Presentations and colourings

7.1 Introduction to colourings

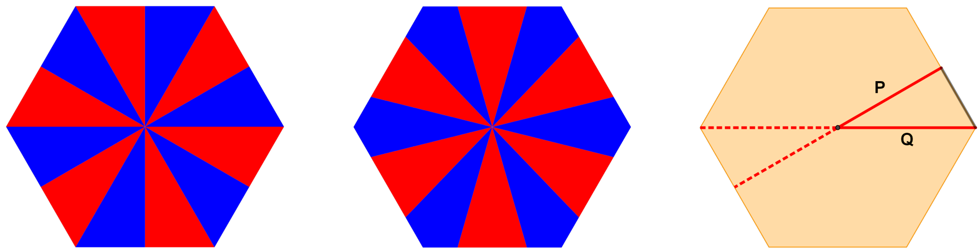


Figure 7.1: A 2-colouring (left); NOT a colouring (centre); Generators (right)

7.1.1 By a *colouring* of a pattern with initial symmetry group G, we mean the addition of colours in such a way that each element of g permutes the colours in a consistent way. The hexagon on the left in the figure, taken to have initial symmetry group D_6 as usual is 2-coloured: in this case, all the reflection symmetries swap red and blue, while all the rotation symmetries leave them fixed. Once the hexagon is coloured, the symmetry group has been reduced from D_6 to the subgroup C_6 of rotations.

The hexagon in the middle however is not 2-coloured: we will rule out the this trivial case where all the symmetries preserve the colours. To put it another way, the coloured pattern always has a symmetry group that is a proper subgroup of the original.

Our aim is to understand possible colourings. To do that we need a proper definition and some group theory.

7.1.2 Definition An n-colouring of a symmetry group G is a group homomorphism $c : G \longrightarrow S_n$ from G to the symmetric group S_n (where $n \geq 2$) of permutations of n colours such that $c(G)$ acts transitively on the colours. (The last condition means that given any two colours a, b there is a symmetry $g \in G$ such that $c(g)$ takes a to b.)

We will regard two colourings as the same if they differ by a permutation of the colours.

7.1.3 Commentary The map $c : G \longrightarrow S_n$ encapsulates the requirement that each $g \in G$ acts consistently on colours: if g takes one red point in the pattern to a blue point then it takes every red point to a blue point.

The fact that c is a group homomorphism is just a consistency requirement so that the permutation of colours associated with a symmetry gg' is equal to the permutation effected by g' followed by the permutation effected by g.

The transitivity requirement means that the hexagon in the centre is not 2-coloured. It also means that if we were to colour the rim of the hexagon on the left yellow, it remains a 2-colouring.

7.1.4 Theorem The group homomorphism of a colouring $c : G \longrightarrow S_n$ has kernel H where H is the normal subgroup of G that comprises all the symmetries that preserve the colours.

Proof. Should be clear from the previous discussion. □

7.1.5 Example The hexagon on the left above illustrates a 2-colouring. The homomorphism sends all rotations to the identity and all reflections to the non-trivial element of S_2 which exchanges the two colours. We will write it as (RB) using cycle notation.

The subgroup preserving the colours in this case is the cyclic group C_6 of rotational symmetries of the hexagon.

7.1.6 Notation We will sometimes refer to a colouring as in the above theorem as a G/H colouring. This often defines the colouring. For instance, there is only one C_6 subgroup of D_6 and so a D_6/C_6 colouring has to be the one where reflections swap the colours and rotations preserve them.

7.2 Colouring dihedral groups

7.2.1 Definition Let $h_1, \dots, h_k$ be elements of a group G. A *word* in $h_1, \dots, h_k$ is a product of elements of the group where each term in the product is one of the h_j or the inverse of one of the h_j.

Note here and hereafter: the "inverses" can be omitted for generators of finite order because if $h^n = 1$ then h^{-1} can be rewritten as = h^{n-1}.

7.2.2 Definition We say that the group G *is generated by* $h_1, \dots, h_k$ and write $G = \langle h_1, \dots, h_k \rangle$ if every element of G can be expressed as a word in $h_1, \dots, h_k$.

7.2.3 Proposition The dihedral group D_6 is generated by the two reflections P, Q in Figure 7.1.

7.2.4 Exercise (NM) Prove the proposition. (It may help to identify the symmetry PQ first.)

7.2.5 Exercise (NM) This continues from the previous proposition.

1. Show that $P^2 = Q^2 = (PQ)^6 = 1$

2. Deduce from part 1 (without geometric reasoning) that also $(QP)^6 = 1$.
3. Show that the symmetries

$$1, P, QP, PQP, QPQP, PQPQP, QPQPQP, PQPQPQP,$$
$$QPQPQPQP, PQPQPQPQP, QPQPQPQPQP, PQPQPQPQPQP$$

constitute the whole group D_6.
4. Show that elements of D_6 such as $QPQPQ$ that terminate in a "Q" are equal to one of the elements in part 3 using only the relations in part 1.

Thus we know everything about D_6 if we know it is generated by two elements satisfying the "relations" in part 1.

7.2.6 Definition Suppose that the group G is generated by $h_1, \dots, h_k$ and we have a finite number of *relations* $R_1, \dots, R_l$ which are equations relating the generators. We say that the generators and relations constitute a *presentation* of G and write $G = \langle h_1, \dots, h_k \mid R_1, \dots, R_l \rangle$ if two words in the generators are equal as elements of G if and only if one can be obtained from the other by repeated use of the relations.

