Workshop 3 – Uniform continuity

The purpose of this workshop is to study an auxiliary topic that we won't cover in the lectures, but which provides one very important result that we shall need in our study of integration.

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. We know that it is continuous at a for all $a \in \mathbb{R}$. So, for every a, for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x - a| < \delta$ imples $|f(x) - f(a)| < \epsilon$. For $a > 1$ and $\epsilon = 1$ find the best possible δ. Is this best possible δ independent of $a > 1$? (**Hint:** Draw the graph of the function and include the horizontal lines $y = a^2 \pm 1$.)

2. Consider the same function, but now on $[0, 1]$. Prove that for all $\epsilon > 0$, if we take $\delta = \epsilon/2$ we have that $|x - a| < \delta$ (and $x, a \in [0, 1]$) implies $|f(x) - f(a)| < \epsilon$. So the "best" δ in the definition of continuity at $a \in [0, 1]$ can be taken to be independent of a in this case.

Definition. Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** on I if for every $\epsilon > 0$ there is a $\delta > 0$ such that $x, y \in I$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

So with $f(x) = x^2$ and $I = [0, 1]$ we have that f is uniformly continuous, while with the same f but $I = (1, \infty)$, we have that it isn't uniformly continuous. (Note that uniform continuity only makes sense for functions which are already continuous!)

3. Let $f(x) = 1/x$ on $(0, \infty)$. Is f uniformly continuous?

4. Let $f(x) = 1/x$ on $[a, \infty)$ where $a > 0$. Is f uniformly continuous?

5. Let I be an open interval in $\mathbb{R}$. Suppose $f : I \to \mathbb{R}$ is differentiable and its derivative f' is bounded on I. Prove that f is uniformly continuous on I.

6. Show that $f(x) = \sin x$ is uniformly continous on $\mathbb{R}$.

7. Let I be an open interval in $\mathbb{R}$. Prove that a continuous function $f : I \to \mathbb{R}$ is uniformly continuous on I if and only if whenever $s_n, t_n \in I$ are such that $|s_n - t_n| \to 0$, then $|f(s_n) - f(t_n)| \to 0$.

A fundamental fact is that if we are working on a **closed, bounded** interval $[a, b]$, then any continuous function $f : [a, b] \to \mathbb{R}$ is automatically uniformly continuous.

Theorem. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then it is uniformly continuous.

8. Prove this theorem by arguing by contradiction, using the the previous question and the Bolzano–Weierstrass theorem.

9. Find an example of an $f : (0, 1) \to \mathbb{R}$ which is continous but not uniformly continuous. Where exactly did we use the fact that $[a, b]$ was a closed and bounded interval in the proof of the theorem?

Assessment task to be handed in on Wednesday of Week 5 (17/10): Questions 6 and 9.