12/8/18

Lecture 8 : Compact spaces

In a series of lectures we now develop the notion of compactness, first for metric spaces and then more generally for topological spaces. This is an important finiteness condition for spaces, where the closed interval $[a,b]$ counts as "finite" but $\mathbb{R}$ does not. Recall from calculus that the closed interval has various special properties with respect to continuous functions defined on it, for example:

Extreme Value Theorem : if $f : [a,b] \longrightarrow \mathbb{R}$ is continuous then f is bounded and attains its minimum and maximum, i.e. there exist $c, d \in [a,b]$ such that

$$f(c) \geqslant f(x) \geqslant f(d) \qquad \forall x \in [a,b].$$

Uniform Continuity if $f : [a,b] \longrightarrow \mathbb{R}$ is continuous then it is uniformly continuous, i.e. from

$$f \text{ continuous} : \forall x \,\forall \varepsilon > 0 \,\exists \delta > 0 \,\forall y \left(|x-y| < \delta \Rightarrow |fx - fy| < \varepsilon \right)$$

we may deduce

$$f \text{ uniformly cts} : \forall \varepsilon > 0 \,\exists \delta > 0 \,\forall x, y \left(|x-y| < \delta \Rightarrow |fx - fy| < \varepsilon \right).$$

The property of compactness is "responsible" for these and other good properties of the interval, in the sense that these results generalise to any compact subset of a metric space (and in a suitable form, to any compact subspace of a topological space). We study compactness for metric spaces first, but the deep theorems will use only the topology.

Recall the following definitions from real analysis:

<u>Defn</u> A subset $X \subseteq \mathbb{R}$ is <u>bounded</u> if $X \subseteq [-M, M]$ for some $M > 0$.

<u>Defn</u> Let $X \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. We call x an <u>adherent point</u> of X if either of the following two equivalent conditions are met:

(i) there is a sequence $(a_n)_{n=0}^{\infty}$, with $a_n \in X$ for all n, converging to x.

(ii) $\forall \varepsilon > 0 \; \exists y \in X (\, |x-y| < \varepsilon \,)$.

The set X is <u>closed</u> if it contains all its adherent points.

<u>Lemma L8-1</u> X is closed in this sense iff. it is closed in the metric topology on $\mathbb{R}$.

<u>Proof</u> Suppose X is closed in the metric topology, and that $x \in \mathbb{R}$ is an adherent point of X. We have to show $x \in X$. Suppose not. Since $\mathbb{R} \setminus X$ is open, there is a ball $x \in B_\varepsilon(x) \subseteq \mathbb{R} \setminus X$. But by (ii) above, there exists $y \in X$ with $|x-y| < \varepsilon$ and thus $y \in B_\varepsilon(x)$. But this is a contradiction.

If X is closed in the above sense and $x \notin X$ then x is <u>not</u> an adherent point, so $\exists \varepsilon > 0 \; \forall y \in X (\, |x-y| \geq \varepsilon \,)$, which says $\exists \varepsilon > 0 \; B_\varepsilon(x) \subseteq \mathbb{R} \setminus X$, which shows $\mathbb{R} \setminus X$ is open. $\square$

<u>Theorem</u> (Heine-Borel) A subset $X \subseteq \mathbb{R}$ is closed and bounded if and only if every sequence $(a_n)_{n=0}^{\infty}$ in X contains a subsequence which converges to some element of X.