

## Lecture 2 : Examples of spaces

(updated 26/7/18)

The first lecture was a lightning tour of the standard abstract notions of space grounded in the example  $\mathbb{R}^n$ , with an extended look at the connection between inner products and rotations. Today we will focus on examples.

Def<sup>n</sup> A metric space is a pair  $(X, d)$  consisting of a set  $X$  and a function

$$d: X \times X \longrightarrow \mathbb{R}$$

satisfying the axioms:

- |      |                                  |                         |                       |
|------|----------------------------------|-------------------------|-----------------------|
| (M1) | $d(x, y) \geq 0$                 | $\forall x, y \in X$    | (non-negativity)      |
| (M2) | $d(x, y) = 0 \iff x = y$         | $\forall x, y \in X$    | (separation)          |
| (M3) | $d(x, y) = d(y, x)$              | $\forall x, y \in X$    | (symmetry)            |
| (M4) | $d(x, y) + d(y, z) \geq d(x, z)$ | $\forall x, y, z \in X$ | (triangle inequality) |

Remark L2-1 (i) The empty set  $\emptyset$  is a metric space ( $\emptyset \times \emptyset = \emptyset$ )

(ii) We can consider metrics taking values in other totally ordered fields (e.g.  $\mathbb{Q}$ ), this is important for constructing the  $p$ -adic numbers, or  $\mathbb{R}$  itself from  $\mathbb{Q}$ , but we will only consider real-valued metrics in this course.

(iii) Usually we abuse notation and say " $X$  is a metric space" where we really mean " $(X, d)$  is a metric space". For this reason, we also often write  $d_X$  for the metric on  $X$ , if this needs clarification.

Example L2-1 The singleton  $X = \{*\}$  is a metric space with  $d(*, *) = 0$ .

Exercise L2-1 Prove that any set  $X$  becomes a metric space with the discrete metric defined by

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y. \end{cases}$$

Lemma L2-1 The following functions define metrics on  $\mathbb{R}^n$

$$d_2(x, y) = \left\{ \sum_{i=1}^n (x_i - y_i)^2 \right\}^{1/2}$$

$$d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$$

$$d_\infty(x, y) = \max \left\{ |x_i - y_i| \right\}_{i=1}^n$$

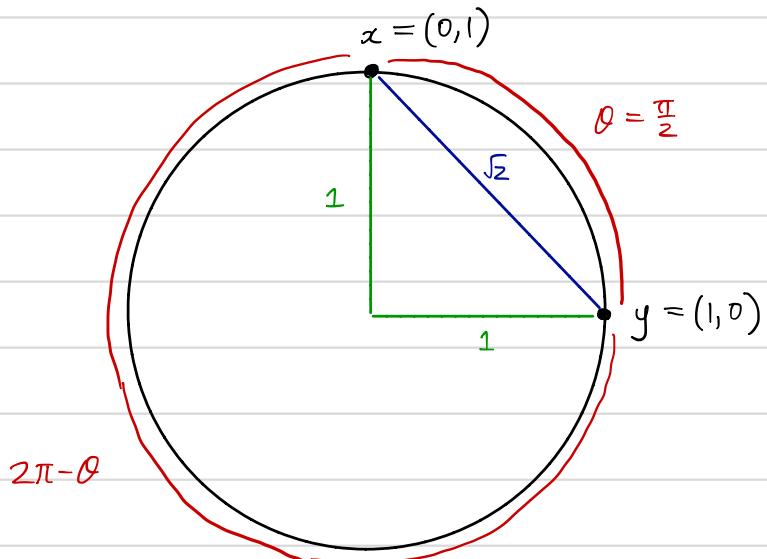
Proof Deferred to next lecture.  $\square$

Remark L2-2  $\mathbb{R}^0 = \{*\}$  is trivially a metric space.

Exercise L2-2 Given a metric space  $(X, d)$  and a subset  $A \subseteq X$  prove that  $(A, d|_A)$  is a metric space with the induced metric

$$d|_A : A \times A \rightarrow \mathbb{R}, \quad d|_A(a, a) = d(a, a).$$

Example L2-2 The circle  $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  becomes a metric space under the restriction of  $d_2$ ,  $d_1$  or  $d_\infty$ . But there is clearly something unsatisfactory about these metrics, see the diagram overleaf.



$$d_2(x, y) = \sqrt{2}$$

$$d_1(x, y) = 2$$

$$d_\infty(x, y) = 1$$

What about  $\frac{\pi}{2}$ ?

