Symmetry and Geometry

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Preamble

0.1 Introduction

It was love at first sight. The first time I saw the snub dodecahedron I fell madly in love with it. I had heard of it before but while creating this course I realised that it could be made using some plastic pieces designed for schools.

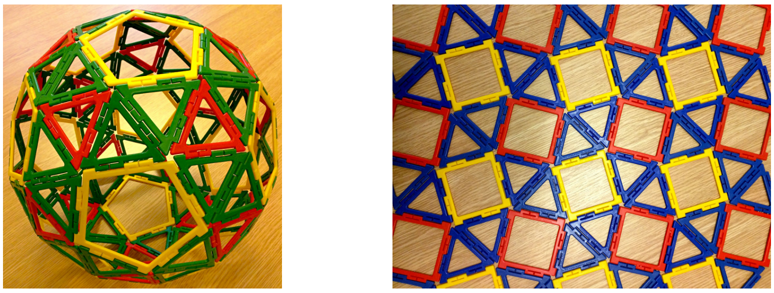


Figure 1: Snub things: dodecahedron (left); square tessellation (right)

The picture does not do it justice; there are better images online but nothing beats handling an object to appreciate its structure. We will get to understand symmetric objects in space in this course and you will soon be able to explain why this object exhibits 532 symmetry.

Something else I had not fully appreciated before was tilings of the plane by regular polygons and the way one can think of them as "infinite radius" versions of polyhedra. On the right above is the "snub square" tessellation which is a "flat" cousin of the snub dodecahedron. As a pattern in the plane, its symmetries provide two examples of "wallpaper groups": one case where only symmetries preserving colours are considered and one where we imagine all the colours to be the same. Colourings of patterns will also be a topic for us and by the end of the course you should also be able to explain why this provides an example of a 2-coloured wallpaper group of type $4{*}2/442$ (and why it is a 2-colouring despite seeming to involve three colours).

This year, I have become even more interested in such tilings and "Archimedean solids" such and they will feature rather more. I hope you will join me in working to understand them better.

What I am mainly trying to say in this introduction is that it was both instructive and fun preparing this course, that I have learned a lot from students taking it in previous years and I hope it will interesting and rewarding to study.

0.1.1 Problem (making a start) What symmetries can you find in the snub square tessellation in the two cases?

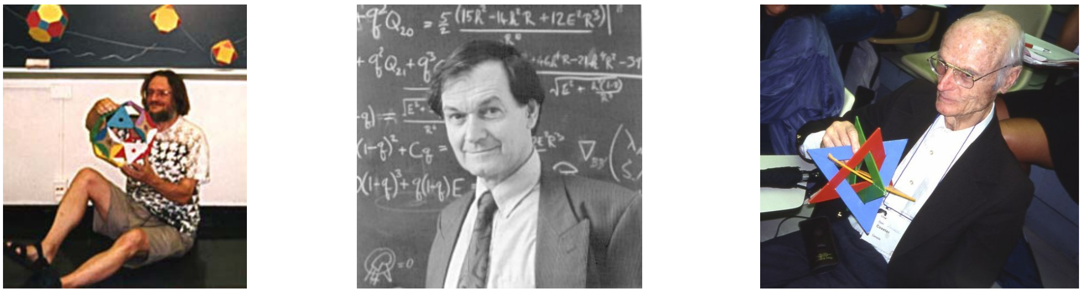


Figure 2: Conway, Penrose and Coxeter (left to right)

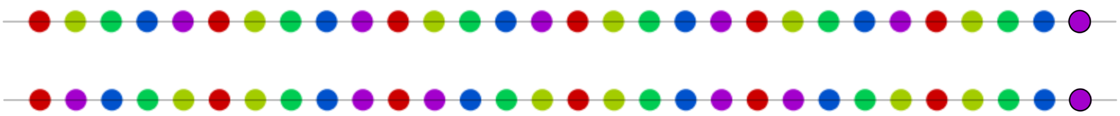
I would like to mention a few mathematicians who have influenced me or the content of the course. When I was an undergraduate I was lectured algebra by John Horton Conway (the inventor of “Life”); one of the few things I still have from lectures in those days is from his course on algebra, “everything you need to remember” summarised on two sides of a piece of paper and including a poem. I recall him saying that a group is not just some soulless, algebraic thing: if it is interesting it is the symmetries of something. This course was inspired by a book written by Conway (and collaborators) and takes a similar view in emphasising the visual side of things and the interplay between algebra and geometry.

