HONOURS COMPLEX VARIABLES 2018–2019

RICHARD GRATWICK

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1. Holomorphic functions

1.1. Complex numbers.

Definition 1.1.1. A *complex number* is an object of the form $z = x+iy$, where $x, y \in \mathbb{R}$. Expressed in this form it is said to be in *cartesian form.* Consider two complex numbers $z = x + iy$ and $w = a + ib$. Then

(i) $z = w$ if and only if $x = a$ and $y = b$;
(ii) the *real part* $\mathrm{Re}(z)$ of z and the *imaginary part* $\mathrm{Im}(z)$ of z are $\mathrm{Re}(z) := x$ and $\mathrm{Im}(z) := y$;
(iii) the *modulus* $|z|$ of z is $|z| := \sqrt{x^2 + y^2}$;
(iv) the *complex conjugate* $\overline{z}$ of z is $\overline{z} := x - iy$;
(v) the operations of addition and multiplication are given by
$$(x + iy) + (a + ib) := (x + a) + i(y + b);$$
$$(x + iy)(a + ib) := (xa - yb) + i(xb + ya);$$
and
(vi) the set of all complex numbers $\mathbb{C}$ is the set $\mathbb{C} := \{\, x + iy : x, y \in \mathbb{R}\,\}$.

Remark 1.1.2. (i) If $z = x + i0$, then we write $z = x$ and say z is real, and if $z = 0 + iy$, then we write $z = iy$ and say z is imaginary.
(ii) The arithmetical operations imply the familiar rule that $i^2 = -1$.
(iii) Note that $\mathrm{Im}(z)$, $\mathrm{Re}(z)$, and $|z|$ are all real numbers.

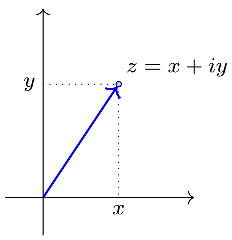
Lemma 1.1.3. Let $u, w, z \in \mathbb{C}$, where $z = x + iy$ for $x, y \in \mathbb{R}$. Then

(i) $z + w = w + z$ and $zw = wz$ (commutativity of addition and multiplication);
(ii) $u + (z + w) = (u + z) + w$ and $u(zw) = (uz)w$ (associativity of addition and multiplication);
(iii) $u(z + w) = uz + uw$ (distributivity);
(iv) $z + 0 = z$, and there exists a unique complex number $-z := -x + i(-y)$ such that $z + (-z) = 0$; and
(v) $1z = z$, and if $z \neq 0$ then there exists a unique complex number $1/z = z^{-1} := x(x^2 + y^2)^{-1} - iy(x^2 + y^2)^{-1}$ such that $z(1/z) = 1$.

Proof. Exercise. □

Remark 1.1.4. This asserts that $\mathbb{C}$, equipped with these operations, is a *field*, and that complex numbers satisfy the usual algebraic rules.

The set of complex numbers is naturally identified with the plane $\mathbb{R}^2$. This is often called the *complex plane* or an *Argand diagram.* A complex number $z = x+iy$ corresponds to the point (x, y) in $\mathbb{R}^2$. We will sometimes abuse notation and regard a set S as both a subset of $\mathbb{R}^2$ and of $\mathbb{C}$, depending on convenience.



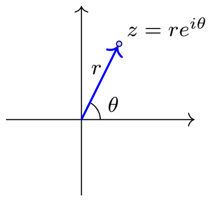
Addition of complex numbers and multiplication of complex numbers by real numbers are preserved under this correspondence, in the sense that $(x + iy) + (a + ib) = (x + a) + i(y + b)$ is represented by the point $(x+a, y+b) = (x, y)+(a, b)$, and $\alpha(x+iy) = (\alpha x)+i(\alpha y)$ is represented by the point $(\alpha x, \alpha y) = \alpha(x, y)$, for $\alpha \in \mathbb{R}$. Furthermore, the modulus of $z = x + iy$ is given by the Euclidean norm of the vector (x, y).

A non-zero point (x, y) in $\mathbb{R}^2$ can be represented by polar coordinates (r, θ), where $r > 0$ is the (Euclidean) distance from (x, y) to the origin and $\theta \in \mathbb{R}$ is the anticlockwise angle made by the ray from the origin $(0, 0)$ to the point (x, y) and the positive x-axis. Elementary trigonometry shows that $x = r\cos\theta$ and $y = r\sin\theta$. Since the point (x, y) represents the complex number $z = x+iy$, we then gain a polar representation of complex numbers.

Definition 1.1.5. The *polar form* of a non-zero complex number $z = x + iy$ is a representation

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

for $r > 0$ and $\theta \in \mathbb{R}$. Thus $x = r\cos\theta$ and $y = r\sin\theta$.



We shall adopt the notation

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{1.1}$$

although we emphasize that we are not yet defining rigorously the exponential function on the complex numbers. At the moment, this is just convenient notation, although we may check (e.g. using the Taylor series expansions for exp, sin, and cos centred at 0) that this is compatible with our usual understanding of the terms, and therefore such expressions can indeed be manipulated using the laws of exponentials with which you are familiar.

Lemma 1.1.6. Let $\theta, \phi \in \mathbb{R}$, and $n \in \mathbb{Z}$. Then
 (i) $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$;
 (ii) $e^{-i\theta} = \frac{1}{e^{i\theta}}$; and
 (iii) $e^{in\theta} = \left(e^{i\theta}\right)^n$.

The last statement is de Moivre's formula

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n .$$

Proof. Exercise. □

Definition 1.1.7. The *exponential form* of a non-zero complex number $z = x + iy$ is a representation

$$z = re^{i\theta}$$

where $r > 0$ and $\theta \in \mathbb{R}$ satisfy $x = r\cos\theta$ and $y = r\sin\theta$.

Remark 1.1.8. (i) The number r is the modulus of z, i.e. $r = \sqrt{x^2 + y^2}$.
 (ii) Since the functions sin and cos are periodic, the value of θ is *not* unique. Hence each representation in the preceding two definitions is described as "a representation", not "the representation".

Exercise 1.1.9. Prove the following identities:
 (i) $(1 + i)^4 = -4$;
 (ii) $(1 + i)^{13} = -2^6(1 + i)$;
 (iii) $(1 + i\sqrt{3})^6 = 2^6$;
 (iv) $\frac{(1+i\sqrt{3})^3}{(1-i)^2} = -4i$; and
 (v) $\frac{(1+i)^5}{(\sqrt{3}+i)^2} = -\sqrt{2}e^{-i\pi/12}$.

Exercise 1.1.10. Show the position of the points $2 - 2i$, $1 + \sqrt{3}i$, $-i$, -3, and $\frac{\sqrt{3}}{2} - \frac{3}{2}i$ on the complex plane, and express them in polar form.

Exercise 1.1.11. Show the position of the points $2e^{3\pi i/4}$, $2e^{7\pi i/6}$, $3e^{4\pi i/3}$, and $4e^{3\pi i/2}$ on the complex plane, and represent them in cartesian form.

Exercise 1.1.12. Find the square roots of the complex numbers $5 - 12i$ and $8 + 4\sqrt{5}i$.

Exercise 1.1.13. Find all the fourth roots of 1, and all the seventh roots of -1, and sketch them in the complex plane.

We collect some elementary properties of complex numbers.

Lemma 1.1.14. Let $z, w \in \mathbb{C}$. Then

(i) $|z| = 0$ if and only if $z = 0$;
(ii) $|\overline{z}| = |z|$;
(iii) $|zw| = |z||w|$;
(iv) $\overline{\overline{z}} = z$;
(v) $|z|^2 = z\overline{z}$;
(vi) $\overline{z+w} = \overline{z} + \overline{w}$;
(vii) $\overline{(zw)} = (\overline{z})(\overline{w})$;
(viii) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$; and
(ix) $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.

Proof. Exercise. □

Lemma 1.1.15 (Triangle inequality)**.** Let $z, w \in \mathbb{C}$. Then

$$|z + w| \leq |z| + |w| . \tag{1.2}$$

Proof. We use several of the facts from lemma 1.1.14 to see that since

$$z\overline{w} + w\overline{z} = z\overline{w} + \overline{\overline{w}z} = 2\mathrm{Re}(\overline{w}z) \leq 2\,|\overline{w}z| = 2\,|w|\,|z|\,,$$

we have that

$$|z+w|^2 = (z+w)\overline{(z+w)} = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \leq |z|^2 + 2\,|w|\,|z| + |w|^2 = (|z| + |w|)^2\,.$$

Since both $|z + w|$ and $|z| + |w|$ are non-negative, taking square roots of both sides gives the result. □

Lemma 1.1.16 (Reverse triangle inequality)**.** Let $z, w \in \mathbb{C}$. Then

$$||z| - |w|| \leq |z - w|\,.$$

Proof. Applying the triangle inequality to w and $z-w$ implies that $|z| = |w + (z - w)| \leq |w| + |z - w|$, so $|z| - |w| \leq |z - w|$. Swapping the roles of z and w then shows that $|w| - |z| \leq |w - z| = |z - w|$. Thus, since $||z| - |w||$ equals either $|z| - |w|$ or $|w| - |z|$, we have that $||z| - |w|| \leq |z - w|$, as required. □

It will be useful at different times to consider both all the possible values of the angle θ in the polar and exponential forms of a complex number, and a uniquely-defined value which lies in a given range.

Definition 1.1.17. Let $z \in \mathbb{C}$ be a non-zero complex number. Then in any exponential representation $z = re^{i\theta}$, we say θ is an *argument* of z. We define $\arg(z)$ be the set of all possible arguments of z, and we define $\mathrm{Arg}(z)$ to be the unique argument θ which lies in the interval $(-\pi, \pi]$. Thus

$$\arg(z) = \left\{\, \theta : z = |z|\, e^{i\theta} \,\right\} = \{\, \mathrm{Arg}(z) + 2\pi k : k \in \mathbb{Z} \,\}\,, \tag{1.3}$$

where $-\pi < \mathrm{Arg}(z) \leq \pi$ satisfies $z = |z|\, e^{i\,\mathrm{Arg}(z)}$. We call $\mathrm{Arg}(z)$ the *principal value* of the argument.

Remark 1.1.18. (i) Note carefully that $\arg(z)$ is a *set* of values, and that $\mathrm{Arg}(z)$ is one well-defined value in the range $(-\pi, \pi]$.
(ii) The value $\mathrm{Arg}(z)$ as a function of z has a discontinuity at all points z on the negative real axis, since for $x, \varepsilon > 0$, we have that $\lim_{\varepsilon\to 0}(-x \pm i\varepsilon) = -x$, but $\lim_{\varepsilon\to 0} \mathrm{Arg}(-x \pm i\varepsilon) = \pm\pi$.

The next result gives the expected formulae for arguments, based on their role as exponents, with appropriate caveats.

Proposition 1.1.19. Let $z, w \in \mathbb{C}$ be non-zero. Then

(i) $\arg(zw) = \arg(z) + \arg(w)$ and $\arg(\overline{z}) = -\arg(z)$, where for any subsets $A, B \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$, we write

$$A + B = \{\, a + b : a \in A,\ b \in B \,\} \quad \text{and} \quad \lambda A = \{\, \lambda a : a \in A \,\};$$

and

(ii) $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$ and $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$, where both equations hold modulo 2π; that is, there exist $k, l \in \mathbb{Z}$ such that

$$\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w) + 2k\pi, \quad \text{and} \quad \operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z) + 2l\pi.$$

Proof. Exercise. □

1.2. Topology of the complex plane. We shall need some definitions of certain types of subsets of the complex plane.

Definition 1.2.1. Let $z_0 \in \mathbb{C}$, and $\varepsilon > 0$.

- The *open ε-disc centred at z_0* is the set

$$D_\varepsilon(z_0) = \{\, z \in \mathbb{C} : |z - z_0| < \varepsilon \,\}.$$

- The *closed ε-disc centred at z_0* is the set

$$\overline{D}_\varepsilon(z_0) = \{\, z \in \mathbb{C} : |z - z_0| \leq \varepsilon \,\}.$$

  (The bar over $D_\varepsilon(z_0)$ denotes (topological) closure, not complex-conjugation.)
- The *punctured ε-disc centred at z_0* is the set

$$D'_\varepsilon(z_0) = \{\, z \in \mathbb{C} : 0 < |z - z_0| < \varepsilon \,\} = D_\varepsilon(z_0) \setminus \{z_0\}.$$

Given these we define the notion of an open set in $\mathbb{C}$, which should be familiar from the corresponding definition in $\mathbb{R}^n$.

Definition 1.2.2. A subset U of $\mathbb{C}$ is *open* if for all $z_0 \in U$, there exists $\varepsilon > 0$ (depending on z_0) such that $D_\varepsilon(z_0) \subseteq U$. A subset F of $\mathbb{C}$ is *closed* if its complement $\mathbb{C} \setminus F$ is open. A *neighbourhood* of a point $z_0 \in \mathbb{C}$ is an open set which contains z_0.

Lemma 1.2.3. Let $z_0 \in \mathbb{C}$, and $\varepsilon > 0$. Then $D_\varepsilon(z_0)$ and $D'_\varepsilon(z_0)$ are open, and $\overline{D}_\varepsilon(z_0)$ is closed.

Proof. Exercise. □

Definition 1.2.4. Let $S \subseteq \mathbb{C}$.

- A point $z_0 \in \mathbb{C}$ is a *limit point of S* if for all $\varepsilon > 0$ the set $D'_\varepsilon(z_0) \cap S$ is non-empty.
- The *closure of S* is the set $\overline{S}$ which contains all the elements of S and all the limit points of S.

Remark 1.2.5. (i) Notice that it is not true in general that $z \in S$ implies that z is a limit point of S.

(ii) That a point z is a limit point of a set S means that there are points arbitrarily close but not equal to z, in the set S.

Lemma 1.2.6. Let $S \subseteq \mathbb{C}$. Then S is closed if and only if $S = \overline{S}$.

Proof. Exercise. □

Definition 1.2.7. Let $S \subseteq \mathbb{C}$. We say that S is *bounded* if there exists $M > 0$ such that $|z| < M$ for all $z \in S$. We say that a set is *unbounded* if it is not bounded.

The definitions for complex sequences $z_n \in \mathbb{C}$ are entirely analogous to those for real sequences. We shall index sequences from $n = 0$, so for our purposes $\mathbb{N}$ includes zero.

Definition 1.2.8. Let $z_n \in \mathbb{C}$ be a complex sequence, and $z \in \mathbb{C}$. Then z *is the limit of* z_n *as* n *tends to* ∞, written $z_n \to z$ as $n \to \infty$, or $\lim_{n\to\infty} z_n = z$, if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon \text{ whenever } n \geq N.$$

Lemma 1.2.9. Let $z_n \in \mathbb{C}$ be a complex sequence, where $z_n = a_n + ib_n$ for $a_n, b_n \in \mathbb{R}$, for each n. Then $z = \lim_{n\to\infty} z_n$ if and only if $\mathrm{Re}(z) = \lim_{n\to\infty} a_n$ and $\mathrm{Im}(z) = \lim_{n\to\infty} b_n$. In particular a complex sequence converges if and only if the (real) sequences of its real and imaginary parts converge.

Proof. Exercise. □

Lemma 1.2.10. Let $S \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. Then $z \in \overline{S}$ if and only if there exists a sequence $z_n \in S$ such that $z = \lim_{n\to\infty} z_n$.

Proof. Exercise. □

Definition 1.2.11. Let $z_n \in \mathbb{C}$ be a sequence. Then z_n is *Cauchy* if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|z_n - z_m| < \varepsilon \text{ whenever } n, m \geq N.$$

Lemma 1.2.12. Let $z_n \in \mathbb{C}$ be a sequence. Then z_n has a limit if and only if z_n is Cauchy.

Proof. Exercise, using the completeness of the real numbers. □

Remark 1.2.13. The forwards implication of this lemma is straightforward and unsurprising. The reverse is a much deeper result, and amounts to the assertion that the complex numbers are complete.

Definition 1.2.14. Let $z_n \in \mathbb{C}$ be a sequence. We say that z_n is *bounded* if the set $\{ z_n : n \in \mathbb{N} \}$ is a bounded set, i.e. if there exists $M > 0$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$.

Lemma 1.2.15 (Bolzano–Weierstrass)**.** Let $z_n \in \mathbb{C}$ be a bounded sequence. Then z_n has a convergent subsequence, i.e. there exist indices $n_k \in \mathbb{N}$ for $k \in \mathbb{N}$ and $z \in \mathbb{C}$ such that $z_{n_k} \to z$ as $k \to \infty$.

Proof. Exercise, using the real Bolzano–Weierstrass Theorem. □

1.3. Complex-valued functions. Consider a function $f\colon \mathbb{C} \to \mathbb{C}$. Then for each complex number $z = x + iy$, the value $f(z)$ is a complex number, with a real part and an imaginary part. These will evidently depend on the value of z, i.e. on the values of the real and imaginary parts x and y of z. Thus the function f can in fact be described in terms of two real-valued functions, each depending on two real variables:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y),$$

where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ define the real and imaginary parts of f respectively. We abbreviate this as $f = u + iv$. Similar considerations apply to functions defined only on a subset S of $\mathbb{C}$.

We discuss complex functions $f\colon \mathbb{C} \to \mathbb{C}$ in much the same way we discuss real functions from $\mathbb{R}$ to $\mathbb{R}$. In particular we can discuss boundedness, limits, and continuity.

Definition 1.3.1. Let $S \subseteq \mathbb{C}$, and $f\colon S \to \mathbb{C}$. Then f is *bounded* if the set $f(S) = \{ f(z) : z \in S \}$ is a bounded set, i.e. if there exists $M > 0$ such that $|f(z)| < M$ for all $z \in S$.

Definition 1.3.2. Let $S \subseteq \mathbb{C}$, $z_0 \in \overline{S}$, $f \colon S \to \mathbb{C}$, and $a_0 \in \mathbb{C}$. Then a_0 is the *limit of $f(z)$ as z tends to z_0* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - a_0| < \varepsilon \text{ whenever } z \in S \text{ satisfies } 0 < |z - z_0| < \delta.$$

We use the usual alternative terminologies for limits, saying $\lim_{z\to z_0} f(z) = a_0$, $f(z) \to a_0$ as $z \to z_0$, and $f(z)$ tends, or converges, to a_0 as z tends to z_0.

Lemma 1.3.3. Let $S \subseteq \mathbb{C}$, $z_0 = x_0 + iy_0 \in \overline{S}$, $f \colon S \to \mathbb{C}$ be of the form $f = u + iv$, and $a_0 \in \mathbb{C}$. Then $a_0 = \lim_{z\to z_0} f(z)$ if and only if $\operatorname{Re}(a_0) = \lim_{(x,y)\to(x_0,y_0)} u(x, y)$ and $\operatorname{Im}(a_0) = \lim_{(x,y)\to(x_0,y_0)} v(x, y)$.

Proof. Exercise. □

Lemma 1.3.4. Let $S \subseteq \mathbb{C}$, $z_0 \in \overline{S}$, $f \colon S \to \mathbb{C}$ satisfy $\lim_{z\to z_0} f(z) = a_0$ for some $a_0 \in \mathbb{C}$, and $w_n \in S \setminus \{z_0\}$ be a sequence such that $w_n \to z_0$. Then $\lim_{n\to\infty} f(w_n) = a_0$.

Proof. Exercise. □

Lemma 1.3.5 (Algebra of limits)**.** Let $S \subseteq \mathbb{C}$, $z_0 \in \overline{S}$, and $f, g \colon S \to \mathbb{C}$ be such that $\lim_{z\to z_0} f(z) = a_0$ and $\lim_{z\to z_0} g(z) = b_0$. Then

(i) $\lim_{z\to z_0}(f(z) + g(z)) = a_0 + b_0$;
(ii) $\lim_{z\to z_0}(f(z)g(z)) = a_0 b_0$; and
(iii) if $b_0 \neq 0$, then $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{a_0}{b_0}$.

Proof. Exercise. □

As with real functions, limits may or may not exist, and may or may not equal the value of the function at that point, if the function is defined at that point at all.

Definition 1.3.6. Let $S \subseteq \mathbb{C}$, $f \colon S \to \mathbb{C}$, and $z_0 \in S$. Then f is *continuous at z_0* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z_0) - f(z)| < \varepsilon \text{ whenever } z \in S \text{ satisfies } |z_0 - z| < \delta.$$

This precisely says that the limit of $f(z)$ as z tends to z_0 exists and equals $f(z_0)$.

We say that f is *continuous on S*, or just *continuous*, if f is continuous at z_0 for all $z_0 \in S$.

Lemma 1.3.7. Let $f \colon \mathbb{C} \to \mathbb{C}$ be such that $f = u + iv$, and $z_0 = x_0 + iy_0 \in \mathbb{C}$. Then f is continuous at z_0 if and only if u and v are continuous at (x_0, y_0).

Proof. Exercise. □

Lemma 1.3.8. Let $f \colon \mathbb{C} \to \mathbb{C}$. Then f is continuous if and only if the preimage $f^{-1}(U) = \{\, z \in \mathbb{C} : f(z) \in U \,\}$ is open for all open $U \subseteq \mathbb{C}$.

Proof. Exercise. □

Lemma 1.3.9. Let $f, g \colon \mathbb{C} \to \mathbb{C}$ be continuous at $z_0 \in \mathbb{C}$. Then

(i) the function $f + g$ is continuous at z_0;
(ii) the function fg is continuous at z_0; and
(iii) if $g(z_0) \neq 0$, then the function $\frac{f}{g}$ is continuous at z_0.

Proof. Exercise. □

Lemma 1.3.10. Let $S \subseteq \mathbb{C}$ be a closed and bounded set, and $f \colon S \to \mathbb{C}$ be continuous. Then $f(S)$ is closed and bounded.

Proof. Exercise. □

1.4. Complex differentiability and holomorphicity. Similarly we can discuss differentiability of complex functions. Whereas continuity did not see much change from the real case, differentiability turns out to be a fount of riches. We make the analogous definition to the real case.

Definition 1.4.1. Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, and $f\colon U \to \mathbb{C}$. Then f is *differentiable at* z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. If this limit exists, then we define the *derivative of* f *at* z_0, $f'(z_0)$, to be the value of this limit.

Lemma 1.4.2. Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, and $f\colon U \to \mathbb{C}$ be differentiable at z_0. Then f is continuous at z_0.

