Mathematics 3 **Honours Analysis**

Notes on Riemann Integration

1. **Introduction.** We are going to see how to define the **integral** of certain functions $f : \mathbb{R} \to \mathbb{R}$ in such a way that it clearly represents the notion of "area under the curve". So this is more like a **definite** than an indefinite integral, and as such the integral of f – which we will denote by $\int f$ – will be a real number rather than a function. We will try to incorporate certain desirable features into our definition. These include **linearity** (i.e. $\int(\alpha f + \beta g) = \alpha \int f + \beta \int g$ for real numbers α and β)and **positivity** (i.e. if $f \geq 0$ then $\int f \geq 0$). We will calculate, from first principles, the integrals of certain functions, and then show that the usual rules for calculating integrals – recognition of anti-derivatives, product rule, substitution etc. all hold in a rigorous way. Thus our usual techniques for calculating integrals will be validated. We shall then see under what circumstances the order of integration and summation can be changed.

2. **Step functions and their integrals.** We need some notation. Recall that the bounded intervals in $\mathbb{R}$ have the form $[a, b], [a, b), (a, b]$ or (a, b). Here, $a < b$ (and in the case of closed intervals we admit the possibility $[a, a] = \{a\}$ too). If $E \subseteq \mathbb{R}$, we define its characteristic function $\chi_E : \mathbb{R} \to \mathbb{R}$ by $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. Let I be a bounded interval. According to our principles, there is only one way to define the integral of χ_I in such a way as it represents that "area under the curve":

$$\int \chi_I := \text{length (I)}.$$

If I is a bounded interval we shall write $|I|$ as a shorthand for its length, i.e. $|I| = b - a$ if $I = [a, b], [a, b), (a, b]$ or (a, b).

We wish to extend this definition to a wider class of functions.

Definition 1. We say that $\phi : \mathbb{R} \to \mathbb{R}$ is a **step function** if there exist real numbers $x_0 < x_1 < \ldots < x_n$ (for some $n \in \mathbb{N}$) such that

(i) $\phi(x) = 0$ for $x < x_0$ and $x > x_n$

(ii) ϕ is constant on (x_{j-1}, x_j) $1 \leq j \leq n$.

We shall use the phrase "ϕ is a *step function with respect to* $\{x_0, x_1, \ldots, x_n\}$" to describe this situation.

Comment. Why do we use the terminology "step function"?

In other words, ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$ iff there exist $c_0, c_1, \ldots, c_n$ such that

$$\phi(x) = \sum_{j=1}^{n} c_j \chi_{(x_{j-1}, x_j)}(x)$$

for $x \neq x_0, x_1, \ldots x_n$. Notice that we make no assumption on the values $\phi(x_j)$, $0 \leq j \leq n$. Notice also that if ϕ is a step function, then ϕ is a bounded function, and that there exists a bounded interval I such that $\phi(x) = 0$ for $x \notin I$. (That is, ϕ has *bounded suport.*) Finally, notice that ϕ is continuous at all points of $\mathbb{R}$ except possibly $\{x_0, x_1, \ldots, x_n\}$, members of which are therefore called **potential jump points** of ϕ. (They may or may not be *actual* jump points.)

Exercise. Show that the class of step functions is a vector space (i.e. that if ϕ and ψ are step functions and α and β are real numbers, then $\alpha\phi + \beta\psi$ is a step function), and that if ϕ and ψ are step functions, then $\max\{\phi, \psi\}, \min\{\phi, \psi\}, |\phi|$ and $\phi\psi$ are also step functions. (**Hint:** If ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$ and ψ is a step function with respect to $\{y_0, y_1, \ldots, y_m\}$ consider $\{x_0, x_1, \ldots, x_n\} \cup \{y_0, y_1, \ldots, y_m\}$.)

Exercise. Show that ϕ is a step function if and only if is of the form

$$\phi(x) = \sum_{j=1}^{n} c_j \chi_{I_j}$$

where each I_j is a bounded interval.