7.2.7 Example A cyclic group of size n has a presentation $\langle a \mid a^n = 1 \rangle$.

7.2.8 Example Generalising what we saw in the previous example,

$$D_n = \langle P, Q \mid P^2 = Q^2 = (PQ)^n = 1 \rangle.$$

7.2.9 Exercise (NM) Generators and relations are not unique. For example, if we write $R = PQ$ in D_n then another presentation of D_n is

$$D_n = \langle P, R \mid P^2 = R^n = (PR)^2 = 1 \rangle.$$

Check you agree with this.

7.2.10 Exercise (NM) Consider a lattice L of translations (such as is contained in every wallpaper group). Let S, T be two translations that generate the lattice. Then we have $L = \langle S, T | STS^{-1}T^{-1} = 1 \rangle$. Explain the relation with a suitable picture.

7.2.11 Examples Consider the group $D_6 = \langle P, Q \mid P^2 = Q^2 = (PQ)^6 = 1 \rangle$. We will try and construct the 2-colouring at the top of this Chapter by means of a group homomorphism $c : D_6 \longrightarrow S_2$. We start by deciding, inspired by inspection of the pattern, that $c(P) = c(Q) = (RB)$ where R (red) and B (blue) define the colours and the parentheses denote the 2-cycle that permutes them.

Since every element of the group can be written in terms of P, Q this determines the value of c on every element by the group laws. (In fact, it says that $c(g) = (RB)$ if g can be written as a product of an odd number of generators and $c(g) = 1$ otherwise.)

For this to be consistent, we need to check that this respects the relations. Clearly $c(P)c(P) = c(Q)c(Q) = 1$. Also $c(P)c(Q) = 1$ and so $(c(P)c(Q))^6 = 1$. These things have to be true for c to be well defined. The theorem encapsulating this is stated after the next exercise.

7.2.12 Exercise (NM) Consider the group $D_n = \langle P, Q \mid P^2 = Q^2 = (PQ)^n = 1\rangle$. Consider the following three possible recipes for 2-colourings.

- $c(P) = c(Q) = (RB)$
- $c(P) = (RB), \quad c(Q) = 1$
- $c(P) = 1, \quad c(Q) = (RB)$

Determine for what values of n each of these exists and where it does sketch a representative example. (A good way of generating an example is to take a fundamental domain and colour it red and see how that pans out.) Why are we not including the possibility $c(P) = c(Q) = 1$?

7.2.13 Theorem Let $G = \langle h_1, \dots, h_k \mid R_1, \dots, R_l\rangle$ and let H be a group. Then there is a one to one correspondence between homomorphisms $G \longrightarrow H$ and maps f from the set of generators to H such that for each relation R_j, the equation defined by replacing each generator h_i by $f(h_i)$ is satisfied in H.

Proof. A sketch is to observe first that a group homomorphism clearly gives rise to such a map f on the generators. Secondly, given a map f defined for the generators, one can extend it uniquely to a map on the whole group. It is well-defined because of the condition imposed on the relations. Making this completely rigorous requires a better definition of the group defined by some generators and relations. □

7.2.14 Exercise (NM) Consider 3-colourings (with colours R,G,B) of the group D_n. Explain why for a 3-colouring $c : D_n \longrightarrow S_3$ we cannot have $c(P)$ or $c(Q)$ being a 3-cycle of the colours. Deduce that up to relabelling the colours the only possibility is that $c(P) = (RB)$ and $c(Q) = (GB)$.

Determine for what values of n this exists and where it does sketch a representative example.

7.3 Generators and relations for wallpaper and other groups

To understand colourings of wallpaper and spherical groups we need to obtain generators and relations. As usual, we will start with the results and justify them afterwards. What follows applies to the wallpaper, spherical and hyperbolic cases.

7.3.1 The general scheme As we will see below, each feature in the signature of a symmetry group gives rise to one “Greek” generator and zero or more “Latin” ones. The Greek generators tie everything together via the overall relation that the product of all the Greek generators is the identity.

Kaleidoscopes

7.3.2 The general case A kaleidoscope $*ab\dots c$ with n mirror meetings (or equivalently, n corner points on the corresponding orbifold boundary) corresponds to one Greek generator α

and $n+1$ Latin ones $P, Q, \dots, T$. The relations are

$$\begin{array}{ll} P^2 = Q^2 = \cdots = S^2 = T^2 = 1 & \text{Since these are reflections} \\ (PQ)^a = (QR)^b = \cdots = (ST)^c = 1 & \text{Rotations around mirror meetings} \\ \alpha^{-1}P\alpha = T & \end{array}$$

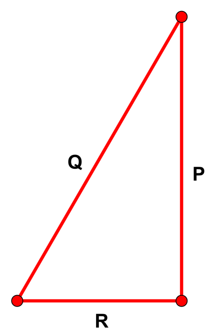
7.3.3 Kaleidoscope on its own If the symmetry of a pattern is a single kaleidoscope with no extra features $*ab\dots c$, then since there is only one Greek generator in the whole pattern, the overall relation becomes just $\alpha = 1$. Thus we have $P = T$ and the generators α and T can be omitted to get:

$$\begin{array}{ll} P^2 = Q^2 = \cdots = S^2 = 1 & P, Q, \dots, S \text{ are reflections} \\ (PQ)^a = (QR)^b = \cdots = (SP)^c = 1 & \text{Rotations around mirror meetings} \end{array}$$

7.3.4 Example Choose you favourite $*632$ pattern. Identify a fundamental region, which should be a triangle similar to that in the diagram. The full symmetry group is generated by P, Q, R with relations

$$P^2 = Q^2 = R^2 = (PQ)^6 = (QR)^3 = (RP)^2 = 1.$$

The last three relations tell us the order of rotational symmetry about the three vertices.