As a postgraduate mathematician I was influenced a great deal by Roger Penrose whose whole approach to mathematics is geometrical. I was interested too by the relationship between some mathematical ideas and the art of Escher, an interplay he was much involved in. We will look at some of Escher’s more mathematical pictures later in the course.

Siobhan Roberts, who visited the School recently, wrote a biography of Conway (see the reading list) and also one of Coxeter, a mathematician I did not meet but who brought geometry back into the mathematical mainstream when the fashion was very much for abstraction.

I should mention William (“Bill”) Thurston too, a mathematician I never met. With Conway, he developed and promoted the “orbifold” approach to geometric symmetry that we will use. He also won the Fields Medal for realising that geometry provides a way in to understanding the classification of 3-manifolds: the analogues of surfaces one dimension up. Conway attributes a commandment to Thurston that we will be heeding: ”thou shalt know no geometrical group save by understanding its orbifold”.

0.2 Patterns on a line



Are the two patterns above equivalent as far as their symmetries are concerned? (We are of course imagining them continuing indefinitely in both directions.) Are there other fundamentally different repeating patterns in the line? The answer to the first question is easily seen to be "no" (one of the patterns has reflection symmetries in vertical lines and the other does not) but to answer the second question, we need first to think about the symmetries of an undecorated line and what "fundamentally different" might mean. While the pictures show "beads", I am imagining everything as one dimensional. A better model would be thinking of an infinitely long and thin coloured, straight rod.

0.2.1 Definition The *Euclidean group in one dimension* $E(1)$ consists of two types of symmetry of the line $\mathbb{R}$: the translations $T_a : x \mapsto x + a$ and the reflections given by $M_a : x \mapsto a - x$. In each case $a \in \mathbb{R}$. Note that T_a and M_a are *distance preserving* in that they preserve distances between pairs of points. Note that T_0 is the identity.

0.2.2 Exercise (BS) What point does M_a reflect in?

0.2.3 Exercise (BM) What happens when you compose translations and reflections with themselves and each other? Complete the following multiplication table:

$\circ$	T_b	M_b
T_a	T_{a+b}	$??$
M_a	$??$	$??$

Note that the entry already completed in the table essentially tells us that the translations for a subgroup of $E(1)$.

0.2.4 Proposition The map p from $E(1)$ to the 2-element group $\{\pm 1\}$ which sends translations to 1 and reflections to -1 is a group homomorphism. The translations are the kernel and thus form a normal subgroup of $E(1)$.

Proof. This essentially follows from the "multiplication table" constructed in Exercise 0.2.3 □

0.2.5 Definition (a little vague) A *discrete subgroup* of a group G is a subgroup that does not contain elements arbitrarily close to the identity 1_G. (This is vague because it is not clear what we mean by "close", but this should be unambiguous in practice for us. In this case, it just means that there are not arbitrarily small translations in the subgroup.)

0.2.6 Exercise (BS) Find an example of a non-discrete (or should that be "indiscrete") proper subgroup of $E(1)$.

0.2.7 Henceforth in this discussion, G is a discrete subgroup of $E(1)$. Restricting the group homomorphism of Proposition 0.2.4 to the subgroup G, we obtain a group homomorphism

$$G \to \{\pm 1\}.$$

The image of the composition is either the trivial subgroup (in which case G contains only translations) or the whole group in which case there are reflections also.

0.2.8 Proposition Every nontrivial discrete subgroup of translations is of the form $\{T_{ka} \mid k \in \mathbb{Z}\}$ for some fixed $a > 0$. We call such a discrete subgroup a 1-dimensional *lattice of translations*.

Proof. Given such a subgroup G, choose $a > 0$ as small as possible such that $T_a \in G$. Then the existence of elements of G not of the given form leads to a contradiction. (Exercise). □

0.2.9 Exercise (BM) Complete the proof of the above proposition. Where in the proof is the condition of discreteness used?