There is only one way of defining the integral of a step function which is consistent with the notion of area under the curve:

Definition 2. If ϕ is a step function with respect to $\{x_0, x_1, \ldots, x_n\}$ which takes the value c_j on (x_{j-1}, x_j), then

$$\int \phi := \sum_{j=1}^{n} c_j(x_j - x_{j-1}).$$

Notice that the values $\{\phi(x_j)\}$ have no effect on the value of $\int \phi$, as one would expect.

So far so good. But before we go any further, there is some housekeeping to be done. Indeed, it's quite possible that a step function ϕ can be a step function with respect to $\{x_0, x_1, \ldots, x_n\}$ *and* also with repect to $\{y_0, y_1, \ldots, y_m\}$. Do we get the same answer for $\int \phi$ calculated both ways?

Exercise. Write the step function $\phi = \chi_{[0,1]}$ in two different ways according to Definition 1. Calculate the two possible candidate values for $\int \phi$ arising from Definition 2.

To guarantee the well-definedness of $\int \phi$, note first that if ϕ is a step function with respect to $\{x_0, \ldots, x_n\}$, and if $x_{j-1} < \xi < x_j$, then ϕ is also a step function with respect to $\{x_0, \ldots, x_{j-1}, \xi, x_j, \ldots, x_n\}$ and the two definitions of $\int \phi$ agree. (Why?) Thus if ϕ is a step function with respect to $\{x_0, \ldots, x_n\}$ and also with respect to $\{y_0, \ldots, y_m\}$, upon ordering $\{x_0, \ldots, x_n\} \cup \{y_0, \ldots, y_m\}$ as $z_0 < \ldots < z_k$ (with $k \leq m + n$) we see that the definitions of $\int \phi$ with respect to $\{x_0, \ldots, x_n\}$ and $\{z_0, \ldots, z_k\}$ and $\{y_0, \ldots, y_m\}$ all agree.

Thus for a step function ϕ, the value of $\int \phi$ is independent of the particular set $\{x_0, \ldots, x_n\}$ with respect to which it is a step function; hence Definition 2 is unambiguous.

Proposition 1. If ϕ and ψ are step functions and α and $\beta \in \mathbb{R}$, then

$$\int (\alpha\phi + \beta\psi) = \alpha \int \phi + \beta \int \psi.$$

Proof. List all the potential jump points of either ϕ or ψ together as $\{x_0 < \ldots < x_n\}$. Suppose $\phi(x) = \sum_{j=1}^{n} c_j \chi_{(x_{j-1}, x_j)}(x)$ and $\psi(x) = \sum_{j=1}^{n} d_j \chi_{(x_{j-1}, x_j)}(x)$ for $x \neq x_0, x_1, \ldots x_n$. Then the left hand side is $\sum_{j=1}^{n}(\alpha c_j + \beta d_j)(x_j - x_{j-1}) = \alpha \sum_{j=1}^{n} c_j(x_j - x_{j-1}) + \beta \sum_{j=1}^{n} d_j(x_j - x_{j-1}) = \alpha \int \phi + \beta \int \psi$. ■

Exercise. Show that if $\phi = \sum_{j=1}^n c_j \chi_{I_j}$ where each I_j is a bounded interval, then $\int \phi = \sum_{j=1}^n c_j |I_j|$.

Exercise. If ϕ and ψ are step functions and $\phi \geq \psi$, (meaning $\phi(x) \geq \psi(x)$ for all x), show that $\int \phi \geq \int \psi$.

3. **Riemann-integrable functions and their integrals.**

Definition 3. Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is **Riemann-integrable** if for every $\epsilon > 0$ there exist step functions ϕ and ψ such that

$$\phi \leq f \leq \psi$$

and

$$\int \psi - \int \phi < \epsilon.$$

Comment. Note that $\psi - \phi \geq 0$ and is a step function, hence its integral is nonnegative.

