

# A Short Introduction to Orbifolds

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## 0: Introduction

In this project we introduce the notion of an orbifold. Manifolds are very strong and useful mathematical objects. They are a nice generalization of Euclidean spaces but still preserves many of their geometrical properties. One of their drawbacks, however, is their weak algebraic properties. By that I mean that the category of manifolds is not closed under limits and colimits. Orbifolds are one way of generalizing manifolds to a more general structure which behaves nicely with at least one algebraic manipulation, quotients, yet retains a relatively close relation to the spaces we are most comfortable with, Euclidean spaces.

This short note is a concise introduction to orbifolds. First, we define orbifolds as generalizations of manifolds. Then, after looking at several examples, we go over to standard homotopical properties of orbifolds, such as the existence of universal covers and the notion of the fundamental group of orbifolds.

After that we consider a special case: 2-dimensional orbifolds or 2-orbifolds. Our goal will be to classify 2-orbifolds. In order to do that we will first look at all possible local structures of 2-orbifolds. Then we introduce an important algebraic invariant which will help us in our classification process, the Euler number. With the help of this information, we will split 2-orbifolds into four types: elliptic, hyperbolic, parabolic and bad orbifolds.

Finally, I wanted to point out that the material is almost completely from the book, “The Geometry and Topology of Three-Manifolds”, by William P. Thurston. In particular chapter 13 of mentioned book. Lack of reference for any statement does not imply any originality on my part.

## 1: Motivation & Definitions

Before we give the definition of a orbifold, let us take look at couple examples for motivational purposes. Consider, the 2-manifold  $\mathcal{M} = \mathbb{R}^2$ . The group  $\mathcal{G} = \mathbb{Z} \times \mathbb{Z}$  acts on  $\mathcal{M}$  by  $(n, m) \cdot (\alpha, \beta) = (\alpha + n, \beta + m)$ , where  $(n, m) \in \mathcal{G}$  and  $(\alpha, \beta) \in \mathcal{M}$ . Now, this action is clearly properly discontinuous and free and so the topological space coming from the quotient of  $\mathcal{M}$  by the group action, from here on denoted by  $\mathcal{M}/\mathcal{G}$ , has a well-defined smooth structure coming from the smooth structure of  $\mathcal{M}$ . Indeed, it is not hard to see that the quotient space is just the torus  $\mathbb{T}$ .

Now, let  $\mathcal{M} = \mathbb{R}^2$  again, but this time let us use a different action of a different group. Let  $\mathcal{G} = \mathbb{Z}/3$  and let it act on  $\mathcal{M}$  by  $n \in \mathbb{Z}/3$  rotating every point in  $\mathcal{M}$  by  $2n\pi/3$ . Clearly, this action is properly discontinuous as  $\mathcal{G}$  is finite. However, the action is not free. In fact, it is free everywhere except for the point  $(0, 0)$ . This means that we cannot a priori conclude that the quotient topological space has a smooth structure at the point  $(0, 0)$  (note this point is its own orbit) and so we have to make a careful analysis. First of all, topologically, the space is homeomorphic to  $\mathcal{M}$ . However, the metric around  $(0, 0)$  has been influenced in quotient. In order to see that, note that the area of the unit ball  $\{x : d(x, (0, 0)) \leq 1\}$  has been divided by 3 and is now just  $2\pi/3$ . Therefore, the point becomes a singularity and the metric makes it look like a cone. Clearly, a cone cannot have a smooth structure on it and so this can never be a smooth manifold.

However, the structure does not seem to be fundamentally different from a manifold and so we would like to expand the notion of a manifold to a topological space which has still most properties of smooth manifolds, but also includes examples like the one above. This brings us the notion of an orbifold. Let's start with the definition:

**Definition 1.** An orbifold  $O$  is a pair  $(X_O, \{U_i\})$ , where  $X_O$  is a Hausdorff topological space and  $\{U_i\}$  is a covering of  $X_O$  i.e.  $X_O = \cup_i \{U_i\}$ , which is closed under finite intersections. Moreover, for each element of the open cover  $U_j$  there exists an open subset of  $\mathbb{R}^n$ ,  $V_j \subset \mathbb{R}^n$  and a finite group  $\Gamma_j$  acting on  $V_j$  and a homeomorphism  $\varphi_j : V_j/\Gamma_j \rightarrow U_j$ . Also, there are equivariancy conditions: Whenever,  $U_i \subset U_j$ , there is an injective group homomorphism  $f_{ij} : \Gamma_i \rightarrow \Gamma_j$  and an embedding  $\varphi_{ij} : V_i \rightarrow V_j$ , which is equivariant with respect to  $f_{ij}$  i.e. for each  $\gamma \in \Gamma_i$ ,  $\varphi_{ij}(\gamma x) = f_{ij}(\gamma)\varphi_{ij}(x)$ , such that the following diagram commutes:

