Workshop 8 – Compactness of the unit square

The purpose of this workshop is to prove that the unit square in $\mathbb{R}^2$, with the usual metric, is compact. We shall then examine what is needed to extend the argument to k dimensions and to more general metric spaces.

We first take the metric space $Q = [0,1]^2 = [0,1] \times [0,1] \subseteq \mathbb{R}^2$, with the usual euclidean metric $d_2(x,y) = |x-y|$. We wish to show that it is compact. (Believe it or not, it's actually easier to visualise the argument in two dimensions than in one!) We do this in a sequence of steps. The argument should be very reminiscent of that given to establish the Bolzano-Weierstrass theorem in FPM.

1. Write down what it means for Q to be compact, and also its negation (which can be used to begin a proof by contradiction). (**Hint:** There exists an $\ldots\ldots \{U_\alpha\}$ such that $\ldots\ldots$)

Solution: For every collection of open sets $\{U_\alpha\}$ in $\mathbb{R}^2$ such that $Q \subseteq \bigcup_\alpha U_\alpha$, there is a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ such that $Q \subseteq \bigcup_{j=1}^k U_{\alpha_j}$; i.e. for every open cover of Q there is a finite subcover. The negation is that there exists some open cover $\{U_\alpha\}$ of Q which has no finite subcover.

We shall work through questions 2–8 under the standing assumption that Q is NOT compact, (i.e. your second answer to question 1), finally deriving a contradiction in question 8.

2. Under our standing assumption, what can be said about *at least one* of the four closed subsquares

$$[0,1/2] \times [0,1/2],\ [0,1/2] \times [1/2,1],\ [1/2,1] \times [0,1/2],\ [1/2,1] \times [1/2,1]$$

of Q? Call such a square Q_1.

Solution: At least one of these 4 subsquares is not coverable by any finite subcollection of $\{U_\alpha\}$ – because if all 4 were coverable by a finite subcollection of the $\{U_\alpha\}$, we could put the 4 subcollections together to form a finite subcollection of the $\{U_\alpha\}$ which covers Q.

3. Repeat the argument of Step 2 on Q_1 to obtain a closed square $Q_2 \subseteq Q_1$ of sidelength $\ldots\ldots$ (fill in the blank) with the property that $\ldots\ldots$ (fill in the blank).

Solution: We obtain a closed square $Q_2 \subseteq Q_1$ of sidelength 2^{-2} with the property that it is not coverable by any finite subcollection of $\{U_\alpha\}$.

4. Now repeat successively, obtaining a closed square $Q_n \subseteq Q_{n-1}$, of sidelength ...... (fill in the blank) with the property that ...... (fill in the blank).

Solution: We obtain a closed square $Q_n \subseteq Q_{n-1}$ of sidelength 2^{-n} with the property that it is not coverable by any finite subcollection of $\{U_\alpha\}$.

5. Pick $x_n \in Q_n$. What can you say about $|x_m - x_n|$ if $m \geq n$?

Solution: If $m \geq n$, then as $x_m \in Q_m \subseteq Q_n$, we have that both x_m and x_n are in Q_n. Hence $|x_m - x_n| \leq \sqrt{2}2^{-n}$.

6. Show that the sequence (x_n) converges to some $x \in \cap_{n=1}^{\infty} Q_n$. Be careful and precise with your argument here, stating clearly any key properties of $\mathbb{R}^2$ that you use.

Solution: By the previous step, (x_n) is a Cauchy sequence and so converges to some $x \in \mathbb{R}^2$ by the **completeness of** $\mathbb{R}^2$**.** Since for all n we have $x_m \in Q_n$ for all $m \geq n$, and since Q_n is closed, we must have that $x \in Q_n$ for all n.

7. Deduce that for some α_0, $x \in U_{\alpha_0}$ (one of the members of the original cover) and hence that there is some $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0}$.

Solution: Since $x \in Q \subseteq \bigcup_\alpha U_\alpha$, there must be some α_0 such that $x \in U_{\alpha_0}$. Since U_{α_0} is open, there is some $r > 0$ such that $B(x, r) \subseteq U_{\alpha_0}$.

8. Show that for n such that $\sqrt{2}2^{-n+1} \leq r$, $Q_n \subseteq B(x, r)$, and hence finally derive a contradiction.

