Workshop 4 – Rigorous Trigonometry via Power Series

The purpose of this workshop is to consider the informal notions of trigonometry that we've become used to since high school – sines, cosines, addition formulae, graphs etc. – and put them on a rigorous basis. We do this via power series. So for the purposes of this workshop, you need to temporarily "forget" any facts you know about trigonometry. You'll develop them from scratch in this workshop and then see that they do correspond to the ones you're already informally familiar with. Pay attention to proper justification and rigour in your arguments.

1. What is the radius of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots ?$$

What is its interval of convergence I?

2. Let $S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ for $x \in I$. Prove that S is differentiable on I and that for $x \in I$,

$$S'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

3. Define $C(x) := S'(x)$. Show that $C'(x) = -S(x)$. Prove that $C(x)^2 + S(x)^2 = 1$ for all x and deduce that for all x we have $|S(x)| \leq 1$ and $|C(x)| \leq 1$.

4. Prove that for all real x and y we have

$S(x+y) = S(x)C(y) + C(x)S(y)$ and that $C(x+y) = C(x)C(y) - S(x)S(y)$.

(**Hint:** Fix y, first simplify the function

$$(S(x+y) - (S(x)C(y) + C(x)S(y)))^2 + (C(x+y) - (C(x)C(y) - S(x)S(y)))^2,$$

and then consider its derivative.)

5. (i) Prove that $S(x) > 0$ for $0 < x \leq \sqrt{6}$.
(ii) Prove that $C(x) > 0$ for $0 \leq x \leq \sqrt{2}$.
(iii) Prove that for $0 \leq x \leq \sqrt{56}$, if $1 - x^2/2! + x^4/4! < 0$, then $C(x) < 0$, and deduce that $C(8/5) < 0$.
(**Hint for whole question:** Group together terms in the power series for S and C in a helpful way. For example, $S(x) = (x - \frac{x^3}{3!}) + (\frac{x^5}{5!} - \frac{x^7}{7!}) + \ldots$)

6. Deduce that there is a unique number $\varpi/2$ satisfying $\sqrt{2} < \varpi/2 < 8/5$ such that $C(\varpi/2) = 0$. Show that $S(\varpi/2) = 1$. (**Remark.** Of course the number ϖ is none other than our old friend π, but we give it a nonstandard name to remind you that you mustn't use anything you already know about π in this question.)

7. Sketch the graphs of $C(x)$ and $S(x)$ over the range $0 \leq x \leq \varpi/2$ taking care to note where the graphs cross the axis or where there is a critical point.

8. Prove that $S(x+\varpi/2) = C(x)$, $S(x+\varpi) = -S(x)$, $S(x+3\varpi/2) = -C(x)$ and that $S(x + 2\varpi) = S(x)$ for all x. Sketch the graph of S on the whole real line. Similarly for C.

So we have established that S and C are 2ϖ-periodic and have graphs very similar to those we expect for sine and cosine. Furthermore the various identities that we are taught (in elementary courses) for sine and cosine are also obeyed by S and C. We haven't yet pinned down the mysterious number ϖ except to show that it is between $2\sqrt{2} \approx 2 \cdot 8$ and $3 \cdot 2$. Why is it related to the circumference or the area of a circle?

9. Consider the plane curve $\gamma : [0, 2\varpi) \to \mathbb{R}^2$ given by $\gamma(t) = (C(t), S(t))$. Show that the image of γ is precisely the unit circle and that γ is injective. Calculate the speed of the curve γ. What can you deduce about the length of the circumference of the unit circle?

Assessment task to be handed in on Wednesday 17/10 of Week 5: Questions 5(i) and (ii), and 6.

Some supplementary activites concerning trigonometry:

(A) Show that a reasonable interpretation of the area of the unit circle is $4\int_0^1(1-t^2)^{1/2}\mathrm{d}t$. Consider the substitution $t = C(s)$ noting that $C : [0,\varpi/2] \to [0,1]$ is a bijecton and indeed has strictly negative derivative $-S(s)$ except at 0. So this is a valid substitution and since $1-C(s)^2 = S(s)^2$ and $S \geq 0$ on $[0,\varpi/2]$ we see that the integral has become

$$\int_{\varpi/2}^{0} S(s)C'(s)\mathrm{d}s = \int_0^{\varpi/2} S(s)^2\mathrm{d}s.$$

By the symmetries we have established above,

$$\int_0^{\varpi/2} S(s)^2\mathrm{d}s = \frac{1}{4}\int_0^{2\varpi} S(s)^2\mathrm{d}s = \frac{1}{4}\int_0^{2\varpi} C(s)^2\mathrm{d}s$$

$$= \frac{1}{8}\int_0^{2\varpi}(C(s)^2 + S(s)^2)\mathrm{d}s = \frac{\varpi}{4}.$$

Multiplying by 4 gives the area of the unit circle as ϖ.

(B) We can define the **angle subtended at** 0 between a point (x, y) in the unit circle and the point $(1, 0)$ as the unique $t \in [0, 2\varpi)$ such that $(x, y) = (C(t), S(t))$, and extend this in the obvious way to pairs of vectors not necessarily of unit length. Show that we have $\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}||\mathbf{y}|\, C(\theta)$ where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Deduce the cosine rule from trigonometry by considering a vector triangle with sides $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$. Use the symmetry of the three choices in the cosine rule and the fact that $C^2 + S^2 = 1$ to deduce the sine rule from trigonometry.