$$\begin{array}{ccc}
V_i & \xrightarrow{\quad} & V_j \\
\downarrow & & \downarrow \\
V_i/\Gamma_i & \xrightarrow{\quad} & V_j/\Gamma_i \\
\uparrow & & \downarrow \\
& & V_j/\Gamma_j \\
& & \uparrow \\
U_i & \xrightarrow{\quad} & U_j
\end{array}$$

Now with the definition out of the way, let's take a look at couple interesting examples:

**Example 1.** If we take all groups  $\Gamma_i$  to be the trivial groups then the orbifold is actually locally Euclidean, which means it is just a manifold. So, we can see that every manifold is an orbifold by taking the trivial group for every cover.

**Example 2.** The second example mentioned before the definition turns also to be an orbifold. Just take  $V = \mathbb{R}^2$ ,  $\Gamma = \mathbb{Z}/3$  and  $U = V/\Gamma$ . Note that we only need one cover and so the other conditions are satisfied by default.

**Example 3.** Let us consider a more interesting example. Let  $\mathcal{M}$  be an  $n$ -manifold with boundary and let  $U_i$  be a cover of  $M$ . Now, for those  $i$  for which  $U_i$  is homeomorphic to  $\mathbb{R}^n$ , let  $V_i = U_i$  and  $\Gamma_i = 1$ . For the rest let  $\Gamma_i = \mathbb{Z}/2$  and let  $V_i$  be the open set whose quotient space under the mirror action of  $\Gamma_i$  is equal to  $U_i$  (it is not hard to show that such an open set always exists). Then the manifold with boundary becomes an example of an orbifold, without ever mentioning the notion of a boundary!

In the beginning we stated that we want an orbifold to be closed under non-free group actions. The next proposition shows that we have indeed reached this goal. So, we can really think of orbifolds as some sort of quotients of manifolds (although you should be careful that this is not true for every possible orbifolds, which will be discussed later).

**Proposition 1.** *Let  $\mathcal{M}$  be a manifold and  $\Gamma$  a group acting on  $\mathcal{M}$  properly discontinuous. Then,  $\mathcal{M}/\Gamma$  is an orbifold.*

*Proof.* Let  $x \in \mathcal{M}/\Gamma$ . Choose  $\tilde{x} \in \mathcal{M}$  such that the projection of  $\tilde{x}$  is  $x$ . As  $\Gamma$  acts properly discontinuous on  $\mathcal{M}$ ,  $\tilde{x}$  has a euclidean neighbourhood  $V_x$ , which does not intersect any of its translations by the group action. This means that the projection from  $V_x$  to  $U_x = V_x/\Gamma$  is an homeomorphism. So, we have a cover  $\{U_x\}$ , which is homeomorphic to a quotient by the group action of a subset of Euclidean space and satisfies almost all the mentioned conditions. The only one left is to make sure that the element of the cover are closed under intersection. In order to ensure that we just add all possible intersections, we just consider a nonempty intersection use the fact that the action is properly discontinuous to find an intersection of the corresponding open sets in  $\mathcal{M}$  and so get the wanted result.  $\square$

So, we proved that orbifolds are basically the closures of manifolds under the quotients of properly discontinuous actions. Note that if we add freeness to our action then our orbifold is guaranteed to be a manifold. And so the we have to pay special attention to the points on which the action is not free, as those are our singular points which prevent smoothness and the rest would allow a smooth structure. As an example we can go back to Example 2 where  $\mathbb{Z}/3$  acted freely on everywhere except  $(0,0)$ , which was exactly the only singularity. Note that the set of singularities is always closed and nowhere dense and as we mentioned is empty if and only if the quotient is an actual manifold.

## 2: The Fundamental Group of an Orbifold

Up until now, we defined the notion of orbifold and gave couple examples. Now, as with any other mathematical object, we would like to classify it or at least have ways to differentiate between them. As any orbifold is a topological space, we may try to apply the machinery of algebraic topology to it. In this note, we only mention the example of the fundamental group to get a better handle at orbifolds.

