Exercises for Honours Analysis lectures – mainly from Wade's book

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Outline

Problem 1

Suppose A and B are two nonempty subsets of $\mathbb{R}$ such that for all $a \in A$ and $b \in B$ we have $a \leq b$.

(i) Show that $\sup A \leq \inf B$.

(ii) Suppose that for every $\epsilon > 0$ there exist $a \in A$ and $b \in B$ such that $|b - a| < \epsilon$. Show that $\sup A = \inf B$.

Solution. (i) Fix $a \in A$. Since $a \leq b$ for all $b \in B$ it follows that a is a lower bound of the set B. Since inf B is the greatest lower bound it follows that $a \leq \inf B$. This holds for all $a \in A$. Thus $\inf B$ is the upper bound of the set A. As $\sup A$ is the least upper bound, clearly $\sup A \leq \inf B$.

(ii) Suppose $\varepsilon = \inf B - \sup A > 0$. The for all $a \in A$ and $b \in B$

$$a \leq \sup A < \inf B \leq b,$$

and hence $b - a \geq \inf B - \sup A = \varepsilon > 0$. Thus it is impossible to have $b - a < \varepsilon$. This is a contradiction and hence

$$\inf B - \sup A = 0.$$

Problem 2

Let (a_n) be a sequence of real numbers and $a \in \mathbb{R}$. Suppose $a_n \to a$.

Show that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \to a.$$

Solution. The sequence (a_n) must be bounded. Let $|a_n| \leq M$ for all n. Given $\varepsilon > 0$ find N such that $\forall n \geq N$:

$$|a_n - a| < \varepsilon.$$

It follows that for all $n \geq N$ we have

$$\left|\frac{a_1 + a_2 + \cdots + a_n}{n} - a\right| \leq \frac{1}{n}\sum_{k=1}^{n} |a_k - a|.$$

$$\leq \frac{1}{n}\sum_{k=1}^{N-1} |a_k - a| + \frac{1}{n}\sum_{k=N}^{n} |a_k - a| \leq \frac{2(N-1)M}{n} + \frac{(n-N+1)\varepsilon}{n}.$$

Here M, N are fixed. We let $n \to \infty$. The first term goes to zero, while the second is $< \varepsilon$. Hence for all very large n we have

$$\left|\frac{a_1 + a_2 + \cdots + a_n}{n} - a\right| < \varepsilon.$$

From this the claim follows.

Problem 3

(i) Suppose that (a_n) is a convergent sequence of real numbers and let A be the set of all its terms, i.e.

$$A = \{a_1, a_2, \dots a_n, \dots\}.$$

Prove that A has a maximum or minimum (or both).

(ii) Give an example of a bounded sequence which has neither a maximum nor a minimum.

Solution. (i) We need to consider 3 numbers, $\inf A$, $\sup A$ and $a = \lim a_n$. There are 3 options. If $\inf A = \sup A = a$ the sequence is constant and the claim is obvious. If $\inf A < a$ if follows that for some N we have

$$\forall n \geq N : \qquad |a_n - a| < a - \inf A.$$

From this $\inf A < a_n$ for all $n \geq N$. Thus

$$\inf A = \min\{a_1, a_2, \dots, a_{N-1}\},$$

and hence minimum exists. The third option is that $a < \sup A$ and by a similar argument we can see that

$$\sup A = \max\{a_1, a_2, \dots, a_{N-1}\},$$

where N is chosen such that

$$\forall n \geq N : \qquad |a_n - a| < \sup A - a.$$

(ii) Take

$$a_n = (-1)^n[1 - 1/n].$$

Problem 4

Let (a_n) be a sequence of real numbers and let $a \in \mathbb{R}$.

Suppose that every subsequence of (a_n) has a sub-subsequence converging to a.

Prove that $a_n \to a$.

Solution. We argue by contradiction. Suppose that (a_n) does not converge to a. Then

$$\exists \varepsilon > 0 \text{ such that for all } N \; \exists n \geq N \text{ such that } |a_n - a| \geq \varepsilon.$$

It follows that there is a subsequence $n_1 < n_2 < n_3 < dots$ such that

$$|a_{n_k} - a| \geq \varepsilon, \qquad \forall k = 1, 2, 3, \ldots.$$

Clearly, any sub-subsequence of this subsequence cannot converge to a as the distance to a is always at least ε. Thus we have a contradiction, and it follows that the whole sequence $a_n \to a$.