0.2.10 Definition By a (symmetric) *pattern in* $\mathbb{R}$ we mean one that is invariant under the action of a discrete subgroup of $E(1)$ that includes a lattice of translations. (The last condition is limiting ourselves to repeating patterns.)

0.2.11 If we take either of our patterns and stretch the underlying line uniformly by some factor, we have a pattern which is to all intents and purposes "the same". Our notion of equivalence of patterns is that if a pattern can be smoothly deformed into another while keeping the same symmetry group, we regard them as equivalent. This idea of "smooth isotopy" will not be defined any more exactly.

This means that we do not lose generality by assuming that the smallest translation in G is of magnitude 1.

0.2.12 Theorem Every symmetric pattern has its group G of symmetries one of the following two possibilities: the first is $T_{\mathbb{Z}}$ the lattice of translations $\{T_k \mid k \in \mathbb{Z}\}$ and the second is D_∞, the *infinite dihedral group* consisting of $T_{\mathbb{Z}}$ together with the reflections $\{M_k \mid k \in \mathbb{Z}\}$.

Proof. A sketch is as follows.

1. If G contains no reflections, then we know it must be $T_{\mathbb{Z}}$ from previously.

2. If G contains a reflection, consider first its subgroup of translations. (The translations do form a subgroup of G since they are the intersection of $G \subseteq E(1)$ with the subgroup of translations in $E(1)$.) From above, shrinking or expanding as necessary, we know this must be $T_{\mathbb{Z}}$.

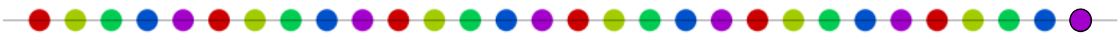
3. Now choose the origin so that M_0 is in G. Deduce by messing around with products that M_k is in G whenever $k \in \mathbb{Z}$.

4. Now show that there can be no further reflections in G. (Hint: the composition of two reflections in G has to be one of the translations in G.)

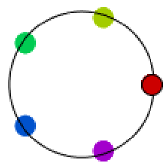
□

0.2.13 Exercise (NM) Complete the proof.

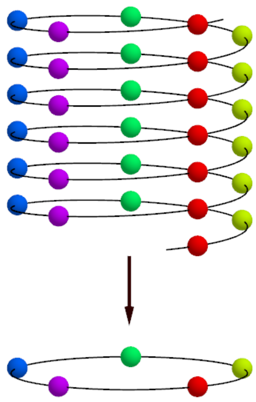
0.2.14 The translation group $T_{\mathbb{Z}}$



0.2.15 The pattern above has only translation symmetries. The whole pattern is generated by repeating  over and over again. We call this little section D a *fundamental domain*: it is a closed, connected region that contains one point from every orbit of $T_{\mathbb{Z}}$ and contains only one such point except possibly where the point in D is on its boundary.



0.2.16 We have a map from the whole line $\mathbb{R}$ with its coloured pattern to D which maps each point on the line to the element of D in its equivalence class. The trouble is that this map is not continuous. We can fix this by connecting the ends of D together to make a circle as above on the right. (I have changed the spacing of the beads for convenience.)



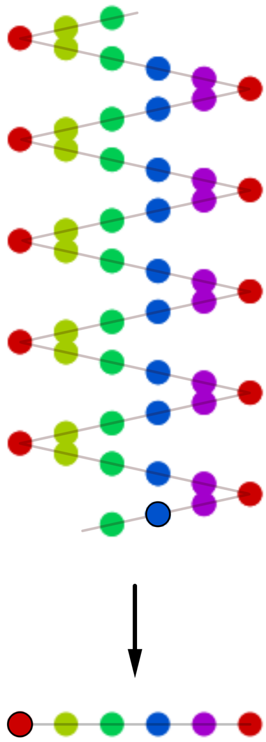
0.2.17 On the right, the real line with its pattern has been wound into a helix and the map that sends each point to its colour in the circle is projection vertically downwards and it is continuous. (Of course, the helix is meant to continue infinitely in both directions.) The circle is our first example of an *orbifold*. It completely defines the pattern in the sense that a 1-dimensional creature walking around on the orbifold would see exactly the same sequence of colours as if it was walking around on the original line. The whole pattern can be recovered from the orbifold by taking its "universal cover", and this idea may be familiar to some of you from Topology. We will think more about it later.