The standard way to define the fundamental group is to look at homotopy classes of loops into the orbifold. Normally, in the context of manifolds, we limit ourselves to smooth loops, which is not an option in orbifolds as they are not smooth everywhere and so complications might arise. Fortunately, there is a second way to define the fundamental group using covering spaces,

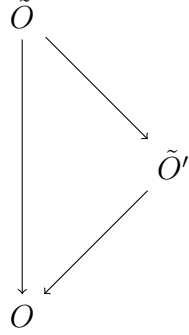
which is also much more compatible with the quotient structures we are dealing with and so will use this approach. First, let us generalize the notion of covering manifold.

**Definition 2.** A covering orbifold of an orbifold  $O$  is an orbifold  $\tilde{O}$ , with a projection  $p : \tilde{O} \rightarrow O$  between the underlying spaces, such that each point  $x \in O$  has a neighborhood  $U = V/\Gamma$  (where  $V$  is an open subset of  $\mathbb{R}^n$ ) for which each component  $C_i$  of  $p^{-1}(U)$  is isomorphic to  $V/\Gamma_i$ , where  $\Gamma_i \subset \Gamma$  is some subgroup. Also, the isomorphism must respect the projections.

As a very standard example, let  $\mathcal{M}$  be a manifold and  $\Gamma$  act properly discontinuous. Then  $\mathcal{M}/\Gamma$  is an orbifold and  $\mathcal{M}$  is a covering orbifold (recall every manifold is an orbifold).

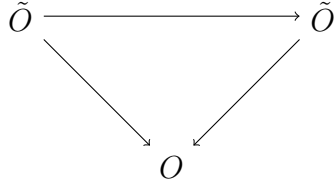
From classical algebraic topology, we recall that among all possible covers there might be one which is larger than any other cover i.e. is itself a cover of any other cover. This cover is normally called the universal cover (for obvious reasons) and do not exist for every topological space. Following any standard reference on this topic there is theorem which specifies two necessary and sufficient conditions for a space to have a universal cover: 1- locally path-connectedness, 2- semi-locally simply connectedness (there is a third condition that the space has to be path-connected, but we resolve that issue by only looking at pointed orbifolds and so automatically giving special attention to the path component of the basepoint). Clearly, every manifold satisfies both conditions as it is locally Euclidean and both conditions are local in nature. Now, a properly discontinuous function does not influences these two conditions. Thus, we conclude that every orbifold has a universal cover. So, we just proved the following proposition.

**Proposition 2.** *Every orbifold  $O$  with fixed based point  $*$  among the nonsingular points has a universal cover  $p : \tilde{O} \rightarrow O$  with basepoint  $\tilde{*}$  and basepoint preserving map, which means it is a connected covering orbifold of  $O$  such that for any other basepoint preserving covering orbifold  $q : \tilde{O}' \rightarrow O$ , there is a basepoint preserving lifting  $p' : \tilde{O} \rightarrow \tilde{O}'$  such that the following map commutes:*



*Proof.* The proof is outlined in the paragraph before the statement of the theorem. The only missing detail is to make all maps respect the basepoint and be basepoint preserving. However, this detail does not change any of the mentioned steps.  $\square$

Now, for fixed basepointed orbifold  $(O, *)$ , with universal cover  $p : (\tilde{O}, \tilde{*}) \rightarrow (O, *)$ , we get the group  $Aut_p(\tilde{O}, \tilde{*})$  of base point preserving automorphisms of  $\tilde{O}$ ,  $q$ , fitting into the following diagram:



These are called Deck transformations. Finally, using all of these facts, we have the following definition:

**Definition 3.** Let  $(O, *)$  be an orbifold with universal cover  $p : (\tilde{O}, \tilde{*}) \rightarrow (O, *)$ . Then we define the fundamental group of  $O$  by

$$\pi_1(O, *) = Aut_p(\tilde{O}, \tilde{*}).$$

Note, as a direct first result, the fundamental group of any universal cover is trivial and so every universal cover is simply connected.

Now, one positive attribute an orbifold can have is that the universal cover is an actual manifold. Orbifolds which have this property are called *good* and all the others *bad*. Note that good orbifolds are those which are of

the form  $M/\Gamma$  for some manifold  $M$  and some group  $\Gamma$ , which was exactly the case we had initially in mind. However, there are also some orbifolds, which do not come from this construction and those are the bad orbifolds. In the next section, when we want to classify 2-orbifolds, we see that this will play an important role.

