A virtual Kawasaki formula

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Abstract

Kawasaki's formula is a tool to compute holomorphic Euler characteristics of vector bundles on a compact orbifold $\mathcal{X}$. Let $\mathcal{X}$ be an orbispace with perfect obstruction theory which admits an embedding in a smooth orbifold. One can then construct the virtual structure sheaf and the virtual fundamental class of $\mathcal{X}$. In this paper we prove that Kawasaki's formula "behaves well" with working "virtually" on $\mathcal{X}$ in the following sense: if we replace the structure sheaves, tangent and normal bundles in the formula by their virtual counterparts then Kawasaki's formula stays true. Our motivation comes from studying the quantum K-theory of a complex manifold X (see [GT]), with the formula applied to Kontsevich' moduli spaces of genus 0 stable maps to X.

1 Introduction

Given a manifold $\mathcal{X}$ and a vector bundle V on $\mathcal{X}$ then Hirzebruch-Riemann-Roch formula states that:

$$\chi(\mathcal{X}, V) = \int_{\mathcal{X}} ch(V) Td(T_{\mathcal{X}}).$$

In [Ka] Kawasaki generalized this formula to the case when $\mathcal{X}$ is an orbifold. He reduces the computation of Euler characteristics on $\mathcal{X}$ to computation of certain cohomological integrals on *the inertia orbifold* $I\mathcal{X}$:

$$\chi(\mathcal{X}, V) = \sum_{\mu} \frac{1}{m_\mu} \int_{\mathcal{X}_\mu} Td(T_{\mathcal{X}_\mu}) ch\left(\frac{Tr(V)}{Tr(\Lambda^\bullet N_\mu^*)}\right). \tag{1}$$

We explain below the ingredients in the formula:
$I\mathcal{X}$ is defined as follows: around any point $p \in \mathcal{X}$ there is a local chart $(\widetilde{U}_p, G_p)$ such that locally $\mathcal{X}$ is represented as the quotient of $\widetilde{U}_p$ by G_p. Consider the set of conjugacy classes $(1) = (h_p^1)$, (h_p^2), …, $(h_p^{n_p})$ in G_p. Define:

$$I\mathcal{X} := \{\left(p, (h_p^i)\right) \quad | \quad i = 1, 2, \dots, n_p\}.$$

Pick an element h_p^i in each conjugacy class. Then a local chart on $I\mathcal{X}$ is given by:

$$\coprod_{i=1}^{n_p} \widetilde{U}_p^{(h_p^i)}/Z_{G_p}(h_p^i),$$

where $Z_{G_p}(h_p^i)$ is the centralizer of h_p^i in G_p. Denote by $\mathcal{X}_\mu$ the connected components of the inertia orbifold (we'll often refer to them as Kawasaki strata). The multiplicity m_μ associated to each $\mathcal{X}_\mu$ is given by:

$$m_\mu := \left| ker\left(Z_{G_p}(g) \to Aut(\widetilde{U}_p^g)\right)\right|.$$

For a vector bundle V we will denote by V^* the dual bundle to V. The restriction of V to $\mathcal{X}_\mu$ decomposes in characters of the g action. Let $E_r^{(l)}$ be the subbundle of the restriction of E to $\mathcal{X}_\mu$ on which g acts with eigenvalue $e^{\frac{2\pi il}{r}}$. Then the trace $Tr(V)$ is defined to be the orbibundle whose fiber over the point $(p,(g))$ of $\mathcal{X}_\mu$ is :

$$Tr(V) := \sum_l e^{\frac{2\pi il}{r}} E_r^{(l)}.$$

Finally, $\Lambda^\bullet N_\mu^*$ is the K-theoretic Euler class of the normal bundle N_μ of $\mathcal{X}_\mu$ in $\mathcal{X}$. $Tr(\Lambda^\bullet N_\mu^*)$ is invertible because the symmetry g acts with eigenvalues different from 1 on the normal bundle to the fixed point locus. We call the terms corresponding to the identity component in the formula *fake Euler characteristics*:

$$\chi^f(\mathcal{X}, V) = \int_{\mathcal{X}} ch(V)Td(T_{\mathcal{X}}).$$

In the case where $\mathcal{X}$ is a global quotient formula (1) is the Lefschetz fixed point formula.

