Analysis Hand in Four

William A. Bevington

Question One - Workshop 7, Q7

Suppose that $d(x, y)$ and $\rho(x, y)$ are metrics on some set X with $x, y \in X$. First I will show that $\rho(x, y)$ and $d(x, y)$ are equivalent iff $B_\rho(x_0; r) = B_d(x_0; r)$, where $B_\rho(x_0; r)$ is a ball with centre x_0 and radius r in the $\rho(x, y)$ metric and $B_d(x_0; r)$ is the same in the $d(x, y)$ metric, then we can conclude that $d(x, y)$ and $\rho(x, y)$ are equivalent iff for any open subset $H \subset_{\text{open}} X$ in (X, d) we have also $H \subset_{\text{open}} X$ in (X, ρ).

Now, we have the two open balls in (X, d) and (X, ρ) given by

$$B_d(x_0; r) := \{y \in X : d(x, y) < r\}, \qquad B_\rho(x_0; r) := \{y \in X : \rho(x, y) < r\}. \tag{1}$$

If $d(x, y)$ and $\rho(x, y)$ are equivalent then by definition we have that $\forall x, x_0 \in X : \forall \varepsilon > 0 : \exists \delta > 0$ such that

$$d(x, x_0) < \delta \Rightarrow \rho(x, x_0) < \varepsilon$$
$$\text{and}$$
$$\rho(x, x_0) < \delta \Rightarrow d(x, x_0) < \varepsilon$$

We can define the sets on which this holds and say that $d(x, y)$ and $\rho(x, y)$ are equivalent if

$$x \in \{y \in X : d(y, x_0) < \delta\} \Rightarrow x \in \{y \in X : \rho(y, x_0) < \varepsilon\}$$
$$\text{and}$$
$$x \in \{y \in X : \rho(y, x_0) < \delta\} \Rightarrow x \in \{y \in X : d(y, x_0) < \varepsilon\},$$

choosing $r = \min\{\delta, \varepsilon\}$ (which exists because we've assumed $d(x, y)$ and $\rho(x, y)$ are equivalent), we can make this an 'if and only if' statement, and using (1) we get that if $d(x, y)$ and $\rho(x, y)$ are equivalent then $B_d(x_0; r) = B_\rho(x_0; r)$ for any r. This does in fact work for any r because we have an arbitrary choice for ε and an arbitrary choice for $d(x, y)$ and $\rho(x, y)$, which determine δ, so we can make $\min\{\delta, \varepsilon\}$ as large or small as we like.

Now we have to show that $B_d(x_0; r) = B_\rho(x_0; r)$ for arbitrary $x_0 \in X$ and $r > 0$ implies that $d(x, y)$ and $\rho(x, y)$ are equivalent. This will essentailly be the same argument in reverse. If $B_d(x_0; r) = B_\rho(x_0, r)$ for an arbitrary choice of x_0 and r, then $x \in B_d(x_0; r) \Rightarrow x \in B_\rho(x_0; r)$, and so $d(x, x_0) < r \Rightarrow \rho(x, x_0) < r$, so we let $\varepsilon = \delta = r$ and we get that

$$\forall x, y \in X : \forall \varepsilon > 0 : \exists \delta > 0 : d(x, x_0) < \delta \Rightarrow \rho(x, x_0) < \varepsilon \text{ and } \rho(x, x_0) < \delta \Rightarrow d(x, x_0) < \varepsilon,$$

and so $d(x, y)$ and $\rho(x, y)$ are equivalent iff $B_d(x_0; r) = B_\rho(x_0; r)$.

Every open subset of a metric space X can be realised as a union of open balls. Thus if $H \subset X$ is an open subset of (X, d) then $H = \cup B_d(x_0; r) = \cup B_\rho(x_0; r) \subset (X, \rho)$, and visa-versa. So $d(x, y)$ and $\rho(x, y)$ are equivalent on X if and only if every subset H which is open with respect to ρ is also open with respect to $d(x, y)$.

Question Two - Workshop 7, Q9

We are asked to show that

$$d_1(f, g) := \int_0^1 |f - g|$$

does not give rise to a complete metric space on $C([0, 1])$. We will construct a counter-example which gives a Cauchy series $f_n \to f$ but where $f \notin C([0, 1])$ and so f_n is not convergent in $C([0, 1])$. Let the sequence $(f_n)_{n \in \mathbb{N}}$ be defined

$$f_n = \frac{1}{1 + (x - \frac{1}{2})^{2n}} \tag{2}$$

where I've shifted $\frac{1}{1+x^{2n}}$ to the left by a half. Now, I will prove that each f_n is continuous on $[0, 1]$ and hence $\forall n \in \mathbb{N} : f_n \in C([0, 1])$, but that $\lim_{n \to \infty} f_n \notin C([0, 1])$.

First of all, each f_n is continuous. To see this note that each of the functions

$$g_1(x) = \frac{1}{1+x}$$
$$g_2(x) = x^{2n}$$
$$g_3(x) = x - \frac{1}{2}$$

are continuous on the domain $[0,1]$, and that $f_n(x) = g_1 \circ g_2 \circ g_3(x)$, and that the composition of continuous functions is continuous. Thus each f_n is continuous.