*Remark 1.* An orbifold is good if and only if it has *some* cover, which is a manifold. Indeed, if  $\tilde{O}'$  is a cover of  $O$  and a manifold, then the universal cover is also a cover of  $\tilde{O}'$ . But every cover of a manifold (covering preserve the locally Euclidean property), which means the universal cover must be a manifold. The other side is trivial.

### 3: The two-dimensional Case

Now, we focus on 2 dimensional orbifolds. From this point on every orbifold will be a smooth 2-dimensional orbifold. The goal is to classify all 2-orbifolds. This involves two steps, in the first step we classify all orbifolds (recall every orbifold is now 2-dimensional), with only one cover,  $\mathbb{R}^2$ . In the case of manifolds, there is only one such manifold (up to diffeomorphism), but in the case of orbifolds the situation is completely different because it also has a group action which influences the metric and the smoothness. The second part is to look at all different ways we can build orbifolds out of those and classify all possible 2-dimensional orbifolds.

We start with the first step. So, let  $\Gamma$  be finite and act on  $\mathbb{R}^2$ . For every group action on  $\mathbb{R}^2$  there is a Riemannian metric invariant under the group action. Now, the exponential map gives a diffeomorphism with the tangent space (which is also  $\mathbb{R}^2$ ), which commutes with the group action and so is also a diffeomorphism after quotienting. However, the group action on the side of the image is always a finite subgroup of  $O(2)$  and so we only need to consider the case of  $\mathbb{R}^2$  being acted on by  $\Gamma$  which is a finite subgroup of  $O(2)$ , and so we have to classify all finite subgroups of  $O(2)$ . But all finite orthogonal transformations in  $\mathbb{R}^2$  consist of finite rotation and reflection. The reflection gives us a  $\mathbb{Z}/2$  subgroup. The finite rotation of degree  $2\pi/n$  gives us a  $\mathbb{Z}/n$  subgroup and a combination of both gives us a  $D_{2n}$ , the dihedral subgroup. All we have to do is to understand the metric and smooth structure of  $\mathbb{R}^2$  under the action of these three types of groups.

The first group,  $\mathbb{Z}/2$ , gives us the half-space i.e. the points where  $x \geq 0$ , similar to Example 3, where we looked at manifolds with boundary. The



second type,  $\mathbb{Z}/n$ , is a generalization to Example 2, with  $n$  instead of 3 and the quotient gives us a cone with an elliptic point at the origin. The third type is a combination of the the two first types and so is a combination of reflection and cone point at the origin and so gives us a corner reflectors.

Thus, we conclude that we have four possible local data for two-dimensional orbifolds:

1.  $\Gamma = 0$ : The trivial cover  $\mathbb{R}^2$  itself
2.  $\Gamma = \mathbb{Z}/2$ : reflection of  $\mathbb{R}^2$  along an axis
3.  $\Gamma = \mathbb{Z}/n$ : cone coming from rotation
4.  $\Gamma = D_{2n}$ : half of a cone being a reflection of the cone

So, an orbifold is locally of these four shapes. This means that every orbifold is topologically a surface i.e. a topological manifold of dimension 2. One very efficient algebraic invariant, which helps in the classification of surfaces is the Euler characteristic. Any triangulation (or even more arbitrary Cell-structure) would give us an integer called the Euler characteristic of a manifold. Moreover, homoemorphic manifolds must have the same Euler characteristic and so it can help us differentiate between different manifolds.

The goal is to generalize this to orbifolds. Clearly, an orbifold is not only determined by topological information, but also needs the algebraic information related to the group action. So, the new triangulation and the new invariant have to reflect this fact. This leads us to the following definition:

**Definition 4.** Let  $O$  be an orbifold, with a cell division on  $X_O$ , which is small enough so that the group acting on the interior of every cell is the same. Then we define the Euler number of an orbifold,  $\chi(O)$ , by the formula:

$$\chi(O) = \sum_{c_i} (-1)^{\dim(c_i)} \frac{1}{|\Gamma(c_i)|}$$

where  $c_i$  is the set of cells of the cell decomposition of  $X_O$  and  $|\Gamma(c_i)|$  is the order of the group associated to the cell  $c_i$ .