Now let $\mathcal{X}$ be a compact, complex orbispace (Deligne-Mumford stack) with a perfect obstruction theory $E^{-1} \to E^0$. This gives rise to the intrinsic normal cone, which is embedded in E_1 - the dual bundle to E^{-1} (see [LT], also [BF]). The virtual structure sheaf $\mathcal{O}_{\mathcal{X}}^{vir}$ was defined in [L] as the K-theoretic pull-back by the zero section of the structure sheaf of this cone. Let $I\mathcal{X} = \coprod_\mu \mathcal{X}_\mu$ be the inertia orbifold of $\mathcal{X}$. We denote by i_μ the inclusion of a stratum $\mathcal{X}_\mu$ in $\mathcal{X}$. For a bundle V on $\mathcal{X}$ we write $i_\mu^* V = V_\mu^f \oplus V_\mu^m$ for its decomposition as the direct sum of the fixed part and the moving part under the action of the symmetry associated to $\mathcal{X}_\mu$. To avoid ugly notation we will often simply write V^m, V^f. The virtual normal bundle to $\mathcal{X}_\mu$ in $\mathcal{X}$ is defined as $[E_0^m] - [E_1^m]$. We will in addition assume that $\mathcal{X}$ admits an embedding j in a smooth compact orbifold $\mathcal{Y}$. This is always true for the moduli spaces of genus 0 stable maps $X_{0,n,d}$ because an embedding $X \hookrightarrow \mathbb{P}^N$ induces an embedding $X_{0,n,d} \hookrightarrow (\mathbb{P}^N)_{0,n,d}$.

Theorem 1.1. Denote by N_μ^{vir} the virtual normal bundle of $\mathcal{X}_\mu$ in $\mathcal{X}$. Then

$$\chi\left(\mathcal{X}, j^*(V) \otimes \mathcal{O}_{\mathcal{X}}^{vir}\right) = \sum_\mu \frac{1}{m_\mu} \chi^f \left(\mathcal{X}_\mu, \frac{Tr(V_\mu \otimes \mathcal{O}_{\mathcal{X}_\mu}^{vir})}{Tr\left(\Lambda^\bullet (N_\mu^{vir})^*\right)} \right). \tag{2}$$

Remark 1.2. A perfect obstruction theory $E^{-1} \to E^0$ on $\mathcal{X}$ induces canonically a perfect obstruction theory on $\mathcal{X}_\mu$ by taking the fixed part of the complex $E_\mu^{-1,f} \to E_\mu^{0,f}$. The proof is the same as that of Proposition 1 in [GP]. This is then used to define the sheaf $\mathcal{O}_{\mathcal{X}_\mu}^{vir}$.

Remark 1.3. It is proved in [FG] that if $\mathcal{X}$ is a scheme, the Grothendieck-Riemann-Roch theorem is compatible with virtual fundamental classes and virtual fundamental sheaves i.e.:

$$\chi^f(\mathcal{X}, V \otimes \mathcal{O}_{\mathcal{X}}^{vir}) = \int_{[\mathcal{X}]} ch(V \otimes \mathcal{O}_{\mathcal{X}}^{vir}) \cdot Td(T^{vir})$$

where $[\mathcal{X}]$ is the virtual fundamental class of $\mathcal{X}$ and T^{vir} is its virtual tangent bundle. Their arguments carry over to the case when $\mathcal{X}$ is a stack.

Remark 1.4. The bundles V to which we apply Theorem 1.1 in [GT] are (sums and products of) cotangent line bundles L_i and evaluation classes $ev_i^*(a_i)$. They are pull-backs of the corresponding bundles on $(\mathbb{P}^N)_{0,n,d}$.

Acknowledgements. I would like to thank Alexander Givental for suggesting the problem and for useful discussions. Thanks are also due to Yuan-Pin Lee who patiently answered my questions on the material in his work [L] and to Hsian-Hua Tseng who read a preliminary draft of the paper.

2 Proof of Theorem 1.1

Before proving Theorem 1.1 we recall a couple of background facts and lemmata on K-theory which we will use.

Let $K_0(X)$ be the Grothendieck group of coherent sheaves on X. Given a map $f : X \to Y$, the K-theoretic pullback $f^*(\mathcal{F}) : K_0(Y) \to K_0(X)$ is defined as the alternating sum of derived functors $Tor^i_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_X)$, provided that the sum is finite. This is always true for instance if f is flat or if it is a regular embedding.

