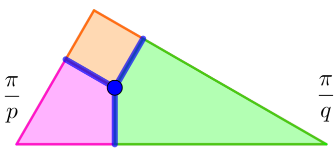
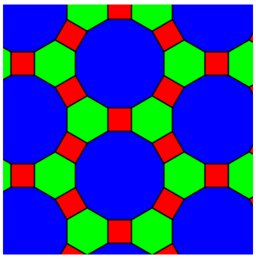
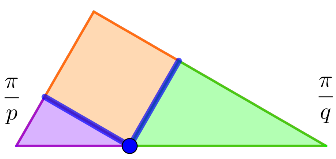
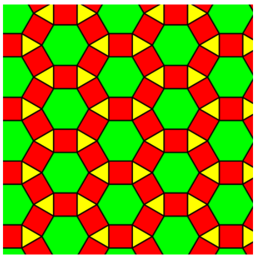
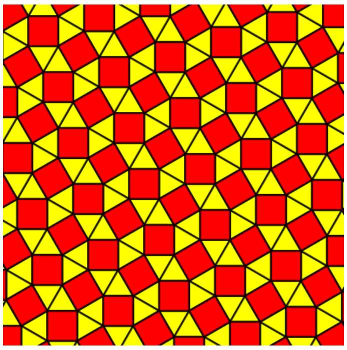
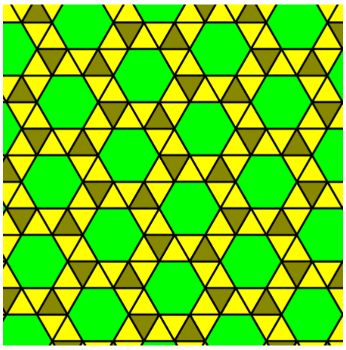
Construction	**Face code / Name**	**The $*632$ case**
	3^6 Deltille	
	6^3 Hextille	
	$(6)^2 6$ Truncated deltille	= coloured hextille Relative
	$(12)^2 3$ Truncated hextille	
	$(3)(6)(3)(6)$ Hexadeltille	
	$(4)(6)(12)$ Truncated hexadeltille	
	$(4)(3)(4)(6)$ Rhombihexadeltille	

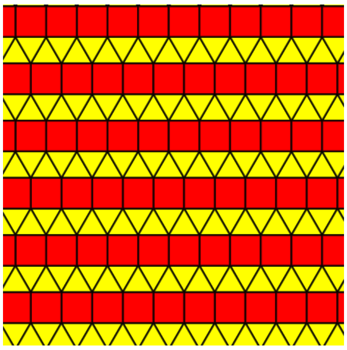
Table 2.1: The Wythoff construction (tiling images from Wikipedia)



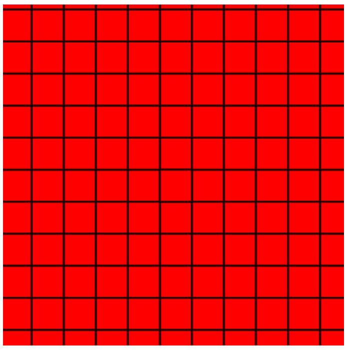
Snub quadrille



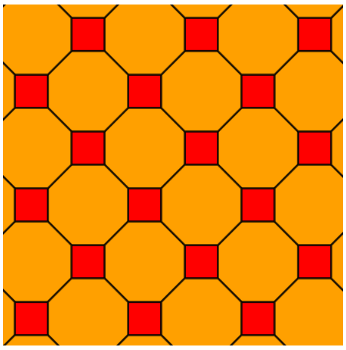
Snub hextille



Isosnub quadrille



Quadrille



Truncated quadrille

On the left are the two Wythoff $*442$ cases. Of the other 5, two are duplicates because of symmetry and the other three are relative: they correspond to colourings of tilings already in the list.

Table 2.2: Two Wythoff $*442$ and three non-Wythoff v-regular tilings (images from Wikipedia)

2.3.13 Exercise () Of the Archimedean tilings, one is also edge-regular. Which one, and what feature of the Wythoff construction corresponds to this property?

2.3.14 Exercise () For each of the seven Wythoff cases, consider the corresponding Catalan tiling. What is the shape of the tile? Which cases give rise to Catalan tilings by Rhombuses and why?