Analysis Hand In Two

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Question One - Worksheet 3, Q6

$$f(x) = \sin(x), \quad x \in \mathbb{R} \tag{1}$$

We wish to show that $\forall \varepsilon > 0 : \exists \delta > 0 : \forall x, y \in \mathbb{R}$: we have that $|x - y| < \delta \Rightarrow |\sin(x) - \sin(y)| < \varepsilon$, note that in the case of uniform continuity ε depends only on δ. Since $\sin(x)$ is differentiable on $I = \mathbb{R}$ with a bounded derivative, $\left|\frac{d}{dx}\sin(x)\right| = |\cos(x)| \leq 1$ for any $x \in \mathbb{R}$, I will instead prove that for any differentiable function $g : \mathbb{R} \to \mathbb{R}$ with a bounded derivative is uniformly continuous, the result will then follow.

Since $g : \mathbb{R} \to \mathbb{R}$ has a bounded derivative by assumption we have that the limit

$$g'(y) := \lim_{x \to y} \frac{g(x) - g(y)}{x - y} \tag{2}$$

exists and is bounded by some $M \in \mathbb{R}$ so that $\forall x \in \mathbb{R} : |g'(x)| < M$. In other words; if $|x - y| < \delta$ then $\left|\frac{g(x)-g(y)}{x-y} - g'(y)\right| < \epsilon$. Let $\varepsilon = \frac{M\delta}{1+\delta}$, so we have that $\delta = \frac{\varepsilon}{M-\varepsilon}$ and since $|x - y| < \delta$

$$\begin{aligned}
\epsilon > \left|\frac{g(x) - g(y)}{x - y} - g'(y)\right| &\geq \left|\frac{g(x) - g(y)}{x - y} - M\right| && \text{since } \forall x : |g'(x)| < M \\
&\geq \left|\left|\frac{g(x) - g(y)}{x - y}\right| - M\right| && \text{by the reverse triangle inequality} \\
\Rightarrow M - \epsilon < \left|\frac{g(x) - g(y)}{x - y}\right| &< M + \epsilon && \text{using } |a - b| < c \Rightarrow -c < a - b < c
\end{aligned}$$

and so we have that $|g(x) - g(y)| < |x - y|(M - \epsilon) < \delta(M - \epsilon) = \frac{\varepsilon}{M-\varepsilon}(M - \varepsilon) = \varepsilon$, that is; g is uniformly continuous. Thus if $g : \mathbb{R} \to \mathbb{R}$ has a bounded derivative on $\mathbb{R}$ then g is uniformly continuous, hence $f(x) = \sin(x)$ is uniformly continuous on $\mathbb{R}$.

Question Two - Worksheet 3, Q9

We will aim to find two sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ for which $|s_n - t_n| \to 0$ but that $\lim_{n\to\infty} |f(s_n) - f(t_n)| \neq 0$ for some continuous function f, then by question seven of the workshop f will not be uniformly continuous.

Let $f : (0, 1) \to \mathbb{R} : x \mapsto 1/x$, then f is continuous on $(0, 1)$ since if $|x - y| < \delta$ and $\varepsilon = \frac{\delta}{|xy|}$ then

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| < \frac{\delta}{|xy|} = \varepsilon.$$

Now we must show that f isn't uniformly continuous. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence defined by $s_n = 1/2n$ so that $s_n \to 0$ and $\forall n \in \mathbb{N} : s_n \in (0, 1/2] \subset (0, 1)$, note that $f(s_n) = f(1/2n) = 2n$. Now, let $(t_n)_{n\in\mathbb{N}}$ be a sequence defined by $t_n = 1/3n$ so that $t_n \to 0$ as $n \to \infty$, and we have $\forall n \in \mathbb{N} : t_n \in (0, 1/3] \subset (0, 1)$, and $f(t_n) = 3n$. Then $|f(t_n) - f(s_n)| = |3n - 2n| = |n| \to \infty \neq 0$, thus $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.

The theorem in the workshop stated that if $f : [a, b] \to \mathbb{R}$ is continuous then it's uniformly continuous. This isn't true for open intervals (a, b) since f can asymptotically approach $\pm\infty$ at either of the end points, however if f is defined on $[a, b]$ then in order for f to be defined, $f(a)$ and $f(b)$ must be finite.

Question Three - Worksheet 4, Q5

We have that

$$S(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad C(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \tag{3}$$

and we wish to show that for $x \in (0, \sqrt{6}] : S(x) > 0$ and that for $x \in (0, \sqrt{2}] : C(x) > 0$. For $S(x)$, consider equation (3):

$$\begin{aligned} S(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \left(\frac{(-1)^{2k} x^{4k+1}}{(4k+1)!} - \frac{(-1)^{2k} x^{4k+3}}{(4k+3)!} \right) \\ &= \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} \left(1 - \frac{x^2}{(4k+2)(4k+3)} \right) \end{aligned}$$

so if $x \in (0, \sqrt{6}]$ then $x^2 \in (0, 6]$ and $x^{4k+1} \in (0, 36^k \sqrt{6}]$. Notice that $1 - \frac{x^2}{(4k+2)(4k+3)} > 0$ precisely when $1 > \frac{x^2}{(4k+2)(4k+3)} \in \left(0, \frac{6}{(4k+2)(4k+3)}\right]$, and so certainly $1 - \frac{x^2}{(4k+2)(4k+3)} > 0$. Thus each term $\frac{x^{4k+1}}{(4k+1)!}\left(1 - \frac{x^2}{(4k+2)(4k+3)}\right)$ is positive and thus the sum of terms $S(x) > 0$ is positive for $x \in (0, \sqrt{6}]$.