0.2.18 The infinite dihedral group D_∞

0.2.19 The case of D_∞ is rather different. There are two different sorts of red ball: those surrounded by green ones and those surrounded by violet ones. Therefore no symmetry of the pattern can take a red ball of one type to one of the other type. But for the other colours, every ball of that colour in the pattern is "the same". Correspondingly, there are two different types of reflection symmetry, centred in the two different types of red ball. When we talk about reflection points being "the same" or "different", the distinction we are making is whether there is a symmetry taking one to the other.

0.2.20 Exercise (BS) Choose a blue ball somewhere in the middle of the picture. What symmetries of the pattern takes this ball to each of the other pictured blue balls?

0.2.21 Exercise (NM) Set the scale and origin so that the red balls are at integer and half-integer points. Then the translation symmetries are by integer amounts. Show that if M is a reflection in a red ball and T_k is a translation by k then $T_k\, M\, T_k^{-1}$ is a reflection in a red ball distance k away. (In which direction?) So all the reflections about integer points are conjugate in the group D_∞, as are all those at half-integer points.



0.2.22 A fundamental domain for D_∞ looks like  and adjoining regions can be obtained by reflecting in the end-points. The orbifold is a closed interval *which we define to have reflecting ends*.

We already have a continuous map from $\mathbb{R}$ with its patterns to D that folds it up in a zigzag. (Hence the name "orbifold": it is the folded orbits of the symmetry group.) Given the orbifold, one can unwrap it to obtain the pattern.

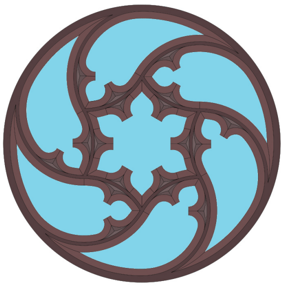
Again, a 1-dimensional creature crawling around on the orbifold, reflecting off the mirrors at the ends, sees exactly what it would crawling around on the original pattern.

0.2.23 Conclusion Note how our analysis reveals strong limitations in possible symmetries: one can have no reflectional symmetries, or two different types. But one cannot have just one type or more than two.

A final thought about this is that we classified the possible symmetry groups and then found their orbifolds. But if we accept that reflections in the pattern give rise to mirror boundaries in the orbifold then what other possible orbifolds are there? If we assume they are compact and connected, then a closed interval with reflecting boundaries or a circle seem to be the only possibilities. We might be able to deduce the completeness of our classification of groups from that. That is what we will seek to do in two dimensions.

0.2.24 Exercise (SN) If we allowed non-compact (but still closed) orbifolds, we would also have the whole real line $\mathbb{R}$ and a semi-infinite closed interval like $[0, \infty)$. To what subgroups of $E(1)$ do these correspond?

0.3 Patterns on a disc



0.3.1 Now we turn to patterns in a closed disc $K = \{(x, y) \mid x^2 + y^2 \leq 1\}$. The matrices

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

rotate the disc by an angle θ anticlockwise about the origin and reflect the plane in the line through the origin making an angle of $\theta/2$ with the positive x-axis respectively.

0.3.2 Definition The *orthogonal group* $O(2) = \{R_\theta\} \cup \{M_\theta\}$. The *special orthogonal group* $SO(2)$ is the subgroup consisting of all the rotations. The rotations are the kernel of the determinant homomorphism $\det : O(2) \to \{\pm 1\}$.

0.3.3 One can think of $SO(2)$ as being a circle with θ as an angular coordinate. The reflections form another separate circle. Because $O(2)$ is compact, a discrete subgroup turns out to be the same thing as a finite subgroup. We define a symmetric pattern on the disc as one which is invariant under a finite (and hence discrete) subgroup of $O(2)$.