Remark. Step functions are Riemann-integrable. (Why?)

Exercise. Show that if f is Riemann-integrable, then f is bounded and has bounded support. If $f(x) = 0$ for $x \notin [a, b]$, show that we may take ϕ and ψ to be zero outside $[a, b]$.

Exercise. Show that neither the function $f(x) = e^{-|x|}$ nor the function $g(x) = x^{-1/2}\chi_{(0,1)}(x)$ is Riemann-integrable.

Exercise. Can you find a countably infinite set $E \subseteq \mathbb{R}$ such that χ_E is Riemann-integrable? Is χ_E Riemann-integrable for every bounded countably infinite set $E \subseteq \mathbb{R}$?

Theorem 1. A function $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable if and only if

$$\sup\{\int \phi \,:\, \phi \text{ is a step function and } \phi \leq f\} = \inf\{\int \psi \,:\, \psi \text{ is a step function and } \psi \geq f\}.$$

Definition 4. If f is Riemann-integrable we define its integral $\int f$ as the common value

$$\int f := \sup\{\int \phi \,:\, \phi \text{ is a step function and } \phi \leq f\} = \inf\{\int \psi \,:\, \psi \text{ is a step function and } \psi \geq f\}.$$

Remark. If f is a step function, this definition agrees with the one previously given. (Why?)

Proof of Theorem 1. Suppose first that f is Riemann-integrable. Then, given $\epsilon > 0$, there exist step functions ϕ_0 and ψ_0 such that $\phi_0 \leq f \leq \psi_0$ and $\int \psi_0 - \int \phi_0 < \epsilon$. Consider the sets of real numbers

$$A = \{\int \phi \,:\, \phi \text{ is a step function and } \phi \leq f\}$$

and

$$B = \{\int \psi \,:\, \psi \text{ is a step function and } \psi \geq f\}.$$

Then A is a nonempty subset of $\mathbb{R}$ (since $\int \phi_0$ belongs to it) which is bounded above by $\int \psi_0$ or indeed any member of B. So it has a least upper bound U which satisfies $\int \phi_0 \leq U \leq \int \psi_0$. Similarly, the greatest lower bound L of B satisfies $\int \psi_0 \geq L \geq \int \phi_0$, and moreover $U \leq L$. Hence $0 \leq L - U < \epsilon$. Since this is true for arbitrary $\epsilon > 0$ we deduce that $U = L$, i.e. $\sup A = \inf B$.

Now suppose that $\sup A = \inf B := I$. By the approximation property of sup and inf, given any $\epsilon > 0$ there exist step functions ϕ and ψ with $\phi \leq f$ and $\psi \geq f$ and $\int \phi > I - \epsilon/2$, and $\int \psi < I + \epsilon/2$. Hence $\int \psi - \int \phi < \epsilon$. ■

Note the connection in the above proof with Problem 1 of the first revision lecture.

Theorem 2. A function $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable if and only if there exist sequences of step functions ϕ_n and ψ_n such that

$$\phi_n \leq f \leq \psi_n \text{ for all } n, \text{ and } \int \psi_n - \int \phi_n \to 0. \tag{1}$$

If ϕ_n and ψ_n are any sequences of step functions satisfying (1), then

$$\int \phi_n \to \int f \text{ and } \int \psi_n \to \int f$$

as $n \to \infty$.

Proof. Supose first that f is Riemann-integrable. Then, taking $\epsilon = 1/n$ in the definition, there exist step functions ϕ_n and ψ_n such that $\phi_n \leq f \leq \psi_n$ and $\int \psi_n - \int \phi_n < 1/n$. Hence $\int \psi_n - \int \phi_n \to 0$.

Now suppose that there exist ϕ_n and ψ_n as in the statement of the theorem. Given $\epsilon > 0$ choose N such that $\int \psi_n - \int \phi_n < \epsilon$ for all $n \geq N$; then ϕ_N and ψ_N do the job.