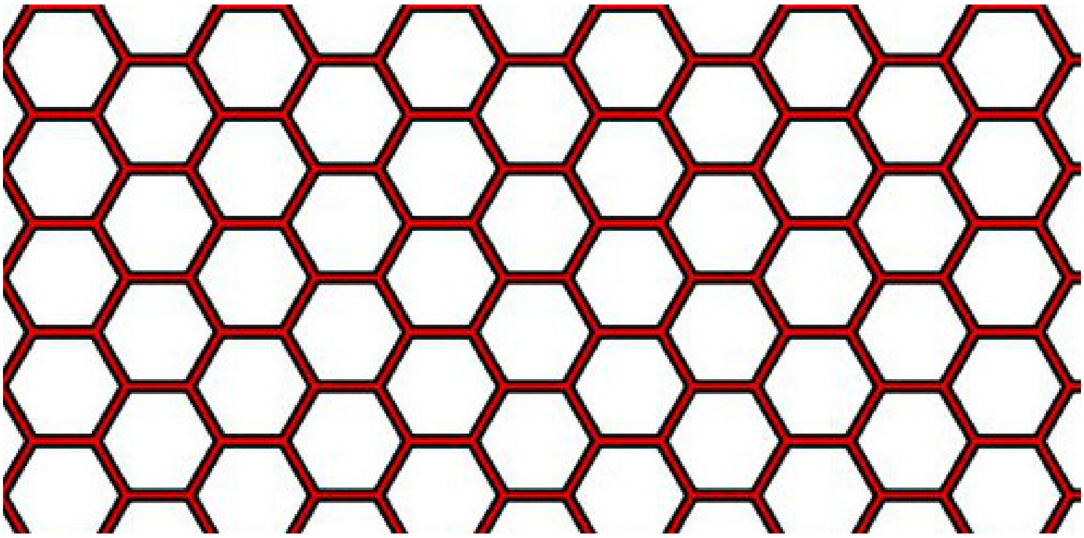
7.3.5 Exercise (NM) Let us consider 3-colourings of $*632$, calling our colours A, B, C. The first three relations tell us that each of P, Q, R must be colour transpositions or the identity. The identity $(PQ)^6 = 1$ tells us nothing since the size of S_3 is 6 and so every element to the 6th power is the identity.

Consider the consequences of $(QR)^3 = 1$. If one of Q, R acts by the identity then so must the other and then we cannot act transitively on the colours.

1. Suppose Q, R act by the same transposition: say $Q \mapsto (AB)$ and $R \mapsto (AB)$. For a transitive action, P must act by a different transposition, but then PR acts by a 3-cycle contradicting $(PR)^2 = 1$. Thus there is no colouring.

2. We can assume therefore that Q, R act by distinct transpositions: without loss of generality, we can assume $Q \mapsto (AB)$, $R \mapsto (BC)$. It is easy to check that the relations are satisfied if P acts by the identity or by (BC) but not otherwise. Thus we have the 3-colourings
$$P \mapsto 1, \qquad Q \mapsto (AB), \quad R \mapsto (BC).$$
and
$$P \mapsto (BC), \qquad Q \mapsto (AB), \quad R \mapsto (BC).$$

Illustrate these colourings on the hexagonal grid. You can start by choosing a fundamental region and hence identifying P, Q and R. Colour it with one of the colours and see how it works out.



7.3.6 Exercise (NM) Investigate the 2-colourings of $*632$ — you should find there are three. Draw examples.

7.3.7 Exercise (NL) Consider the wallpaper group $**$. Each $*$ gives a Greek and Latin generator, and since we have two features the product of the Greek generators must be 1.

$$\begin{aligned} P^2 = 1, &\quad \alpha^{-1}P\alpha = P \\ Q^2 = 1, &\quad \beta^{-1}Q\beta = Q \\ \alpha\beta &= 1 \end{aligned}$$

Now, the more complicated relations on the right of the top two lines just say that the Greek generator commutes with its Latin companion. And since the product of the Greek generators is 1, we can eliminate one of them in favour of the other. We arrive at

$$P^2 = Q^2 = 1, \quad P\alpha = \alpha P, \quad Q\alpha = \alpha Q.$$

The generators P, Q correspond to two adjacent mirrors of different sorts and the generator α corresponds to a translation symmetry parallel to the lines of reflection.

Find the 2-colourings and 3-colourings of this. (There are seven 2-colourings but only two 3-colourings.) Give examples in all cases.

7.3.8 Exercise (NM) What 2-colourings and 3-colourings of the full symmetry group $*432$ of the cube are there?

The true blues

7.3.9 Gyrations A gyration, corresponding to n gives rise to a single Greek generator β and the single relation

$$\beta^n = 1$$

7.3.10 Exercise (CM) Consider the group 333. According to the rules, it has three generators satisfying

$$\alpha^3 = \beta^3 = \gamma^3 = \alpha\beta\gamma = 1.$$

Show that there are no 2-colourings of this group and determine the possible 3-colourings, giving examples. Take care to account fully for equivalence under relabelling of the colours.

7.3.11 Handles A handle (i.e. an **o**) gives rise to two one Greek and two Latin generators satisfying

$$X^{-1}Y^{-1}XY = \alpha.$$

In the case of the group **o** with just one handle, $\alpha = 1$ and so we simply have two generators satisfying

$$X^{-1}Y^{-1}XY = 1 \quad \text{or equivalently} \quad XY = YX.$$

The generators X and Y are then just two generators of the lattice of translations.