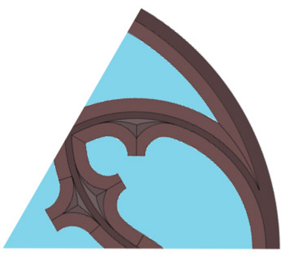
0.3.4 Proposition The finite nontrivial subgroups of $SO(2)$ are the groups

$$C_n = \{R_{2k\pi/n} \mid 0 \leq k < n\}, \qquad n \in \mathbb{Z}, \quad n \geq 2.$$

Proof. The same idea as for the classification of translation subgroups of $E(1)$ works. See Proposition 0.2.8. ☐

0.3.5 Exercise (NS) Complete the proof.

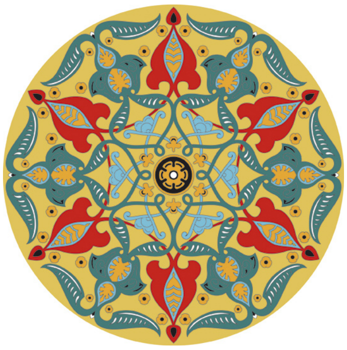
Rotational symmetry



0.3.6 The gothic tracery window at the top of this discussion has C_6 symmetry (by which we mean a cyclic group of rotations of order six). The obvious choice for its fundamental domain is a sector such as that above on the left, but the map from the disc to that would not be continuous. The fix this time is to wrap the sector into a cone, as above on the right. The original curved boundary is not reflecting because the pattern does not extend beyond: it is simply the boundary of our universe.

A 2-dimensional creature crawling around the orbifold experiences the same pattern as if they were on the original disc, and a special sort of unwinding recovers the disc from the orbifold.

Dihedral Symmetry



0.3.7 The Islamic-style decoration above has 6-fold rotational symmetry but, providing you don't look too hard, it also has reflectional symmetries. Its symmetry group is D_6, the rotational and reflectional symmetries of a regular hexagon.

0.3.8 In general, we write D_n for the symmetries of a regular n-gon when $n \geq 3$ and set D_2 to be the 4-element group of symmetries of an oblong (the "mattress group" - why so named?) and D_1 to be the group containing the identity and a single reflection.

0.3.9 Proposition A finite subgroup G of $O(2)$ has either no reflections or exactly as many reflections as rotations.

Proof. Consider the restriction of the determinant map to give a homomorphism $G \longrightarrow \{\pm 1\}$. □

0.3.10 Exercise (BS) Complete the proof. Remember that in a homomorphism of finite groups, every element in the image has the same number of preimages.

0.3.11 Theorem Every finite subgroup of $O(2)$ is either a cyclic group of rotations C_n (including the case $n = 1$ of the trivial subgroup) or a dihedral group D_n, $n \geq 1$.

Proof. This is broadly analogous to the proof of Proposition 0.2.12. Consider the subgroup of rotations. Then consider the consequences of having a reflection. □

0.3.12 Exercise (NM) Complete the proof.

0.3.13 Reflecting boundaries The orbifold is a sector bounded by the two adjacent reflection lines as on the right above. But the straight edge boundaries here should be taken as reflecting, just as for the earlier example of D_∞. For that reason they are marked in red, which is our preferred colour for mirror lines.

0.3.14 Slogans These phenomena that we have just seen will be central for us as we explore 2-dimensional geometric patterns.

1. Reflection lines (which we sometimes call "mirrors") lead to reflecting boundaries in the orbifold.

2. A meeting of reflection lines leads to a "corner" in the reflecting boundary.

3. At a meeting point of mirrors, one always has rotational symmetry. We will call a rotational symmetry whose centre is not on a mirror a *gyration*. These lead to *cone points* in the orbifold.

0.3.15 Where are we headed? First of all, we are going to familiarise ourselves with symmetries of the Euclidean plane and explore the zoo of 17 different "wallpaper groups". But to understand their orbifolds and, in the end, see why there are just 17 we are going to need to learn quite a lot about Euler characteristics. (If you remember the famous formula $V - E + F = 2$ for polyhedra, that "2" at the end is the Euler characteristic of the sphere.)

We will need to understand and classify all possible surfaces including ones with holes in. To get a flavour of this, look at the "frieze pattern" below. (And yes, we will classify these as well.) To make its orbifold, take a length of the strip equal to the distance between a flower and an upside-down one, then fasten the ends together with a twist to make a Möbius band. (You really need a longer, thinner version of the picture to do this comfortably in three dimensional space.) If the pattern is printed on both sides of the paper, it will join up exactly right. We are going to have to understand surfaces including things like Möbius bands, toruses, Klein bottles and projective planes that may also have holes with reflecting boundaries, corners in their boundaries and cone points.