Finally, in such a case, by the definition of $\int f$ (in Definition 4) we have $\int \phi_n \leq \int f \leq \int \psi_n$, and so $|\int \phi_n - \int f| = \int f - \int \phi_n \leq \int \psi_n - \int \phi_n \to 0$. Similary $\int \psi_n \to \int f$. ■

Example. Let $f(x) = x$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise. Show that f is Riemann-integrable and calculate $\int f$ (using either the definition or Theorem 2).

Example. Let $f(x) = x^2$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise. Show that f is Riemann-integrable and calculate $\int f$ (using either the definition or Theorem 2).

Example. Prove, by induction on n, that for any integer $m \geq 1$ we have

$$\frac{n^{m+1}}{m+1} \leq \sum_{j=1}^{n} j^m \leq \frac{(n+1)^{m+1}}{m+1}.$$

Let $f(x) = x^m$ for $0 \leq x \leq 1$ and $f(x) = 0$ otherwise. Calculate the value of $\int f$.

The following is a very useful criterion for Riemann-integrability:

Lemma 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function with bounded support $[a, b]$. The following are equivalent:

(i) f is Riemmann-integrable.

(ii) for every $\epsilon > 0$ there exist $a = x_0 < \ldots < x_n = b$ such that, if M_j and m_j denote the supremum and infimimum values of f on $[x_{j-1}, x_j]$ respectively, then

$$\sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon.$$

(iii) for every $\epsilon > 0$ there exist $a = x_0 < \ldots < x_n = b$ such that, with $I_j = (x_{j-1}, x_j)$ for $j \geq 1$,

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)||I_j| < \epsilon.$$

For the purposes of this lemma it's convenient to introduce some notation. For $f : \mathbb{R} \to \mathbb{R}$ a bounded function with bounded support $[a, b]$, and for $a = x_0 < \ldots < x_n = b$ we let $I_j = (x_{j-1}, x_j)$, $m_j := \inf_{x \in I_j} f(x)$ and $M_j := \sup_{x \in I_j} f(x)$. We define the *lower step function of f with respect to* $\{x_0, \ldots, x_n\}$ as

$$\phi_*(x) = \sum_{j=1}^{n} m_j \chi_{I_j} + \sum_{j=0}^{n} f(x_j)\chi_{\{x_j\}}$$

and the *upper step function of f with respect to* $\{x_0, \ldots, x_n\}$ as

$$\phi^*(x) = \sum_{j=1}^{n} M_j \chi_{I_j} + \sum_{j=0}^{n} f(x_j)\chi_{\{x_j\}}.$$

Notice that ϕ_* and ϕ^* are step functions, and that $\phi_* \leq f \leq \phi^*$.

Proof.

(i) implies (ii). Suppose first that f is Riemann-integrable, let $\epsilon > 0$ and let ϕ and ψ be step functions as in the definition of Riemann-integrability, with $\int \psi - \int \phi < \epsilon$. We may assume that ϕ and ψ are zero outside $[a, b]$. Enumerate the potential jump points of ϕ and ψ together as $a = x_0 < \ldots < x_n = b$. Let ϕ_* and ϕ^* be the lower and upper step functions of f with respect to $\{x_0, \ldots, x_n\}$. Then

$$\phi \leq \phi_* \leq f \leq \phi^* \leq \psi \text{ and } \int \phi^* - \int \phi_* \leq \int \psi - \int \phi < \epsilon.$$

But $\int \phi^* - \int \phi_* = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1})$, so (ii) holds.