7.3.12 Exercise (NM) What can you say about 2-colourings and 3-colourings of **o**? Give examples.

Miracles

7.3.13 Miracles A miracle has one Latin and one Greek generator satisfying

$$Z^2 = \delta.$$

In essence, Z is the glide and δ the translation which is its square.

7.3.14 Exercise (NM) Consider $\times\times$. We have generators Y, Z and γ, δ satisfying

$$Y^2 = \gamma, \qquad Z^2 = \delta, \qquad \gamma\delta = 1.$$

We can simplify this: we do not need explicitly to mention γ and δ. The whole thing reduces to a group generated by Y, Z subject only to

$$Y^2 Z^2 = 1.$$

Find the 2-colourings and 3-colourings of this.

7.3.15 Exercise (NM) Write down standard generators and relations for $*\times$ and simplify them. What can you say about 2-colourings and 3-colourings?

Gyroscopic groups

7.3.16 Gyroscopic groups are those with rotational symmetry about the centre of what would otherwise be a simple kaleidoscope. The colourings can be a little confusing!

7.3.17 Exercise (NM) Consider the group $3{*}3$. Following the rules, we have generators and relations as follows.

$$P^2 = Q^2 = 1; \quad (PQ)^3 = 1; \quad \alpha^{-1}P\alpha = Q$$
$$\beta^3 = 1$$
$$\alpha\beta = 1$$

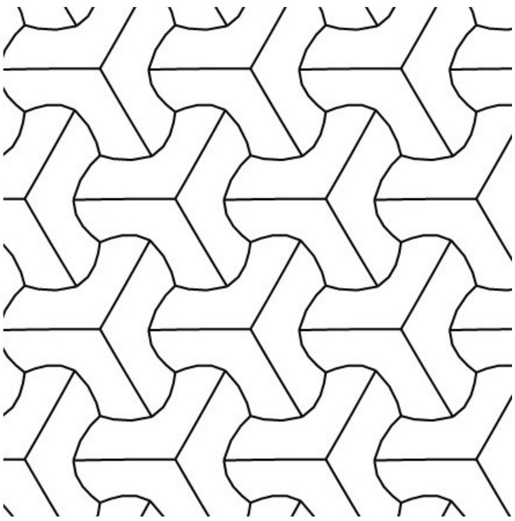
The last two lines imply that $\alpha = \beta^2$ and so we can eliminate α. We arrive at

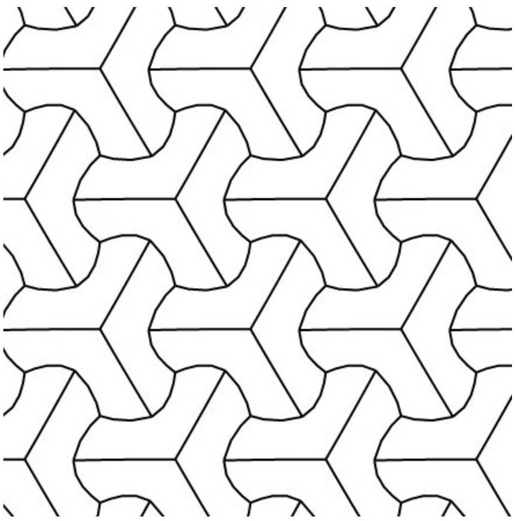
$$P^2 = Q^2 = 1, \quad (PQ)^3 = 1, \quad Q = \beta P \beta^2.$$

Next we note that $Q^2 = 1$ follows from $P^2 = 1$ and the final relation and so it can be dropped. Then we can eliminate Q altogether to arrive at

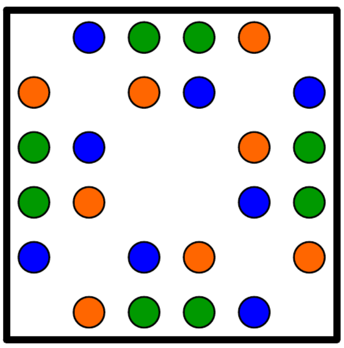
$$P^2 = 1, \quad \beta^3 = 1, \quad (P\beta P\beta^2)^3 = 1.$$

The generator β is a gyration and P is a reflection in a chosen side of the triangle surrounding β. Investigate the 2-colourings and 3-colourings. Draw some pictures!

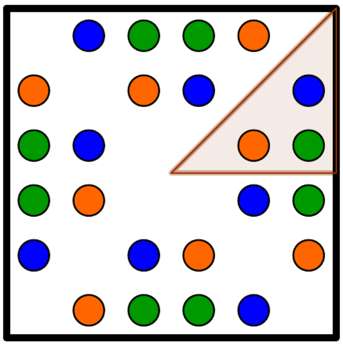




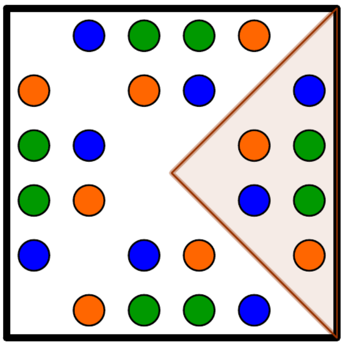
More theory of colourings



7.3.18 Example Just a reminder of the fact that not every part of a pattern needs to be coloured to have an n-colouring, and indeed there may be extra colours in the pattern. The little pattern on the right is a 2-colouring of D_4.



7.3.19 Example Neglecting colours in the pattern, the "full" symmetry group is D_4 and a fundamental domain is the little flagstone on the right. The coloured pattern has symmetry C_4 (the subgroup of rotations). So we have the surjective group homomorphism $c : D_4 \longrightarrow S_2$ with kernel C_4.
Note also that when colouring, you do not have to start by colouring a whole fundamental domain in one colour.