Problem 1

(i) Let $a_n = \sqrt{n}$. Show that

$$|a_n - a_{n-1}| \to 0.$$

(ii) Is (a_n) a Cauchy sequence?

(iii) If b_n is such that

$$|b_n - b_{n-1}| \leq \frac{1}{2^n},$$

show that (b_n) is a Cauchy sequence.

Solution. (i) Clearly,

$$\sqrt{n}-\sqrt{n-1}=\frac{(\sqrt{n}-\sqrt{n-1})(\sqrt{n}+\sqrt{n-1})}{\sqrt{n}+\sqrt{n-1}}=\frac{1}{\sqrt{n}+\sqrt{n-1}}.$$

From this the claim follows.

(ii) No. $\sqrt{n}\to\infty$ as $n\to\infty$ so it's not convergent. If it were a Cauchy sequence it would have to be convergent.

(iii) Let $n<m$ and let's look at

$$|a_m-a_n|\leq|a_m-a_{m-1}|+|a_{m-1}-a_{m-2}|+\cdots+|a_{n+1-a_n}|\leq$$

$$\frac{1}{2^m}+\frac{1}{2^{m-1}}+\cdots+\frac{1}{2^n}\leq\sum_{k=n}^{\infty}\frac{1}{2^k}=\frac{1}{2^{n-1}}.$$

Hence given $\varepsilon>0$ if we choose N such that $2^{N-1}<\varepsilon$ we have $|a_m-a_n|<\varepsilon$ for all $n,m\geq N$.

Problem 2

Let (a_n) be a sequence of real numbers.

If the series

$$\sum_n a_n^2$$

converges, show that the series

$$\sum_n \frac{a_n}{n}$$

converges.

Is the converse true?

Solution. (i) We use Cauchy-Schwarz inequality. we have

$$\sum_n \frac{a_n}{n} \leq \left(\sum_n a_n^2\right)^{1/2} \left(\sum_n \frac{1}{n^2}\right)^{1/2}.$$

Since both sum on the right hand side are finite we have the claim.

(ii) No converse is false. Take $a_n = \frac{1}{n^{1/2}}$.

Problem 3

Prove that the sequence

$$a_n = \sum_{k=1}^{n-1} \frac{1}{k} - \log n$$

converges.

Hint: Show it is monotone and bounded.

Solution. To see that a sequence is monotone it's useful to look at

$$a_{n+1} - a_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) - \sum_{k=1}^{n-1} \frac{1}{k} + \log n =$$

$$= \frac{1}{n} - \log(1 + \frac{1}{n}).$$

We claim that this difference is positive since $\log(1+x) < x$ for all $x > 0$. This can be checked for example by using the mean value theorem.

(ii) We prove the sequence is bounded. First, we have

$$\frac{1}{k} \leq \int_{k-1}^{k} \frac{dx}{x} = \log k - \log(k-1).$$

Using this we see that

$$a_n = 1 + \sum_{k=2}^{n-1} \frac{1}{k} - \log n \leq$$

$$\leq 1 + (\log 2 - \log 1) + (\log 3 - \log 2) + \cdots + (\log(n-1) - \log(n-2)) - \log n$$

$$= 1 + \log(n-1) - \log n \leq 1.$$

Hence the sequence is monotone (increasing) and bounded from above. Thus it is convergent.

Problem 4

Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq |x - y|^{\alpha}$$

for all x and y in $\mathbb{R}$.

Show that if $\alpha > 1$ then f is constant.

Is the same conclusion valid when $\alpha = 1$?

Solution. We calculate $f'(x)$. Clearly,

$$|f'(x)| = \left| \lim_{y \to x} \frac{f(x) - f(y)}{x - y} \right| \leq \lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|}$$

$$\leq \lim_{y \to x} \frac{|x - y|^{\alpha}}{|x - y|} = \lim_{y \to x} |x - y|^{\alpha - 1} = 0,$$

if $\alpha > 1$. From this we see that f must be constant as its derivative is zero everywhere.

