Analysis Hand In One

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Question One

$$f_n(x) = \frac{x^n}{1+x^n} \tag{1}$$

To show that (1) converges pointwise we must establish that there is a limit-function f where $\forall \epsilon > 0 : \exists N \in \mathbb{N}$ such that $\forall n \in N : |f_n(x) - f(x)| < \epsilon$. We break this down into the four cases where $x = 0$, $x = 1$, $x \in (0, 1)$ and $x > 1$. If $x = 0$ then $f_n(x) = 0$ for all n, and so f_n clearly converges to 0. If $x = 1$ then $f_n(x) = \frac{1}{2}$ for all n, so clearly $f_n \to \frac{1}{2}$.

If $x \in (0, 1)$ then $x^n \to 0$ as $n \to \infty$ so I claim that $f_n(x) \to 0$ as $n \to \infty$ also. Notice that $\forall x \in (0, 1) : |\frac{x^n}{1+x^n}| \leq |\frac{x^n}{1}| = |x^n|$, so letting $\epsilon = x^n$ we see that $\forall x > 0 : \exists N \in \mathbb{N}$ such that $\forall n > N : |f_n(x) - 0| \leq |x^n| = \epsilon$.

If $x \in (1, \infty)$ then $|\frac{x^n}{1+x^n} - 1| = |\frac{x^n}{1+x^n} - \frac{1+x^n}{1+x^n}| = |\frac{-1}{1+x^n}| \leq \frac{1}{x^n}$. Thus if we let $\epsilon = \frac{1}{x^n}$ then, $\forall x \in (1, \infty) : \forall \epsilon > 0 : \exists N \in \mathbb{N}$ such that $\forall n > N : |f_n(x) - 1| < \epsilon$. In other words, $f_n \to 1$ pointwise on $(1, \infty)$, and so

$$f_n \to f = \begin{cases} 0 & x \in [0, 1) \\ \frac{1}{2} & x = 1 \\ 1 & x \in (1, \infty) \end{cases}$$

Next, I claim that $f_n \to f$ uniformly on $x \in [0, a)$ if $0 \leq a < 1$. First I'll show that for intervals $[0, b)$, where $b \geq 1$, f_n does not converge uniformly. We must calculate $\lim_{n\to\infty} \sup_{x\in[0,b)} |f_n(x)|$, and show that this does not equal zero. We need only concern ourselves with $b > 1$ since if $b = 1$ then we can choose $x = b = 1$ to get $f_n(1) = \frac{1}{2}$ and $\forall x \in [0, 1) : f_n(x) \to 0$, so f_n cannot converge uniformly for $b = 1$.

We have that $\forall x \in [0, b) : x < b$ and if $n > 1$ then, as $b > 1$ we know that $x^n < b^n$. I will show than b^n is the supremum by contradiction: if $1 < M < b^n$ is another upper bound to x^n for $M \in \mathbb{R}$ then $M^{\frac{1}{n}} < b$, so $M \in [0, b)$ and so isn't an upper bound; contradiction! Thus $\sup_{x\in[0,b)} |f_n| = b^n$. Now, $b > 1$, so $b^n > 1$ and so clearly $\lim_{n\to\infty} \sup_{x\in[0,b)} |f_n| = \lim_{n\to\infty} b^n \neq 0$, thus for $b > 1$ we have that f_n cannot converge uniformly.

Finally, I will prove that for $a < 1$, $f_n \to f$ uniformly for $x \in [0, a)$. Note that $\forall x \in [0, a) : x < a$ so $x^n < a^n$ and $\lim_{n\to\infty} a^n = 0$ (since $a < 1$). Letting $\epsilon = a^n$ we have that

$$\forall \epsilon > 0 : \exists N \in \mathbb{N} : \forall x \in [0, a) : |f_n(x) - 0| = \left|\frac{x^n}{1+x^n}\right| \leq \left|\frac{a^n}{1}\right| = \epsilon,$$

and so $f_n \to 0$ uniformly on $[0, a)$ so long as $0 \leq a < 1$.

Question Two

$$f_n(x) = nx(1-x^2)^n, \quad x \in [0, 1]. \tag{2}$$

We will begin by using the M-test to show that $g(x) = \sum_n f_n(x) < \infty$, then deduce that $f_n \to f$ pointwise for some f.

To begin, we will use the Ratio test on f_n to test convergence of $\sum_n f_n$ on the interval $(0, 1)$, noting that f_n converges pointwise for the endpoints $x = 0, 1$ since $\forall x : f_n(0) = f_n(1) = 0$, and certainly $|f_n(0) - 0| = 0 < \epsilon$ for any $\epsilon > 0$, thus $f_n \to 0$ for $x = 0, 1$. Testing

$$\frac{f_{n+1}}{f_n} = \frac{(n+1)x(1-x^2)^{n+1}}{nx(1-x^2)^n} = \frac{n+1}{n}(1-x^2),$$

we see that $L := \lim_{n\to\infty} \left|\frac{f_{n+1}}{f_n}\right| = 1 - x^2$, hence $0 < L < 1$ as $x > 0$, so $\sum_n f_n$ converges by the ratio test. By the Divergence Theorem, then, we have that $\lim_{n\to\infty} f_n = 0$, and hence $f_n \to 0$ at least pointwise.