(ii) implies (iii). Since for a nonempty bounded subset $A \subseteq \mathbb{R}$ we have $\sup\{|a - b| : a, b \in A\} = \sup A - \inf A$, it follows that $\sup_{x,y \in I_j} |f(x) - f(y)| = M_j - m_j$. So, assuming (ii),

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)||I_j| = \int \sum_j M_j \chi_{I_j} - \int \sum_j m_j \chi_{I_j} = \int \phi^* - \int \phi_* < \epsilon.$$

(iii) implies (i). If there exist $a = x_0 < \ldots < x_n = b$ such that

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)||I_j| < \epsilon$$

holds, then the lower and upper step functions ϕ_* and ϕ^* of f with respect to $\{x_0, \ldots, x_n\}$ verify the definition of Riemann-integrability of f. ■

The main general properties of the Riemann integral are summarised in:

Theorem 3. Suppose f and g are Riemann-integrable and α and β are real numbers. Then

(a) $\alpha f + \beta g$ is Riemann-integrable and

$$\int (\alpha f + \beta g) = \alpha \int f + \beta \int g$$

(b) If $f \geq 0$ then $\int f \geq 0$; if $f \leq g$ then $\int f \leq \int g$.

(c) $|f|$ is Riemann-integrable and $|\int f| \leq \int |f|$

(d) $\max\{f, g\}$ and $\min\{f, g\}$ are Riemann-integrable.

(e) fg is Riemann-integrable.

Proof. By Theorem 2 there are sequences $\phi_n, \psi_n, \varphi_n$ and θ_n of step functions such that

$$\phi_n \leq f \leq \psi_n \text{ and } \varphi_n \leq g \leq \theta_n$$

and

$$\int \phi_n, \int \psi_n \to \int f \text{ and } \int \varphi_n, \int \theta_n \to \int g.$$

(a) Let us first prove it for α and $\beta \geq 0$. Then

$$\alpha\phi_n + \beta\varphi_n \leq \alpha f + \beta g \leq \alpha\psi_n + \beta\theta_n$$

and both $\int(\alpha\phi_n + \beta\varphi_n)$ and $\int(\alpha\psi_n + \beta\theta_n)$ converge to $\alpha \int f + \beta \int g$. In particular, the difference $\int(\alpha\psi_n + \beta\theta_n) - \int(\alpha\phi_n + \beta\varphi_n) \to 0$. By Theorem 2, $\alpha f + \beta g$ is Riemann-integrable and $\int(\alpha f + \beta g) = \alpha \int f + \beta \int g$.

Now let us prove it for $\alpha = -1$ and $\beta = 0$. In this case we have $-\psi_n \leq -f \leq -\phi_n$ and $\int(-\psi_n)$ and $\int(-\phi_n)$ both converge to $-\int f$. So $-f$ is Riemann-integrable and $\int -f = -\int f$.

Combining these two cases gives the general case. (Why?)

(b) If $f \geq 0$ then each $\psi_n \geq 0$, so that each $\int \psi_n \geq 0$ and hence $\lim \int \psi_n = \int f \geq 0$. For the second part apply what's just been proved to $g - f$.

(c) If f is Riemann-integrable, then, by Lemma 1, so is $|f|$ since by the triangle inequality we have $||f(x)| - |f(y)|| \leq |f(x) - f(y)|$. Now use part (b). (Why is this enough?)

(d) $\max\{f, 0\} = (f + |f|)/2$, then use (a) and (c), while $\min\{f, 0\} = (f - |f|)/2$ likewise; then $\max\{f, g\} = \max\{f - g, 0\} + g$ – use the firts part of this item, and part (a) – and $\min\{f, g\} = -\max\{-f, -g\}$ – use the result for max and part (a).

(e) Let us first see that f^2 is Riemann-integrable. Since f is bounded there is an M such that $|f(x)| \leq M$ for all x. We have $|f(x)^2 - f(y)^2| = |f(x) - f(y)||f(x) + f(y)| \leq 2M|f(x) - f(y)|$. Using Lemma 1 we see that f Riemann-integrable implies f^2 Riemann-integrable. (Why?) Finally, $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$. Now use part (a) and what we have just proved about squares. ■

Exercise. If $|f|$ is Riemann-integrable, need f be?