Proof. Exercise. □

Lemma 1.4.3. Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, and $f, g\colon U \to \mathbb{C}$ be differentiable at z_0. Then

(i) the function $f + g$ is differentiable at z_0, with
$$(f + g)'(z_0) = f'(z_0) + g'(z_0);$$
(ii) the function fg is differentiable at z_0, with
$$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0);$$
and

(iii) if $g(z_0) \neq 0$, then the function $\frac{f}{g}$ is differentiable at z_0, with
$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Proof. Exercise. □

Lemma 1.4.4 (Chain rule)**.** Let $z_0 \in \mathbb{C}$, U be a neighbourhood of z_0, $g\colon U \to \mathbb{C}$ be such that $g(U)$ is a neighbourhood of $g(z_0)$, and $f\colon g(U) \to \mathbb{C}$. Suppose furthermore that g is differentiable at z_0 and f is differentiable at $g(z_0)$. Then the composition $f \circ g\colon U \to \mathbb{C}$ is differentiable at z_0 and

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

Proof. Exercise. □

The fact that in the definition of the derivative, z can approach z_0 from any direction in the complex plane turns out to have serious consequences.

Theorem 1.4.5 (Cauchy–Riemann equations)**.** Let $z_0 = x_0 + iy_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, and $f\colon U \to \mathbb{C}$ be differentiable at z_0, where $f = u + iv$. Then

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0). \tag{1.4}$$

These are the *Cauchy–Riemann equations.*

Proof. We consider taking the limit in the definition of the derivative in two ways, from both the real and imaginary directions. For $h > 0$ we consider first the real direction,

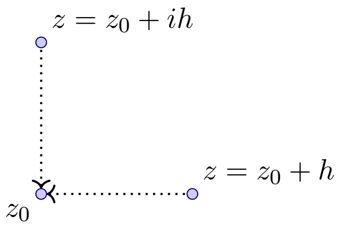


FIGURE 1. Two ways for z to approach z_0

approaching z_0 via $z = z_0 + h$, and see that

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{z_0 + h - z_0} \\
&= \lim_{h\to 0} \frac{f((x_0+h) + iy_0) - f(x_0+iy_0)}{h} \\
&= \lim_{h\to 0} \frac{(u(x_0+h,y_0) + iv(x_0+h,y_0)) - (u(x_0,y_0) + iv(x_0,y_0))}{h} \\
&= \lim_{h\to 0} \left(\frac{u(x_0+h,y_0) - u(x_0,y_0)}{h} + i \frac{v(x_0+h,y_0) - v(x_0,y_0)}{h} \right) \\
&= \lim_{h\to 0} \frac{u(x_0+h,y_0) - u(x_0,y_0)}{h} + i \lim_{h\to 0} \frac{v(x_0+h,y_0) - v(x_0,y_0)}{h} \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0).
\end{aligned}$$

These two partial derivatives exist by lemma 1.3.3. This gives us one expression for $f'(z_0)$ in terms of the partial derivatives of the real functions u and v. We now approach z_0 from the imaginary direction via $z = z_0 + ih$, for $h > 0$, and similarly see that

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+ih) - f(z_0)}{z_0 + ih - z_0} \\
&= \lim_{h\to 0} \frac{f(x_0 + i(y_0+h)) - f(x_0+iy_0)}{ih} \\
&= \lim_{h\to 0} \frac{(u(x_0,y_0+h) + iv(x_0,y_0+h)) - (u(x_0,y_0) + iv(x_0,y_0))}{ih} \\
&= \lim_{h\to 0} \left(-i\, \frac{u(x_0,y_0+h) - u(x_0,y_0)}{h} + \frac{v(x_0,y_0+h) - v(x_0,y_0)}{h} \right) \\
&= -i \lim_{h\to 0} \frac{u(x_0,y_0+h) - u(x_0,y_0)}{h} + \lim_{h\to 0} \frac{v(x_0,y_0+h) - v(x_0,y_0)}{h} \\
&= -i \frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0).
\end{aligned}$$

Again, the two partial derivatives exist by lemma 1.3.3. This gives us a different expression for $f'(z_0)$ in terms of the partial derivatives of the real functions u and v.

Since the derivative of f at z_0 is well-defined, the two expressions for $f'(z_0)$ must agree. Comparing the real parts and the imaginary parts of the two expressions for $f'(z_0)$, we see respectively that indeed

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0), \quad \text{and} \quad \frac{\partial v}{\partial x}(x_0,y_0) = -\frac{\partial u}{\partial y}(x_0,y_0). \qquad \square$$

Thus if a complex function is differentiable at a point $z_0 = x_0 + iy_0$, then the Cauchy–Riemann equations hold at (x_0, y_0). The converse is not true—we need an extra condition of continuity of the partial derivatives at the appropriate point.

Theorem 1.4.6. Let $z_0 = x_0 + iy_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, and $f : U \to \mathbb{C}$ be such that $f = u + iv$, for functions u and v which are continuously differentiable on a neighbourhood of (x_0, y_0), and satisfy the Cauchy–Riemann equations at (x_0, y_0). Then f is differentiable at z_0.

Proof. Omitted. □

Definition 1.4.7. Let $z_0 \in \mathbb{C}$. We say that a complex function f is *holomorphic at* z_0 if there exists a neighbourhood U of z_0 on which f is defined and differentiable. If a function $f : U \to \mathbb{C}$ is holomorphic at z_0 for every $z_0 \in U$, then we say f is *holomorphic on* U, or just *holomorphic*.

Remark 1.4.8.
(i) Note carefully that being holomorphic at a point is a stronger condition than being differentiable at a point. For a function to be holomorphic at a point z_0, it must be differentiable at all points in some neighbourhood of z_0, not just at z_0 itself.
(ii) It follows from the definition that if a function is holomorphic at a point, then it is holomorphic in a neighbourhood of that point.

Example 1.4.9. The function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = z$ is holomorphic. Let $z_0 \in \mathbb{C}$. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1,$$

so $f'(z_0)$ exists and equals 1 for all z_0.

Example 1.4.10. The function $g : \mathbb{C} \to \mathbb{C}$ defined by $g(z) = \overline{z}$ is not differentiable anywhere. We have that $g(z) = g(x + iy) = x - iy$, so $g = u + iv$ where $u(x, y) = x$ and $v(x, y) = -y$. Thus at any point $z = x + iy \in \mathbb{C}$ we have that

$$\frac{\partial u}{\partial x}(x, y) = 1 \neq -1 = \frac{\partial v}{\partial y}(x, y),$$

so the Cauchy-Riemann equations do not hold, and so g is not differentiable.

Example 1.4.11. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = |z|^2$. In the form $f = u + iv$, this function has $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Therefore for all $z = x + iy \in \mathbb{C}$, we have that

$$\frac{\partial u}{\partial x}(x, y) = 2x, \quad \frac{\partial u}{\partial y}(x, y) = 2y, \quad \text{and} \quad \frac{\partial v}{\partial y}(x, y) = 0 = \frac{\partial v}{\partial x}(x, y).$$

Thus the Cauchy–Riemann equations hold only when $x = 0$ and $y = 0$, i.e. at $z = 0$. Therefore f can only be differentiable at the origin. This immediately implies that f is nowhere holomorphic, since there is no non-empty open set on which f is differentiable. Since the partial derivatives are continuous everywhere, theorem 1.4.6 implies that f is indeed differentiable at the origin.

Holomorphic functions are intimately related to harmonic functions.

Definition 1.4.12. Let $h : \mathbb{R}^2 \to \mathbb{R}$. Then h is *harmonic* if it satisfies for all $(x, y) \in \mathbb{R}^2$ the partial differential equation known as Laplace's equation:

$$\frac{\partial^2 h}{\partial x^2}(x, y) + \frac{\partial^2 h}{\partial y^2}(x, y) = 0. \tag{1.5}$$

Lemma 1.4.13. Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable, i.e. all the second partial derivatives of u and v exist and are continuous, and suppose the function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic on $\mathbb{C}$. Then u and v are harmonic.

Proof. We simply apply the Cauchy–Riemann equations (1.4) at any point $(x, y) \in \mathbb{R}^2$:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = \frac{\partial}{\partial x}\frac{\partial u}{\partial x}(x,y) + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}(x,y) &= \frac{\partial}{\partial x}\frac{\partial v}{\partial y}(x,y) - \frac{\partial}{\partial y}\frac{\partial v}{\partial x}(x,y) \\ &= \frac{\partial^2 v}{\partial x\,\partial y}(x,y) - \frac{\partial^2 v}{\partial y\,\partial x}(x,y) \\ &= 0.\end{aligned}$$

The last line follows because the mixed partial derivatives are equal whatever order the derivatives are taken, because v is twice differentiable. In fact, we will see later that this assumption is redundant, because the fact that f is holomorphic at a point z implies that it is in fact infinitely differentiable at that point, and hence so are its real and imaginary parts.

Hence u is harmonic. Applying this argument to the holomorphic function $-if = -i(u + iv) = v - iu$ shows that v is harmonic. □

Definition 1.4.14. Let $U \subseteq \mathbb{R}^2$ be open, and let $u\colon U \to \mathbb{R}$ be harmonic. We say that a harmonic function $v\colon U \to \mathbb{R}$ is the *harmonic conjugate of* u if the complex-valued function $f = u + iv$ is holomorphic on U.

Exercise 1.4.15. Let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $u(x, y) = x^3 - 3xy^2 + y$.

(i) Prove that u is harmonic.
(ii) Prove that u is the real part of a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ by constructing a harmonic conjugate $v\colon \mathbb{R}^2 \to \mathbb{R}$.
(iii) Write this holomorphic function f explicitly as a function of z, where $z = x + iy$.

1.5. Polynomials and rational functions. We now build up a collection of holomorphic functions. We have already seen that the function $f(z) = z$ is holomorphic on the whole complex plane $\mathbb{C}$. In this section we will show that this is true also of any polynomial $P(z)$. We will also see that ratios of polynomials are holomorphic everywhere except for a finite set of points where the denominator equals 0.

Definition 1.5.1. Suppose $P\colon \mathbb{C} \to \mathbb{C}$ is a polynomial. The *degree* of P, written $\deg(P)$, is the highest power of the variable in the definition of P. Thus if $P(z) = \sum_{n=0}^{N} a_n z^n$ for some $N \in \mathbb{N}$ and complex numbers $a_0, \ldots, a_N$, where $a_N \neq 0$, then $\deg(P) = N$.

Lemma 1.5.2. Let $z_0 \in \mathbb{C}$, and suppose f and g are complex functions which are holomorphic at z_0. Then

(i) the function $f + g$ is holomorphic at z_0;
(ii) the function fg is holomorphic at z_0; and
(iii) if $g(z_0) \neq 0$, then the function $\frac{f}{g}$ is holomorphic at z_0.

Proof. By definition there exist neighbourhoods U_1 and U_2 of z_0 such that f and g are defined and differentiable on U_1 and U_2 respectively. Define $U = U_1 \cap U_2$. Then U is a neighbourhood of z_0 such that both f and g are defined and differentiable on U. Then for any point z in U, lemma 1.4.3 implies that the functions $f + g$ and fg are differentiable at z. Hence $f + g$ and fg are holomorphic at z_0, by definition.

Suppose that $g(z_0) \neq 0$. We have to take a little more care when finding a neighbourhood of z_0 on which the function f/g is defined, since g could take the value 0 at certain points of U_2 and hence of U defined in the previous paragraph. However, g is differentiable at z_0, hence is continuous at z_0, by lemma 1.4.2. So since $g(z_0) \neq 0$ by assumption, there is a neighbourhood V of z_0 such that $g(z) \neq 0$ for all $z \in V$. Define $W = U \cap V$. Then W is a neighbourhood of z_0 on which f and g are defined, differentiable, and g is non-zero. Therefore for any point z in W, lemma 1.4.3 implies that the function f/g is differentiable at z. Hence f/g is holomorphic at z_0, by definition. □

By an easy induction it follows that any finite sum or product of holomorphic functions is holomorphic at all common points of holomorphicity.

Corollary 1.5.3. Let $N \in \mathbb{N}$, and $a_0, \dots, a_N \in \mathbb{C}$. Then the complex polynomial $P(z) = \sum_{n=0}^{N} a_n z^n$ is holomorphic on $\mathbb{C}$, and $P'(z_0) = \sum_{n=1}^{N} n a_n z_0^{n-1}$.

Proof. Holomorphicity follows by the preceding remark and since constant functions and the function $f(z) = z$ are holomorphic (see example 1.4.9). The formula for the derivative follows by induction and the algebra of derivatives, lemma 1.4.3. □

Definition 1.5.4. Let $P, Q \colon \mathbb{C} \to \mathbb{C}$ be two polynomials, Then the function $R \colon \{\, z \in \mathbb{C} : Q(z) \neq 0 \,\} \to \mathbb{C}$ defined by $R(z) = P(z)/Q(z)$ is a *rational function*.

Remark 1.5.5. The domain of a rational function is always an open set, since it is of the form $Q^{-1}(\mathbb{C} \setminus \{0\})$ for some polynomial Q. The set $\mathbb{C} \setminus \{0\}$ is an open set, and a polynomial is continuous, so by lemma 1.3.8 the set $Q^{-1}(\mathbb{C} \setminus \{0\})$ is an open set.

Lemma 1.5.6. Let $P, Q \colon \mathbb{C} \to \mathbb{C}$ be polynomials. Then the rational function $R = P/Q$ is holomorphic on $\{\, z \in \mathbb{C} : Q(z) \neq 0 \,\}$.

Proof. This follows by lemma 1.5.2. □

Lemma 1.5.7. Let $U \subseteq \mathbb{C}$ be open, g be holomorphic on U, and let f be holomorphic on $g(U)$. Then the composition $f \circ g$ is holomorphic on U.

Proof. This follows by the chain rule, lemma 1.4.4. □

Finally we record a convenient way of abbreviating the Cauchy–Riemann equations. Define the first-order differential operators

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Lemma 1.5.8. Let $U \subseteq \mathbb{R}^2$ be open, and $u, v \colon U \to \mathbb{R}$. Then u and v satisfy the Cauchy–Riemann equations if and only if the function $f \colon U \to \mathbb{C}$ defined by $f = u + iv$ satisfies $\overline{\partial} f = 0$.

Proof. We simply calculate that

$$\begin{aligned} 2\overline{\partial} f = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} \ &= \frac{\partial}{\partial x}(u+iv) + i\frac{\partial}{\partial y}(u+iv) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\ &= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \end{aligned}$$

This expression is zero if and only if both the real and imaginary parts are zero, which holds if and only if the Cauchy–Riemann equations are satisfied. □

Exercise 1.5.9. Determine the sets on which the following functions are holomorphic:

(i) $f(z) = 1/(z^2 - 1)$;
(ii) $f(z) = z^2 + i\,|z|^2$;
(iii) $f(z) = z/(z^2 + 1)$;
(iv) $f(z) = \operatorname{Re}(z^2)$; and
(v) $f(z) = (z + 2i)/(z^2 + 4)$.

Exercise 1.5.10. Let $P \colon \mathbb{C} \to \mathbb{C}$ be the polynomial $P(z) = \sum_{n=0}^{N} a_n z^n$, where all the coefficients $a_0, \dots, a_N$ are real, and let $z_0 \in \mathbb{C}$ satisfy $P(z_0) = 0$. Prove that $P(\overline{z_0}) = 0$.

1.6. The complex exponential and related functions.

Definition 1.6.1. The *(complex) exponential function* is the function $\exp\colon \mathbb{C} \to \mathbb{C}$ defined by

$$\exp(z) = \exp(x+iy) := e^x \cos y + ie^x \sin y = e^x (\cos y + i \sin y). \tag{1.6}$$

Proposition 1.6.2. (i) The exponential function is holomorphic on $\mathbb{C}$, with $\exp'(z) = \exp(z)$ for all $z \in \mathbb{C}$.

(ii) For all $z, w \in \mathbb{C}$ we have that

$$\exp(z+w) = \exp(z)\exp(w). \tag{1.7}$$

(iii) For all $z \in \mathbb{C}$ we have that

$$\exp(z + 2\pi i) = \exp(z). \tag{1.8}$$

Proof. (i) We see that $\exp(z) = \exp(x+iy) = u(x,y) + iv(x,y)$ for $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$. These two functions u and v are continuously differentiable on $\mathbb{R}^2$ and we easily compute that

$$\frac{\partial u}{\partial x}(x,y) = e^x \cos y = \frac{\partial v}{\partial y}(x,y) \quad \text{and} \quad \frac{\partial v}{\partial x}(x,y) = e^x \sin y = -\frac{\partial u}{\partial y}(x,y),$$

for any $(x,y) \in \mathbb{R}^2$, hence the Cauchy–Riemann equations hold at all points $z = x+iy$. Therefore theorem 1.4.6 implies that exp is differentiable at all points z, hence that it is holomorphic on $\mathbb{C}$. Since we now know that the derivative is well-defined, we can evaluate the derivative by evaluating the limit of the difference quotient however we like, e.g. by considering real h only, to see that the derivative is given by

$$\exp'(z) = \exp'(x+iy) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = e^x \cos y + ie^x \sin y = \exp(z).$$

(ii) This is a straightforward calculation using properties of the real exponential and trigonometric functions. Applying the statement and induction we see that $\exp(in\theta) = (\exp(i\theta))^n$, which by definition gives the formula

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n.$$

(iii) By definition $\exp(2\pi i) = 1$, which combined with (1.7) implies the stated periodicity.

□

Remark 1.6.3. Equation (1.8) means in particular that the exponential is not one-to-one, in sharp contrast with the real exponential function.

Definition 1.6.4. We define the *complex cosine* and *complex sine* functions by

$$\cos(z) := \frac{\exp(iz) + \exp(-iz)}{2} \quad \text{and} \quad \sin(z) := \frac{\exp(iz) - \exp(-iz)}{2i}.$$

Lemma 1.6.5. Let $z = x+iy \in \mathbb{C}$. Then

(i) The complex sine and cosine functions are holomorphic at z with derivatives

$$\cos'(z) = -\sin(z) \quad \text{and} \quad \sin'(z) = \cos(z);$$

(ii) $\cos^2(z) + \sin^2(z) = 1$; and

(iii) $\cos(z+2\pi) = \cos(z)$ and $\sin(z+2\pi) = \sin(z)$.

Proof. Exercise. □

The usual functions

$$\tan(z) := \frac{\sin(z)}{\cos(z)}, \quad \cot(z) := \frac{1}{\tan(z)}, \quad \sec(z) := \frac{1}{\cos(z)}, \quad \text{and} \quad \csc(z) := \frac{1}{\sin(z)}$$

are defined and holomorphic whenever the relevant denominator is non-zero.

Lemma 1.6.6. Let $z, w \in \mathbb{C}$. Then

(i) $\sin(z + \pi/2) = \cos(z)$;
(ii) $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$; and
(iii) $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$.

Proof. Exercise. □

Lemma 1.6.7. Let $z = x + iy \in \mathbb{C}$. Then

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y \quad \text{and} \quad \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

Proof. Exercise. □

Remark 1.6.8. In contrast to their real counterparts, the complex trigonometric functions are *not* bounded. Using lemma 1.6.7, we see that fixing x and letting $y \to \pm\infty$ makes $|\sin(x + iy)| \to \infty$ and $|\cos(x + iy)| \to \infty$.

Definition 1.6.9. We define the *complex hyperbolic cosine* and the *complex hyperbolic sine* functions by

$$\cosh(z) := \frac{\exp(z) + \exp(-z)}{2} \quad \text{and} \quad \sinh(z) := \frac{\exp(z) - \exp(-z)}{2}.$$

The usual functions

$$\tanh(z) := \frac{\sinh(z)}{\cosh(z)}, \quad \coth(z) := \frac{1}{\tanh(z)}, \quad \operatorname{sech}(z) := \frac{1}{\cosh(z)}, \quad \text{and} \quad \operatorname{csch}(z) := \frac{1}{\sinh(z)}$$

are defined whenever the relevant denominator is non-zero.

Lemma 1.6.10. Let $z \in \mathbb{C}$. Then

$$\sinh(iz) = i \sin(z) \quad \text{and} \quad \cosh(iz) = \cos(z).$$

Proof. Exercise. □

Remark 1.6.11. It follows that the complex hyperbolic functions are holomorphic on $\mathbb{C}$.

Exercise 1.6.12. Determine the sets on which the following functions f are holomorphic:

(i) $f(z) = \exp(z^3)$; and
(ii) $f(z) = \exp(-|z|^2)$.

Exercise 1.6.13. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = f(x + iy) = e^{(x^2 - y^2)} \left(\cos(2xy) + i \sin(2xy)\right).$$

(i) Use the Cauchy–Riemann equations to prove that f is holomorphic on $\mathbb{C}$.
(ii) Give an alternative proof that f is holomorphic on $\mathbb{C}$ by writing $f(z)$ explicitly in terms of z.

1.7. The complex logarithm. This section introduces the logarithm of a complex number. We will see that in contrast with the real logarithm function, which is only defined for positive real numbers, the complex logarithm is defined for all non-zero complex numbers, but at a price: the function is not single-valued.

A warning about notation: we will use the notation "log" for the complex analogue of the natural logarithm, *not* for the logarithm base 10 (which will never appear). We will retain the notation "ln" for the real natural logarithm of a positive real number.

By analogy with the real natural logarithm, we define the (complex) logarithm as an inverse to the (complex) exponential function. The main difference is that whereas the real exponential is one-to-one, so that it has an inverse on its image (the positive real axis), the complex exponential, as we saw above, is periodic. This means that we cannot talk about *the* logarithm but of *a* logarithm, for a non-zero complex number will have, as we will see below, an infinite number of logarithms.

Definition 1.7.1. Let $z \in \mathbb{C}$ be non-zero. We define the multivalued function $\log(z)$ by

$$\log(z) := \{\, w \in \mathbb{C} : \exp(w) = z \,\}.$$

We call any element w of $\log(z)$ a *logarithm* of z.

Remark 1.7.2. Note carefully that $\log(z)$ is a *set* of values.