Similarly we want to show that $C(x) > 0$ for $x \in (0, \sqrt{2}]$. Splitting the sum as we did above we get

$$\begin{aligned} C(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\ &= \sum_{k=0}^{\infty} \left(\frac{(-1)^{2k} x^{4k}}{(4k)!} - \frac{(-1)^{2k+2} x^{4k+2}}{(4k+2)!} \right) \\ &= \sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} \left(1 - \frac{x^2}{(4k+1)(4k+2)} \right), \end{aligned}$$

and clearly $\frac{x^{4k}}{(4k)!}$ is positive. Now, $x \in (0, \sqrt{2}] \Rightarrow x^2 \in (0, 2]$, and since $k \geq 0$ we have that $(4k+1)(4k+2) \geq (0+1)(0+2) = 2$ and so $1 - \frac{x^2}{(4k+1)(4k+2)} > 0$, and since each term is positive the sum of all the terms will be positive too, thus $C(x) > 0$ for $x \in (0, \sqrt{2}]$.

Finally we wish to prove that if $x \in [0, \sqrt{56}]$ and $1 - x^2/2! + x^4/4! < 0$ then $C(x) < 0$. Rewriting this as a quadratic in $a = x^2$ we have that $a^2 - 12a + 24 < 0$, and we get critical values $a = 6 \pm 2\sqrt{3}$ and so the critical values for the quartic in x are $\left\{\sqrt{6+2\sqrt{3}}, \sqrt{6-2\sqrt{3}}, -\sqrt{6+2\sqrt{3}}, -\sqrt{6-2\sqrt{3}}\right\}$, which gives (by looking at the graph) that $x^4 - 12x^2 + 24 < 0$ iff $x \in (-\sqrt{6+2\sqrt{3}}, -\sqrt{6-2\sqrt{3}}) \cup (\sqrt{6-2\sqrt{3}}, \sqrt{6+2\sqrt{3}})$, but by assumption we have that $x \in (0, \sqrt{56}]$ so certainly $x > 0$ and $\sqrt{6+\sqrt{3}} < \sqrt{56}$ thus $(\sqrt{6-2\sqrt{3}}, \sqrt{6+2\sqrt{3}}) \subset (0, \sqrt{56})$ and

$$x \in \left(0, \sqrt{56}\right] \text{ and } 1 - x^2/2! + x^4/4! < 0 \quad \Rightarrow \quad x \in \left(\sqrt{6 - 2\sqrt{3}}, \sqrt{6+2\sqrt{3}} \right),$$

and so $x^2 \in \left(6 - 2\sqrt{3}, 6 + 2\sqrt{3}\right)$. Note that

$$C(x) = \sum_{k=0}^{\infty} \left(\frac{x^{4k}}{(4k)!} - \frac{x^{4k+2}}{(4k+2)!} \right) = \sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!} \tag{4}$$

where $x \in \left(\sqrt{6-2\sqrt{3}}, \sqrt{6+2\sqrt{3}}\right)$, and $1 - x^2/2! + x^4/4! = 0$ so the first two terms of the first sum $1 + x^4/4! = x^2/2!$; the first term of the second sum, thus

$$C(x) = \sum_{k=2}^{\infty} \frac{x^{4k}}{(4k)!} - \sum_{k=1}^{\infty} \frac{x^{4k+2}}{(4k+2)!} = \sum_{k=2}^{\infty} \left(\frac{x^{4k}}{(4k)!} - \frac{x^{4k-2}}{(4k-2)!} \right),$$

which is clearly negative, as each term is negative.

Now, we have that for $x \in \left(\sqrt{6-2\sqrt{3}}, \sqrt{6+2\sqrt{3}}\right)$ then $C(x) < 0$. Since $1.59 \approx \sqrt{6-2\sqrt{3}} < 8/5 = 1.6 < \sqrt{6+2\sqrt{3}} \approx 3.08$ by direct calculation, we have that $C(8/5) < 0$ by the above.

Question Four - Worksheet 4, Q6

From the above question we have that $C(8/5) < 0$, and $\frac{d}{dx}C(x) = -S(x)$, so $C(x)$ is a continuous funtion on some interval I (in this case $I = \mathbb{R}$, which we proved in the workshop), thus we can use the intermediate value theorem:

> **Intermediate Value Theorem:**
> If f is continuous on some interval $[a, b]$ and takes values $f(a) = y_a$ and $f(b) = y_b$, then for any y between y_a and y_b, $\exists c \in [a, b]$ such that $y = f(c)$.

In our case, $[a, b]$ can be any closed subset of $I = \mathbb{R}$, but we'll choose $[\sqrt{2}, 8/5]$. We have that $C(8/5) < 0$ and that $C(\sqrt{2}) > 0$ from the above, and thus there must be some $\omega/2 \in [\sqrt{2}, 8/5]$ such that $C(\omega/2) = 0$ since 0 is between $C(8/5) < 0 < C(\sqrt{2})$.

Now we wish to show that $S(\omega/2) = 1$, note that $\frac{d}{dx}S(x) = C(x)$, and that $\forall x : S(x) \leq 1$, and so 1 is a global maximum for $S(x)$. At any global maximum strictly inside some interval (ie, not one of the endpoints) has a derivative of zero, thus if $S(c) = 1$ for some c then $\frac{dS}{dx}|_{x=c} = C(c) = 0$. There is a unique point $\omega/2$ in $(\sqrt{2}, 8/5)$ (note the lack of end-points) in which $C(\omega/2) = 0$ and given that there is a unique point $c \in (\sqrt{2}, 8/5)$ in which $S(c) = 1$ so that $C(c) = 0$, it must be that $c = \omega/2$, thus there is a unique $\omega/2 \in (\sqrt{2}, 8/5)$ in which $C(\omega/2) = 0$ and $S(\omega/2) = 1$.