Consider $\int_0^1 f_n(x)dx = \int_0^1 nx(1-x^2)^n = \left[-\frac{1}{2}\frac{n}{n+1}(1-x^2)^{n+1}\right]_0^1 = \frac{1}{2}\frac{n}{n+1}$. Since $f_n \to 0$ as $n \to \infty$ we have that

$$\int_0^1 \lim_{n\to\infty}(f_n(x))dx = \int_0^1 f(x)dx = \int_0^1 0dx = 0 \neq \frac{1}{2} = \lim_{n\to\infty}\left(\frac{1}{2}\frac{n}{n+1}\right) = \lim_{n\to\infty}\int_0^1 f_n(x)dx,$$

but if $f_n \to f = \lim_{n\to\infty} f_n$ uniformly then $\forall x : \lim \int f_n dx = \int \lim f_n dx$, and so clearly the convergence cannot be uniform on $[0,1]$.

Now we test to see whether (2) converges uniformly on $[a,1]$ for $a \in (0,1)$. I claim that it does not uniformly converge, which I'll show by proving that $\lim_{n\to\infty} \sup_{x\in[a,1]} |f_n(x)| \neq 0$. Differentiating (2) we get that $\frac{d}{dx} f_n(x) = n(1-x^2)^n + n^2 x(-2x)(1-x^2)^{n-1}$, so

$$\frac{df_n}{dx} = 0 \Rightarrow 1 - x^2 = 2nx^2, \tag{3}$$

where I've divided through by $n(1-x^2)^{n-1}$, which means I must assume $x \neq 1$, however if $x = 1$ then $f_n(1) = 0$ so we can see if this is a maximum later. so by (3) we have that $(2n+1)x^2 = 1$, and so $x = \sqrt{\frac{1}{2n+1}}$ produces the maximum value of $f_n(x)$, where I took the positive root since $x \in [a,1] \subset [0,1]$. This gives

$$\begin{aligned} f_n\left(\sqrt{\frac{1}{2n+1}}\right) &= n\sqrt{\frac{1}{2n+1}}\left(1 - \left(\sqrt{\frac{1}{2n+1}}\right)^2\right)^n \\ &= \frac{n}{\sqrt{2n+1}}\left(\frac{2n}{2n+1}\right)^n \\ &= \frac{2^n n^{n+1}}{(2n+1)^{n+\frac{1}{2}}} \\ &\geq \frac{2^n n^{n+1}}{(2n+0)^n} \\ &= n. \end{aligned}$$

Thus $n \leq \lim_{n\to\infty} \sup_{x\in\mathbb{R}}(f_n(x)) = \infty$ and so is certainly not zero, and hence f_n doesn't converge uniformly.

Question Three

Knowing that $f_n \to f$ uniformly, and that $x_n \to x$ as $n \to \infty$ we aim to show that $f_n(x_n) \to f(x)$, that is; that $\lim_{n\to\infty}(f_n(x_n)) = f(x)$.

Since $x_n \to x$ we know that $\forall \varepsilon_x > 0 : \exists N_x \in \mathbb{N}$ such that $\forall n > N_x : |x_n - x| < \varepsilon_x$, we also know that $f_n \to f$ uniformly means that

$$\forall \varepsilon_f > 0 : \exists N_f \in \mathbb{N} : \forall n > N_f : \forall x \in \mathbb{R} : |f_n(x) - f(x)| < \varepsilon_f. \tag{4}$$

And, finally we are told that each f_i is continuous, so that

$$\forall i : \forall \varepsilon_i > 0 : \exists \delta > 0 : \forall x, y \in \mathbb{R} : |x - y| < \delta \Rightarrow |f_i(x) - f_i(y)| < \varepsilon_i, \tag{5}$$

and now we just use the triangle inequality, let $N = \max\{N_i, N_f, N_x\}$ and $\varepsilon = 2\min\{\varepsilon_i, \varepsilon_f, \varepsilon_x\}$. If we let $\delta = \varepsilon_x$ then $\exists N \in \mathbb{N} : |x - x_n| < \delta$ then:

$$\begin{aligned} |f_n(x_n) - f(x)| &= |f_n(x_n) - f_n(x) + f_n(x) - f(x)| \\ &\leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| && \text{by the Triangle Inequality} \\ &< \varepsilon_i + \varepsilon_f < \varepsilon && \text{by (4) and (5).} \end{aligned}$$

Where we've used that $|f_n(x_n) - f_n(x)| < \varepsilon_i$ since f_n is continuous, and that $|f_n(x) - f(x)| < \varepsilon_f$ by the uniform convergence of $f_n \to f$. Thus we get that:

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n > N : \ |f_n(x_n) - f(x)| < \varepsilon,$$

and thus $f_n(x_n) \to f(x)$ as $n \to \infty$.