For any fiber square:

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & V \arrow[d] \\ B' \arrow[r, "i"] & B \end{tikzcd}$$

with i a regular embedding one can define K-theoretic refined Gysin homomorphisms $i^! : K_0(V) \to K_0(V')$ (see [L]). One way to define the map $i^!$ is the following: the class $i_*(\mathcal{O}_{B'}) \in K^0(B)$ has a finite resolution of vector bundles, which is exact off B'. We pull it back to V and then cap (i.e. tensor product) with classes in $K_0(V)$, to get a class on $K_0(V)$ with homology supported on V', which we can regard as an element of $K_0(V')$, because there is a canonical isomorphism between complexes on V with homology supported on V' and $K_0(V')$.

In the following two lemmata X, Y, Y' are assumed DM stacks. We will use the following result:

Lemma 2.1. *Consider the diagram:*

$$\begin{tikzcd} \iota^* C_{X/Y} \arrow[r] \arrow[d] & C_{X/Y} \arrow[d] \\ X' \arrow[r, "\iota"] \arrow[d] & X \arrow[d, "j"'] \\ Y' \arrow[r, "i"] & Y \end{tikzcd}$$

with i a regular embedding and j an embedding, $C_{X/Y}$ is the normal cone of X in Y and both squares are fiber diagrams. Then:

$$i^![\mathcal{O}_{C_{X/Y}}] = [\mathcal{O}_{C_{X'/Y'}}] \in K_0(\iota^* C_{X/Y}). \tag{3}$$

This is stated and proved in [L] (Lemma 2). The proof is based on a more general statement (Lemma 1 of [L]), which has been worked out in [Kr] on the level of Chow rings. Since K-theoretic statements are stronger, we give below the key-ingredient which allows one to carry over Kresch's proof to K-theory:

Lemma 2.2. *Let $f : X \to Y$ be a closed embedding let $g : Y \to \mathbb{P}^1$ be a surjection such that $g \circ f$ is flat. Denote by X_0 and Y_0 the fibers over 0 of $g \circ f$ and g respectively. Moreover assume that the restriction of f to $X \setminus X_0$ is an isomorphism. Then if i is the inclusion of $\{0\}$ in $\mathbb{P}^1$, $i^!(\mathcal{O}_Y) = \mathcal{O}_{X_0} \in K_0(Y_0)$.*

Proof: the skyscraper sheaves at all points of $\mathbb{P}^1$ represent the same element in $K_0(\mathbb{P}^1)$, hence if we pull-back a resolution of any point $P \in \mathbb{P}^1$ by g we get the same elements of $K_0(Y)$. On the other hand since f is an isomorphism above $\mathbb{P}^1 \setminus \{0\}$, pulling-back by g of the structure sheaf of a point $P \neq 0$ is the same as pulling back by $g \circ f$ followed by f_*. By what we said above we can replace P with 0. Now from the flatness of $g \circ f$ above 0 the pull-back of the structure sheaf of 0 by $g \circ f$ is the structure sheaf of the fiber X_0. The result then follows from the definition of $i^!$.

Remark 2.3. Lemma 2.2 allows one to show Lemma 2.1: intermediately one shows, following [Kr],(notation is as in Lemma 2.1) that $[\mathcal{O}_{C_1}] = [\mathcal{O}_{C_2}]$ in $K_0(C_{X'}Y \times_Y C_XY)$ where $C_1 := C_{i^*C_XY}(C_XY)$ and $C_2 := C_{j^*C_{Y'}Y}(C_{Y'}Y)$.