Lemma 1.7.3. Let $z, w \in \mathbb{C}$ be non-zero, $z = re^{i\theta}$ in exponential form. Then

(i)

$$\begin{aligned}\log(z) = \log(re^{i\theta}) = \ln|z| + i\arg(z) &= \{\, \ln|z| + i\psi : \psi \in \arg(z) \,\} \\ &= \{\, \ln|z| + i\operatorname{Arg}(z) + 2\pi i k : k \in \mathbb{Z} \,\} \\ &= \{\, \ln r + i\theta + i2\pi k : k \in \mathbb{Z} \,\};\end{aligned}$$

(ii) $\log(zw) = \log(z) + \log(w)$; and

(iii) $\log(1/z) = -\log(z)$;

where the last two statements are understood as equalities between sets.

Proof. (i) Let $a + ib \in \log(z)$. Then by definition

$$re^{i\theta} = z = \exp(a + ib) = e^a(\cos b + i\sin b) = e^a e^{ib},$$

which implies, by comparing polar forms, that $r = e^a$ and $e^{i\theta} = e^{ib}$. Solving the first equation implies that $a = \ln|z| = \ln r$. The second implies that $b \in \arg(z)$, i.e. that $b = \operatorname{Arg}(z) + 2\pi k$ for some $k \in \mathbb{Z}$, and that $\theta - b = 2\pi l$ for some $l \in \mathbb{Z}$, by (1.3). Clearly any complex number of this form is indeed a logarithm of z.

(ii) This follows by additivity of the exponential, (1.7).

(iii) This follows as a special case of (ii), since

$$\log(1/z) + \log(z) = \log(1) = \ln|1| + i\arg(1) = \{\, 0 + 2\pi i k : k \in \mathbb{Z} \,\},$$

so if $u \in \log(1/z)$, then there exists $v \in \log(z)$ and $k \in \mathbb{Z}$ such that $u + v = i2\pi k$, from which it follows that $-u + i2\pi k = v \in \log(z)$, i.e. that $-u \in \log(z)$. Thus $\log(1/z) \subseteq -\log(z)$. To prove the reverse set inclusion, we note that if $u \in \log(z)$, then since $\log(z) = \log(1/(1/z))$, by definition and the inclusion we just proved we have that $-u \in \log(1/(1/z)) \subseteq -\log(1/z)$, which implies that $u \in \log(1/z)$, as required. Hence the two sets are indeed equal. $\square$

We now discuss holomorphicity of the complex logarithm, for which we need to investigate its differentiability. To even begin taking the difference quotient, we need to choose a representative of the multivalued function for the two terms in the numerator. This must be the "same" representative, in some sense, for there to be any hope of the limit of the difference quotient existing. This choice we have to make is known as choosing a *branch* of the logarithm,

Definition 1.7.4. The *principal branch* of the logarithm function is the function $\mathrm{Log}\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}$ defined by

$$\mathrm{Log}(z) := \ln|z| + i\,\mathrm{Arg}(z)$$

for non-zero $z \in \mathbb{C}$, where Arg is the principal value of the argument function.

Remark 1.7.5. The function $\mathrm{Log}(z)$ is single-valued, but at a price: it is not continuous on the whole complex plane, since $\mathrm{Arg}(z)$ is not continuous on the whole complex plane. As observed in remark 1.1.18, the principal value $\mathrm{Arg}(z)$ of the argument function is discontinuous at all points on the non-positive real axis. Indeed, let $x, \varepsilon > 0$. Then $\lim_{\varepsilon\to 0}(-x \pm i\varepsilon) = -x$, whereas $\lim_{\varepsilon\to 0}\mathrm{Log}(-x \pm i\varepsilon) = \ln x \pm i\pi$. The non-positive real axis is an example of a *branch cut* for this function, and the origin is a *branch point*.

Definition 1.7.6. A *branch cut* is a subset L of the complex plane which is removed so that a holomorphic branch of a multivalued function may be defined on the remaining *cut plane* $\mathbb{C}\setminus L$. We shall only consider branch cuts that are half-lines or line segments. An endpoint of a branch cut is a *branch point*.

We shall most commonly consider branch cuts that are half-lines, so we shall introduce some notation for these. Let $z_0 \in \mathbb{C}$ and $\phi \in \mathbb{R}$. Then

$$L_{z_0,\phi} = \{\, z \in \mathbb{C} : z = z_0 + re^{i\phi} \text{ for } r \geq 0 \,\}$$

denotes the half-line from the point z_0 at angle ϕ. The notation $D_{z_0,\phi}$ will denote the cut plane with a branch point at z_0 and a branch cut along $L_{z_0,\phi}$, i.e. $D_{z_0,\phi} = \mathbb{C}\setminus L_{z_0,\phi}$. If $z_0 = 0$ we shall not notate it, i.e. we shall write $L_{0,\phi} = L_\phi$ and $D_{0,\phi} = D_\phi$. The cut plane associated with the principal branch of the argument and logarithm functions will just be denoted by D, i.e. $D = D_{-\pi}$.

Example 1.7.7.

(i) $L_{0,0} = L_0 = \{\, z \in \mathbb{C} : z = x \text{ for } x \geq 0 \,\}$.
(ii) $L_{i,\pi/2} = \{\, z \in \mathbb{C} : z = iy \text{ for } y \geq 1 \,\}$.

Definition 1.7.8. Let $\phi \in \mathbb{R}$. We define the branch $\mathrm{Arg}_\phi(z)$ of the argument function to take the values

$$\phi < \mathrm{Arg}_\phi(z) \leq \phi + 2\pi.$$

So $\mathrm{Arg} = \mathrm{Arg}_{-\pi}$. This gives rise to a branch $\mathrm{Log}_\phi(z)$ of the logarithm function defined by

$$\mathrm{Log}_\phi(z) = \ln|z| + i\,\mathrm{Arg}_\phi(z).$$

Lemma 1.7.9. Let $\phi \in \mathbb{R}$. Then the branch Log_ϕ of the logarithm is holomorphic on the cut plane D_ϕ, and its derivative is given by $\mathrm{Log}'_\phi(z) = \frac{1}{z}$ for all z in D_ϕ.

Proof. We can check that at all points in D_ϕ the function $\mathrm{Log}_\phi(z) = \ln|z| + i\,\mathrm{Arg}(z)$ has real and imaginary parts u and v which are continuously differentiable and satisfy the Cauchy–Riemann equations. Therefore theorem 1.4.6 implies that Log_ϕ is differentiable everywhere on D_ϕ, and therefore is holomorphic on D_ϕ.

To derive the formula for the derivative, we observe that for all $z \in D_\phi$, we have by definition that $z = \exp(\mathrm{Log}_\phi(z))$. Differentiating this expression using the chain rule, lemma 1.4.4, implies that

$$1 = \exp'(\mathrm{Log}_\phi(z))\,\mathrm{Log}'_\phi(z) = \exp(\mathrm{Log}_\phi(z))\,\mathrm{Log}'_\phi(z) = z\,\mathrm{Log}'_\phi(z),$$

which implies that $\mathrm{Log}'_\phi(z) = 1/z$, as required. □

The choice of branch is immaterial for many properties of the logarithm, although it is important that a choice be made if we need a function and not just a multivalued function. Different applications will in general require different branches.

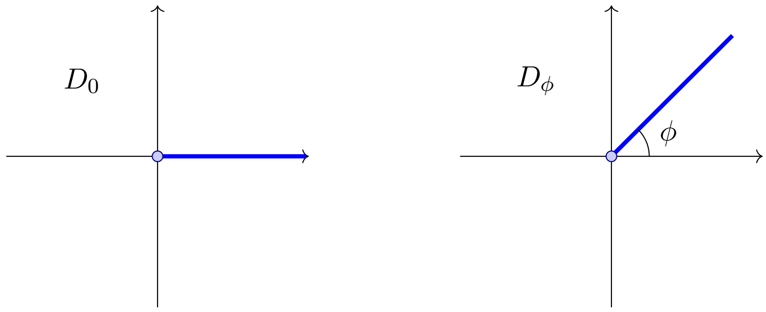


FIGURE 2. Two branch cuts with a branch point at 0.

Often we will want to choose a holomorphic branch of logarithm Log_ϕ so that the composition $\mathrm{Log}_\phi(g(z))$ is holomorphic on a certain set, where g is some given function g. Understanding where this composition is holomorphic just amounts to an application of the chain rule.

Lemma 1.7.10. Let $\phi \in \mathbb{R}$, $U \subseteq \mathbb{C}$ be open, and $g\colon U \to \mathbb{C}$ be holomorphic on U. Then the function $\mathrm{Log}_\phi(g(z))$ is holomorphic on $U \cap g^{-1}(D_\phi)$. In particular, if g is holomorphic on $\mathbb{C}$, then $\mathrm{Log}_\phi(g(z))$ is holomorphic on $g^{-1}(D_\phi)$, i.e. at points z such that $g(z) \in D_\phi$.

Proof. This is just a direct consequence of lemma 1.4.4 and the fact that Log_ϕ is holomorphic on D_ϕ. □

Example 1.7.11. (i) The function $\mathrm{Log}(z-1)$ is of the form $\mathrm{Log}(g(z))$ where $g(z) = z-1$ is holomorphic on $\mathbb{C}$. So by lemma 1.7.10 this is holomorphic on the set

$$g^{-1}(D) = \{\, z \in \mathbb{C} : g(z) \in D \,\} = \{\, z \in \mathbb{C} : z - 1 \in D \,\} = D_{1,-\pi}.$$

(ii) The function $\mathrm{Log}_{-\pi/4}(z-(1+i))$ is of the form $\mathrm{Log}_{-\pi/4}(g(z))$ where $g(z) = z-(1+i)$ is holomorphic on $\mathbb{C}$. So by lemma 1.7.10 this is holomorphic on the set

$$g^{-1}(D_{-\pi/4}) = \left\{\, z \in \mathbb{C} : g(z) \in D_{-\pi/4} \,\right\} = \left\{\, z \in \mathbb{C} : z-(1+i) \in D_{-\pi/4} \,\right\} = D_{1+i,-\pi/4}.$$

(iii) The function $\mathrm{Log}(1/z)$ is of the form $\mathrm{Log}(g(z))$ where $g(z) = 1/z$ is holomorphic on $\mathbb{C}\backslash\{0\}$. So by lemma 1.7.10 this function is holomorphic on the set $g^{-1}(D)\backslash\{0\}$, i.e. the set of all non-zero complex numbers z such that $1/z \in D$. This is the set of all non-zero complex numbers such that $1/z$ is not a non-positive real number. The value $1/z$ is a negative real number if and only if z is a negative real number, so the excluded values of z are exactly the negative real values and zero. So the function $\mathrm{Log}(1/z)$ is holomorphic on D. This is basically the argument that inversion $(z \mapsto 1/z)$ maps the negative real axis to itself.

Example 1.7.12. Let us try to find branches of the following logarithms that are holomorphic on $D_1(0)$.

(i) First consider $\log(z-1)$. Let $g(z) = z-1$. By lemma 1.7.10 the function $\mathrm{Log}_\phi(g(z))$ is holomorphic on $g^{-1}(D_{0,\phi}) = D_{1,\phi}$. If $\phi \in [-\pi/2, \pi/2]$, then $D_1(0) \subseteq D_{1,\phi}$, so any such ϕ will do, e.g. $\phi = 0$.

(ii) Now consider $\log(z+1)$, and let $g(z) = z+1$. Similarly, $\mathrm{Log}_\phi(g(z))$ is holomorphic on $g^{-1}(D_{0,\phi}) = D_{-1,\phi}$. If $\phi \in [\pi/2, 3\pi/2]$, then $D_1(0) \subseteq D_{-1,\phi}$, so any such ϕ will do, e.g. $\phi = \pi$.

(iii) Now let us try to find a holomorphic branch of $\log(z^2-1)$ on $D_1(0)$. We are in fact nearly done. By the previous two parts, the function $f(z) = \mathrm{Log}_0(z-1) + \mathrm{Log}_\pi(z+1)$ is holomorphic on $D_1(0)$ since it is a sum of two functions that are holomorphic there. Moreover, by lemma 1.7.3(ii),

$$\mathrm{Log}_0(z-1) + \mathrm{Log}_\pi(z+1) \in \log(z-1) + \log(z+1) = \log((z-1)(z+1)) = \log(z^2-1),$$

so $f(z) \in \log(z^2-1)$, and thus is a branch of $\log(z^2-1)$ that is holomorphic on $D_1(0)$.

Example 1.7.13. Consider differentiating the (multivalued) function $f(z) = \log(z^2 + 2iz + 2)$ at the point $z = i$. We need to choose a branch of the logarithm which is holomorphic at the point $i^2 + 2ii + 2 = -1$. Choose $\operatorname{Log}_0(z)$. Then, by the chain rule

$$f'(i) = \operatorname{Log}_0'(-1)(2i+2i) = \frac{4i}{-1} = -4i.$$

Any other valid branch will give the same result.

Exercise 1.7.14. Determine the sets on which the following functions are holomorphic:

(i) (a) $\operatorname{Log}(z-i)$;
    (b) $\operatorname{Log}(z-(1+i))$;
    (c) $\operatorname{Log}_0(z-1)$;
    (d) $\operatorname{Log}_0(z-i)$;
    (e) $\operatorname{Log}_0(z-(1+i))$;
    (f) $\operatorname{Log}_{-\pi/4}(z-1)$;
    (g) $\operatorname{Log}_{-\pi/4}(z-i)$;
(ii) (a) $2\operatorname{Log}(1/z)$;
    (b) $\operatorname{Log}(1/z^2)$;
(iii) (a) $\operatorname{Log}\left(1-\frac{1}{z}\right)$; and
    (b) $\operatorname{Log}\left(1-\frac{1}{z^2}\right)$.

Exercise 1.7.15.

(i) (a) Find a branch of $\log(z-1)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) < 1 \}$.
    (b) Find a branch of $\log(z+1)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > -1 \}$.
    (c) Find a branch of $\log(z^2-1)$ that is holomorphic on $\{ z \in \mathbb{C} : -1 < \operatorname{Re}(z) < 1 \}$.
(ii) (a) Find a branch of $\log(z-i)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$.
    (b) Find a branch of $\log(z+i)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$.
    (c) Find a branch of $\log(z^2+1)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$.

Exercise 1.7.16. Determine a branch of the function $f(z) = \log(z^2+2z+3)$ that is holomorphic at $z=-1$, and find its derivative at that point.

Exercise 1.7.17. Find all solutions $z \in \mathbb{C}$ of the equation $\cos(z) = \sin(z)$.

1.8. Complex powers. Having defined the complex logarithm, we may now define complex powers of complex numbers. In general, these too will be multivalued.

Definition 1.8.1. Let $\alpha, z \in \mathbb{C}$, with $z \neq 0$. Then we define the *α-th power of z* by

$$z^\alpha = \{ \exp(\alpha w) : w \in \log(z) \}.$$

Remark 1.8.2. From lemma 1.7.3 and (1.7), we may rewrite this as the following alternative expressions:

$$z^\alpha := \{ \exp(\alpha \ln|z| + i\alpha \operatorname{Arg}(z) + i\alpha 2\pi k) : k \in \mathbb{Z} \} = \{ \exp(\alpha \operatorname{Log}(z)) \exp(i\alpha 2\pi k) : k \in \mathbb{Z} \}. \tag{1.9}$$

Depending on the value of α, we will have either one, finitely many, or infinitely many values of z^α.

Theorem 1.8.3. Let $\alpha, z \in \mathbb{C}$, with $z \neq 0$. Then

(i) if $\alpha \in \mathbb{Z}$ there is exactly one value of z^α;
(ii) if $\alpha = p/q$ where p, q are coprime integers, with $q \neq 0$, there are exactly q values of z^α;
(iii) if α is irrational or non-real, there are infinitely many values of z^α.

Proof. (i) For any integer k, since α is an integer, so too is αk, and therefore $\exp(i2\pi\alpha k) = 1$. Therefore by (1.9) there is only one element in the set z^α, viz

$$\exp(\alpha \operatorname{Log}(z)) = \exp(\alpha \ln|z| + i\alpha \operatorname{Arg}(z)) = e^{\alpha \ln|z|}(\cos(\alpha \operatorname{Arg}(z)) + i \sin(\alpha \operatorname{Arg}(z)))$$
$$= \begin{cases} 1 & \alpha = 0, \\ \underbrace{z\, z\, \cdots\, z}_{\alpha \text{ times}} & \alpha > 0, \\ \frac{1}{z^{-\alpha}} & \alpha < 0. \end{cases}$$

(ii) Consider first, for a non-zero integer q, the expression (1.9), which, since $\operatorname{Log}(1) = 0$, gives that

$$1^{1/q} = \{\, \exp(\operatorname{Log}(1)/q) \exp(i2\pi k/q) : k \in \mathbb{Z} \,\} = \{\, \exp(i2\pi k/q) : k \in \mathbb{Z} \,\}.$$

The $2\pi i$-periodicity of exp implies that there are exactly q distinct values of $\exp(i2\pi k/q)$, given by

$$1^{1/q} = \{\, \exp(i2\pi k/q) : k \in \mathbb{Z} \,\} = \left\{1, \omega, \omega^2, \ldots, \omega^{q-1}\right\},$$

where $\omega := \exp(i2\pi/q)$. Since we can write $z = |z| \exp(i \operatorname{Arg}(z))$, we have that

$$\exp(\operatorname{Log}(z)/q) = \exp((\ln|z| + i \operatorname{Arg}(z))/q) = \exp(\ln|z|/q) \exp(i \operatorname{Arg}(z)/q)$$
$$= |z|^{1/q} \exp(i \operatorname{Arg}(z)/q),$$

hence

$$z^{1/q} = \{\, \exp(\operatorname{Log}(z)/q) \exp(i2\pi k/q) : k \in \mathbb{Z} \,\}$$
$$= \left\{ |z|^{1/q} \exp(i \operatorname{Arg}(z)/q)\omega^k : k = 0, \ldots, q-1 \right\}.$$

That is, $z^{1/q}$ has q distinct values given by multiplying any one value of $z^{1/q}$ by the q values ω^k for $k = 0, 1, \ldots, q-1$. The p/q-th powers of z are then found by taking the p powers of the values of $z^{1/q}$:

$$z^{p/q} = \left\{ |z|^{p/q} \exp(ip \operatorname{Arg}(z)/q)\omega^{pk} : k = 0, \ldots, q-1 \right\},$$

which remain q distinct values since p and q are coprime.

(iii) Suppose α is irrational and that there are integers k and k' such that $\exp(i\alpha 2\pi k) = \exp(i\alpha 2\pi k')$. Then by (1.7), $\exp(i\alpha 2\pi(k - k')) = 1$, which implies that $\alpha(k - k')$ is an integer, which since α is irrational implies that $k = k'$. Thus each different integer gives a different value for $\exp(i\alpha 2\pi k)$, so there are infinitely many values of z^α.

For $\alpha = a + ib$, where $b \neq 0$, we have, for any $k \in \mathbb{Z}$, that

$$\exp(i\alpha 2\pi k) = \exp(i(a + ib)2\pi k) = \exp(i2\pi ka) \exp(-2\pi kb),$$

where the real value $\exp(-2\pi kb)$ is the modulus of the complex number, and evidently takes infinitely many values as k varies through the integers.

□

As we have already started doing, we shall sometimes abuse notation and regard z^α as being one of the elements of the set as which it is formally defined, particularly in the case in which α is an integer, and therefore there is indeed only one value of z^α.

Definition 1.8.4. Let q be a positive integer. Then the q values

$$1^{1/q} = 1, \omega, \ldots, \omega^{q-1},$$

where $\omega := \exp(i2\pi/q)$, are the q *roots of unity*.

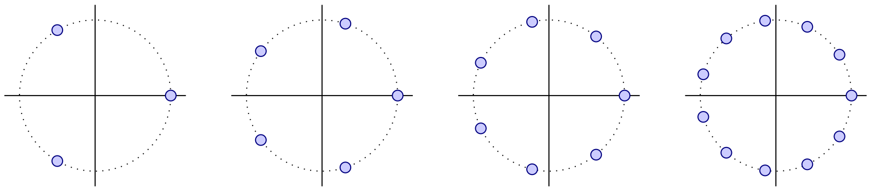


FIGURE 3. Some roots of unity.

Example 1.8.5. (i) Using the addition property (1.7) of the exponential,

$$z^2 = \exp(2\operatorname{Log}(z)) = \exp(\operatorname{Log}(z) + \operatorname{Log}(z)) = \exp(\operatorname{Log}(z))\exp(\operatorname{Log}(z)) = zz.$$

(ii) By definition,

$$\begin{aligned} i^i = \{\exp(iw) : w \in \log(i)\} &= \{\exp(i(\operatorname{Log}(i))\exp(i2\pi ik)) : k \in \mathbb{Z}\} \\ &= \{\exp(i(i\pi/2))\exp(-2\pi k) : k \in \mathbb{Z}\} \\ &= \left\{ e^{-\pi/2}e^{-2\pi k} : k \in \mathbb{Z} \right\}. \end{aligned}$$

Note that all these values are real.

Remark 1.8.6. We now explain why we use the notation $\exp(z)$ and not e^z for the exponential function. The former is a single-valued function defined as in equation (1.6), whereas the latter is the number $e = \exp(1)$ raised to the complex power z, which is a multivalued function.

Every branch of the logarithm gives rise to a branch of z^α.

Definition 1.8.7. For z in the domain D on which the principal branch of the logarithm is defined, we define the *principal branch of the function* z^α by $z^\alpha = \exp(\alpha \operatorname{Log}(z))$.

Lemma 1.8.8. Let $\alpha, \beta, z \in \mathbb{C}$, with $z \neq 0$. Then $z^\alpha z^\beta = z^{\alpha+\beta}$, where the principal branch of logarithm is chosen for each power function.

Proof. Exercise. □

Lemma 1.8.9. A branch of z^α is holomorphic on the domain D_ϕ on which the associated branch Log_ϕ of the logarithm is holomorphic, and for all $z \in D_\phi$,

$$\frac{d}{dz}z^\alpha = \frac{d}{dz}(\exp(\alpha \operatorname{Log}_\phi(z))) = \exp(\alpha \operatorname{Log}_\phi(z))\frac{\alpha}{z} = \alpha\frac{z^\alpha}{z} = \alpha z^{\alpha-1}.$$

Proof. The chain rule implies that the function $\exp(\alpha \operatorname{Log}_\phi(z))$ is differentiable at all points of D_ϕ, with the derivative given. Therefore it is holomorphic on D_ϕ. □

Exercise 1.8.10. Find the derivative of the principal branch of z^{1+i} at the point $z = i$.

Exercise 1.8.11. Show that it is not true in general that $(zw)^\alpha = z^\alpha w^\alpha$, for the principal branch of the power function in each case.