Let us now examine whether arc length gives a metric on  $S^1$ . Firstly, there is not a unique arc connecting two points, and secondly, to avoid explicitly talking about inverse trigonometric functions, let us use the bijection

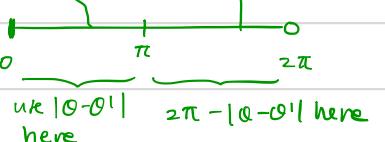
$$\Phi: [0, 2\pi) \longrightarrow S^1, \quad \Phi(\theta) = (\cos \theta, \sin \theta).$$

Then we define ("a" for "arc") for  $x, y \in S^1$  lies in  $(0, 2\pi]$

$$d_a(x, y) = \min \{ |\Phi^{-1}(x) - \Phi^{-1}(y)|, 2\pi - |\Phi^{-1}(x) - \Phi^{-1}(y)| \}$$

$$\text{i.e. } d_a(\Phi\theta, \Phi\theta') = \min \{ |\theta - \theta'|, 2\pi - |\theta - \theta'| \}$$

Lemma L2-2 The pair  $(S^1, d_a)$  is a metric space.



Proof (M1) is clear. For (M2), we have for  $x, y \in S^1$  with  $x = \Phi\theta, y = \Phi\theta'$

$$d_a(\Phi\theta, \Phi\theta') = 0 \iff |\theta - \theta'| = 0 \quad (2\pi - |\theta - \theta'| \text{ cannot be zero})$$

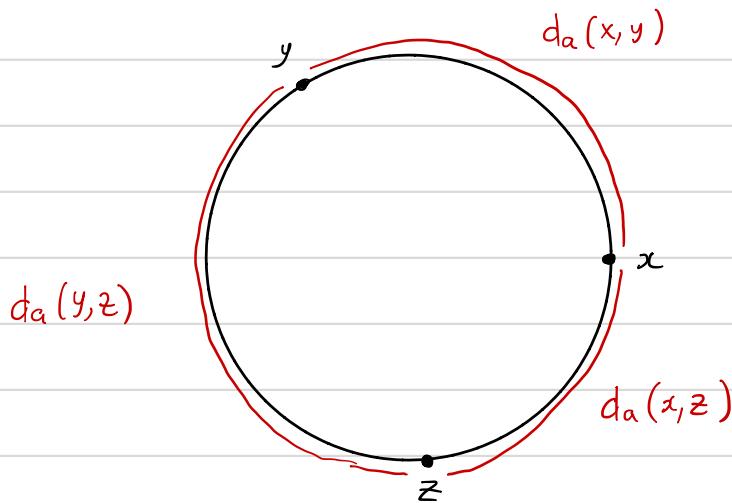
$$\iff \theta = \theta'$$

$$\iff \Phi\theta = \Phi\theta' \quad (\text{as } \Phi \text{ is a bijection})$$

For (M3) consider

$$\begin{aligned}
 d_\alpha(\theta, \theta') &= \min\{|\theta - \theta'|, 2\pi - |\theta - \theta'|\} \\
 &= \min\{|\theta' - \theta|, 2\pi - |\theta' - \theta|\} \\
 &= d_\alpha(\theta', \theta).
 \end{aligned}$$

For (M4), we have a situation like the following:



To prove that for all  $x, y, z \in S^1$

$$d_\alpha(x, y) + d_\alpha(y, z) \geq d_\alpha(x, z)$$

we can just do a case by case analysis. This is fine but a bit lame. Let us instead observe that we can reduce to the case  $x = (1, 0)$ . Let

$$R_y : S^1 \rightarrow S^1$$

be the restriction of the rotation by  $\gamma$  on  $\mathbb{R}^2$  in Lecture 1. Observe that

$$R_\theta(\cos\theta, \sin\theta) = (\cos(\theta+\varphi), \sin(\theta+\varphi))$$

i.e.  $R_\theta \circ \varphi(\theta) = \varphi(\theta+\varphi)$  where  $\theta+\varphi$  is taken "modulo  $2\pi$ ".

Now notice that  $|\theta-\theta'|$  does not change if  $\theta \rightarrow \theta+\varphi$ ,  $\theta' \rightarrow \theta'+\varphi$ , so

$$d_a(R_\theta x, R_\theta y) = d_a(x, y) \quad \forall x, y \in S^1$$

Hence to prove, for some fixed  $x, y, z \in S^1$  that

$$d_a(x, y) + d_a(y, z) \geq d_a(x, z)$$

it is equivalent to prove

$$d_a(R_\theta x, R_\theta y) + d_a(R_\theta y, R_\theta z) \geq d_a(R_\theta x, R_\theta z).$$

But  $R_\theta x = (1, 0)$ , and so it suffices to prove (M4) in the case where  $x = (1, 0)$  and  $y, z$  are arbitrary.