7.3.20 Example The coloured pattern has symmetry group C_4 which has a fundamental domain as in the picture on the right. The fundamental domain is twice the size of that for the uncoloured pattern.

7.3.21 The general case In general, given a colouring $c : G \longrightarrow S_n$ with kernel H (the symmetry group of the coloured pattern) and image $K \leq S_n$, the size of a fundamental domain for H is $\#K$ times that for G.

Furthermore, the orbifold for H can be obtained by "unfolding" the orbifold for G.

Thinking about the proof of the "Magic Theorems" this tells us also that the orbifold Euler characteristic of H must be $\#K$ times that of G. Of course, this only tells us anything in the case of spherical or hyperbolic geometry: in the Euclidean case both Euler characteristics must be zero.

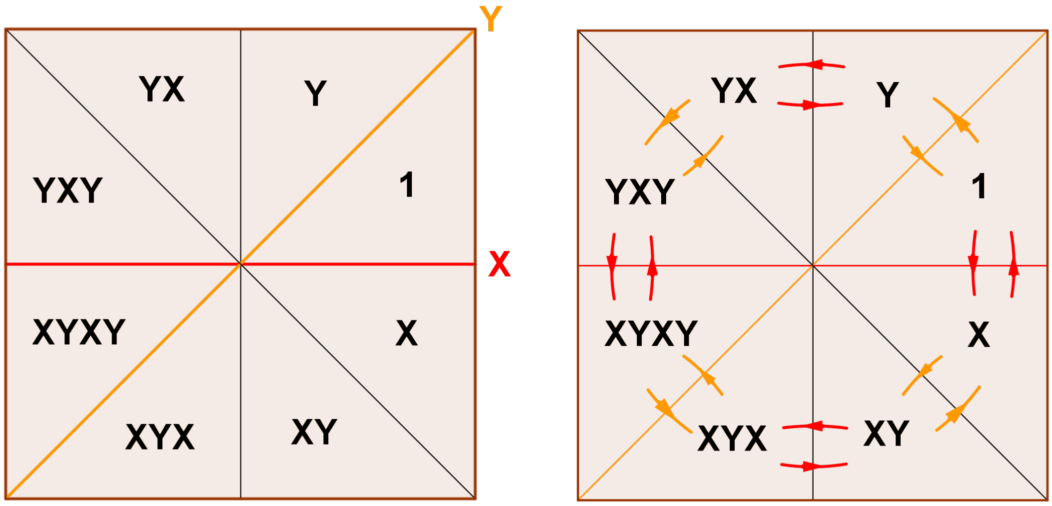
7.3.22 Restrictions on K Finally, note that for a colouring the image of $c : G \longrightarrow S_n$ the image K must be a subgroup of S_n that acts transitively on the n colours. For $n = 2$ the only possibility is $K = S_2$ but for $n = 3$ there are two possibilities: the image may be S_3 or just the subgroup of even permutations. So the orbifold for H may be 6 or 3 times the size of that for G.

7.4 Understanding the presentations

It remains to convince ourselves that our recipes for the presentations of the groups are correct. We will try and do that here.

Dihedral groups revisited

7.4.1 Generators for D_4 In the left-hand figure, two generators X, Y for D_4 have been identified. Once we identify one "reference" fundamental domain as corresponding to $1 \in D_4$, then the other seven corresponding domains can be labelled by the unique group element that takes our reference domain to the given one.



Note that, for instance, the domain labelled YX is mapped by the action of X to the domain labelled $X(YX) = XYX$. Do not get confused by noticing, for example, that Y sends $XYXY$ to XYX: in fact $YXYXY = XYX$ (exercise).

7.4.2 Right multiplication In the right-hand picture, red arrows have been drawn indicating all the reflections conjugate in D_4 to the original X and the same with orange arrows for Y. Notice how the the red arrows correspond to multiplication *on the right* by X and similarly for the orange arrows and Y. (Take care again not to be confused by the fact that there are different ways of writing an element in terms of the generators.)

There are several things to notice about this: starting with a group element like YX, multiplying on the left by a generator often takes that element to one on the far side of the pattern. On the other hand, multiplying on the right by a generator takes you to an adjacent fundamental domain.

Notice that this makes it easy to write down a formula for the group element corresponding to a final domain in terms of the generators: simply find a path along the arrows to the domain and write down the generators you have used in order from left to right.

We sum up the situation in the following proposition.

7.4.3 Proposition Consider a generator $Q \in G$ of a group G. Let $g \in G$. Then

$$(gQg^{-1})g = gQ.$$

That is, multiplying g on the left by the conjugate of Q by g leads to the same result as multiplying g on the right by Q.

7.4.4 Definition In Figure 7.2, we have thrown away the geometric parts of the picture to leave just the *Cayley graph* for this presentation of D_4. This is a directed graph with one vertex for each group element and directed edges that tell us the result of multiplying each group element on the right by each of the generators.

7.4.5 Relations Every closed loop of arrows that one can find in the graph corresponds to a word in the generators that is equal to the identity. So, the trivial loops that occur from

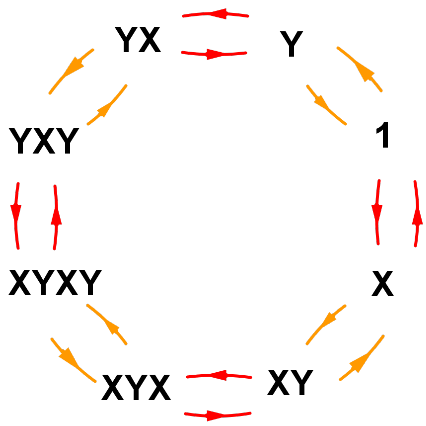


Figure 7.2: The Cayley graph of this presentation of D_4.

following a red or orange edge and then returning immediately by the same coloured loop to our starting point correspond to $X^2 = 1$ and $Y^2 = 1$. And the loop starting at 1 and travelling clockwise round the circle by loops of alternating colour to return to its starting place tells us that $(XY)^4 = XYXYXYXY = 1$.