We now go on to prove Theorem 1.1. We have:

$$\chi(\mathcal{X}, j^*V \otimes \mathcal{O}_{\mathcal{X}}^{vir}) = \chi(\mathcal{Y}, V \otimes j_*\mathcal{O}_{\mathcal{X}}^{vir}).$$

Kawasaki's formula applied to the sheaf $V \otimes j_*\mathcal{O}_{\mathcal{X}}^{vir}$ on $\mathcal{Y}$ gives:

$$\chi(\mathcal{Y}, V \otimes j_*\mathcal{O}_{\mathcal{X}}^{vir}) = \sum_\mu \frac{1}{m_\mu}\chi^f\left(\mathcal{Y}_\mu, \frac{Tr(V_\mu \otimes i_\mu^* j_*\mathcal{O}_{\mathcal{X}}^{vir})}{Tr(\Lambda^\bullet N_\mu^*)}\right). \tag{4}$$

From the fiber diagram:

$$\begin{array}{ccc} \mathcal{X}_\mu & \xrightarrow{i'_\mu} & \mathcal{X} \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ \mathcal{Y}_\mu & \xrightarrow{i_\mu} & \mathcal{Y} \end{array}$$

and Theorem **6.2** in [FL] (where this is proved for Chow rings) we have $i_\mu^* j_*\mathcal{O}_{\mathcal{X}}^{vir} = j'_* i_\mu^! \mathcal{O}_{\mathcal{X}}^{vir}$. Plugging this in (4) gives:

$$\chi^f\left(\mathcal{Y}_\mu, \frac{Tr\left(V_\mu \otimes i_\mu^* j_*\mathcal{O}_{\mathcal{X}}^{vir}\right)}{Tr(\Lambda^\bullet N_\mu^*)}\right) = \chi^f\left(\mathcal{Y}_\mu, \frac{Tr\left(V_\mu \otimes j'_* i_\mu^! \mathcal{O}_{\mathcal{X}}^{vir}\right)}{Tr(\Lambda^\bullet N_\mu^*)}\right). \tag{5}$$

Let G_μ be the cyclic group generated by one element of the conjugacy class associated to $\mathcal{X}_\mu$. Then we will show that:

$$Tr\left(\frac{i_\mu^! \mathcal{O}_{\mathcal{X}}^{vir}}{\Lambda^\bullet(N_\mu^*)}\right) = Tr\left(\frac{\mathcal{O}_{\mathcal{X}_\mu}^{vir}}{\Lambda^\bullet(N_\mu^{vir})^*}\right) \tag{6}$$

in the G_μ-equivariant K-ring of $\mathcal{X}_\mu$. This is essentially the computation of Section 3 in [GP] carried out in $\mathbb{C}^*$-equivariant K-theory. Relation (6) then follows by embedding the group G_μ in the torus and specializing the value of the variable t in the ground ring of $\mathbb{C}^*$-equivariant K-theory to a $|G_\mu|$-root of unity.

If we define a cone $D := C_{\mathcal{X}/\mathcal{Y}} \times_{\mathcal{X}} E_0$, then this is a $T\mathcal{Y}$ cone (see [BF]). The virtual normal cone D^{vir} is defined as $D/T\mathcal{Y}$ and $\mathcal{O}^{vir}_{\mathcal{X}}$ is the pull-back by the zero section of the structure sheaf of D^{vir}. Alternatively there is a fiber diagram:

$$\begin{array}{ccc} T\mathcal{Y} & \longrightarrow & D \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{0_{E_1}} & E_1 \end{array}$$

whre the bottom map is the zero section of E_1. Then one can define $\mathcal{O}^{vir}_{\mathcal{X}}$ as $0^*_{T\mathcal{Y}} 0^!_{E_1}[\mathcal{O}_D]$. We'll prove formula (6) following closely the calculation in [GP]. First by definition of $\mathcal{O}^{vir}_{\mathcal{X}}$ and by commutativity of Gysin maps we have :

$$i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}} = i^!_\mu 0^*_{T\mathcal{Y}} 0^!_{E_1}[\mathcal{O}_D] = 0^*_{T\mathcal{Y}} 0^!_{E_1} i^!_\mu [\mathcal{O}_D]. \tag{7}$$

We pull-back relation (3) to $(i'_\mu)^* D = (i'_\mu)^*(C_{\mathcal{X}/\mathcal{Y}} \times E_0)$ to get:

$$i^!_\mu[\mathcal{O}_D] = [\mathcal{O}_{D_\mu} \times (E^m_0)^*]. \tag{8}$$

In the equality above we have used the fact that $D_\mu = C_{\mathcal{X}_\mu/\mathcal{Y}_\mu} \times E^f_0$ and we identified the sheaf of sections of the bundle E^m_0 with the dual bundle $(E^m_0)^*$. Plugging (8) in (7) we get:

$$i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}} = 0^*_{T\mathcal{Y}} 0^!_{E_1}[\mathcal{O}_{D_\mu} \times (E^m_0)^*]. \tag{9}$$