Harder Exercise. Show that if $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous and $\Phi(0) = 0$, and if $f : \mathbb{R} \to \mathbb{R}$ is Riemann-integrable, then so is $\Phi \circ f$. [See Rudin's Principles of Mathematical Analysis, Theorem 6.11.] Why is the condition $\Phi(0) = 0$ needed?

For the next result we'll need to use the fact that a continuous function $g : [a, b] \to \mathbb{R}$ is in fact *uniformly continuous.*

Theorem 4. If $g : [a, b] \to \mathbb{R}$ is continuous, and f is defined by $f(x) = g(x)$ for $a \leq x \leq b$, $f(x) = 0$ for $x \notin [a, b]$, then f is Riemann-integrable.

Proof. Note that f is bounded (by the extreme value theorem) and has bounded support. Let $\epsilon > 0$. Uniform continuity of f on $[a,b]$ tells us that there is a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon/(b-a)$. Choose $a = x_0 < \ldots < x_n = b$ such that $x_i - x_{i-1} < \delta$. Then for all x and y in (x_{i-1}, x_i) we have $|f(x) - f(y)| < \epsilon/(b-a)$. Turning to Lemma 1, we see that

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)||I_j| \leq \sum_{j=1}^{n} \frac{\epsilon}{(b-a)}|I_j| \leq \epsilon,$$

and so by Lemma 1, f is Riemann-integrable. ■

Suppose that $f : I \to \mathbb{R}$ where I is a bounded interval of the form $[a,b]$, $[a,b)$, $(a,b]$ or (a,b). Define $\tilde{f}(x) = f(x)$ for $x \in I$ and $f(x) = 0$ for $x \notin I$. If $\tilde{f}$ is Riemann-integrable we say that f is **Riemann-integrable on** I and define the **definite integral of** f **on** I as

$$\int_I f = \int_a^b f = \int_a^b f(x)\mathrm{d}x := \int \tilde{f}.$$

Note that if α is a constant then $\int_a^b \alpha = \alpha(b-a)$.

So f is Riemann-integrable on $[a,b]$ if and only if there exist sequences of step functions $\phi_n \leq f \leq \psi_n$ with ϕ_n and ψ_n zero off $[a,b]$, and with $\int \phi_n$ and $\int \psi_n$ converging to the common value $\int_I f$, if and only if f is bounded and for every $\epsilon > 0$ there exist $a = x_0 < \ldots < x_n = b$ such that, with $I_j = (x_{j-1}, x_j)$ for $j \geq 1$,

$$\sum_{j=1}^{n} \sup_{x,y \in I_j} |f(x) - f(y)||I_j| < \epsilon.$$

Moreover Theorem 4 tells us that if $g : [a,b] \to \mathbb{R}$ is continuous, then g is Riemann-integrable on $[a,b]$.

If f is Riemann-integrable on $[a,c]$ and $a < b < c$ then f is Riemann-integrable on $[a,b]$ and on $[b,c]$, and we have

$$\int_a^c f = \int_a^b f + \int_b^c f.$$

(Why?) Consequently piecewise continuous functions of bounded support are Riemann-integrable.

When $b < a$ we adopt the convenient convention that $\int_a^b f = -\int_b^a f$.

4. **Fundamental Theorem of Calculus and Practical Integration**

Let $g : [a,b] \to \mathbb{R}$ be Riemann-integrable. For $a \leq x \leq b$ define

$$G(x) = \int_a^x g.$$

One version of the fundamental theorem of calculus tells us that, at points of continuity of g, G is differentiable on (a,b) and its derivative is $g(x)$.

Theorem 5. Let $g : [a,b] \to \mathbb{R}$ be Riemann-integrable. For $a \leq x \leq b$ let $G(x) = \int_a^x g$. Suppose g is continuous at x for some $x \in [a,b]$. [If x is an endpoint we mean one-sided continuous.] Then G is differentiable at x and $G'(x) = g(x)$. [If x is an endpoint we mean one-sided differentiable.]