Solution: Since $x \in Q_n$ we have that $|x - y| \leq \sqrt{2}2^{-n}$ for any $y \in Q_n$ so that $Q_n \subseteq B(x, \sqrt{2}2^{-n+1})$. Thus for any such n, Q_n is covered by the *single* member U_{α_0} of the original collection, in contradiction to the statement that no Q_n can be covered by finitely many of the U_α's.

9. What would we have to change to make the above argument work for the unit cube in $\mathbb{R}^k$?

Solution: Instead of 4 subsquares at each stage we have 2^k; replace the $\sqrt{2}$'s by $\sqrt{k}$'s.

10. What properties of an otherwise arbitrary complete metric space (X, d) would be needed to make the above argument work in order to establish compactness of X?

Solution: We'd still suppose there is some open cover $\{U_\alpha\}$ with no finite subcover. We'd need to build a nested sequence of closed subsets Q_n, none of which can be covered by finitely many members of the open cover, with the diameter of Q_n tending to zero as $n \to \infty$, ensuring that if $x_n \in Q_n$, then (x_n) will be a Cauchy sequence in X which would converge to some $x \in X$ by completeness of X, and for which some ball around x would then contain a Q_n but be contained in some U_{α_0}, contradiction. What we'd need to do this is that **for all $r > 0$ we can cover X by finitely many closed balls of radius $r > 0$.**

Indeed, suppose that X is complete and has the property that for all $r > 0$ we can cover X by finitely many closed balls of radius $r > 0$. Suppose there is some open cover $\{U_\alpha\}$ of X with no finite subcover. Take $r = 1$. Then for some closed ball Q_0 of radius 1, no finite subcollection of $\{U_\alpha\}$ covers it. Then for some closed ball B of radius 2^{-1}, $B \cap Q_0 := Q_1$ has the property that no finite subcollection of $\{U_\alpha\}$ covers it. Continuing, for some closed ball B' of radius 2^{-2}, $B' \cap Q_1 := Q_2$ has the property that no finite subcollection of $\{U_\alpha\}$ covers it. Hence we find a sequence of closed sets Q_n, with $Q_n \subseteq Q_{n-1}$ and $\text{diam}(Q_n) \leq 2^{n-1}$ such that for all n, no finite subcollection of $\{U_\alpha\}$ covers Q_n. We arrive at:

Theorem. *Suppose (X, d) is a complete metric space which has the property that for all $r > 0$ we can cover X by finitely many closed balls of radius $r > 0$. Then X is compact.*

11. Show that if a metric space X is compact then it has the property that for all $r > 0$ we can cover X by finitely many closed balls of radius $r > 0$.

Solution: Suppose X is compact. Consider the open cover of X given by $\{B(x, r) : x \in X\}$. This has a finite subcover $\{B(x_j, r) : 1 \leq j \leq n\}$. Then the closed balls of radius r with centres at the x_j cover X.

In the lectures – or see Exercise 10.4.10 (a) from Wade – we'll prove that if X is compact then it is sequentially compact, i.e. every sequence in X has a convergent subsequence. Use this freely below.

12. Show that if a metric space X is compact then it is complete, and deduce that a metric space is compact *if and only if* it is complete and has the property that for all $r > 0$ we can cover X by finitely many closed balls of radius $r > 0$.

Solution: Take (x_n) to be any Cauchy sequence, it has a convergent subsequence, and hence converges.

13. Show that a metric space is compact if and only if it is sequentially compact.

Solution: The forward implication is given for free above. For the reverse, it is enough by the conclusion of Exercise 10 to show that it is complete and has the property that for all $r > 0$ we can cover X by finitely many closed balls of radius $r > 0$. First, let (x_n) be Cauchy. By hypothesis, it has a convergent subsequence, and since (x_n) was Cauchy, it converges. So X is complete. For the second, suppose that for some $r > 0$, there is *no* cover of X by finitely many closed balls of radius r. Pick $x_1 \in X$. Now pick x_2 such that $d(x_2, x_1) > r$. Now pick x_3 such that $d(x_3, x_2) > r$ and $d(x_3, x_1) > r$. Continuing, we get a sequence x_n such that $d(x_m, x_n) > r$ for all m and n, and hence with no convergent subsequence.