Notice that the action of $T\mathcal{Y}_\mu$ leaves $D_\mu \times (E^m_0)^*$ invariant (it acts trivially on $(E^m_0)^*$). Now we can write $0^*_{T\mathcal{Y}} = 0^*_{T\mathcal{Y}^f_\mu} \times 0^*_{T\mathcal{Y}^m_\mu}$ and since $D^{vir}_\mu = D_\mu/T\mathcal{Y}_\mu$ we rewrite (9) as:

$$i^!_\mu \mathcal{O}^{vir}_{\mathcal{X}} = 0^*_{T\mathcal{Y}^m_\mu} 0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times (E^m_0)^*]. \tag{10}$$

The proof of Lemma 1 in [GP] works in our set-up as well: it uses excess intersection formula which holds in K-theory. It shows that the following relation holds in the $\mathbb{C}^*$-equivariant K-ring of $\mathcal{X}_\mu$:

$$0^*_{T\mathcal{Y}^m_\mu} 0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times (E^m_0)^*] = 0^*_{E^m_0}\left(0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times (E^m_0)^*]\right) \cdot \frac{\Lambda^\bullet (T\mathcal{Y}^m)^*}{\Lambda^\bullet (E^m_0)^*}. \tag{11}$$

The class $0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times E^m_0]$ lives in the $\mathbb{C}^*$-equivariant K-ring of E^m_0. The class doesn't depend on the bundle map $E^m_0 \to E^m_1$ so we can assume this map to be 0. Then by excess intersection formula and the definition of $\mathcal{O}^{vir}_{\mathcal{X}_\mu}$ we get :

$$0^*_{E^m_0}\left(0^!_{E_1}[\mathcal{O}_{D^{vir}_\mu} \times (E^m_0)^*]\right) = \mathcal{O}^{vir}_{\mathcal{X}_\mu} \cdot \Lambda^\bullet (E^m_1)^*. \tag{12}$$

Formula (12) holds because $D^{vir}_\mu \times (E^m_0) \subset E^f_1 \times E^m_0$ and $0^!_{E_1}$ acts as $0^!_{E^f_1} \times 0^!_{E^m_1}$ on factors. $0^!_{E^f_1}[\mathcal{O}_{D^{vir}_\mu}] = \mathcal{O}^{vir}_{\mathcal{X}_\mu}$ by definition of $\mathcal{O}^{vir}_{\mathcal{X}_\mu}$. By excess intersection formula applied to the fiber square:

$$\begin{array}{ccc} E^m_0 & \longrightarrow & E^m_0 \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ \mathcal{X}_\mu & \xrightarrow{0_{E^m_1}} & E^m_1 \end{array}$$

we have $0^*_{E^m_0} 0^!_{E^m_1}[(E^m_0)^*] = 0^*_{E^m_0}\pi^*\Lambda^\bullet(E^m_1)^* = \Lambda^\bullet(E^m_1)^*$. Plugging formula (12) in (11) (note that $N_\mu = T\mathcal{Y}^m_\mu$ and $N^{vir}_\mu = [E^m_0] - [E^m_1]$) and taking traces proves (6). We now plug (6) in (5) and then pull-back to $\mathcal{X}_\mu$ to get:

$$\begin{aligned}\chi^f\left(\mathcal{Y}_\mu, \frac{Tr(V_\mu \otimes j_* i^*_\mu \mathcal{O}^{vir}_{\mathcal{X}})}{Tr(\Lambda^\bullet N^*_\mu)}\right) &= \chi^f\left(\mathcal{Y}_\mu, Tr(V_\mu) \otimes j'_* \frac{Tr(\mathcal{O}^{vir}_{\mathcal{X}_\mu})}{Tr(\Lambda^\bullet (N^{vir}_\mu)^*)}\right) = \\ &= \chi^f\left(\mathcal{X}_\mu, \frac{Tr(V_\mu \otimes \mathcal{O}^{vir}_{\mathcal{X}_\mu})}{Tr(\Lambda^\bullet (N^{vir}_\mu)^*)}\right).\end{aligned} \tag{13}$$

This concludes the proof of the proposition.

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