This picture gives an alternative proof that the three relations above are sufficient to define the group. Consider a loop in the graph corresponding to a word that is equal to 1. We can eliminate any "backtracking" by using $X^2 = Y^2 = 1$ and reduce to a loop that travels k times round the circle. But that reduces the word to some power of $(XY)^4$.

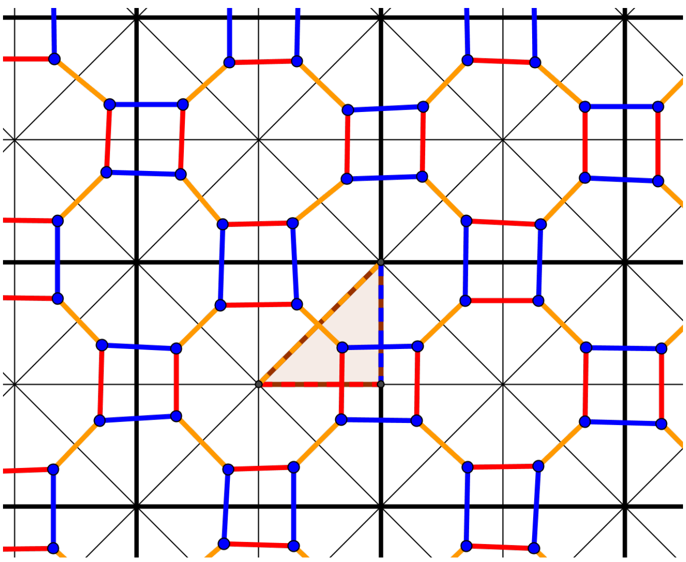


Figure 7.3: generators and relations for $*442$

7.4.6 Presentation of a kaleidoscope Figure 7.3 shows a square grid in black. Its symmetry group is $*442$, a kaleidoscope based on the shaded right-angled triangle near the centre. We take the reference fundamental domain to be the shaded one and the generators to be reflections in the boundary of that region: P (red), Q (gold) and R (blue). All the conjugate reflections have been added in the appropriate colour. Each edge in the picture should be thought of as two directed edges, one in each direction; this simplification is reasonable because reflections are of order two and so they are their own inverse.

Our simplified presentation for $*442$ is that P, Q, R satisfy

$$P^2 = Q^2 = R^2 = (PQ)^4 = (QR)^4 = (RP)^2 = 1.$$

The fact that P, Q, R are order two is implicit in the picture. The remaining three relations correspond exactly to the three different sorts of tiles (two octagonal, one square) in the coloured tiling.

Now consider a closed loop of coloured edges in the graph, corresponding to a word in the generators which equals 1. We can reduce the loop by cutting out one polygon at a time, which corresponds to a simplification of the word using one of the known relations. Thus eventually one can reduce the loop to going round a single coloured tile and hence to nothing: correspondingly, we have shown that the word can be simplified to 1 using just the relations we have stated.

We can check the other kaleidoscopes similarly. The argument works for wallpaper, spherical and hyperbolic groups. The critical feature is all three underlying spaces are simply connected, and so every closed loop in them can be shrunk to a point.

7.4.7 Exercise (NM) Find the smallest translation symmetry of the square lattice of the example. Identify the image of the reference fundamental domain under it. Hence write down the translation in terms of the given generators.

7.4.8 Exercise (NL) Take two other kaleidoscopic groups, at least one of which is not a "wallpaper" example and construct the analogous tiling that proves the relations are complete.

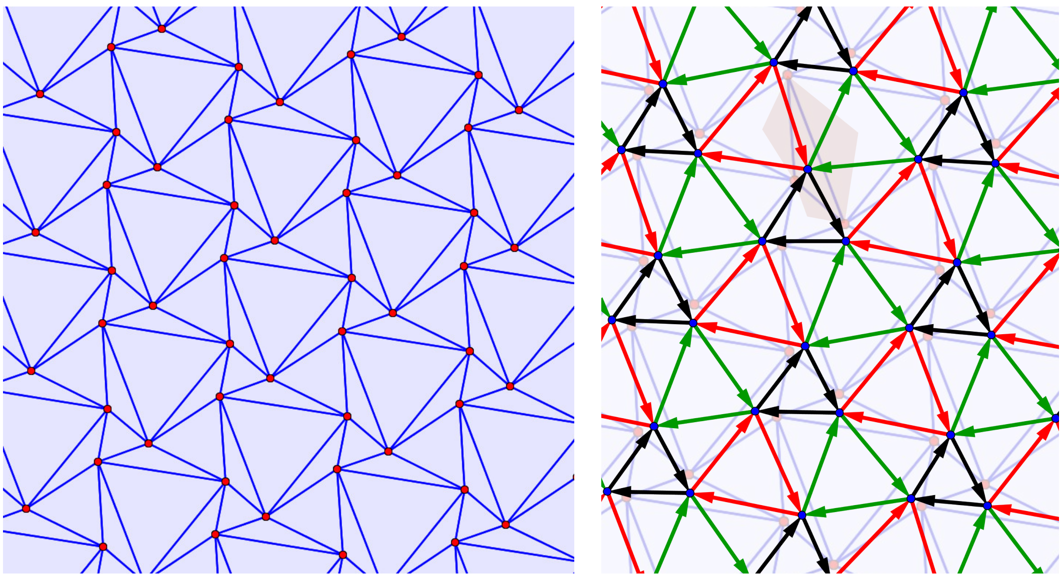


Figure 7.4: Presentation of 333

7.4.9 A gyratory example In Figure 7.4 we see a wallpaper with 333 symmetry. A fundamental domain can be taken to be a whole scalene triangle with one third of each adjacent equilateral one, as marked rather faintly on the right-hand picture.