Example 1.8.12. The function $z^z = \exp(z \operatorname{Log}(z))$ is holomorphic on the set D on which Log is holomorphic, and has derivative, using the chain rule and product rule, given by

$$\frac{d}{dz}z^z = \frac{d}{dz}(\exp(z \operatorname{Log}(z)) = \exp(z \operatorname{Log}(z))\frac{d}{dz}(z \operatorname{Log}(z)) = \exp(z \operatorname{Log}(z))(\operatorname{Log}(z) + 1).$$

Example 1.8.13. Consider the multivalued function $f(z) = ((z-1)/(z+1))^{1/2}$. To define a branch of $f(z)$ holomorphic on a set $U \subseteq \mathbb{C}$, we need to find a holomorphic function g on U satisfying

$$g(z) \in \log\left(\frac{z-1}{z+1}\right). \tag{1.10}$$

(i) Suppose we want a branch of f that is holomorphic on $D_1(0)$. Using lemma 1.7.3(ii) and (iii), we have that

$$\log\left(\frac{z-1}{z+1}\right) = \log(z-1) + \log\left(\frac{1}{z+1}\right) = \log(z-1) + (-\log(z+1)).$$

So *any* two branches of logarithm Log_{ϕ_1} and Log_{ϕ_2} will satisfy, for $z \neq \pm 1$,

$$\mathrm{Log}_{\phi_1}(z-1) - \mathrm{Log}_{\phi_2}(z+1) \in \log\left(\frac{z-1}{z+1}\right).$$

So if ϕ_1 and ϕ_2 are chosen such that $g_1(z) = \mathrm{Log}_{\phi_1}(z-1)$ and $g_2(z) = \mathrm{Log}_{\phi_2}(z+1)$ are both holomorphic on $D_1(0)$, the function $g(z) = g_1(z) - g_2(z)$, a sum of holomorphic functions, is also holomorphic on $D_1(0)$. Then $f(z) = \exp(\frac{1}{2}g(z))$ is a holomorphic function on $D_1(0)$, where $g(z) \in \log((z-1)/(z+1))$. In other words it is a branch of $f(z)$ which is holomorphic on $D_1(0)$.

The function $g_1(z) = \mathrm{Log}_{\phi_1}(z-1)$ is holomorphic on D_{1,ϕ_1}. Choosing any $\phi_1 \in [-\pi/2, \pi/2]$ implies $D_1(0) \subseteq D_{1,\phi_1}$, so g_1 is holomorphic on $D_1(0)$. Similarly $g_2(z) = \mathrm{Log}_{\phi_2}(z+1)$ is holomorphic on D_{-1,ϕ_2}, so choosing $\phi_2 \in [\pi/2, 3\pi/2]$ implies that $D_1(0) \subseteq D_{-1,\phi_2}$, and so g_2 is holomorphic on $D_1(0)$.

(ii) Suppose now we want a branch of f that is holomorphic on $\mathbb{C} \setminus \overline{D}_1(0)$. Rewrite, for $z \neq 0$,

$$\frac{z-1}{z+1} = \frac{1-\frac{1}{z}}{1+\frac{1}{z}}.$$

By the same argument as above, we see that

$$\log\left(\frac{z-1}{z+1}\right) = \log\left(\frac{1-\frac{1}{z}}{1+\frac{1}{z}}\right) = \log(1-\tfrac{1}{z}) + (-\log(1+\tfrac{1}{z})).$$

So now any branches Log_{ϕ_3} and Log_{ϕ_4} of logarithm will satisfy

$$\mathrm{Log}_{\phi_3}(1-\tfrac{1}{z}) - \mathrm{Log}_{\phi_4}(1+\tfrac{1}{z}) \in \log(1-\tfrac{1}{z}) - \log(1+\tfrac{1}{z}) = \log\left(\frac{z-1}{z+1}\right).$$

So now we just need to make sure we choose holomorphic branches Log_{ϕ_3} and Log_{ϕ_4} so that the functions $g_3(z) = \mathrm{Log}_{\phi_3}(1-\frac{1}{z})$ and $g_4(z) = \mathrm{Log}_{\phi_4}(1+\frac{1}{z})$ are both holomorphic on $\mathbb{C} \setminus \overline{D}_1(0)$. The composition $g_3(z) = \mathrm{Log}_{\phi_3}(1-\frac{1}{z})$ is non-holomorphic on the set $\{\, z \in \mathbb{C} : 1-\frac{1}{z} \in L_{\phi_3} \,\}$. So we choose ϕ_3 so that everything in that set lies in $\overline{D}_1(0)$, because this is where we're allowed to be non-holomorphic. That is, we choose ϕ_3 such that $1-\frac{1}{z} \in L_{\phi_3}$ implies $|z| \le 1$, or, equivalently, that $\left|\frac{1}{z}\right| \ge 1$. But $1-\frac{1}{z} \in L_{\phi_3}$ implies that $1-\frac{1}{z} = re^{i\phi_3}$ for some $r \ge 0$, i.e. that $\frac{1}{z} = 1 - re^{i\phi_3}$. Then we can see geometrically that $\phi_3 \in [\pi/2, 3\pi/2]$ implies that $\left|\frac{1}{z}\right| = \left|1-re^{i\phi_3}\right| \ge 1$, as required. Similarly g_4 will only be non-holomorphic on the set $\{\, z \in \mathbb{C} : 1+\frac{1}{z} \in L_{\phi_4} \,\}$. So we choose ϕ_4 such that $1+\frac{1}{z} \in L_{\phi_4}$ implies that $\left|\frac{1}{z}\right| \ge 1$. Since $1+\frac{1}{z} \in L_{\phi_4}$ implies that $1+\frac{1}{z} = re^{i\phi_4}$ for some $r \ge 0$, this in turn implies that $\frac{1}{z} = re^{i\phi_4} - 1$, which in fact gives us exactly the same condition on ϕ_4 as on ϕ_3.

Given two appropriate branches of logarithm Log_{ϕ_3} and Log_{ϕ_4}, we then continue as before: define $g(z) = g_3(z) - g_4(z)$, which as a sum of functions holomorphic on $\mathbb{C} \setminus \overline{D}_1(0)$ is also holomorphic on $\mathbb{C} \setminus \overline{D}_1(0)$. Then again we're done: we just define $f(z) = \exp(\frac{1}{2}g(z))$, and this is a holomorphic function on $\mathbb{C} \setminus \overline{D}_1(0)$, where $g(z) \in \log((z-1)/(z+1))$. In other words it is a branch of $f(z)$ which is holomorphic on $\mathbb{C} \setminus \overline{D}_1(0)$.

Exercise 1.8.14. Find branches of the following multifunctions that are holomorphic on the given set:

(i) $(z-i)^{1/2}$ on $\{\, z \in \mathbb{C} : \mathrm{Im}(z) < 1 \,\}$; and
(ii) $\left(\frac{z-1}{z+1}\right)^{3/4}$ on $\mathbb{C} \setminus [-1,1]$;

Exercise 1.8.15. For each of the values $z = -1$, $1+i$, 0, and i, calculate the following values, if defined:

(i) the modulus $|z|$;
(ii) the argument $\arg(z)$;
(iii) the real and imaginary parts;
(iv) the exponential $\exp(z)$;
(v) the complex logarithm $\log(z)$; and
(vi) the complex power z^z.

2. Conformal maps and Möbius transformations

2.1. Conformal maps.

Definition 2.1.1. Let $U \subseteq \mathbb{C}$ be open, and let $f \colon U \to \mathbb{C}$. We say f is *conformal* if f preserves angles: i.e. if the angle between the images under f of two straight lines in U is equal to the angle between the two straight lines themselves. By applying this definition to tangents of differentiable curves, more generally we can say the same about the angles between curves at certain points.

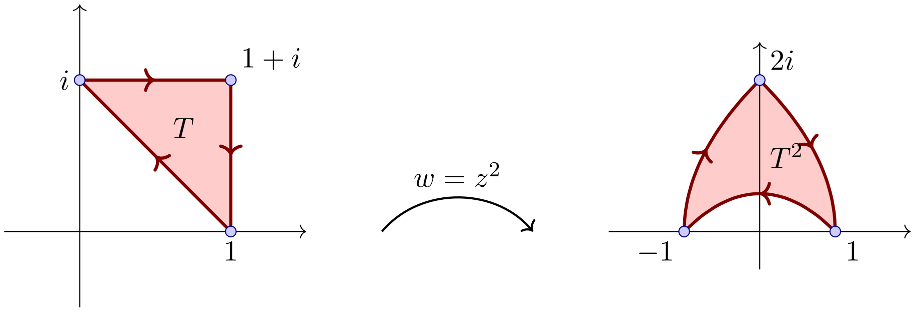
Theorem 2.1.2. Let $U \subseteq \mathbb{C}$ be open, and let $f \colon U \to \mathbb{C}$ be holomorphic. Then f preserves angles at every $z_0 \in U$ where $f'(z_0) \neq 0$. That is, if C_1 and C_2 are differentiable curves in U intersecting at a point $z_0 \in U$, then the angle between the tangents to C_1 and C_2 at z_0 is equal to the angle between the tangents to $f(C_1)$ and $f(C_2)$ at $f(z_0)$.

Proof. Let C be a regular curve in U, i.e. be the image of a continuously differentiable function $z \colon [t_0, t_1] \to U$, for some subinterval $[t_0, t_1] \subseteq \mathbb{R}$, where $z'(t) \neq 0$ for all t. Its image under f is a curve in the complex plane defined by $t \mapsto w(t) := f(z(t))$. Let $s_0 \in [t_0, t_1]$, $z_0 = z(s_0)$ and $w_0 = w(s_0) = f(z_0)$. Suppose $f'(z_0) \neq 0$. By the chain rule we have that $w'(s_0) = f'(z_0)z'(s_0)$. Since $z'(s_0) \neq 0$, we have that $w'(s_0) \neq 0$. Then the tangent vector $z'(s_0)$ to the curve C at z_0 and the tangent vector $w'(s_0)$ to the curve $f(C)$ at w_0 are related by multiplication by the non-zero complex number $f'(z_0)$. Writing $f'(z_0) = r_0 e^{i\theta_0}$ in polar form, we see that multiplication by $f'(z_0)$ consists of a rotation by an angle θ_0 and a dilation by r_0.

So now let C_1 and C_2 be two curves intersecting at z_0. Their images under f are the curves $f(C_1)$ and $f(C_2)$ which intersect at $f(z_0)$ in such a way that their tangents are obtained from the tangents of C_1 and C_2 at z_0 by a dilation and a rotation. But both of these operations preserve the angle between the two tangents. $\square$

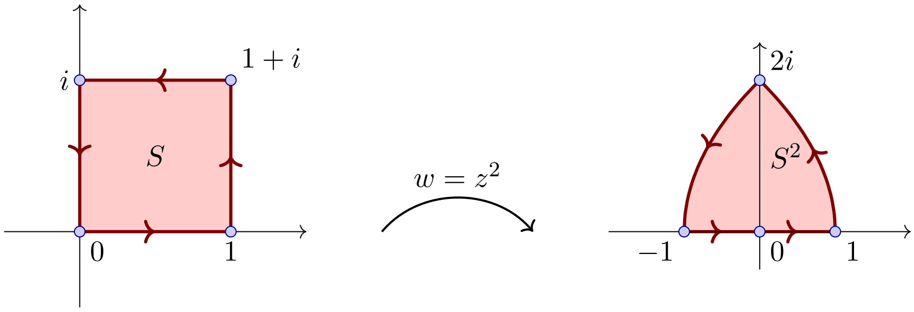
Example 2.1.3. Let T be the triangular region with vertices $(1, i, 1+i)$ in the complex plane, and consider its image under $w = z^2$. The vertices are mapped to $(1, -1, 2i)$, respectively. The edges of the triangle are parametrized by, for $t \in [0,1]$,

- $z_1(t) = t + i$, which is mapped to $w_1(t) = (z_1(t))^2 = (t^2 - 1) + 2it$, so $z_1'(t) = 1$ and $w_1'(t) = 2t + 2i$;
- $z_2(t) = 1 + i(1-t)$, which is mapped to $w_2(t) = (z_2(t))^2 = (1 - (1-t)^2) + 2i(1-t)$; so $z_2'(t) = -i$ and $w_2'(t) = 2(1-t) - 2i$; and
- $z_3(t) = (1-t) + it$, which is mapped to $w_3(t) = (z_3(t))^2 = (1-2t) + 2it(1-t)$, so $z_3'(t) = -1 + i$, and $w_3'(t) = -2 + 2i(1-2t)$.



Notice that indeed the internal angles (calculated from the tangent lines) at the vertices of the region T^2 in the w-plane agree with those of the triangular region T in the z-plane: the angles at $z=1$ and at $w=1$ equal $\frac{\pi}{4}$, and similarly the angles at $z=i$ and at $w=-1$, whereas the angles at $z=1+i$ and at $w=2i$ are right angles.

To see that this can fail if the derivative of f is zero, consider the square S with vertices $(0,1,1+i,i)$. In the same way we work out that it gets mapped to the region S^2 in the complex plane which is depicted in the figure below.



But now notice that whereas the interior angle of region S at the origin in the complex z-plane is a right angle, the corresponding angle of region S^2 is a straight angle.

Exercise 2.1.4. Sketch the images under the holomorphic function $f(z)=\exp(z)$ of the following subsets of $\mathbb{C}$:

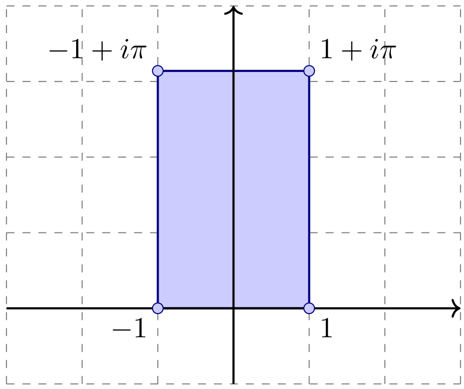
(i) the strip $\{\, z\in\mathbb{C} : 0<\operatorname{Im}(z)<\pi \,\}$;
(ii) the half-strip $\{\, z\in\mathbb{C} : \operatorname{Re}(z)<0 \text{ and } 0<\operatorname{Im}(z)<\pi \,\}$;
(iii) the half-strip $\{\, z\in\mathbb{C} : \operatorname{Re}(z)>0 \text{ and } 0<\operatorname{Im}(z)<\pi \,\}$;
(iv) the rectangle $\{\, z\in\mathbb{C} : 1<\operatorname{Re}(z)<2 \text{ and } 0<\operatorname{Im}(z)<\pi \,\}$; and
(v) the half-planes $\{\, z\in\mathbb{C} : \operatorname{Re}(z)>0 \,\}$ and $\{\, z\in\mathbb{C} : \operatorname{Re}(z)<0 \,\}$.

Exercise 2.1.5. Sketch the images under the holomorphic function $f(z)=\cos(z)$ of the following subsets of $\mathbb{C}$:

(i) the half-strip $\{\, z\in\mathbb{C} : 0<\operatorname{Re}(z)<\pi \text{ and } \operatorname{Im}(z)<0 \,\}$;
(ii) the half-strip $\{\, z\in\mathbb{C} : 0<\operatorname{Re}(z)<\pi \text{ and } \operatorname{Im}(z)>0 \,\}$; and
(iii) the strip $\{\, z\in\mathbb{C} : 0<\operatorname{Re}(z)<\pi \,\}$.

Exercise 2.1.6. Sketch the image under the holomorphic function $f(z)=z^3$ of the set $\{\, z\in\mathbb{C} : |z|\le 1, \text{and } 0\le \operatorname{Arg}(z)\le \pi/2 \,\}$.

Exercise 2.1.7. Sketch the image under the holomorphic function $f(z)=\exp(z)$ of the set shown in the figure:



Exercise 2.1.8. For $\theta \in \mathbb{R}$, sketch the image under the holomorphic function $f(z) = \mathrm{Log}(z)$ of the set $\{z \in \mathbb{C} : 0 \leq \mathrm{Arg}(z) \leq \theta\}$.

Exercise 2.1.9. Find a holomorphic function f which maps the set $\{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$ onto the set $\{ z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1 \}$.

2.2. Definition of Möbius transformations.

Definition 2.2.1. A *Möbius transformation* is a function of the form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ are such that $ad \neq bc$.

Remark 2.2.2. If f is a Möbius transformation defined by $a, b, c, d \in \mathbb{C}$, and $\lambda \in \mathbb{C}$ is non-zero, then λa, λb, λc, and λd define the same Möbius transformation:

$$\frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{\lambda(az + b)}{\lambda(cz + d)} = \frac{az + b}{cz + d}.$$

Given this, we can always divide by the factor $\sqrt{ad - bc}$, which is non-zero by assumption, and impose the condition $ad - bc = 1$, rather than just $ad - bc \neq 0$.

There is a natural identification between Möbius transformations and 2×2 complex matrices with determinant one.

Lemma 2.2.3. To a complex matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant $ad - bc = 1$ we associate the Möbius transformation $f_M(z) = \frac{az+b}{cz+d}$. Under this correspondence we have that

$$f_{M_1 M_2} = f_{M_1} \circ f_{M_2} \quad \text{and} \quad f_{M^{-1}} = f_M^{-1}.$$

Proof. Exercise. □

Exercise 2.2.4. Let $f(z) = \frac{2z+1}{3z-2}$. Calculate $f(1/z)$ and $f(f(z))$.

2.3. The extended complex plane and the Riemann sphere. Möbius transformations are not well-defined on the whole complex plane, since the transformation $f(z) = \frac{az+b}{cz+d}$, for $c \neq 0$, is not defined at the point $z = -d/c$. It turns out we can deal with this not by removing points from the complex plane but by adding an extra point.

Definition 2.3.1. The *extended complex plane* is the set $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where ∞ here is just some object $\infty \notin \mathbb{C}$. We extend the usual arithmetical operations in the following way: for $a \in \mathbb{C}$ and non-zero $b \in \mathbb{C}$,

$$a + \infty = \infty, \quad b \cdot \infty = \infty, \quad \frac{b}{0} = \infty, \quad \text{and} \quad \frac{b}{\infty} = 0.$$

We then define for $f(z) = \frac{az+b}{cz+d}$ the values $f(-d/c) = \infty$, and $f(\infty) = a/c$. Note that we do not assign meaning to the terms $0 \cdot \infty$ or $\infty + \infty$, since there are no consistent ways of doing this which are compatible with the usual rules of arithmetic.

So far this definition is just nothing more than formal manipulation of symbols. However, we now see that the extended complex plane can in fact be understood geometrically as a sphere, via a technique called stereographic projection.

Definition 2.3.2. Consider coordinates (X, Y, Z) describing $\mathbb{R}^3$. We identify the complex plane with the plane defined by $Z = 0$, and a complex number $X + iY$ with the point $(X, Y, 0)$. The *Riemann sphere* is the unit sphere S^2 in $\mathbb{R}^3$ defined by

$$S^2 := \left\{ (X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1 \right\},$$

and we consider the "north pole" to be the point $N := (0, 0, 1)$.

We define $\phi\colon \mathbb{C} \to S^2$ by stipulating that the three points N, z, and $\phi(z)$ are colinear. It is clear that $\lim_{|z|\to\infty} \phi(z) = (0, 0, 1)$, thus we define also $\phi(\infty) = N$ and thereby consider ϕ as being defined on the extended complex plane $\phi\colon \tilde{\mathbb{C}} \to S^2$.

The map $\phi\colon \tilde{\mathbb{C}} \to S^2$ is evidently bijective, so it has an inverse $\psi\colon S^2 \to \tilde{\mathbb{C}}$. This function ψ is the *stereographic projection.*

Remark 2.3.3. Calculation shows that

$$\phi(z) = \phi(x + iy) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right)$$

and

$$\psi(X, Y, Z) = \begin{cases} \frac{X+iY}{1-Z} & (X, Y, Z) \neq N, \\ \infty & (X, Y, Z) = N. \end{cases}$$

Lemma 2.3.4. Stereographic projection maps a circle to either a circle or a straight line (a "circline").

Proof. A circle on the Riemann sphere is the intersection of a plane with the sphere. The equation of a plane in $\mathbb{R}^3$ is $\alpha X + \beta Y + \gamma Z = \delta$, for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. By remark 2.3.3, stereographic projection associates $z = x + iy$ with the point (X, Y, Z) given by

$$X = \frac{2x}{x^2 + y^2 + 1}, \; Y = \frac{2y}{x^2 + y^2 + 1}, \; Z = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$

Substituting this into the equation of the plane gives that

$$\frac{2\alpha x + 2\beta y + \gamma(x^2 + y^2 - 1)}{x^2 + y^2 + 1} = \delta,$$

which is equivalent to $(\gamma - \delta)(x^2 + y^2) + 2\alpha x + 2\beta y = \gamma + \delta$. If $\gamma = \delta$, then substitution shows that the point $N = (0, 0, 1)$ lies on the plane, and our derived equation is the equation of a straight line $\alpha x + \beta y = \gamma$. If $\gamma \neq \delta$, then N does not lie on the plane, and the derived equation is that of a circle:

$$\left(x + \frac{\alpha}{\gamma - \delta} \right)^2 + \left(y + \frac{\beta}{\gamma - \delta} \right)^2 = \frac{\alpha^2 + \beta^2 + \gamma^2 - \delta^2}{(\gamma - \delta)^2}.$$ □

As is suggested by this proof, a straight line in the extended complex plane is the image of a circle passing through the north pole on S^2, which itself is mapped to the point ∞. Thus we regard straight lines as circles of infinite radius.

2.4. Deconstructing Möbius transformations. We consider four classes of Möbius transformations.

Definition 2.4.1. (i) A *translation* is a Möbius transformation of the form $f(z) = z + b$ for some $b \in \mathbb{C}$, which corresponds to the matrix $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$.

(ii) A *rotation* is a Möbius transformation of the form $f(z) = az$, where $|a| = 1$, so that $a = e^{i\theta}$ for some θ, which corresponds to the matrix $\left(\begin{smallmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{smallmatrix}\right)$.

(iii) A *dilation* is a Möbius transformation of the form $f(z) = rz$, where $r > 0$, which corresponds to the matrix $\left(\begin{smallmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{smallmatrix}\right)$.

(iv) An *inversion* is a Möbius transformation of the form $f(z) = 1/z$, which corresponds to the matrix $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$.