Next, consider the function

$$T: S^1 \longrightarrow S^1 \quad T(a_1, a_2) = (a_1, -a_2).$$

and observe that

$$d_a(x, y) = d_a(Tx, Ty) \quad \forall x, y \in S^1$$

because  $T(\cos\theta, \sin\theta) = (\cos(-\theta), \sin(-\theta))$  and  $|\theta-\theta'| = |-\theta-(-\theta')|$ .

OK, so we have to prove for all  $y, z \in S^1$  that

$$d_a((1,0), y) + d_a(y, z) \geq d_a((1,0), z). \quad (6.1)$$

This is equivalent to

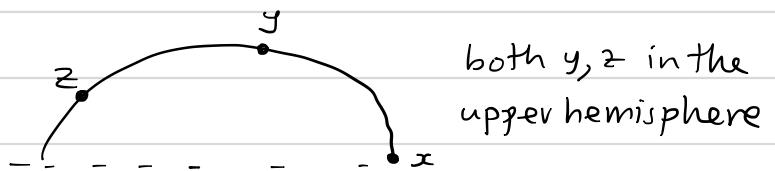
$$d_a((1,0), Ty) + d_a(Ty, Tz) \geq d_a((1,0), Tz) \quad (6.2)$$

So we may assume  $y$  lies in the upper hemisphere! i.e. that  $\theta' = \Phi^{-1}(y)$  lies in  $[0, \pi]$ , because if it doesn't then  $Ty$  does, and we can just prove (6.2) instead. But if  $\theta' \in [0, \pi]$  then

$$d_a((1,0), y) = \theta'$$

and we just have two cases depending on  $\theta'' = \Phi^{-1}(z)$

Case 1  $0 \leq \theta'' \leq \pi$ , i.e.



Then  $0 \leq |\theta' - \theta''| \leq \pi$  so

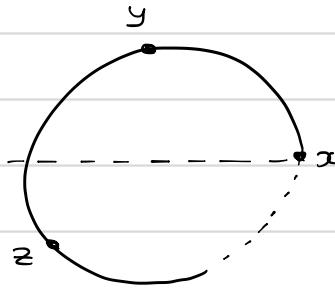
$$d_a((1,0), y) + d_a(y, z) = \theta' + |\theta' - \theta''|$$

Now either  $\theta' \geq \theta''$  in which case

$$\theta' + |\theta' - \theta''| = 2\theta' - \theta'' \geq \theta'' = d_a((1,0), z)$$

or  $\theta' < \theta''$  in which case  $\theta' + |\theta' - \theta''| = \theta'' = d_a((1,0), z)$ .

Case 2

 $\pi < \theta'' < 2\pi$  i.e.Now  $|\theta' - \theta''| = \theta'' - \theta'$  and so

$$d_a((1,0),y) + d_a(y,z) = \theta' + \min\{\theta'' - \theta', 2\pi - (\theta'' - \theta')\}$$

$$= \min\{\theta'', 2\pi + 2\theta' - \theta''\}.$$

$$\geq 2\pi - \theta'' = d_a((1,0),z).$$

which completes the proof that  $d_a$  is a metric.  $\square$ 

Exercise L2-3 Give a direct proof of Lemma L2-2 by dividing into cases, as follows: given  $x, y, z \in S^1$  set  $\theta = \underline{\theta}^{-1}(x)$ ,  $\theta' = \underline{\theta}^{-1}(y)$  and  $\theta'' = \underline{\theta}^{-1}(z)$ . Consider the following three statements:

$$P_1 \quad |\theta - \theta'| \leq \pi$$

$$P_2 \quad |\theta' - \theta''| \leq \pi$$

$$P_3 \quad |\theta - \theta''| \leq \pi$$

Each is either true or false for a particular triple  $(x, y, z)$ , and this means there are  $2^3 = 8$  cases (e.g.  $P_1, P_2$  true but  $P_3$  false).

Prove each case individually, and in this way prove the lemma.

Remark L2-3 Observe that  $d_a(x, y) \leq \pi$  for all  $x, y \in S^1$ , in contradistinction to  $\mathbb{R}^n$  where the metric ( $d_1, d_2$  or  $d_\infty$ ) has no upper bound. In this sense  $(S^1, d_a)$  is really a new example.

Exercise L2-4 Is it true that for distinct  $x, y, z$  that

$$d_a(x, y) + d_a(y, z) + d_a(z, x) = 2\pi ?$$

(i.e. don't be fooled by pictures like the one on p.4)