Proof. We will deal with the case $a < x < b$. Let $h > 0$ be sufficiently small so that $x+h < b$ and consider $|\dfrac{G(x+h) - G(x)}{h} - g(x)|$. (The argument for $h < 0$ is similar.) This quantity equals

$$|\frac{1}{h}\int_x^{x+h} [g(t) - g(x)]\mathrm{d}t| \le \frac{1}{h}\int_x^{x+h} |g(t) - g(x)|\mathrm{d}t$$

by Theorem 3 (b). Now, as g is continuous at x, if $\epsilon > 0$, there exists a $\delta > 0$ such that if $x < t < x + h$ and $h < \delta$, then $|g(t) - g(x)| < \epsilon$. So for such h,

$$\frac{1}{h}\int_x^{x+h} |g(t) - g(x)|\mathrm{d}t \le \epsilon$$

by Theorem 3 (b) once again. Thus $h < \delta$ implies $\left|\dfrac{G(x+h) - G(x)}{h} - g(x)\right| < \epsilon$, and so $G'(x)$ exists and equals $g(x)$. ■

The other version of the Fundamental Theorem of Calculus tells us that we can integrate derivatives to get back to where we started:

Theorem 6. Suppose $f : [a, b] \to \mathbb{R}$ has continuous derivative f' on $[a, b]$. Then

$$\int_a^b f' = f(b) - f(a).$$

Proof. Let $G(x) = \int_a^x f'$. Then by Theorem 5, $G'(x)$ exists for all x in (a, b) and $G'(x) = f'(x)$. Thus $G - f$, being continuous on $[a, b]$ and differentiable on (a, b), must be constant on $[a, b]$ by Rolle's theorem. So $\int_a^b f' = G(b) = G(a) + f(b) - f(a) = f(b) - f(a)$. ■

The three main elementary methods for evaluating integrals are (i) recognising an antiderivative, (ii) integration by parts, and (iii) substitution. Now that we have the fundamental theorem of calculus on a rigorous footing, these methods become rigorous too as they consist of applications of the fundamental theorm of calculus either (i) alone or (ii) together with the product rule for differentiation or (iii) together with the chain rule for differentiation.

Exercise. Give clear statements of and arguments for results justifiying the methods of integration by parts and by substitution.

Exercise. Carry out the following giving full justifications:

(i) By recognising an antiderivative, calculate

$$\int_0^1 14(1 + t^2)\mathrm{d}t.$$

(ii) By using a substitution, check that

$$\int_0^1 \frac{\mathrm{d}t}{1 + t^2} = \frac{\pi}{4}.$$

(iii) Using integration by parts, calculate

$$\int_0^1 te^{-t}\mathrm{d}t.$$

Exercise. Define $L(x) = \int_1^x \frac{\mathrm{d}t}{t}$ for $x > 0$. Show that $L'(x) = 1/x$. Show that $E \circ L$ is the identity on $\{x : x > 0\}$ and that $L \circ E$ is the identity on $\mathbb{R}$. [E is the exponential function introduced in the Notes on Power Series.]

5. **Integrals and uniform limits of sequences and series of functions.**

Theorem 7. Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of Riemann-integrable functions which converges uniformly to a function f. Suppose that f_n and f are zero outside some common interval $[a, b]$. Then f is Riemann-integrable and

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof. Let $\epsilon > 0$. There is an N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$. For each such n there exist step functions ϕ_n and ψ_n such that

$$\phi_n \leq f_n \leq \psi_n \text{ and } \int \psi_n - \int \phi_n < 1/n.$$

Since

$$f_n(x) - \epsilon < f(x) < f_n(x) + \epsilon \tag{2}$$

we have

$$\phi_n(x) - \epsilon < f(x) < \psi_n(x) + \epsilon$$

for $x \in [a, b]$ and $n \geq N$. Let $\varphi_n = \phi_n - \epsilon\chi_{[a,b]}$ and $\theta_n = \psi_n + \epsilon\chi_{[a,b]}$. Then φ_n and θ_n are step functions and

$$\varphi_n \leq f \leq \theta_n \text{ and } \int \theta_n - \int \varphi_n = \int \psi_n - \int \phi_n + 2\epsilon(b-a) < \frac{1}{n} + 2\epsilon(b-a).$$