Note that this Euler number is not necessarily an integer anymore. However, as a special case, if we think of manifolds as orbifolds with the trivial group acting on them then the Euler number goes back to be the same integer we mentioned before. Again, different Euler numbers mean that the

corresponding orbifolds are not diffeomorphic. In order to get a better handle on Euler numbers we have the following proposition related to Euler numbers and coverings (the proof relies of a some information about Euler characteristics which we won't dive into and so will be omitted).

**Proposition 3.** *If  $p : \tilde{O} \rightarrow O$  is a  $k$ -sheeted covering map of orbifolds, then we have the following:*

$$\chi(\tilde{O}) = k\chi(O)$$

For example, we know the Euler number of  $\mathbb{R}^2$  is 1. Also, it is an  $n$ -fold cover of the orbifold  $O = \mathbb{R}^2/(\mathbb{Z}/n)$  and so  $O$  has Euler number  $\frac{1}{n}$ . With this result at hand and some tedious effort we can prove now the main result about 2-orbifolds, namely, to classify them all completely.

**Theorem 1.** *A closed two-dimensional orbifold has an elliptic, parabolic or hyperbolic structure if and only if it is good. An orbifold  $O$  has a hyperbolic structure if and only if  $\chi(O) < 0$ , and a parabolic structure if and only if  $\chi(O) = 0$ . An orbifold is elliptic or bad if and only if  $\chi(O) > 0$ .*

*All bad, elliptic and parabolic orbifolds are mentioned below, where  $(n_1, \dots, n_k; m_1, \dots, m_l)$  denotes an orbifold with elliptic points of orders  $n_1, \dots, n_k$  (in ascending order) and corner reflectors of orders  $m_1, \dots, m_l$  (in ascending order). Orbifolds not listed are hyperbolic.*

- *Bad orbifolds:*
  - $X_O = S^2 : (n), (n_1, n_2)$  with  $n_1 < n_2$ .
  - $X_O = D^2 : (; n), (; n_1, n_2)$  with  $n_1 < n_2$ .
- *Elliptic orbifolds:*
  - $X_O = S^2 : (), (n, n), (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)$ .
  - $X_O = D^2 : (; ), (; n, n), (; 2, 2, n), (; 2, 3, 3), (; 2, 3, 4), (; 2, 3, 5), (n; ), (2; m), (3; 2)$ .
  - $X_O = \mathbb{P}^2 : (), (n)$ .
- *Parabolic orbifolds:*
  - $X_O = S^2 : (2, 3, 6), (2, 4, 4), (3, 3, 3), (2, 2, 2, 2)$ .
  - $X_O = D^2 : (; 2, 3, 6), (; 2, 4, 4), (; 3, 3, 3), (; 2, 2, 2, 2), (2; 2, 2), (3; 3), (4; 2), (2; 2; )$ .
  - $X_O = \mathbb{P}^2 : (2, 2)$ .
  - $X_O = \mathbb{T}^2 : ()$
  - $X_O = \text{Klein Bottle} : ()$
  - $X_O = \text{Annulus} : (; )$
  - $X_O = \text{Möbius band} : (; )$

*Proof.* A complete proof of this theorem can be found in Thurston's book mentioned in the introduction on page 312 as Theorem 13.3.6.  $\square$

## 4: Conclusion

This basically gives us a complete understanding of all 2-orbifolds. Already, we can see a much bigger generality than we have in the case of manifolds. So, it is reasonable to expect a much more complicated situation for higher dimensional orbifolds and, as a matter of fact, existing classification are limited to specific sets of higher orbifolds, like *Tetrahedral Orbifolds*.

One other way to achieve results for higher dimensions is to look at the special family of orbifolds: orbifolds which are tangent spaces of other orbifolds. Like in the manifold case they enjoy better properties which makes them easier to understand and study. Both of these topics can be found in Thurston's book and the interested reader is encouraged to look it up in Chapter 13 of his book.

From a completely different perspective, there exists a strongly algebraic way to think about orbifolds, which has a far more categorical flavor than what we have seen in this note. In that method, we employ the language of stacks and define an orbifold to be a particular algebraic stack. A good place to look at those is a paper by E. Lerman titled, "Orbifolds as stacks? ".

Finally, as a conclusion, I hope this note has convinced the reader the importance and usefulness of reasonable algebraic generalizations of manifolds, of which orbifolds is a very good example. The extended power and generality it provides offers the opportunity for many new theorems while retaining the power of geometric intuition.