When $\alpha = 1$ the function does not have to be constant, functions like $\sin x$ or x satisfy the condition.

Ex. 7.1.1

a) Prove that $x/n \to 0$ uniformly as $n \to \infty$ on any closed interval $[a, b]$.

b) Prove that $1/nx \to 0$ pointwise but not uniformly on $(0, 1)$ as $n \to \infty$.

Solution. a) Pick $\varepsilon > 0$. Let $f_n(x) = x/n$. We estimate

$$\sup_{x \in [a,b]} |f_n(x) - 0| = \frac{1}{n} \sup_{x \in [a,b]} |x| = max\{|a|, |b|\}/n.$$

Clearly this goes to zero as $n \to \infty$.

(b) For every fixed $x > 0$ we clearly have

$$\lim_{n\to\infty} \frac{1}{nx} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{n} = 0.$$

The convergence fails to be uniform however. Let $x_n = 1/n$ and $f_n(x) = 1/nx$. Clearly,

$$f_n(x_n) = \frac{1}{nn^{-1}} = 1 \quad \text{does not converge to 0.}$$

So the convergence fails to be uniform.

Ex. 7.1.3

A sequence of functions f_n is said to be *uniformly bounded* on a set E if there is an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all $n \in \mathbb{N}$.

Suppose that for each $n \in \mathbb{N}$, $f_n : E \to \mathbb{R}$ is bounded. If $f_n \to f$ uniformly on E, prove that f is a bounded function and that f_n is uniformly bounded on E.

Solution. Since (f_n) converges uniformly to f we have that there exists N such that for all $n \geq N$:

$$|f_n(x) - f(x)| \leq 1, \qquad \forall x \in E.$$

In particular, we know that f_1, $f_2, \ldots$, f_N are all bounded. Say

$$M = \max_{1 \leq i \leq N} \sup_{x \in E} |f_n(x)|.$$

This number is finite as the max is taken over a finite set of numbers. In particular $|f_N| \leq M$ and hence for f we have:

$$|f(x)| \leq |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq 1 + M.$$

Thus f is bounded. It follows that for all $n \geq N$ we then have

$$|f_n(x)| \leq |f_n(x) - f(x) + f(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq 1 + 1 + M = M + 2.$$

Hence the sequence of functions (f_n) is uniformly bounded by $M + 2$.

Ex. 7.1.5

Suppose that $f_n \to f$ and $g_n \to g$ uniformly on E as $n \to \infty$.

Prove that $f_n g_n \to fg$ pointwise on E.

Prove that if f and g are bounded on E, then $f_n g_n \to fg$ uniformly on E.

Show that the previous part may be false when g is unbounded.

Solution. Consider

$$|f(x)g(x)-f_n(x)g_n(x)| \le |f(x)g(x)-f_n(x)g(x)|+|f_n(x)g(x)-f_n(x)g_n(x)|$$

$$\le |g(x)||f_n(x)-f(x)|+|f_n(x)||g(x)-g_n(x)|.$$

For a fixed $x \in E$ clearly $|g(x)|, |f_n(x)|$ are bounded (any convergent sequence is bounded), say by $M(x)$. hence

$$|f(x)g(x)-f_n(x)g_n(x)| \le M(x)(|f_n(x)-f(x)|+|g_n(x)-g(x)|),$$

from which the claim on pointwise convergence follows. We also see

that if both f and g are bounded then (by the previous problem) $|g|, |f_n|$ are uniformly bounded and hence M in the estimate above does not depend on x. Thus

$$|f(x)g(x)-f_n(x)g_n(x)| \le M(|f_n(x)-f(x)|+|g_n(x)-g(x)|),$$

for all $x \in E$. This estimate implies uniform convergence, given $\varepsilon > 0$ we find N such that for all $n \geq N$ and for all $x \in E$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2M}, \qquad |g_n(x) - g(x)| < \frac{\varepsilon}{2M}.$$

This combined with the previous estimate given the desired result.

If we take $E = \mathbb{R}$ and $f_n(x) = g_n(x) = x + \frac{1}{n}$ then clearly, $f_n = g_n \to x$ uniformly on E but the convergence $f_n g_n \to x^2$ is uniform only on bounded sets but not on the whole $\mathbb{R}$.