We say that a Möbius transformation f *fixes the point at infinity* if $f(\infty) = \infty$. Translations, rotations, and dilations fix the point at infinity. Inversions do not.

These four classes make, in the following precise sense, all the Möbius transformations.

Theorem 2.4.2. Let f be a Möbius transformation. Then f is a composition of a finite number of translations, rotations, dilations, and, if and only if f does not fix the point at infinity, one inversion.

Proof. Let $f(z) = (az + b)/(cz + d)$ be a Möbius transformation. Notice that since by definition $f(\infty) = a/c$, f fixes the point at infinity if and only if $c = 0$. Suppose indeed that $c = 0$. In that case, the condition $ad - bc \neq 0$ says that $ad \neq 0$, and $f(z) = (a/d)z + (b/d)$ is the composition $f_2 \circ f_1$ of the two functions

- $f_1(z) = (a/d)z$, which is a dilation and a rotation; and
- $f_2(z) = z + b/d$, which is a translation.

On the other hand, if $c \neq 0$, then we may write $f(z)$ in the following way

$$f(z) = \frac{az + b}{cz + d} = \frac{(bc - ad)/c^2}{z + \frac{d}{c}} + \frac{a}{c},$$

which shows that it is the composition $g_4 \circ g_3 \circ g_2 \circ g_1$ of the four functions

- $g_1(z) = z + d/c$, which is a translation;
- $g_2(z) = 1/z$, which is inversion;
- $g_3(z) = ((bc - ad)/c^2)z$, which is a dilation and a rotation; and
- $g_4(z) = z + (a/c)$, which is a translation. □

The following is a nice and surprising application of this result.

Corollary 2.4.3. Möbius transformations map circlines to circlines.

Proof. By theorem 2.4.2 it suffices to show that each of the four specific types of Möbius transformations described in definition 2.4.1 preserves circlines. This is geometrically clear for translations, dilations, and rotations, but not obvious for inversions. For inversions, it is more useful to look at how they behave on the Riemann sphere. Since this is equivalent to the extended complex plane under stereographic projection, and stereographic projection itself preserves circlines, by lemma 2.3.4, it suffices to show that inversions map circles on the Riemann sphere to circles on the Riemann sphere.

If $z = x + iy \in \mathbb{C}$, then by remark 2.3.3, $\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{x}{|z|^2} - i\frac{y}{|z|^2}$ is associated with the point

$$\phi(1/z) = \left(\frac{2x/|z|^2}{(1/|z|^2) + 1}, \frac{-2y/|z|^2}{(1/|z|^2) + 1}, \frac{1/|z|^2 - 1}{(1/|z|^2) + 1} \right) = \left(\frac{2x}{|z|^2 + 1}, \frac{-2y}{|z|^2 + 1}, \frac{1 - |z|^2}{|z|^2 + 1} \right).$$

But this is obtained from $\phi(z)$ be applying the rotation $(X, Y, Z) \mapsto (X, -Y, -Z)$. Thus inversion on the Riemann sphere corresponds to a rotation of the sphere, which evidently preserves circles. □

Exercise 2.4.4. Let $f(z) = \frac{1+z}{1-z}$. Sketch the images under f of the real and imaginary axes.

2.5. The cross-ratio. An important fact that we shall establish in this section is that we can always map three distinct points in the extended complex plane to the three distinct values $1, 0, \infty$ by a Möbius transformation. We begin the proof of this with a lemma.

Lemma 2.5.1. Suppose $f(z) = \frac{az+b}{cz+d}$ is a Möbius transformation and $z_2, z_3, z_4 \in \tilde{\mathbb{C}}$ are three distinct points for which $f(z_2) = z_2$, $f(z_3) = z_3$, and $f(z_4) = z_4$. Then f is the identity.

Proof. The equation $f(z) = z$ is equivalent to the quadratic equation $cz^2+(d-a)z-b = 0$, which therefore has at most two solutions in $\mathbb{C}$, unless it is an identity. By assumption there are three solutions in $\tilde{\mathbb{C}}$ of $f(z) = z$, so one solution must be ∞, i.e. we must have that $\frac{a}{c} = f(\infty) = \infty$, which implies that $c = 0$. This reduces our quadratic equation to $(d - a)z - b = 0$, which must have two distinct solutions, which is only possible if $d = a$ and $b = 0$. Then indeed $f(z) = z$ for all $z \in \tilde{\mathbb{C}}$, as required. □

Theorem 2.5.2. Let $z_2, z_3, z_4 \in \tilde{\mathbb{C}}$ be three distinct points. Then there exists a unique Möbius transformation f such that

$$f(z_2) = 1, \quad f(z_3) = 0, \quad \text{and} \quad f(z_4) = \infty.$$

Proof. We first prove uniqueness. Suppose that f and g are two such Möbius transformations. Then the Möbius transformation $g^{-1} \circ f$ fixes the three distinct points z_2, z_3, z_4, which implies by lemma 2.5.1 that $g^{-1} \circ f$ is the identity, which in turn implies that $g^{-1} = f^{-1}$, i.e. that $g = f$, as required.

Now we prove existence, by constructing a suitable transformation f. First suppose that $z_2, z_3, z_4 \in \mathbb{C}$. Then we see that

$$f(z) := \lambda \frac{z - z_3}{z - z_4}$$

will satisfy $f(z_3) = 0$ and $f(z_4) = \infty$, for any $\lambda \neq 0$. We therefore define $\lambda := \frac{z_2-z_4}{z_2-z_3}$, which is well-defined and non-zero since the three points are distinct, and which ensures that the final condition $f(z_2) = 1$ is satisfied.

In the case in which one of the three values is ∞, we see that the following definitions suffice:

$$f(z) := \begin{cases} \frac{z-z_3}{z-z_4} & z_2 = \infty, \\ \frac{z_2-z_4}{z-z_4} & z_3 = \infty, \\ \frac{z-z_3}{z_2-z_3} & z_4 = \infty. \end{cases}$$ □

Corollary 2.5.3. Let $(z_2, z_3, z_4), (w_2, w_3, w_4) \in \tilde{\mathbb{C}}$ be two triplets of distinct points. Then there is a unique Möbius transformation f such that

$$f(z_2) = w_2, \quad f(z_3) = w_3, \quad \text{and} \quad f(z_4) = w_4.$$

Proof. By theorem 2.5.2 there exist unique Möbius transformations g, h such that (z_2, z_3, z_4) and (w_2, w_3, w_4) respectively are sent to $(1, 0, \infty)$. Then $f := h^{-1} \circ g$ is a Möbius transformation with the required properties. □

Exercise 2.5.4. Find Möbius transformations f which satisfy:

(i) $f(0) = i$, $f(1) = 1$, and $f(-1) = -1$;
(ii) $f(0) = -i$, $f(1) = 1$, and $f(-1) = -1$;
(iii) $f(0) = 0$, $f(1) = -1$, and $f(-1) = 1$;
(iv) $f(0) = \infty$, $f(1) = -1$, and $f(\infty) = 1$; and
(v) $f(0) = 1$, $f(1) = \infty$, and $f(\infty) = i$.

Definition 2.5.5. Let $z_1, z_2, z_3, z_4 \in \tilde{\mathbb{C}}$ be distinct points. The *cross-ratio* $[z_1, z_2, z_3, z_4]$ of the four points is the image of z_1 under the Möbius transformation which sends (z_2, z_3, z_4) to $(1, 0, \infty)$.

Remark 2.5.6. Since f was explicitly constructed in theorem 2.5.2, we have the following formulae, for $z_1, z_2, z_3, z_4 \in \mathbb{C}$:

$$\begin{aligned}
[z_1, z_2, z_3, z_4] &= \frac{z_1 - z_3}{z_1 - z_4}\frac{z_2 - z_4}{z_2 - z_3}; \\
[\infty, z_2, z_3, z_4] &= \frac{z_2 - z_4}{z_2 - z_3}; \\
[z_1, \infty, z_3, z_4] &= \frac{z_1 - z_3}{z_1 - z_4}; \\
[z_1, z_2, \infty, z_4] &= \frac{z_2 - z_4}{z_1 - z_4}; \text{ and} \\
[z_1, z_2, z_3, \infty] &= \frac{z_1 - z_3}{z_2 - z_3}.
\end{aligned}$$

A remarkable fact is that Möbius transformations preserve the cross-ratio.

Theorem 2.5.7. Let $z_1, z_2, z_3, z_4 \in \tilde{\mathbb{C}}$ be distinct, and let f be a Möbius transformation. Then

$$[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4].$$

Proof. Define $g\colon \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ by $g(z) = [z, z_2, z_3, z_4]$. Then by definition, $g(z_2) = 1$, $g(z_3) = 0$, and $g(z_4) = \infty$. Therefore $g \circ f^{-1}$ sends $(f(z_2), f(z_3), f(z_4))$ to $(1, 0, \infty)$, so by definition of the cross-ratio,

$$[f(z_1), f(z_2), f(z_3), f(z_4)] = (g \circ f^{-1})(f(z_1)) = g(z_1) = [z_1, z_2, z_3, z_4]. \qquad \square$$

3. Complex integration

3.1. Complex integrals. We begin by establishing how to integrate a complex function of a real variable.

Definition 3.1.1. Let $[a, b] \subseteq \mathbb{R}$ be an interval, and $f\colon [a, b] \to \mathbb{C}$ be of the form $f = u + iv$. Then f is *integrable* if its real and imaginary parts $u, v\colon [a, b] \to \mathbb{R}$ are integrable in the usual (real) sense, and we define the integral of f by

$$\int_a^b f(t)\, dt := \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt.$$

We will not develop formal criteria for integrability of complex functions. It will usually suffice to observe that continuous functions are integrable. The following elementary properties of integrable functions should not be a surprise.

Lemma 3.1.2. Let $[a, b] \subseteq \mathbb{R}$ be an interval, and let $f, g\colon [a, b] \to \mathbb{C}$ be integrable, and $\alpha, \beta \in \mathbb{C}$. Then

(i) $\alpha f + \beta g$ is integrable, and

$$\int_a^b (\alpha f(t) + \beta g(t))\ dt = \alpha \int_a^b f(t)\, dt + \beta \int_a^b g(t)\, dt;$$

(ii) if f is continuous and $f = \frac{dF}{dt}$ for a differentiable function $F\colon [a, b] \to \mathbb{C}$, then

$$\int_a^b f(t)\, dt = F(b) - F(a); \tag{3.1}$$

and

(iii)
$$\left| \int_a^b f(t)\, dt \right| \le \int_a^b |f(t)|\, dt.$$

Proof. Exercise. □

3.2. Contour integrals. The definition of integrability for complex functions of a real variable may seem irrelevant, since we are discussing complex functions of a *complex* variable. However, our integration of these functions will be defined via such functions. An integral of the form $\int_{z_0}^{z_1} f(z)\, dz$ for a function $f\colon \mathbb{C} \to \mathbb{C}$ does not have an immediate interpretation, since, in contrast to the real case, there are many (infinitely many) choices of how to go from point z_0 to z_1. We shall have to specify one such path along which to integrate, at which point the integral will reduce to an integral of the form above.

We need to formalize the notion of a path in the complex plane connecting one point with another.

Definition 3.2.1. Let $z_0, z_1 \in \mathbb{C}$ be distinct. Then a *(parametrized) curve* Γ connecting z_0 and z_1 is a continuous function $\gamma\colon [t_0, t_1] \to \mathbb{C}$, for some $t_0, t_1 \in \mathbb{R}$ satisfying $t_0 < t_1$, such that $\gamma(t_0) = z_0$ and $\gamma(t_1) = z_1$. Writing $z_0 = x_0 + iy_0$ and $z_1 = x_1 + iy_1$, we can decompose γ into real and imaginary parts $\gamma = x + iy$ for continuous real functions $x, y\colon [t_0, t_1] \to \mathbb{R}$, so $\gamma(t) = x(t) + iy(t)$ for each $t \in [t_0, t_1]$, $x(t_0) = x_0$, $x(t_1) = x_1$, and $y(t_0) = y_0$, and $y(t_1) = y_1$. We say that the curve Γ is *regular* if γ is continuously differentiable and $\gamma'(t) \neq 0$ for all $t \in (t_0, t_1)$.

Remark 3.2.2. A curve Γ is therefore the image under a continuous function of a closed and bounded interval of the real line, and hence Γ itself is a closed and bounded subset of $\mathbb{C}$, by lemma 1.3.10.

We now define the integral of a complex function along a curve. We shall let Γ represent the curve as a subset of the complex plane, always understood as having a parametrization $\gamma\colon [t_0, t_1] \to \mathbb{C}$, with $\gamma([t_0, t_1]) = \Gamma$.

Definition 3.2.3. Let $z_0, z_1 \in \mathbb{C}$ be distinct, Γ be a regular curve connecting z_0 and z_1, and $f\colon \Gamma \to \mathbb{C}$ be continuous. Then we define *the integral of f along* Γ by
$$\int_\Gamma f(z)\, dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\, dt.$$

Example 3.2.4. Consider the function $f(z) = f(x + iy) = x^2 + iy^2$ integrated along the regular curve Γ parametrized by the function $\gamma\colon [0, 1] \to \mathbb{C}$ given by $\gamma(t) = t + it$. This is the straight line segment joining the origin and the point $1 + i$. We see that $f(\gamma(t)) = t^2 + it^2$ and $\gamma'(t) = 1 + i$, so using complex linearity of the integral and performing the elementary real integral, we find that
$$\int_\Gamma f(z)\, dz = \int_0^1 f(\gamma(t))\gamma'(t)\, dt = \int_0^1 (t^2 + it^2)(1+i)\, dt = \int_0^1 (1+i)^2 t^2\, dt = 2i\, \frac{t^3}{3}\bigg|_0^1 = \frac{2i}{3}.$$
Suppose that we take instead the regular parametrized curve Γ' defined by $\gamma(t) = \sin t + i \sin t$, for $t \in [0, \pi/2]$. This is a different parametrized curve than before, but it traces the same set of points in the complex plane: the straight line segment from the origin to $1 + i$. We now have that $f(\gamma(t)) = \sin^2 t + i \sin^2 t$, and $\gamma'(t) = \cos t + i \cos t$, so the integral is now
$$\int_{\Gamma'} f(z)\, dz = \int_0^{\pi/2} (1+i)\sin^2 t(1+i)\cos t\, dt = \int_0^{\pi/2} 2i \sin^2 t \cos t\, dt = 2i \frac{\sin^3 t}{3}\bigg|_0^{\pi/2} = \frac{2i}{3},$$
as before. This is not a coincidence, and we will see a little later that the integral is invariant under (reasonable) reparametrizations.

Example 3.2.5. Consider now the function $f(z) = 1/z$ integrated along the regular curve Γ with parametrization $\gamma\colon [0,1] \to \mathbb{C}$ given by $\gamma(t) = R\exp(i2\pi t)$, for some $R > 0$. Then Γ is the circle of radius R centred at the origin. We have that $f(\gamma(t)) = (1/R)\exp(-i2\pi t)$ and $\gamma'(t) = 2\pi i R\exp(i2\pi t)$, hence

$$\int_\Gamma f(z)\,dz = \int_0^1 \frac{2\pi i R\exp(i2\pi t)}{R\exp(i2\pi t)}\,dt = 2\pi i\int_0^1 dt = 2\pi i. \tag{3.2}$$

Notice that this is independent of the radius R. This is in sharp contrast with real integrals, which we are used to interpreting physically in terms of area.

Example 3.2.6. Consider integrating the constant function $f(z) = 1$ along *any* regular curve Γ with a parametrization $\gamma\colon [0,1] \to \mathbb{C}$. It may seem that we do not have enough information to compute the integral, but let us see how far we can get with the information given. Using the fundamental theorem of calculus (3.1), we have that

$$\int_\Gamma f(z)\,dz = \int_0^1 \gamma'(t)\,dt = \gamma(1) - \gamma(0),$$

which depends only on the endpoints $\gamma(0)$ and $\gamma(1)$, and not otherwise on the curve used to join them. Notice that this integral is therefore *not* the length of the curve, as one might have expected.

Definition 3.2.7. Let Γ be a regular curve in $\mathbb{C}$. We define the *arclength* $\ell(\Gamma)$ by

$$\ell(\Gamma) := \int_{t_0}^{t_1} |\gamma'(t)|\,dt = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Lemma 3.2.8. Let Γ be an arc of a circle of radius r traced through an angle θ. Then $\ell(\Gamma) = r\theta$.

Proof. Exercise. □

We immediately gain a useful estimate for complex integrals.

Lemma 3.2.9 (M-L lemma)**.** Let Γ be a regular curve in $\mathbb{C}$, and let $f\colon \Gamma \to \mathbb{C}$ be continuous. Then

$$\left|\int_\Gamma f(z)\,dz\right| \le \max_{z\in\Gamma}|f(z)|\ell(\Gamma).$$

Proof. We simply calculate, using lemma 3.1.2, that

$$\begin{aligned}\left|\int_\Gamma f(z)\,dz\right| = \left|\int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\,dt\right| \le \int_{t_0}^{t_1} |f(\gamma(t))||\gamma'(t)|\,dt &\le \max_{z\in\Gamma}|f(z)|\int_{t_0}^{t_1}|\gamma'(t)|\,dt \\ &= \max_{z\in\Gamma}|f(z)|\ell(\Gamma).\end{aligned}$$ □

Remark 3.2.10. Since f is a continuous function on Γ, which is a closed and bounded subset of $\mathbb{C}$, the function f is indeed bounded on Γ, and the values $\max|f|$ and $\min|f|$ are indeed attained on Γ.

We state some useful properties of complex integrals.

Lemma 3.2.11. Let Γ be a regular curve in $\mathbb{C}$, $f, g\colon \Gamma \to \mathbb{C}$ be continuous, and $\alpha, \beta \in \mathbb{C}$. Then

(i)

$$\int_\Gamma (\alpha f(z) + \beta g(z))\,dz = \alpha\int_\Gamma f(z)\,dz + \beta\int_\Gamma g(z)\,dz;$$

(ii) if $\gamma\colon [t_0, t_1] \to \mathbb{C}$ and $\tilde{\gamma}\colon [\tilde{t}_0, \tilde{t}_1] \to \mathbb{C}$ are two parametrizations of Γ, such that there exists an injective differentiable function $\lambda\colon [\tilde{t}_0, \tilde{t}_1] \to [t_0, t_1]$ with $\lambda'(t) > 0$ for all $t \in [\tilde{t}_0, \tilde{t}_1]$ such that $\tilde{\gamma}(t) = \gamma(\lambda(t))$ for all $t \in [\tilde{t}_0, \tilde{t}_1]$, then

$$\int_{\tilde{t}_0}^{\tilde{t}_1} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\, dt = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\, dt;$$

and

(iii) if Γ is parametrized by $\gamma\colon [0,1] \to \mathbb{C}$, then the curve we shall notate as $-\Gamma$, which runs in the opposite direction to Γ but along the same path, parametrized by $\tilde{\gamma}(t)\colon [0,1] \to \mathbb{C}$ defined by $\tilde{\gamma}(t) = \gamma(1-t)$, satisfies

$$\int_{-\Gamma} f(z)\, dz = -\int_{\Gamma} f(z)\, dz. \tag{3.3}$$

Proof. (i) Exercise.

(ii) We simply calculate, using a change of variables in the integral, that

$$\begin{aligned}\int_{\tilde{t}_0}^{\tilde{t}_1} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\, dt = \int_{\tilde{t}_0}^{\tilde{t}_1} f(\gamma(\lambda(t)))\frac{d}{dt}\gamma(\lambda(t))\, dt &= \int_{\tilde{t}_0}^{\tilde{t}_1} f(\gamma(\lambda(t)))\gamma'(\lambda(t))\lambda'(t)\, dt \\ &= \int_{\lambda(\tilde{t}_0)}^{\lambda(\tilde{t}_1)} f(\gamma(\lambda))\gamma'(\lambda)\, d\lambda \\ &= \int_{t_0}^{t_1} f(\gamma(\lambda))\gamma'(\lambda)\, d\lambda,\end{aligned}$$

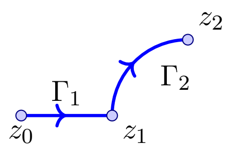
which up to relabelling the variable of integration is the required result.

(iii) Similarly we just calculate, using a change of variables, that

$$\begin{aligned}\int_{-\Gamma} f(z)\, dz = \int_0^1 f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\, dt = \int_0^1 -f(\gamma(1-t))\gamma'(1-t)\, dt &= \int_1^0 f(\gamma(s))\gamma'(s)\, ds \\ &= -\int_0^1 f(\gamma(s))\gamma'(s)\, ds \\ &= -\int_\Gamma f(z)\, dz. \qquad \square\end{aligned}$$

Remark 3.2.12. The second result tells us that integrals are independent of any "reasonable" parametrization chosen. The third result looks like a special case assuming a certain specific parametrization, but in light of the second statement, we see that there is no loss of generality, and it states that reversing the direction of the curve—under any parametrization—just changes the sign of the integral. Since integration is independent of parametrization, we will henceforth choose specific parametrizations as suits our purposes without further comment.

We will not always only want to consider regular curves, i.e. which can be parametrized by a function with a continuous derivative. We will want to consider curves which turn sharp "corners". This is possible, since—assuming only finitely many such corners—these curves are a finite union of regular curves.



If Γ_1 is a curve joining z_0 to z_1 and Γ_2 is a curve joining z_1 to z_2, then we can make a curve Γ joining z_0 to z_2 by first going to the intermediate point z_1 via Γ_1 and then from there via Γ_2 to our destination z_2. The resulting curve Γ is still continuous, but it will generally fail to be regular, since the derivative of the parametrization will not in general exist at the intermediate point z_1, as shown in the figure.

Definition 3.2.13. A curve Γ from z_0 to z_1 in $\mathbb{C}$ is a *contour* if it is a finite union of regular curves, which together join z_0 with z_1, i.e. there exist $\gamma_i\colon [t_0^i, t_1^i] \to \mathbb{C}$ for $i = 1, \ldots, n$, such that each γ_i defines a regular curve Γ_i, the union of Γ_i is the whole curve Γ, and $\gamma_i(t_1^i) = \gamma_{i+1}(t_0^{i+1})$ for each $i = 1, \ldots, n-1$, $\gamma_i(t_0^1) = z_0$, and $\gamma_n(t_1^n) = z_1$. Each Γ_i is a *regular component* of Γ.