The three generators α (green), β (red), γ (black) are one-third turns about the centres of large, medium and small equilateral triangles respectively. Travelling "backwards" along an edge corresponds to right multiplication by the inverse (which is also the square) of the corresponding generator.

We see the "tiles" in the graph correspond to the relations

$$\alpha^3 = \beta^3 = \gamma^3 = \alpha\beta\gamma = 1.$$

Thus the picture shows that 333 is generated by these generators with these relations.

7.5 Where do the tilings come from? (Not examinable)

7.5.1 Introduction For the wallpaper and spherical groups at least, one can construct tilings of the sort studied in the previous section and hence establish that the generators and relations completely describe the group and so justify our analysis of colourings.

But one should of course be curious as to where the formulae for writing down the generators and relations comes from. To do that we need to return to thinking about orbifolds.

7.5.2 Way back in §1.3.15 we saw how for the wallpaper group **o**, the orbifold, obtained by folding the orbits, is a torus. The picture in that section shows a torus with two red curves on it. Thinking of the black dot where they intersect as a reference point in a fundamental region, when you travel around one of the closed curves in the orbifold, in the unfolded picture it brings you back to an adjacent fundamental region. And if you go round the loop the other way, it takes you to a different, neighbouring fundamental domain.

In fact, the fundamental domains are in one to one correspondence with pairs of integers (m, n), representing a loop on the torus that travels m times round one of the loops and n times round the other. One loop on the torus can be deformed into another precisely when they have the same values of m and n. Perhaps surprisingly, it does not matter which order you go round the loops: see the video at `https://www.youtube.com/watch?v=nLcr-DWVEto`.

If you have done some topology, you will probably see (or know already) that what we are doing here is considering the "fundamental group" of the torus and the plane is its "universal cover".

7.5.3 The picture in Fig 7.5 shows the two closed curves on the torus (with a chosen direction) "lifted" to the whole plane. The result, denoting the yellow generator by X and the red one by Y is a tiling like the ones in the previous section. We can see that there is just one sort of "tile" which corresponds to the relation

$$XYX^{-1}Y^{-1} = 1.$$

Thus we see that the tiling with relations is something that is lifted from the orbifold to the plane.

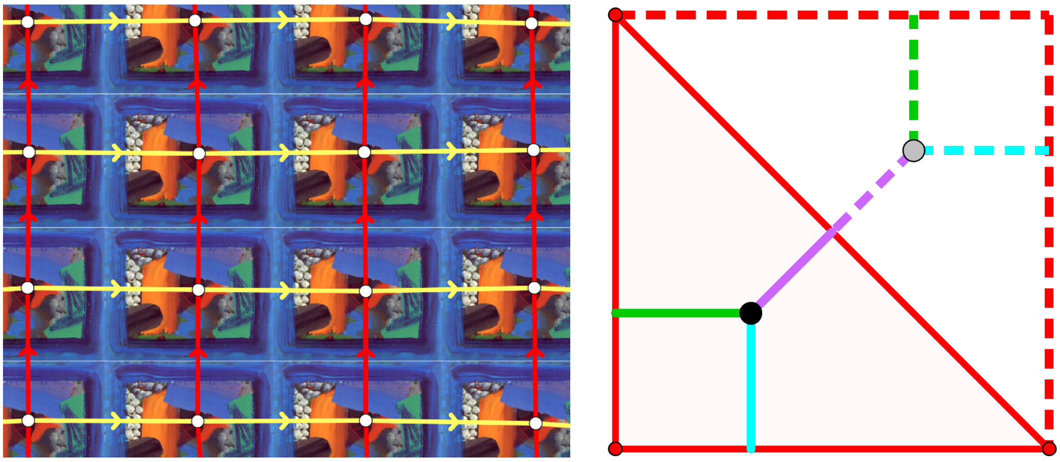


Figure 7.5: Generators for **o** and $*442$

7.5.4 On the right in Fig 7.5 there is a picture of the orbifold for $*442$ with a black dot marking a reference point. The three coloured segments joining the black dots to the sides are curves that one could travel along, bounce off the reflecting boundary and return to the reference point. That curve, lifted to the whole pattern (as shown only for the pink one) carries us to an adjoining fundamental domain. But repeating travel along the curve takes us back where we started. Those little segments therefore give us generators P, Q, R each squaring to the identity — and it is because they square to the identity that we do not need an arrow on them: each is equal to its inverse.