So if $n \geq \max\{N, \epsilon^{-1}\}$,

$$\int \theta_n - \int \varphi_n < (1 + 2(b-a))\epsilon.$$

Hence f is Riemann-integrable. By integrating (2) we have, for $n \geq N$,

$$\int f_n - \epsilon(b-a) \leq \int f \leq \int f_n + \epsilon(b-a)$$

that is,

$$|\int f_n - \int f| \leq \epsilon(b-a)$$

which shows that $\lim_{n\to\infty} \int f_n = \int f$. ■

Corollary. Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of Riemann-integrable functions such that $\sum_n f_n$ converges uniformly to a function f. Suppose that f_n and f are zero outside some common interval $[a, b]$. Then $f = \sum_n f_n$ is Riemann-integrable and

$$\int \sum_n f_n = \sum_n \int f_n.$$

6. **A couple of odds and ends.**

(i) Improper integrals. We'd like to be able to say that the function $f(x) = \dfrac{1}{1+x^2}$ is integrable on $\mathbb{R}$, or that the function $g(x) = e^{-x}$ is integrable on $(0,\infty)$, but as they do not have bounded support, they lie outside the classical Riemann theory. Likewise, the function $h(x) = x^{-1/2}$ is not integrable on $(0,1)$. All this is in spite of the fact that we have

$$\int_{-R}^{R} f \leq \pi \text{ for all } R > 0$$

and

$$\int_{0}^{R} g \leq 1 \text{ for all } R > 0$$

and

$$\int_{r}^{1} h \leq 2 \text{ for all } 0 < r \leq 1.$$

There are some boundaries on what we can hope to do meaningfully. For example if we let $H(x) = 1/x^3$ on $(-1,1)$ we have that $\int_{-1}^{-r} H + \int_{r}^{1} H = 0$ for all $0 < r < 1$ but it's doubtful whether we want to atttribute a number to $\int H$. (Why?)

Given a possibly unbounded function $f : \mathbb{R}$ to $\mathbb{R}$, consider the truncated or chopped-off versions of f defined by

$$f_n(x) = \mathrm{mid}\{-n, f(x), n\}\chi_{[-n,n]}(x).$$

Also consider

$$F_n(x) = \min\{|f(x)|, n\}\chi_{[-n,n]}(x).$$

If we have $\sup_n \int F_n < \infty$, but not otherwise, we define the **improper integral** of f over an interval I as

$$\int_I f := \lim_{n\to\infty} \int_I f_n.$$

Exercise. Show that under the hypothesis $\sup_n \int F_n < \infty$ the limit in this definition exists.

(ii) Integral test. Suppose $(a_n)_{n=1}^{\infty}$ is a non-negative sequence of numbers and $f : [1,\infty) \to (0,\infty)$ is a function such that

(i) $\int_1^n f \leq K$ for some K and all n, and

(ii) $a_n \leq f(x)$ for $n \leq x < n+1$.

Then $\sum_n a_n$ converges to a real number which is at most K.

The reason for this is simply that $\phi := \sum_{k=1}^{n} a_k \chi_{[k,k+1)}$ is a step function which satisfies $\phi \leq f\chi_{[1,n+1)}$ so that

$$\sum_{k=1}^{n} a_k = \int \phi \leq \int f\chi_{[1,n+1)} = \int_1^{n+1} f \leq K.$$

Exercise. Prove that if $p > 1$ then the series $\sum 1/n^p$ converges.

Exercise. Formulate a similar criterion which ensures the *divergence* of a series of nonnegative terms, and use it to show that the harmonic series $\sum 1/n$ diverges.