For a continuous function $f\colon \Gamma \to \mathbb{C}$, we define the *contour integral of f along* Γ by

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^n \int_{\Gamma_i} f(z)\,dz.$$

3.3. Independence of path.

Definition 3.3.1. Let $D \subseteq \mathbb{C}$. We will say that D is a *domain* if D is open and every two points in D can be connected by a contour which lies wholly in D.

Lemma 3.3.2. Let $D \subseteq \mathbb{C}$ be a domain, and suppose $u\colon D \to \mathbb{R}$ is differentiable, with $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$ on D. Then u is constant on D.

Proof. Let $P, Q \in D$. By assumption there exists a contour Γ joining P and Q which lies wholly in D, parametrized by $(x, y)\colon [0,1] \to D$ with $(x(0), y(0)) = P$ and $(x(1), y(1)) = Q$. Then by the fundamental theorem of calculus and the chain rule, we have that

$$\begin{aligned} u(Q) - u(P) &= u(x(1), y(1)) - u(x(0), y(0)) \\ &= \int_0^1 \frac{d}{dt} u(x(t), y(t))\,dt \\ &= \int_0^1 \left(\frac{\partial u}{\partial x}(x(t), y(t)) x'(t) + \frac{\partial u}{\partial y}(x(t), y(t)) y'(t) \right) dt \\ &= \int_0^1 0\,dt \\ &= 0. \end{aligned}$$ □

Definition 3.3.3. Let $D \subseteq \mathbb{C}$ be a domain, and $f\colon D \to \mathbb{C}$ be continuous. We say that f has an *antiderivative* F on D if there exists a function $F\colon D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$.

Remark 3.3.4. Notice that, by definition, F is holomorphic on D.

Theorem 3.3.5 (Fundamental Theorem of Calculus)**.** Let $D \subseteq \mathbb{C}$ be a domain, Γ be a contour in D joining points $z_0, z_1 \in D$, and $f\colon D \to \mathbb{C}$ have an antiderivative F on D. Then

$$\int_\Gamma f(z)\,dz = F(z_1) - F(z_0).$$

Proof. Let us first assume that Γ is a regular curve, parametrized by $\gamma\colon [0,1] \to D$. Then by the chain rule and the fundamental theorem of calculus, (3.1),

$$\begin{aligned} \int_\Gamma f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_0^1 F'(\gamma(t))\gamma'(t)\,dt &= \int_0^1 \frac{d}{dt} F(\gamma(t))\,dt \\ &= F(\gamma(1)) - F(\gamma(0)) \\ &= F(z_1) - F(z_0). \end{aligned}$$

Now consider the general case, in which Γ has regular components Γ_i for $i = 1, \ldots, n$. Suppose each curve Γ_i has endpoints z_0^i and z_1^i. Then by definition, $z_0^1 = z_0$, $z_1^n = z_1$, and $z_1^i = z_0^{i+1}$ for each $i = 1, \ldots, n-1$. So applying the special case above to each Γ_i, we have that

$$\begin{aligned}\int_\Gamma f(z)\,dz = \sum_{i=1}^n \int_{\Gamma_i} f(z)\,dz &= \sum_{i=1}^n \big(F(z_1^i) - F(z_0^i)\big)\\ &= F(z_1^n) - \left(\sum_{i=1}^{n-1} \big(F(z_0^{i+1}) - F(z_1^i)\big)\right) - F(z_0^1)\\ &= F(z_1) - F(z_0).\end{aligned}$$ □

Corollary 3.3.6. Let $D \subseteq \mathbb{C}$ be a domain, and f be holomorphic on D with $f'(z) = 0$ for all $z \in D$. Then f is constant.

Proof. Let $z_0, z_1 \in D$. By definition there exists a contour Γ connecting z_0 and z_1 which lies in D. Then theorem 3.3.5 implies that

$$f(z_1) - f(z_0) = \int_\Gamma f'(z)\,dz = \int_\Gamma 0\,dz = 0.$$ □

Definition 3.3.7. Let Γ be a contour in $\mathbb{C}$. We say that Γ is *closed* if its endpoints are the same point, i.e. the parametrization $\gamma\colon [t_0, t_1] \to \mathbb{C}$ satisfies $\gamma(t_0) = \gamma(t_1)$.

Corollary 3.3.8. Let $D \subseteq \mathbb{C}$ be a domain, Γ be a closed contour in D, and $f\colon D \to \mathbb{C}$ be continuous, with an antiderivative on D. Then

$$\int_\Gamma f(z)\,dz = 0.$$

Proof. Let F be an antiderivative of f on D. Then theorem 3.3.5 implies, since the endpoints z_0, z_1 of the contour Γ are the same, that

$$\int_\Gamma f(z)\,dz = F(z_1) - F(z_0) = 0.$$ □

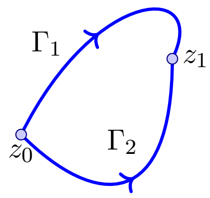
We shall pursue this fact in more detail later.

Lemma 3.3.9 (Path-independence lemma)**.** Let $D \subseteq \mathbb{C}$ be a domain, and $f\colon D \to \mathbb{C}$ be continuous. Then the following are equivalent:

(i) f has an antiderivative F on D;
(ii) $\int_\Gamma f(z)\,dz = 0$ for all closed contours Γ in D; and
(iii) all contour integrals $\int_\Gamma f(z)\,dz$ are independent of the path Γ, and depend only on the endpoints.

Proof. We have proved that (i) implies (ii) in corollary 3.3.8.

We now prove that (ii) implies (iii). Let $z_0, z_1 \in D$, and suppose we have two contours Γ_1, Γ_2 connecting z_0 and z_1 and lying in D. Then consider the contour defined by



composing the contour Γ_1 with the contour $-\Gamma_2$. Since Γ_1 joins z_0 to z_1 and $-\Gamma$ joins z_1 to z_0, this new contour is a closed contour starting at and returning to z_0. Therefore, by condition (ii) and (3.3), we have that

$$0 = \int_\Gamma f(z)\,dz = \int_{\Gamma_1} f(z)\,dz + \int_{-\Gamma_2} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz,$$

which implies that $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$, as required.

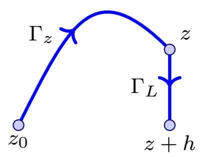
We now prove that (iii) implies (i). Fix $z_0 \in D$. For each $z \in D$, there exists a contour Γ_z connecting z_0 and z, by definition of D being a domain. Since, by assumption, the value of the contour integral is independent of the choice of Γ_z, we can define a function $F\colon D \to \mathbb{C}$ by

$$F(z) := \int_{\Gamma_z} f(w)\,dw,$$

for each $z \in D$. We claim that F is an antiderivative of f on D. So, fix $z \in D$. We want to investigate the limiting behaviour of the quantity

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\left(\int_{\Gamma_{z+h}} f(w)\,dw - \int_{\Gamma_z} f(w)\,dw\right),$$

as the complex number h tends to 0. First notice that since D is open, there exists $h_0 > 0$ such that $D_{h_0}(z) \subseteq D$, so $|h| < h_0$ implies that $z+h \in D$. Since we only care about the limit as $h \to 0$, we can assume that $|h| < h_0$.



Then the straight line segment Γ_L joining z to $z+h$ lies inside D. By our path-independence asumption, we may take our contour Γ_{z+h} to be that contour gained by taking the composition of the two contours Γ_z and Γ_L. Therefore

$$\int_{\Gamma_{z+h}} f(w)\,dw - \int_{\Gamma_z} f(w)\,dw = \int_{\Gamma_z} f(w)\,dw + \int_{\Gamma_L} f(w)\,dw - \int_{\Gamma_z} f(w)\,dw = \int_{\Gamma_L} f(w)\,dw.$$

Consider the parametrization $\gamma\colon [0,1] \to D$ of Γ_L given by $\gamma(t) = z + th$. Then

$$\frac{1}{h}\int_{\Gamma_L} f(w)\,dw = \frac{1}{h}\int_0^1 f(\gamma(t))\gamma'(t)\,dt = \frac{1}{h}\int_0^1 f(z+th)h\,dt = \int_0^1 f(z+th)\,dt.$$

We claim this converges to $f(z)$ as $h \to 0$, but we have to be careful with interchanging limits and integration. Let $\varepsilon > 0$. By continuity of f, there exists $\delta > 0$ such that

$$|f(z+w) - f(z)| < \varepsilon \text{ whenever } w \in D \text{ satisfies } |w| < \delta.$$

In particular, choosing $|h| < \delta$, we have for any $t \in [0,1]$ that $|f(z+th) - f(z)| < \varepsilon$. Then

$$\left|\int_0^1 f(z+th)\,dt - f(z)\right| = \left|\int_0^1 f(z+th) - f(z)\,dt\right| \le \int_0^1 |f(z+th) - f(z)|\,dt < \int_0^1 \varepsilon\,dt = \varepsilon.$$

Hence

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = \lim_{h\to 0}\frac{1}{h}\int_{\Gamma_L} f(w)\,dw = \lim_{h\to 0}\int_0^1 f(z+th)\,dt = f(z). \qquad \square$$

Exercise 3.3.10. Calculate the contour integral of the following functions f along the given contours Γ:

(i) $f(z) = 3z^2 - 5z + i$, where Γ is the straight line segment from i to 1;
(ii) $f(z) = \exp(z)$, where Γ is the upper half circle of radius 1, from 1 to -1;
(iii) $f(z) = 1/z$, where Γ is any contour in the right half-plane from $-3i$ to $3i$; and
(iv) $f(z) = 1/(1+z^2)$, where Γ is the straight line segment from 1 to $1+i$.

Exercise 3.3.11. Evaluate the contour integral $\int_\Gamma |z|^2 \, dz$ where Γ is the square with vertices at 0, 1, $1+i$, and i, traversed anti-clockwise, starting at 0.

3.4. Cauchy's Integral Theorem. We now reach one of the most important results in complex analysis. It tells us that, roughly, the three statements in lemma 3.3.9 hold because f is holomorphic.

Definition 3.4.1. Let Γ be a contour in $\mathbb{C}$. Then Γ is *simple* if it has no self-intersections, except possibly at the endpoints, i.e. $\gamma(t) \neq \gamma(s)$ for all distinct $s, t \in [t_0, t_1]$, unless $s = t_0$ and $t = t_1$ and Γ is a closed contour. A *loop* is a simple, closed contour.

The following result seems obvious, but is surprisingly hard to prove.

Theorem 3.4.2 (Jordan Curve Theorem)**.** Let Γ be a loop in $\mathbb{C}$. Then Γ defines two regions in the complex plane, with Γ as their common boundary: a bounded domain, the *interior* of Γ, and an unbounded domain, the *exterior* of Γ.

Proof. Omitted. □

Definition 3.4.3. Let Γ be a loop in $\mathbb{C}$. We let $\mathrm{Int}(\Gamma)$ and $\mathrm{Ext}(\Gamma)$ denote the interior and the exterior of Γ respectively. Thus $\mathbb{C} = \mathrm{Int}(\Gamma) \cup \Gamma \cup \mathrm{Ext}(\Gamma)$. Any property (e.g. holomorphicity of a function) which holds for points $z \in \Gamma \cup \mathrm{Int}(\Gamma)$ shall be said to hold *inside and on* Γ.

We shall very often consider loops which are circles, so we shall introduce some notation for them. Let $z_0 \in \mathbb{C}$, and $r > 0$. Then the loop $C_r(z_0)$ is the circle of radius r centred at z_0, which is parametrized by $\gamma\colon [0,1] \to \mathbb{C}$, where $\gamma(t) = z_0 + r\exp(2\pi i t)$. Note that we carefully distinguish the circle $C_r(z_0)$, which is a one-dimensional curve in the complex plane, and the disc $D_r(z_0)$, which is a two-dimensional area in the complex plane. The circle $C_r(z_0)$ is the boundary of $D_r(z_0)$, i.e. $C_r(z_0) = \overline{D}_r(z_0) \setminus D_r(z_0)$, and $\mathrm{Int}(C_r(z_0)) = D_r(z_0)$.

Definition 3.4.4. Let Γ be a loop in $\mathbb{C}$. Then we say that Γ is *positively-oriented* if as we move along the curve in the direction of parametrization, the interior is on the left-hand side.

Remark 3.4.5. Unless otherwise stated, all loops shall be positively-oriented.

Definition 3.4.6. Let $D \subseteq \mathbb{C}$ be a domain. Then D is *simply-connected* if $\mathrm{Int}(\Gamma) \subseteq D$ for all loops $\Gamma \subseteq D$.

Example 3.4.7. For $z_0 \in \mathbb{C}$ and $R > 0$, the disc $D_R(z_0)$ is simply-connected, whereas the punctured disc $D'_R(z_0)$ is not. Any circle $C_r(z_0)$ for $0 < r < R$ has $C_r(z_0) \subseteq D'_R(z_0)$, but $z_0 \in \mathrm{Int}(C_r(z_0)) = D_r(z_0) \subseteq D_R(z_0)$, whereas $z_0 \notin D'_R(z_0)$ by definition of the punctured disc.

Theorem 3.4.8 (Cauchy Integral Theorem)**.** Let Γ be a loop, and f be holomorphic inside and on Γ. Then
$$\int_\Gamma f(z) \, dz = 0.$$

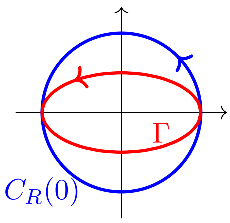
Proof. Omitted. □

Corollary 3.4.9. Let $D \subseteq \mathbb{C}$ be a simply-connected domain, and f be holomorphic on D. Then f has an antiderivative on D.

Proof. By the Cauchy Integral Theorem, the contour integral of f around any loop is zero. By the path-independence lemma, lemma 3.3.9, f has an antiderivative. □

A very important consequence of the Cauchy Integral Theorem is that we can deform a contour without changing the value of the integral of any f over the contour, provided that in deforming the contour we do not let it cross any point where f is not holomorphic. The following two results demonstrate this.

Example 3.4.10. Consider the contour integral $\int_\Gamma \frac{1}{z}\,dz$, where Γ is the ellipse $x^2+4y^2 = 1$. In example 3.2.5 we computed the same integral around a circular contour $C_R(0)$, obtaining $\int_{C_R(0)} \frac{1}{z}\,dz = 2\pi i$. We will argue, using the Cauchy Integral Theorem, that we get the same answer whether we integrate along Γ or along $C_R(0)$. Consider the two domains in the interior of the circle $C_R(0)$ but in the exterior of the ellipse Γ.



The integrand is holomorphic everywhere in the complex plane except for the origin, which lies outside these two regions. The Cauchy Integral Theorem says that the contour integral vanishes along either of the two contours which make up the boundary of these domains. Let us be more explicit and let us call these contours $\Gamma_\pm$ as in the figure below.

Then it is clear that
$$\int_{C_R(0)} \frac{1}{z}\,dz = \int_{\Gamma_+} \frac{1}{z}\,dz + \int_{\Gamma_-} \frac{1}{z}\,dz + \int_{\Gamma} \frac{1}{z}\,dz.$$
By the Cauchy Integral Theorem, $\int_{\Gamma_\pm} \frac{1}{z}\,dz = 0$, whence $\int_\Gamma \frac{1}{z}\,dz = \int_{C_R(0)} \frac{1}{z}\,dz = 2\pi i$.

We now consider a slightly more general example, which it turns out gives a result of some significance.

Theorem 3.4.11. Let $z_0 \in \mathbb{C}$ and Γ be a loop in $\mathbb{C}$ which does not pass through z_0. Then
$$\int_\Gamma \frac{1}{z-z_0}\,dz = \begin{cases} 2\pi i & \text{if } z_0 \in \operatorname{Int}(\Gamma); \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the Jordan Curve Theorem, we must distinguish two possibilities: $z_0 \in \operatorname{Int}(\Gamma)$, or $z_0 \in \operatorname{Ext}(\Gamma)$ ($z_0 \notin \Gamma$ by assumption). If $z_0 \in \operatorname{Ext}(\Gamma)$, then the integral is zero because the integrand is holomorphic everywhere except at z_0, which does not lie in the interior of Γ. Hence Cauchy's Integral Theorem applies, and the integral is zero.

On the other hand, if $z_0 \in \operatorname{Int}(\Gamma)$, then certainly if Γ were the circle of radius $R > 0$ centred at z_0, then the same calculation as in example 3.2.5 yields a value of $2\pi i$ for the integral. The significant assertion to be proved here is that this is true whatever the shape of Γ.

Figure 4 depicts the contour Γ and the circular contour $C_r(z_0)$. Such a picture is certainly possible, since z_0 lies in the interior of Γ, which is an open set, and therefore some circle of positive radius can be centred at z_0 which lies inside the interior of Γ. It also shows two pairs of points (P_1, P_2) and (P_3, P_4), each pair having one point on each contour, as well as straight line segments joining the points in each pair.

Now consider the following loop Γ_1 starting and ending at P_1, as illustrated in figure 5. We start at P_1 and go to P_4 via the top half of Γ, call this Γ_+; then we go to P_3 along

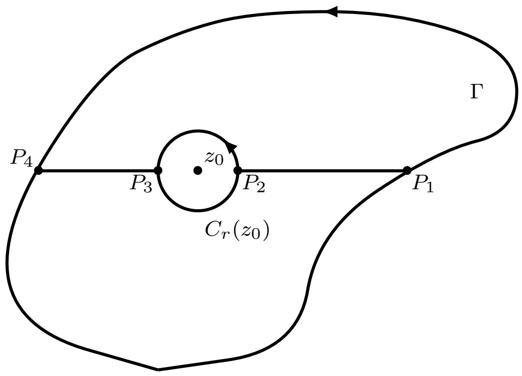


FIGURE 4. The contours Γ and $C_r(z_0)$ and some special points.

the straight line segment joining them, call this $-\gamma_{34}$; then to P_2 via the upper half of $C_r(z_0)$ in the negative sense, call this $-C_r^+(z_0)$; and then back to P_1 via the straight line segment joining P_2 and P_1, call this $-\gamma_{12}$. Cauchy's Integral Theorem and (3.3) says that

$$
\begin{aligned}
0 = \int_{\Gamma_1} \frac{1}{z-z_0}\,dz &= \left(\int_{\Gamma_+} + \int_{-\gamma_{34}} + \int_{-C_r^+(z_0)} + \int_{-\gamma_{12}}\right) \frac{1}{z-z_0}\,dz \\
&= \left(\int_{\Gamma_+} - \int_{\gamma_{34}} - \int_{C_r^+(z_0)} - \int_{\gamma_{12}}\right) \frac{1}{z-z_0}\,dz,
\end{aligned}
$$

from which we deduce that

$$
\int_{\Gamma_+} \frac{1}{z-z_0}\,dz = \left(\int_{\gamma_{34}} + \int_{C_r^+(z_0)} + \int_{\gamma_{12}}\right) \frac{1}{z-z_0}\,dz.
$$

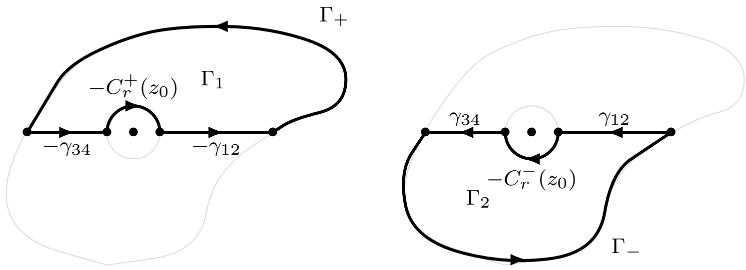


FIGURE 5. The contours Γ_1 and Γ_2.

Similarly, consider the loop Γ_2 starting and ending at P_4. We start at P_4 and go to P_1 along the lower half of Γ, call this Γ_-; then we go to P_2 along γ_{12}; then to P_3 via the lower half of the circular contour in the negative sense $-C_r^-(z_0)$; and then finally back to P_4 along γ_{34}. Again, by the Cauchy Integral Theorem and (3.3),

$$
\begin{aligned}
0 = \int_{\Gamma_2} \frac{1}{z-z_0}\,dz &= \left(\int_{\Gamma_-} + \int_{\gamma_{12}} + \int_{-C_r^-(z_0)} + \int_{\gamma_{34}}\right) \frac{1}{z-z_0}\,dz \\
&= \left(\int_{\Gamma_-} + \int_{\gamma_{12}} - \int_{C_r^-(z_0)} + \int_{\gamma_{34}}\right) \frac{1}{z-z_0}\,dz,
\end{aligned}
$$

from which we deduce that

$$\int_{\Gamma_-} \frac{1}{z-z_0}\,dz = \left(-\int_{\gamma_{34}} + \int_{C_r^-(z_0)} - \int_{\gamma_{12}} \right) \frac{1}{z-z_0}\,dz.$$

Putting the two results together, we find that

$$\begin{aligned}
\int_{\Gamma} \frac{1}{z-z_0}\,dz = \int_{\Gamma_+} \frac{1}{z-z_0}\,dz + \int_{\Gamma_-} \frac{1}{z-z_0}\,dz &= \int_{C_r^+(z_0)} \frac{1}{z-z_0}\,dz + \int_{C_r^-(z_0)} \frac{1}{z-z_0}\,dz \\
&= \int_{C_r(z_0)} \frac{1}{z-z_0}\,dz \\
&= 2\pi i.
\end{aligned}$$

□

In the following section we will generalize this formula in a variety of ways.

We conclude by stating—but not proving—the general result which allows us to deform contours in the manner we have just seen.

Theorem 3.4.12 (Deformation Theorem)**.** Let $\Gamma_1, \Gamma_2 \subseteq \mathbb{C}$ be loops, and f be holomorphic on Γ_1, Γ_2, and $(\mathrm{Int}(\Gamma_1) \setminus \mathrm{Int}(\Gamma_2)) \cup (\mathrm{Int}(\Gamma_2) \setminus \mathrm{Int}(\Gamma_1))$. Then $\int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$.

Proof. Omitted. □

Exercise 3.4.13. Evaluate the contour integral of the function $f(z) = z^2 + 3z$ along the contour Γ which joins 2 to $2i$, defined to be:

(i) the arc of the circle of radius 2 centred at 0, traversed anticlockwise;
(ii) the arc of the circle of radius 2 centred at 0, traversed clockwise; and
(iii) the straight line between the two points.