One sees also in the picture the beginnings of the three resulting relational tiles, each alternating between two of the colours. These give us the relations

$$(PQ)^4 = (QR)^4 = (RP)^2 = 1.$$

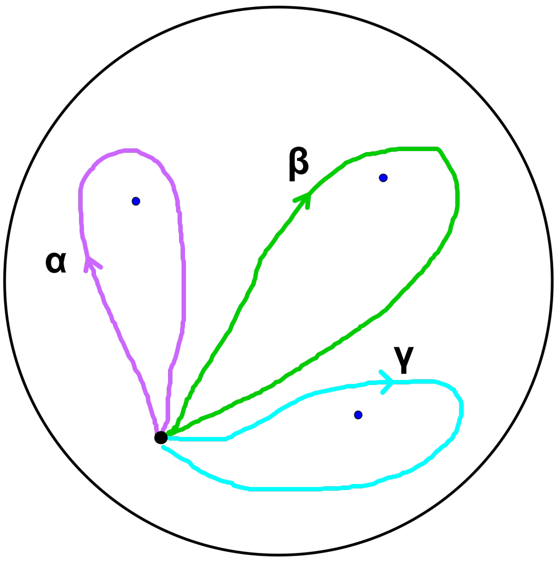
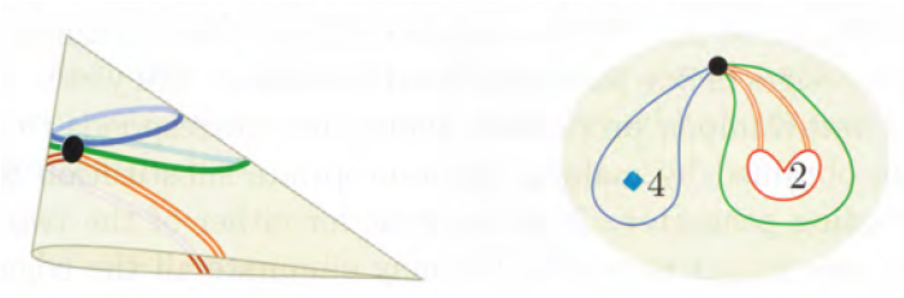


Figure 7.6: Generators for **333**

7.5.5 The Greek generators A Greek generator representing a gyration is clear: in the orbifold it is a curve from the base point winding round the cone point and returning to base. If the gyration has order 3 then we have a Greek generator α with $\alpha^3 = 1$. To reconstruct the pattern, the orbifold has to be "carved open" with cuts originating at cone points so that it can be "laid flat" on the plane.

Supposing now we consider the case of 333. The orbifold is a sphere with three cone points. The picture in the figure is a little symbolic and shows the three generators as loops surrounding one of the three cone points. You have to imagine the picture on the surface of a sphere. When you do that, you see that the curve $\alpha\beta\gamma$ surrounds a portion of the sphere (the "outside" of the three loops) with no symmetry feature inside it. Therefore that loop can be shrunk to nothing and so represents the identity. This accounts for the final relation in the presentation of 333 as

$$\alpha^3 = \beta^3 = \gamma^3 = \alpha\beta\gamma = 1.$$

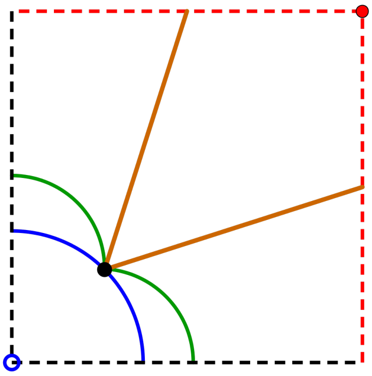


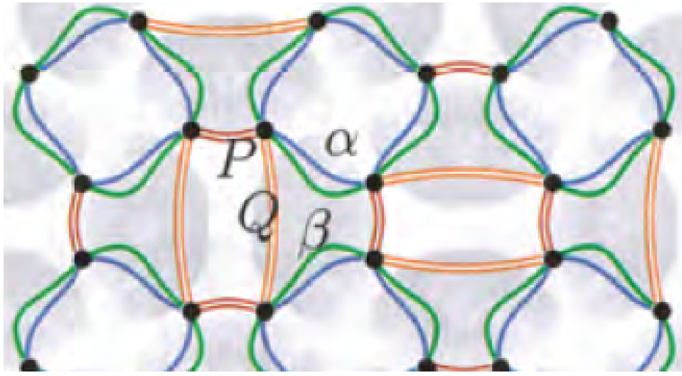
7.5.6 So, finally, the origin of the generators and relations is that one takes the orbifold, thought of as a sphere with added features, chooses a base point and throws a loop around each symmetry feature which one labels as a Greek generator. Within each loop (except in the case of the gyration where the only thing one has is that the correct power of the Greek generator is the identity) one has Latin generators corresponding to generators associated with that feature.

We conclude with an example. In the figure, the orbifold of $4{*}2$ is shown: it is a cone with a reflecting boundary, containing a single corner point. On the right is a symbolic picture of the orbifold where a sphere with a chosen black base point has a blue loop thrown around the cone point and a green one around the kaleidoscope. Suitably oriented, these will become generators α and β with $\alpha\beta = 1$. On the left, the same picture is drawn on the orbifold with the addition of two double red lines leading to the reflecting edge. These represent the reflection generators *P*, *Q*.

7.5.7 To generate the corresponding tiling on a pattern on the plane, the orbifold needs to be carved open by cut running from the cone point to the reflecting boundary, meeting the latter at a point on the boundary between the red generators on the side away from the corner point.

The result is shown on the left in the final figure. The fundamental region is one quarter of a square kaleidoscope — the red dashed edges are the reflecting boundary with the corner point on the top right and the black dashed lines are the cut that carved open the orbifold. The cone point is in the bottom-left corner.





On the right of the last figure, the generators appear on a $4{*}2$ pattern. With orientations consistently chosen one gets precisely the presentation produced by the rules we have been using:

$$P^2 = Q^2 = (PQ)^2 = 1, \quad Q = \alpha^{-1}P\alpha, \quad \beta^4 = 1, \quad \alpha\beta = 1.$$