We now show that (f_n) is Cauchy. Let $z = x - 1/2$ so that $z \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ and

$$\begin{aligned}|f_n(z) - f_m(z)| &= \left|\frac{1}{1+z^{2n}} - \frac{1}{1+z^{2m}}\right| \\ &= \left|\frac{1+z^{2m}-1-z^{2n}}{(1+z^{2n})(1+z^{2m})}\right| \\ &= \left|\frac{z^{2m}-z^{2n}}{(1+z^{2n})(1+z^{2m})}\right|.\end{aligned}$$

Since $-\frac{1}{2} \leq z \leq \frac{1}{2}$ we have $0 \leq z^2 \leq \frac{1}{4} \Rightarrow 1 \leq 1 + z^{2n} \leq 1 + \frac{1}{2^n}$ and so

$$|f_n(z) - f_m(z)| \leq \left|\frac{\frac{1}{4^n} - \frac{1}{4^m}}{(1)(1)}\right| = \left|\frac{1}{4^n} - \frac{1}{4^m}\right|.$$

Without loss of generality, assume $m > n$ and so $\frac{1}{4^m} < \frac{1}{4^n}$, thus $\left|\frac{1}{4^n} - \frac{1}{4^m}\right| \leq \frac{1}{4^n} =: \varepsilon$ so (f_n) is Cauchy.

Now we show that $f(x) := \lim_{n\to\infty} f_n(x)$ isn't continuous on $[0,1]$, as it has a discontinuity at $x = \frac{1}{2}$. Substituting $x = \frac{1}{2}$ into (2) we get that $\forall n : f_n(1/2) = 1$ and so clearly $f(1/2) = 1$. However, if $x \in (\frac{1}{2}, 1]$ then $x - \frac{1}{2} > 0 \Rightarrow 1 + (x - \frac{1}{2})^{2n} > 1$ and so $f_n(x) = \frac{1}{1+(x-\frac{1}{2})^{2n}} \to 0$ as $n \to \infty$ by the p-test. So $\lim_{x\to 1/2^+} f(x) = 0$ and $f(x) = \frac{1}{2}$, thus f is discontinuous at $x = \frac{1}{2}$.

Thus we have shown that there exists a Cauchy sequence on the metric space $\Big(C([0,1]), d_1\Big)$ which is not convergent; so the space isn't complete.

Question Three - Workshop 8

I will prove that if (X,d) is a complete metric space then any closed subset of $A \subset X$ is compact, thus we will get for free that $[0,1] \times [0,1] \subset \mathbb{R}^2$ is compact, since $\mathbb{R}^2$ is complete.

Suppose $A \subset X$ is a closed subset of (X,d) with the restriction of d to A, dentoed $d|_A$, forming a metric (sub)space $(A, d|_A)$, and that $(A, d|_A), (X,d)$ are complete spaces. We say that $Q \subset X$ is compact if given any open cover $\mathcal{U}$ of Q we can find a finite sub-cover $\{\mathcal{U}_\alpha\} \subset \mathcal{U}$ of Q.

Thus, for the sake of contradiction, assume that $\mathcal{U}$ is an open cover of A in which there exists a subcover $\{U_\alpha\}$ which is infinite. We partition $\mathcal{U}$ into h subsets $Q_1 = \mathcal{U}_1 \sqcup \cdots \sqcup \mathcal{U}_h$, so that $\{\mathcal{U}_1, \ldots, \mathcal{U}_h\}$ is a subcover of $\mathcal{U}$, by our assumption it must be the case that at least one of the $\mathcal{U}_i$ has no finite subcover.

Without loss of generality say that $\mathcal{U}_1$ has no finite subcover, and let $Q_1 := \mathcal{U}_1$. We can now partition this into a further four subsets which cover it and then we repeat the steps from the last paragraph, giving the sequence $Q_{n+1} \subset Q_n \subset \cdots \subset Q_1$, where we choose each Q_i to be one with no finite subcover by our assumption.

Now we choose $x_n \in Q_n$ and $x_m \in Q_m$ with $m > n$ so that $Q_m \subset Q_n$, note that the maximum distance between these two points is the maximum distance between any two points in Q_n, given by $k_n = \sup\{d(x_i, x_j) : x_i, x_j \in Q_n\}$. We require now that $(k_n)_{n\in\mathbb{N}}$ is a Cauchy sequence (more on this at the end), which by the completeness of X gives that it is a convergent sequence, so $x_n, x_m \to x$ as $m, n \to \infty$ for some $x \in \bigcap_{i=0}^{\infty} Q_i$.

So we have that $x \in \bigcap_{i\in I} Q_i$ for some finite indexing set I since $Q_m \subset Q_n$ for $m > n$, so we may just choose a finite I as $\bigcap_{i\in I} \subset Q_1$, for instance, and one is finite. Let $B(x,r)$ be an open ball so that $B(x,r) \subseteq \bigcap_{i\in I} Q_i$, we can choose a suitably large (yet still finite) N for which, given $n \geq N$ we have $\sup\{d(x_i, x)\} \leq r$ since the sequence $(x_i)_{n\in\mathbb{N}}$ is convergent to a finite limit. Then $Q_n \subseteq B(x,r)$, and so our ball is a finite subcover of Q_n, which is in contradiction with our statement that each Q_i has no finite subcover.

Now, we required earlier that $(k_n)_{n\in\mathbb{N}}$ be a Cauchy sequence, which may not be true for any choice of sub-division of each Q_i. However we may choose a subdivision each time which suitably decreaces the 'size' of Q_{i+1} as to make (k_n) Cauchy since we have not fixed the number of subdivisions h, and so can choose a suitably large h as to suitably decrease the size of each subdivision, forcing $(k_n)_{n\in\mathbb{N}}$ to be Cauchy. Thus, since $[0,1] \times [0,1] \subset \mathbb{R}^2$ is complete, it is also compact.