Workshop 2 – Uniform convergence of sequences of functions

The purpose of this workshop activity is to provide some practice in the notions of pointwise and uniform convergence of sequences of functions, and in some of the theorems concerning uniform convergence of sequences of functions.

1. Let $f_n(x) = \frac{xn^{1/2}}{1+nx^2}$ for $x \in \mathbb{R}$. Prove that f_n converges pointwise to the zero function. Is the convergence uniform over $\mathbb{R}$? (**Hint:** Fix n and think about $\sup_{x\in\mathbb{R}} |f_n(x)|$. Does this go to zero as $n \to \infty$?)

2. Let $f_n : [0, 1) \to \mathbb{R}$ be defined by $f_n(x) = nx^n$. Show that $f_n \to 0$ pointwise but $\int_0^1 f_n \to 1$. What does this demonstrate?

3. Consider the sequence of functions on $\mathbb{R}$ given by $f_n(x) = (x - 1/n)^2$. Prove that it converges pointwise and find the limit function. Is the convergence uniform on $\mathbb{R}$? Is the convergence uniform on bounded intervals?

4. Let $f_n(x) = x - x^n$. Prove that f_n converges pointwise on $[0, 1]$ and find the limit function. Is the convergence uniform on $[0, 1]$? Is the convergence uniform on $[0, 1)$?

5. Consider the sequence of functions defined on $[0,\infty)$ defined by $f_n(x) = \frac{x^n}{1+x^n}$. Prove that (f_n) converges pointwise and find the limit function. Is the convergence uniform on $[0,\infty)$? Is the convergence uniform on bounded intervals of the form $[0, a)$?

6. Let $f_n(x) = nx(1 - x^2)^n$ for $0 \le x \le 1$. Prove that f_n converges pointwise on $[0, 1]$ and find the limit function. Is the convergence uniform on $[0, 1]$? (**Hint:** Consider the integrals$\int_0^1 f_n$.) Is the convergence uniform on $[a, 1]$ where $0 < a < 1$?

7. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of continuous functions which converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$. Let (x_n) be a sequence of real numbers which converges to $x \in \mathbb{R}$. Show that $f_n(x_n) \to f(x)$.

Assessment task to be handed in on Wednesday of Week 3 (03/10/2018 at noon): Questions 5, 6 and 7.

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   Solution. If $x = 0$ we have $f_n(0) = 0$ for all n and so $f_n(0)$ converges to 0. If $x \neq 0$, then $|f_n(x)| \le \frac{|x|n^{1/2}}{nx^2} = \frac{1}{n^{1/2}|x|}$ which goes to zero as $n \to \infty$. So f_n converges pointwise to 0. But $f_n(n^{-1/2}) = 1/2$ for all n so the convergence is not uniform over $\mathbb{R}$. (If you hadn't spotted that $n^{-1/2}$ is an interesting point, you could have used calculus to find the maximum of the function $|f_n|\ldots$)

2. Let $f_n : [0,1) \to \mathbb{R}$ be defined by $f_n(x) = nx^n$. Show that $f_n \to 0$ pointwise but $\int_0^1 f_n \to 1$. What does this demonstrate?

   Solution. From FPM we know that for $0 \le x < 1$ we have $nx^n \to 0$ as $n \to \infty$. However $\int_0^1 f_n = n/(n+1) \to 1$ as $n \to \infty$. So pointwise convegence on an interval does not imply the corresponding convergence of definite integrals.

3. Consider the sequence of functions on $\mathbb{R}$ given by $f_n(x) = (x - 1/n)^2$. Prove that it converges pointwise and find the limit function. Is the convergence uniform on $\mathbb{R}$? Is the convergence uniform on bounded intervals?

   Solution. For each fixed x we have $x_n := x - 1/n \to x$ as $n \to \infty$; hence, by FPM, $x_n^2 \to x^2$. So the sequence of functions f_n converges pointwise to the function $f(x) = x^2$. What about uniform convergence? We need to consider whether the sequence $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$ goes to zero as $n \to \infty$. Let's look at

$$|f_n(x) - f(x)| = |(x - 1/n)^2 - x^2| = \frac{|2x - 1/n|}{n}.$$

The values of this expression as x ranges over $\mathbb{R}$ are not even *bounded* (to see this let $x \to \infty$), and so the sup does not even *exist*, let alone go to 0 as $n \to \infty$. So we do not have uniform convergence of f_n to f. (Another way of seeing this is to note that $|f_n(n) - f(n)| = \frac{|2n - n^{-1}|}{n} \geq 1$ and so $|f_n(n) - f(n)|$ does not go to zero.)

If however we work on $[-M, M]$ instead of on the whole of $\mathbb{R}$, the above calculation shows that for $|x| \leq M$ we have

$$|f_n(x) - f(x)| = \frac{|2x - 1/n|}{n} \leq \frac{2M + 1/n}{n}$$

so that $\sup_{x \in [-M,M]} |f_n(x) - f(x)|$ goes to zero as $n \to \infty$, and hence the convergence is uniform on bounded intervals.

4. Let $f_n(x) = x - x^n$. Prove that f_n converges pointwise on $[0, 1]$ and find the limit function. Is the convergence uniform on $[0, 1]$? Is the convergence uniform on $[0, 1)$?

   Solution. For $0 \leq x < 1$ we have $f_n(x) \to x$ and for $x = 1$ we have $f_n(1) = 0$. So the limit function is $f(x) = x$ for $0 \leq x < 1$ and $f(1) = 0$. Since each f_n is continuous on $[0, 1]$ and f isn't, the convergence can't be uniform on $[0, 1]$. As for uniform convergence on $[0, 1)$, for $0 \leq x < 1$ we have $|f_n(x) - f(x)| = |x^n|$ so that $\sup_{0 \leq x < 1} |f_n(x) - f(x)| = \sup_{0 \leq x < 1} |x^n| = 1$, so once again the convergence is not uniform.

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