Exercise 3.4.14. Evaluate the contour integral of the function $f(z) = 1/(z-2)$ along the contour $\Gamma = C_r(z_0)$, where

(i) $r = 4$ and $z_0 = 2$;
(ii) $r = 4$ and $z_0 = 10$; and
(iii) $r = 10$ and $z_0 = 0$.

Exercise 3.4.15. What is the value of $\int_\Gamma z^{-n}\,dz$, for $n \neq 1$, where Γ is a loop not passing through 0?

Exercise 3.4.16. For $a > 1$, define $I = \int_0^{2\pi} \frac{1}{a+\cos t}\,dt$. By considering $z = \exp(it)$, whence $\cos t = (z + \overline{z})/2$, convert I_a to a contour integral, and hence evaluate it.

3.5. Cauchy's Integral Formula. The following result is quite remarkable. It says that given a function which is holomorphic on and inside a loop, the value of the function at any point inside the loop is determined by the values of the function on the loop itself.

Theorem 3.5.1 (Cauchy Integral Formula)**.** Let Γ be a loop, z_0 be in the interior of Γ, and f be holomorphic inside and on Γ. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-z_0}\,dz.$$

Proof. By theorem 3.4.11,

$$f(z_0) = \frac{f(z_0)}{2\pi i} \int_\Gamma \frac{1}{z-z_0}\,dz = \frac{1}{2\pi i} \int_\Gamma \frac{f(z_0)}{z-z_0}\,dz,$$

so it suffices to prove that

$$\int_\Gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0.$$

By the Deformation Theorem, we can deform Γ to a small circle $C_r(z_0)$ which lies inside Γ, without changing the value of this integral. So it suffices to prove that, given $\varepsilon > 0$, for small enough $r > 0$ we have that

$$\left|\int_{C_r(z_0)} \frac{f(z) - f(z_0)}{z - z_0}\, dz\right| < \varepsilon.$$

Since as discussed, this equals the contour integral in question, this shows that this contour integral does indeed equal 0. So let $\varepsilon > 0$. Since f is holomorphic, it is continuous, therefore there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \frac{\varepsilon}{2\pi} \text{ whenever } |z - z_0| < \delta.$$

Consider $0 < r < \delta$, and parametrize the circle $C_r(z_0)$ by $\gamma\colon [0,1] \to \mathbb{C}$, where $\gamma(t) = z_0 + r\exp(2\pi i t)$. Then

$$\begin{aligned}
\left|\int_{C_r(z_0)} \frac{f(z) - f(z_0)}{z - z_0}\, dz\right| &= \left|\int_0^1 \frac{f(\gamma(t)) - f(z_0)}{\gamma(t) - z_0} \gamma'(t)\, dt\right| \\
&= \left|\int_0^1 \frac{f(\gamma(t)) - f(z_0)}{r\exp(2\pi i t)} 2\pi i r \exp(2\pi i t)\, dt\right| \\
&\le 2\pi \int_0^1 |f(\gamma(t)) - f(z_0)|\, dt \\
&< 2\pi \frac{\varepsilon}{2\pi} \\
&= \varepsilon.
\end{aligned}$$

□

Theorem 3.5.2. Let $D \subseteq \mathbb{C}$ be a domain, Γ be a contour in D (not necessarily closed), and suppose $g\colon D \to \mathbb{C}$ is continuous on Γ. Then the function $G\colon D \setminus \Gamma \to \mathbb{C}$ defined by

$$G(z) = \int_\Gamma \frac{g(w)}{w - z} dw$$

is holomorphic, and

$$G'(z) = \int_\Gamma \frac{g(w)}{(w - z)^2}\, dw.$$

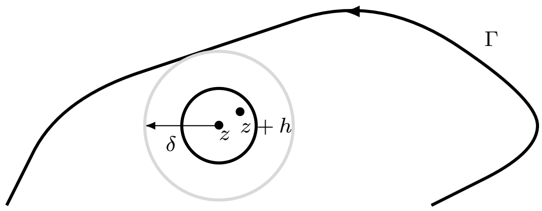
Proof. Fix $z \in D \setminus \Gamma$. We are interested in the limiting behaviour as the complex number h tends to 0 of the quantity

$$\begin{aligned}
\frac{G(z+h) - G(z)}{h} = \frac{1}{h}\int_\Gamma \left(\frac{g(w)}{w - (z+h)} - \frac{g(w)}{w - z}\right) dw &= \frac{1}{h}\int_\Gamma \frac{g(w)h}{(w - z - h)(w - z)}\, dw \\
&= \int_\Gamma \frac{g(w)}{(w - z - h)(w - z)}\, dw.
\end{aligned}$$

Comparing with the claimed value of the derivative, we have that

$$\begin{aligned}
\left|\int_\Gamma \frac{g(w)}{(w - z - h)(w - z)}\, dw - \int_\Gamma \frac{g(w)}{(w - z)^2}\, dw\right| &= \left|\int_\Gamma \left(\frac{g(w)}{(w - z - h)(w - z)} - \frac{g(w)}{(w - z)^2}\, dw\right)\right| \\
&= \left|h\int_\Gamma \frac{g(w)}{(w - z - h)(w - z)^2}\, dw\right|.
\end{aligned}$$

Now, there exists (see remark 3.2.10) $M > 0$ such that $|g(w)| \le M$ for all $w \in \Gamma$. Moreover, since $z \notin \Gamma$, there is $\delta > 0$ such that $|w - z| \ge \delta$ for all $w \in \Gamma$.



Suppose $|h| < \delta/2$. Then by the reverse triangle inequality, we have that $|w - z - h| \geq |w - z| - |h| > \delta - \delta/2 = \delta/2$ for all $w \in \Gamma$. Hence, for all $w \in \Gamma$, we have that $\left|(w - z - h)(w - z)^2\right| \geq \frac{\delta}{2}\delta^2 = \delta^3/2$, so

$$\left|\frac{g(w)}{(w - z - h)(w - z)^2}\right| \leq \frac{2M}{\delta^3}.$$

So, by lemma 3.2.9,

$$\begin{aligned}
\lim_{h \to 0} \left|\frac{G(z + h) - G(z)}{h} - \int_\Gamma \frac{g(w)}{(w - z)^2}\, dw\right| &= \lim_{h \to 0} \left|h \int_\Gamma \frac{g(w)}{(w - z - h)(w - z)^2}\, dw\right| \\
&\leq \lim_{h \to 0} |h| \frac{2M\ell(\Gamma)}{\delta^3} \\
&= 0.
\end{aligned}$$

□

Remark 3.5.3. The same argument shows (glossing over some technical details) that defining

$$H(z) = \int_\Gamma \frac{g(w)}{(w - z)^n}\, dw$$

in a similar way, for a positive integer n, we have that H is holomorphic, and

$$H'(z) = n \int_\Gamma \frac{g(w)}{(w - z)^{n+1}}\, dw.$$

Corollary 3.5.4. Let $D \subseteq \mathbb{C}$ be a domain and f be holomorphic on D. Then f is infinitely differentiable on D, and all of its derivatives are holomorphic on D.

Proof. Let $z_0 \in D$. By definition, there exists $\varepsilon > 0$ such that f is holomorphic on the closed disc $\overline{D}_\varepsilon(z_0)$, which has boundary circle $C_\varepsilon(z_0)$. Then for any $z \in D_\varepsilon(z_0)$, the Cauchy Integral representation holds, from theorem 3.5.1:

$$f(z) = \frac{1}{2\pi i} \int_{C_\varepsilon(z_0)} \frac{f(w)}{w - z}\, dw.$$

But this is of the form given in theorem 3.5.2, hence the derivative exists and is given by

$$f'(z) = \frac{1}{2\pi i} \int_{C_\varepsilon(z_0)} \frac{f(w)}{(w - z)^2}\, dw.$$

This too is of the form given in theorem 3.5.2, for a function $g(w) = f(w)/(w - z)$, hence the derivative exists and is given by

$$f''(z) = (f')'(z) = \frac{2}{2\pi i} \int_{C_\varepsilon(z_0)} \frac{f(w)/(w - z)}{(w - z)^2}\, dw = \frac{2}{2\pi i} \int_{C_\varepsilon(z_0)} \frac{f(w)}{(w - z)^3}\, dw.$$

The higher derivatives follow in the same fashion, and are all holomorphic on $D_\varepsilon(z_0)$. □

Theorem 3.5.5 (Generalized Cauchy Integral Formula)**.** Let Γ be a loop, f be holomorphic inside and on Γ, and z lie inside Γ. Then f is infinitely differentiable at z and, for all positive integers n,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(w)}{(w - z)^{n+1}}\, dw.$$

Proof. Omitted. □

Remark 3.5.6. Interpreting (as is conventional) $0! = 1$ and $f^{(0)} = f$, putting $n = 0$ into the Generalized Cauchy Integral Formula gives the Cauchy Integral Formula.

Example 3.5.7. It now follows that the real and imaginary parts of a holomorphic function are infinitely differentiable, which makes the assumption in lemma 1.4.13 of them being twice continuously differentiable redundant.

Remark 3.5.8. The Generalized Cauchy Integral Formula can also be used in the "other direction," to compute contour integrals if we know the values of the derivatives of an appropriate function. If f is holomorphic on and inside a loop Γ and z_0 is in the interior of Γ, then

$$\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0). \tag{3.4}$$

Example 3.5.9. Let us compute the contour integral

$$\int_{C_1(0)} \frac{\exp(5z)}{z^3}\,dz.$$

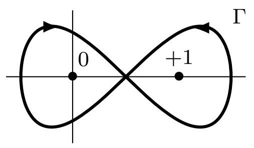
This integral is of the form (3.4) with $n = 2$, $z_0 = 0$, and $f(z) = \exp(5z)$, which is certainly holomorphic in and on the contour. Therefore by (3.4) we have

$$\int_{C_1(0)} \frac{\exp(5z)}{z^3}\,dz = 2\pi i \frac{1}{2!} \frac{d^2}{dz^2}\left(\exp(5z)\right)\bigg|_{z=0} = 2\pi i \frac{1}{2!} 25 = 25\pi i.$$

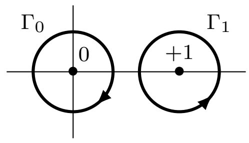
Example 3.5.10. Let us compute the contour integral

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\,dz,$$

where Γ is the figure-8 contour depicted. Two things prevent us from applying the Gener-



alized Cauchy Integral Formula: the contour is not a loop—indeed it is not simple—and the integrand is not of the form $g(z)/(z-z_0)^n$ where g is holomorphic inside the contour. This last problem can be solved by rewriting the integrand using partial fractions, but we are still faced with a contour which is not simple.



This problem can be circumvented by noticing that the regular contour Γ can be written as a piecewise regular contour with two regular components: both starting and ending at the point of self-intersection of Γ. The first such contour is the left lobe of Γ, which is a negatively oriented loop about $z = 0$, and the second is the right lobe of Γ, which is a positively oriented loop about $z = 1$. Because the integrand is holomorphic everywhere but at $z = 0$ and $z = 1$, the Deformation Theorem tells us that we get the same result by integrating around the circular contours Γ_0 and Γ_1 in the figure. In other words,

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\,dz = \int_{\Gamma_0} \frac{2z+1}{z(z-1)^2}\,dz + \int_{\Gamma_1} \frac{2z+1}{z(z-1)^2}\,dz.$$

Since $(2z+1)/(z-1)^2$ is holomorphic inside and on Γ_0, we use the Cauchy Integral Formula to see that

$$\int_{\Gamma_0} \frac{2z+1}{z(z-1)^2}\,dz = \int_{\Gamma_0} \frac{(2z+1)/(z-1)^2}{z}\,dz = -2\pi i,$$

after taking into account that Γ_0 is negatively oriented. Similarly, the Generalized Cauchy Integral formula (with $n = 1$), implies that

$$\int_{\Gamma_1} \frac{2z+1}{z(z-1)^2}\,dz = \int_{\Gamma_1} \frac{(2z+1)/z}{(z-1)^2}\,dz = 2\pi i\,\frac{d}{dz}\left(\frac{2z+1}{z}\right)\bigg|_{z=1} = -2\pi i,$$

Therefore

$$\int_\Gamma \frac{2z+1}{z(z-1)^2}\,dz = -4\pi i.$$

Theorem 3.5.11 (Morera)**.** Let $D \subseteq \mathbb{C}$ be a domain, $f\colon D \to \mathbb{C}$ be continuous, and suppose that $\int_\Gamma f(z)\,dz = 0$ for all loops Γ inside D. Then f is holomorphic.

Proof. By the path-independence lemma, lemma 3.3.9, we know that f has an antiderivative F on D. Therefore F is holomorphic. Corollary 3.5.4 implies that the derivative of a holomorphic function is holomorphic, so we know that $F' = f$ is holomorphic. $\square$

Exercise 3.5.12. Let Γ be the contour parametrized by $\gamma\colon [0, 2\pi] \to \mathbb{C}$ where $\gamma(t) = 2\cos t + i \sin 2t$. Sketch Γ, and evaluate the contour integrals $\int_\Gamma f(z)\,dz$ of the following functions:

(i) $f(z) = 1/(z-1)$;
(ii) $f(z) = 1/(z+1)$;
(iii) $f(z) = 1/(z^2-1)$; and
(iv) $f(z) = (2z^2 - z + 1)/(z-1)^2(z+1)$.

Exercise 3.5.13. Evaluate the contour integral

$$\int_{C_3(0)} \frac{\exp(iz)}{(z^2+1)^2}\,dz.$$

Exercise 3.5.14. Evaluate the contour integral

$$\int_{C_r(z_0)} \frac{z+i}{z^3+2z^2}\,dz,$$

where:

(i) $r = 1$ and $z_0 = 0$;
(ii) $r = 2$ and $z_0 = -2 + i$; and
(iii) $r = 1$ and $z_0 = 2i$.

Exercise 3.5.15. Evaluate the contour integral $\int_{C_2(0)} f(z)\,dz$ where f is defined as:

(i) $f(z) = \sin(3z)/(z - (\pi/2))$;
(ii) $f(z) = z\exp(z)/(2z-3)$;
(iii) $f(z) = \cos(z)/(z^3+9z)$;
(iv) $f(z) = (5z^2+2z+1)/(z-i)^3$; and
(v) $f(z) = \exp(-z)/(z+1)^2$.

Exercise 3.5.16. For each of the following functions, explain why $\int_{C_2(0)} f(z)\,dz = 0$:

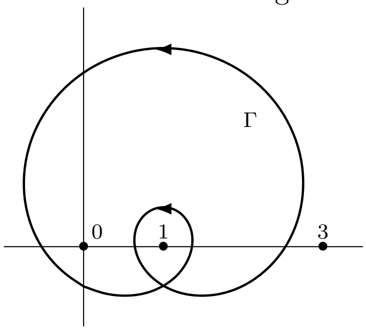
(i) $f(z) = z/(z^2+25)$;

(ii) $\exp(-z)(2z+1)$;
(iii) $\mathrm{Log}(z+3)$;
(iv) $\cos(z)/(z^2-6z+10)$; and
(v) $\sec(z/2)$.

Exercise 3.5.17. Evaluate

$$\int_{C_1(0)} \frac{z^2\exp(z)}{2z+i}\,dz.$$

Exercise 3.5.18. Evaluate the contour integrals of the following functions, where Γ is



the contour depicted:

(i) $f(z)=\cos(z)/z^2(z-1)$; and
(ii) $f(z)=\cos(z)/z^2(z-3)$.

3.6. Liouville's Theorem and its applications.

Theorem 3.6.1. Let $D\subseteq\mathbb{C}$ be a domain, $z_0\in D$ and $R>0$ be such that $\overline{D}_R(z_0)\subseteq D$, f be holomorphic on D, and $M>0$ be such that $|f(z)|\leq M$ for all $z\in D$. Then for all $n\in\mathbb{N}$, we have that

$$|f^{(n)}(z_0)|\leq\frac{n!M}{R^n}.$$

Proof. We simply apply the Generalized Cauchy Integral Formula and lemma 3.2.9 with the contour $C_R(z_0)$ to see that

$$|f^{(n)}(z_0)|=\left|\frac{n!}{2\pi i}\int_{C_R(z_0)}\frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|\leq\frac{n!}{2\pi}\max_{z\in C_R(z_0)}\frac{|f(z)|}{|z-z_0|^{n+1}}\ell(C_R(z_0))\leq\frac{n!M2\pi R}{2\pi R^{n+1}}$$
$$=\frac{n!M}{R^n}.\qquad\square$$

This seems innocuous enough, but has the following rather surprising consequence.

Theorem 3.6.2 (Liouville's Theorem)**.** Let f be holomorphic on $\mathbb{C}$ and be bounded, i.e. satisfy for some $M>0$, $|f(z)|\leq M$ for all $z\in\mathbb{C}$. Then f is constant.

Proof. Let $z_0\in\mathbb{C}$. Then theorem 3.6.1 applies on all circles of radius R centred at z_0, so for all $R>0$ we have that $|f'(z_0)|\leq\frac{M}{R}$. Letting $R\to\infty$, we see that $|f'(z_0)|\leq\lim_{R\to\infty}\frac{M}{R}=0$, hence $f'(z_0)=0$. This is in turn implies that f is constant, by corollary 3.3.6. $\square$

Theorem 3.6.3 (Fundamental Theorem of Algebra)**.** Let $P\colon\mathbb{C}\to\mathbb{C}$ be a polynomial. Then if P is non-constant, P has at least one root, i.e. there exists at least one $z\in\mathbb{C}$ such that $P(z)=0$.

Proof. Suppose that P does not have any roots. Then the function $1/P$ is holomorphic on $\mathbb{C}$. It suffices to prove that $1/P$ is bounded, for then Liouville's Theorem will imply that $1/P$, and hence P itself, is constant.

Without loss of generality we can assume that the polynomial has the form $P(z) = z^N + a_{N-1}z^{N-1} + \cdots + a_1 z + a_0$ for some N and $a_{N-1}, \ldots, a_0 \in \mathbb{C}$. There exists (see exercise 3.6.4) $R > 0$ such that for $|z| \geq R$, we have that $|P(z)| \geq |z|^N/2$. Then, for $|z| \geq R$, we have that

$$\left|\frac{1}{P(z)}\right| = \frac{1}{|P(z)|} \leq \frac{2}{|z|^N} \leq \frac{2}{R^N}.$$

On the other hand, the function $1/P$, being continuous, is bounded on the closed and bounded set $\overline{D}_R(0)$ by some $M > 0$. Therefore $1/|P(z)|$ is bounded on $\mathbb{C}$ by the largest of M and $2/R^N$. Hence $1/P(z)$ is bounded. □

Exercise 3.6.4. Prove the missing assertion from the proof of the Fundamental Theorem of Algebra: that for a (monic) polynomial P of degree N, there exists $R > 0$ such that

$$|P(z)| \geq \frac{1}{2}|z|^N \text{ whenever } |z| \geq R.$$

3.7. The Maximum Modulus Principle.

Theorem 3.7.1. Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$ and $R > 0$ be such that the closed disc $\overline{D}_R(z_0) \subseteq D$, and f be holomorphic on D. Then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\, dt.$$

Proof. Exercise. □

Remark 3.7.2. In particular, notice that if there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in C_R(z_0)$, in the above notation, then $|f(z_0)| \leq M$, by lemma 3.2.9.

Lemma 3.7.3. Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$ and $R > 0$ be such that $\overline{D}_R(z_0) \subseteq D$, f be holomorphic on D, and such that $\max_{z\in\overline{D}_R(z_0)}|f(z)| = |f(z_0)|$. Then $|f(z)|$ is constant on $\overline{D}_R(z_0)$.

Proof. By theorem 3.7.1, for any $0 < r \leq R$, we have that

$$|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt\right| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{it})|\, dt \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\, dt = |f(z_0)|.$$

Hence these inequalities must in fact be equalities, thus

$$\int_0^{2\pi} \left(|f(z_0)| - |f(z_0 + re^{it})|\right)\, dt = 0,$$

but since the integrand is non-negative by assumption, it must be zero. But every $z \in \overline{D}_R(z_0)$ is of the form $z_0 + re^{it}$ for some $r \in (0, R]$ and some $t \in \mathbb{R}$. Hence $|f(z)| = |f(z_0)|$ for every $z \in \overline{D}_R(z_0)$, as required. □

Thus $|f(z)|$ cannot achieve its maximum at the centre of a disc unless it is constant on the disc.

Exercise 3.7.4. Let $D \subseteq \mathbb{C}$ be a domain, and f be holomorphic on D, such that $|f(z)|$ is constant on D. Show, using the Cauchy–Riemann equations, or otherwise, that f is constant on D.

Theorem 3.7.5 (Maximum Modulus Principle)**.** Let $D \subseteq \mathbb{C}$ be a domain, and f be holomorphic and bounded on D, $|f(z)| \leq M$, say, for all $z \in D$, for some $M > 0$. If $|f(z)|$ achieves its maximum at $z_0 \in D$, then f is constant on D.

Proof. By exercise 3.7.4, it suffices to show that $|f(z)|$ is constant on D.

Suppose for a contradiction that there exists $z_1 \in D$ with $|f(z_1)| < |f(z_0)|$. Choose a curve Γ in D which connects z_0 to z_1. Then there is a point w on Γ, possibly equal to z_0, such that $|f(z)| = |f(z_0)|$ for all z preceding w along Γ, and such that there exists $z \in \Gamma$ arbitrarily close to w where $|f(z)| < |f(z_0)|$. By continuity $|f(w)| = |f(z_0)|$. Now since D is open, it contains some open disc $D_\varepsilon(w)$ centred at w. But then by lemma 3.7.3, $|f(z)|$ is constant on that disc, which is a contradiction. □

Remark 3.7.6. It follows that a function which is holomorphic on a bounded domain, and continuous up to and including the boundary, attains its maximum modulus on the boundary.

Example 3.7.7. Let us find the maximum value of $|z^2 + 3z - 1|$ on the unit disc $\overline{D}_1(0)$ Since $z^2 + 3z - 1$ is holomorphic on $\mathbb{C}$, the maximum occurs somewhere on the unit circle. For $|z| = 1$, we have that

$$|z^2 + 3z - 1|^2 = (z^2 + 3z - 1)(\overline{z}^2 + 3\overline{z} - 1) = 11 - 2\mathrm{Re}(z^2).$$

But $|z^2| = 1$ as well, and hence $\mathrm{Re}(z^2) \geq -1$, attaining its minimum at $z^2 = -1$, whence $|z^2 + 3z - 1|^2$ attains its maximum value, 13, at $z = \pm i$. Therefore the maximum value of $|z^2 + 3z - 1|$ is $\sqrt{13}$.

Since holomorphic functions and harmonic functions are intimately related, there is a corresponding principle for harmonic functions.

Theorem 3.7.8 (Maximum/minimum principle for harmonic functions)**.** Let $D \subseteq \mathbb{R}^2$ be a domain, and $\phi\colon D \to \mathbb{R}$ be harmonic, such that ϕ is bounded above or below on D by $M > 0$, and $\phi(z_0) = M$ for some $z_0 \in D$. Then ϕ is constant on D.

Proof. We show that ϕ is the real part of a function f which is holomorphic on D.

Define the function $g\colon D \to \mathbb{C}$ by

$$g(z) = g(x + iy) = \frac{\partial \phi}{\partial x}(x, y) - i\frac{\partial \phi}{\partial y}(x, y).$$

Since ϕ is harmonic, it is twice continuously differentiable, and g satisfies the Cauchy–Riemann equations on D. Hence theorem 1.4.6 implies that g is holomorphic and by corollary 3.4.9 has a holomorphic antiderivative $G = u + iv$, say. Since $G' = g$, we have that

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y},$$

whence u and ϕ have identical partial derivatives and therefore, by lemma 3.3.2, $u - \phi$ is constant in D, which means that $\phi = u + c$, for some $c \in \mathbb{R}$. So $f = G + c$ has real part ϕ, as required.

Now notice that $|\exp(f(z))| = e^{\phi(z)}$ and that since the real exponential is a monotonically increasing function, the maxima of ϕ coincide with the maxima of $|\exp(f)|$. But $\exp(f)$ is holomorphic on D, so theorem 3.7.5 implies that $|\exp(f(z))|$ is constant, hence $\phi(z)$, is constant. The corresponding minimum principle follows from the observation that the minima of ϕ are the maxima of $-\phi$, which is harmonic if ϕ is. □

4. Series expansions for holomorphic functions

In this section we will uncover another surprising feature of holomorphic functions: that functions holomorphic on a disc admit expression as a Taylor series, and functions holomorphic on an annulus admit expression as a Laurent series, which is a generalized Taylor series including also terms in negative powers of the variable.

4.1. Infinite series.

Definition 4.1.1. Let $z_n \in \mathbb{C}$ be a sequence. The expression $z_0+z_1+z_2+\cdots = \sum_{j=0}^{\infty} z_j$ is an *infinite series*, which without further comment may or may not make sense. We say that the series *converges*, or *is convergent*, if the sequence $S_n \in \mathbb{C}$ of partial sums $S_n = \sum_{j=0}^{n} z_j$ is a convergent sequence, with limit $S \in \mathbb{C}$, say, in which case we say that $\sum_{j=0}^{\infty} z_j = S$. Otherwise we say that the series *diverges*, or is *divergent*.

Lemma 4.1.2. Let $\sum_{j=0}^{\infty} z_j$ be a convergent series. Then $z_n \to 0$ as $n \to \infty$.

Proof. Exercise. □

We can use this fact as an easy way of seeing that a series is *not* convergent.

Example 4.1.3. The series $\sum_{j=0}^{\infty} \frac{j}{2j+1}$ is divergent, since $\frac{j}{2j+1} = \frac{1}{2+1/j} \to \frac{1}{2} \neq 0$ as $j \to \infty$.

The next example shows that, while the condition that $z_n \to 0$ is a necessary condition for the series to converge, it is not sufficient.

Example 4.1.4. The series $\sum_{j=1}^{\infty} \frac{1}{j}$ is divergent, although we have that $\frac{1}{j} \to 0$. One way to see this is to notice that for every $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{j} = \sum_{j=1}^{n} \int_{j}^{j+1} \frac{1}{j}\, dx > \sum_{j=1}^{n} \int_{j}^{j+1} \frac{1}{x}\, dx = \int_{1}^{n+1} \frac{1}{x}\, dx = \ln(n+1),$$

and $\lim_{n\to\infty} \ln(n+1) = \infty$. This is the familiar *harmonic series*.

Example 4.1.5. The series $\sum_{j=1}^{\infty} \frac{1}{j^2}$ is convergent. Consider the partial sum

$$\sum_{j=1}^{n} \frac{1}{j^2} = 1 + \sum_{j=2}^{n} \frac{1}{j^2} < 1 + \sum_{j=2}^{n} \frac{1}{j(j-1)} = 1 + \sum_{j=2}^{n} \left(\frac{1}{j-1} - \frac{1}{j} \right) = 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n},$$

hence $\sum_{j=1}^{n} \frac{1}{j^2} < 2 - \frac{1}{n}$, which is bounded above by 2 in the limit as $n \to \infty$. Since the terms of the series are non-negative, the partial sums are a monotonically increasing sequence. Since they are bounded, we therefore see that they converge. A surprising application of contour integration will be to evaluate this series to see that $\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}$, see example 5.6.1.

Lemma 4.1.6 (The Comparison Test)**.** Let $z_n \in \mathbb{C}$ be a sequence such that $|z_n| \leq M_n$ for some non-negative real numbers M_n, for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$, where $\sum_{j=0}^{\infty} M_j$ is a convergent series. Then $\sum_{j=0}^{\infty} z_j$ is a convergent series.

Proof. Exercise. □

Lemma 4.1.7. Let $c \in \mathbb{C}$. Then the series $\sum_{j=0}^{\infty} c^j$ is convergent if and only if $|c| < 1$.

Proof. Suppose that $|c| < 1$, then the partial sums $S_n = \sum_{j=0}^{n} c^j$ satisfy

$$(1-c)S_n = (1-c)\sum_{j=0}^{n} c^j = 1 - c^{n+1},$$

hence

$$\left| S_n - \frac{1}{1-c} \right| = \left| \frac{1-c^{n+1}}{1-c} - \frac{1}{1-c} \right| = \frac{|c|^{n+1}}{|1-c|} \to 0 \quad \text{as } n \to \infty.$$

Hence $\sum_{j=0}^{\infty} c^j = \frac{1}{1-c}$.

On the other hand, if $|c| \geq 1$, then $|c^n| = |c|^n \geq 1^n = 1$, which does not converge to 0, so by (the contrapositive to) lemma 4.1.2, the series does not converge. □

Example 4.1.8. Consider the series

$$\sum_{j=0}^{\infty} \frac{3+2i}{(j+1)^j}. \tag{4.1}$$

Notice that for $j \geq 3$, we have

$$\left|\frac{3+2i}{(j+1)^j}\right| = \frac{\sqrt{13}}{(j+1)^j} < \frac{4}{(j+1)^j} \leq \frac{4}{4^j} = \frac{4}{2^j}\frac{1}{2^j} < \frac{1}{2^j} = \left(\frac{1}{2}\right)^j.$$

Since $\left|\frac{1}{2}\right| < 1$, the geometric series $\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j$ converges. Hence by the comparison test, lemma 4.1.6, the original series (4.1) converges as well.

Lemma 4.1.9 (The Ratio Test)**.** Let $z_n \in \mathbb{C}$ be a sequence, and suppose that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L.$$

Then

(i) if $L < 1$, the series $\sum_{j=0}^{\infty} z_j$ is convergent; and
(ii) if $L > 1$, the series $\sum_{j=0}^{\infty} z_j$ is divergent.

If $L = 1$, then we can conclude nothing, since the series might be convergent or divergent.

Proof. Exercise. □

Exercise 4.1.10. Determine whether the series $\sum_{j=1}^{\infty} z_j$ converges (we index from $j = 1$ to ensure all the terms are well-defined), where the sequence z_n is defined by:

(i) $z_n = 1/(n^4 2^n)$;
(ii) $z_n = (n^2 2^n)/(2n^2 + 2n + 1)$;
(iii) $z_n = (n!)/(n^n)$;
(iv) $z_n = (-1)^n n^2/(3+2i)^n$;
(v) $z_n = ((1+2i)/(1-i))^n$;
(vi) $z_n = (ni^n)/(2n+1)$; and
(vii) $z_n = (n!)/(5n)$.

Exercise 4.1.11. Evaluate the following convergent series $\sum_{j=0}^{\infty} z_j$, where the sequence z_n is defined by:

(i) $z_n = (i/3)^n$;
(ii) $z_n = 3/(1+i)^n$;
(iii) $z_n = (1/2i)^n$;
(iv) $z_n = (1/3)^{2n}$; and
(v) $z_n = (-1)^n(2/3)^n$.

Our interest in sequences and series is motivated by our desire to express holomorphic functions in terms of series, thus we need to consider the case in which the terms of a sequence or series are given by functions of a complex variable.

Definition 4.1.12. Let $S \subseteq \mathbb{C}$, and $f_n \colon S \to \mathbb{C}$ be a sequence of functions. We say that f_n *converge pointwise* to a function $f \colon S \to \mathbb{C}$ if for each $z \in S$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \varepsilon \text{ whenever } n \geq N.$$

Thus the sequence of complex numbers defined by $f_n(z)$ converges to $f(z)$.

Remark 4.1.13. We emphasize that N is chosen for each value of z, thus (in general) depends on the point z. This is in direct contrast to the following situation, in which one value of N satisfies the definition for all $z \in S$ simultaneously.

Definition 4.1.14. Let $S \subseteq \mathbb{C}$, and $f_n \colon S \to \mathbb{C}$ be a sequence of functions. We say that f_n *converge uniformly* to a function $f \colon S \to \mathbb{C}$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $z \in S$,

$$|f_n(z) - f(z)| < \varepsilon \text{ whenever } n \geq N.$$

Thus the sequence of complex numbers defined by $f_n(z)$ converges to $f(z)$, but moreover converges, roughly speaking, at the same rate.

Example 4.1.15. Consider the functions $f_n \colon \mathbb{C} \to \mathbb{C}$ defined by $f_n(z) = \exp(-nz^2)$. Then f_n is holomorphic for all $n \in \mathbb{N}$. Consider the functions restricted to the real axis, for which we use the variable x. Evidently $f_n(0) = 1$ for all $n \in \mathbb{N}$, thus $\lim_{n\to\infty} f_n(0) = 1$. Now consider real $x \neq 0$, and let $\varepsilon > 0$. Then choosing an integer $N > \frac{\ln(1/\varepsilon)}{x^2}$, we have that

$$0 < \exp(-nx^2) < \varepsilon \text{ whenever } n \geq N,$$

so by definition $\lim_{n\to\infty} f_n(x) = 0$. Thus on the real axis, f_n converges pointwise to the function $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0, \end{cases}$$

which is evidently discontinuous at $x = 0$. Thus we see that a sequence of holomorphic functions can converge pointwise to a function which is not even continuous, and thus certainly not holomorphic. The problem here is that the convergence is *not* uniform, since the value of N chosen evidently does depend on the value of the point x.

Definition 4.1.16. Let $S \subseteq \mathbb{C}$, and suppose $f_n \colon S \to \mathbb{C}$ is a sequence of functions. Then the series $\sum_{j=0}^{\infty} f_j(z)$ *converges pointwise* or *converges uniformly* if the corresponding sequence of partial sums $S_n := \sum_{j=0}^{n} f_j$ converges pointwise or uniformly, respectively.

Lemma 4.1.17. Let $S \subseteq \mathbb{C}$, and suppose $f_n \colon S \to \mathbb{C}$ is a sequence of continuous functions, and that f_n converges uniformly to a function $f \colon S \to \mathbb{C}$. Then f is continuous.

Proof. Exercise. □

Example 4.1.18. Consider the geometric series $\sum_{j=0}^{\infty} z^j$ defined on the set $\overline{D}_R(0)$ for some $0 \leq R < 1$. We know that $\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$. Consider the sequence of partial sums $S_n(z) = \sum_{j=0}^{n} z^j$, and let $\varepsilon > 0$. Let $z \in \overline{D}_R(0)$. By the reverse triangle inequality, we have that $|1 - z| \geq 1 - |z| \geq 1 - R$, so

$$\left| S_n(z) - \frac{1}{1-z} \right| = \left| \frac{1 - z^{n+1}}{1-z} - \frac{1}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{R^{n+1}}{|1-z|} \leq \frac{R^{n+1}}{1-R}.$$

The point is that this number on the right-hand side is independent of z, thus if we choose an integer N such that $R^{N+1} < \varepsilon(1-R)$, which is possible since $R < 1$, we see that

$$\left| S_n(z) - \frac{1}{1-z} \right| \leq \frac{R^{n+1}}{1-R} \leq \frac{R^{N+1}}{1-R} < \varepsilon \text{ whenever } n \geq N.$$

The number N is independent of z, so S_n converges uniformly to the function $\frac{1}{1-z}$, hence the series $\sum_{j=0}^{\infty} z^j$ converges uniformly to the function $\frac{1}{1-z}$ on the set $\overline{D}_R(0)$.

The following result is a very convenient way of establishing uniform convergence of series.

Lemma 4.1.19 (Weierstrass M-test)**.** Let $S \subseteq \mathbb{C}$, $f_n \colon S \to \mathbb{C}$ be a sequence of functions, $M_n \geq 0$ be a sequence of non-negative numbers, such that for all $z \in S$ and all $n \geq n_0$, for some $n_0 \in \mathbb{N}$, we have that $|f_n(z)| \leq M_n$, and the series $\sum_{j=0}^{\infty} M_j$ converges. Then the series $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly on S.

Proof. We begin by showing that the series converges pointwise. Let $z \in S$. We show that the sequence of partial sums $S_n(z) = \sum_{j=0}^{n} f_j(z)$ is Cauchy. Let $\varepsilon > 0$. Note that for $m \geq n \geq n_0$,

$$|S_m(z) - S_n(z)| = \left|\sum_{j=n+1}^{m} f_j(z)\right| \leq \sum_{j=n+1}^{m} |f_j(z)| \leq \sum_{j=n+1}^{m} M_j \leq \sum_{j=n+1}^{\infty} M_j$$
$$= \sum_{j=0}^{\infty} M_j - \sum_{j=0}^{n} M_j. \quad (4.2)$$

Since by assumption the series $\sum_{j=0}^{\infty} M_j$ converges, by definition there exists $n_1 \in \mathbb{N}$ such that

$$\left|\sum_{j=0}^{\infty} M_j - \sum_{j=0}^{n} M_j\right| < \varepsilon \text{ whenever } n \geq n_1.$$

Let $N = \max\{n_1, n_0\}$, and let $n, m \geq N$. Without loss of generality, $m \geq n$. Then

$$|S_m(z) - S_n(z)| \leq \left|\sum_{j=0}^{\infty} M_j - \sum_{j=0}^{n} M_j\right| < \varepsilon,$$

as required. So by lemma 1.2.12, the partial sums $S_n(z)$ converge. Define $f\colon S \to \mathbb{C}$ by $f(z) = \lim_{n\to\infty} S_n(z)$. Then by definition $\sum_{j=0}^{\infty} f_j(z)$ converges pointwise to f. It remains to prove that this convergence is uniform. For $n \geq n_0$, we can take the limit as $m \to \infty$ in (4.2) to see that

$$\left|\sum_{j=0}^{\infty} f_j(z) - S_n(z)\right| \leq \left|\sum_{j=0}^{\infty} M_j - \sum_{j=0}^{n} M_j\right|.$$

Then choosing the same N as before, which is independent of z, we see that

$$\left|\sum_{j=0}^{\infty} f_j(z) - S_n(z)\right| < \varepsilon \text{ whenever } n \geq N,$$

for any $z \in S$. □

Example 4.1.20. The Weierstrass M-test gives us a faster approach to proving uniform convergence of the geometric series $\sum_{j=0}^{\infty} z^j$ on $\overline{D}_R(0)$ for $R < 1$. Simply note that on this set, we have for all n that

$$|z^n| = |z|^n \leq R^n,$$

and since $R < 1$, the series $\sum_{j=0}^{\infty} R^j$ converges. Hence the Weierstrass M-test applies to give the conclusion.

Lemma 4.1.21. Let $S \subseteq \mathbb{C}$, $f_n\colon S \to \mathbb{C}$ be continuous functions which converge uniformly to a function $f\colon S \to \mathbb{C}$, and Γ be a contour inside S. Then the sequence of complex numbers $\int_\Gamma f_n(z)\,dz$ converges to $\int_\Gamma f(z)\,dz$.

Proof. Let $\varepsilon > 0$. By uniform convergence, there exists $N \in \mathbb{N}$ such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{\ell(\Gamma)} \text{ whenever } n \geq N,$$

for all $z \in S$, in particular for all $z \in \Gamma$. Then by lemma 3.2.9,

$$\left|\int_\Gamma f_n(z)\,dz - \int_\Gamma f(z)\,dz\right| = \left|\int_\Gamma f_n(z) - f(z)\,dz\right| \leq \max_{z\in\Gamma}|f_n(z)-f(z)|\ell(\Gamma) < \frac{\varepsilon}{\ell(\Gamma)}\ell(\Gamma) = \varepsilon,$$

for all $n \geq N$, as required. □

Lemma 4.1.22. Let $S \subseteq \mathbb{C}$, $f_n \colon S \to \mathbb{C}$ be continuous functions such that the series $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly on S, and Γ be a contour inside S. Then

$$\int_\Gamma \sum_{j=0}^{\infty} f_j(z)\, dz = \sum_{j=0}^{\infty} \int_\Gamma f_j(z)\, dz.$$

Proof. This follows by applying lemma 4.1.21 to the partial sums $\sum_{j=0}^{n} f_j(z)$. □

Theorem 4.1.23. Let $D \subseteq \mathbb{C}$ be a simply-connected domain, and f_n be holomorphic on D and converge uniformly to a function $f \colon D \to \mathbb{C}$. Then f is holomorphic on D.

Proof. First notice that since f_n are holomorphic, they are continuous, so lemma 4.1.17 implies that f is continuous. Let Γ be a loop in D. Then the Cauchy Integral Theorem implies that $\int_\Gamma f_n(z)\, dz = 0$ for all $n \in \mathbb{N}$. Therefore by lemma 4.1.21

$$\int_\Gamma f(z)\, dz = \lim_{n\to\infty} \int_\Gamma f_n(z)\, dz = \lim_{n\to\infty} 0 = 0.$$

Since this holds for any loop, Morera's Theorem implies that f is holomorphic. □

4.2. Power series.

Definition 4.2.1. Let $z_0 \in \mathbb{C}$, and $a_n \in \mathbb{C}$ be a sequence. A *power series* is an infinite series of the form

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

and a_j are the *coefficients* of the power series.

Theorem 4.2.2. Let $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ be a power series. Then there is a number $R \in [0, \infty) \cup \{\infty\}$, such that

(i) the series converges on $D_R(z_0)$;
(ii) the series converges uniformly on $\overline{D}_r(z_0)$ for any $r \in [0, R)$; and
(iii) the series diverges on $\mathbb{C} \setminus \overline{D}_R(z_0)$.

Proof. Omitted. The real case can be found in the Honours Analysis course, and the complex case is not significantly different. □

Definition 4.2.3. Let $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ be a power series. The value R from theorem 4.2.2 is the *radius of convergence* of the power series.

Theorem 4.2.4. Let $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ be a power series, and suppose that the sequence $\left|\frac{a_j}{a_{j+1}}\right|$ has a limit. Then the radius of convergence R is equal to this limit.

Proof. Exercise. □

Remark 4.2.5. Note that this does *not* assert that the radius of convergence can always be evaluated by taking the limit of the sequence $\left|\frac{a_j}{a_{j+1}}\right|$, since this limit does not in general exist. For example, define

$$a_j = \begin{cases} 1 & j \text{ is even,} \\ 2 & j \text{ is odd.} \end{cases}$$

Then

$$\left|\frac{a_j}{a_{j+1}}\right| = \begin{cases} \frac{1}{2} & j \text{ is even,} \\ 2 & j \text{ is odd,} \end{cases}$$

so the sequence $\left|\frac{a_j}{a_{j+1}}\right|$ has no limit.

Theorem 4.2.6. Let $f(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j$ be a power series, with radius of convergence R. Then f is holomorphic on $D_R(z_0)$.

Proof. Each partial sum $\sum_{j=0}^{n} a_j(z - z_0)^j$ is a polynomial, therefore holomorphic. Since for any $0 \leq r < R$, the power series converges uniformly on $\overline{D}_r(z_0)$, applying theorem 4.1.23 to the partial sums, we see that the power series $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ is holomorphic on $D_r(z_0)$ for each $r < R$, hence is holomorphic on $D_R(z_0)$. □

Exercise 4.2.7. Find the radius of convergence and the disc of convergence of the power series $\sum_{j=0}^{\infty} a_j(z - z_0)^j$, where the sequence a_n and $z_0 \in \mathbb{C}$ are given by:

(i) $a_n = n$, $z_0 = 0$;
(ii) $a_n = 2^{-n}$, $z_0 = i$;
(iii) $a_n = 1/(n!)$, $z_0 = 0$;
(iv) $a_n = n^3$, $z_0 = 0$;
(v) $a_n = 2^n$, $z_0 = 1$;
(vi) $a_n = (-1)^n n/3^n$, $z_0 = i$; and
(vii) $a_n = n!$, $z_0 = 0$.

Exercise 4.2.8. Determine the radius of convergence and the disc of convergence of the power series

(i) $\sum_{j=0}^{\infty}(z + 5i)^{2j}(j + 1)^2$; and
(ii) $\sum_{j=0}^{\infty} z^{2j}/(4^j)$.

4.3. Taylor series.

Definition 4.3.1. Let $z_0 \in \mathbb{C}$, and f be holomorphic at z_0. The *Taylor series of f centred at z_0* is the power series

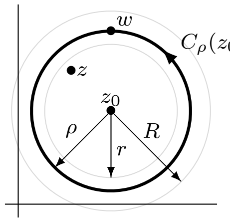
$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j,$$

where, as is conventional, $0! = 1$, and $f^{(0)} = f$.

A priori, there is no reason why the Taylor series should converge at any point z other than z_0, and even if it does, no reason why it should converge to a value at z at all related to $f(z)$. A remarkable fact about holomorphic functions is that the Taylor series does indeed converge on the largest disc on which the function is holomorphic, and moreover to the value $f(z)$.

Theorem 4.3.2 (Taylor Series)**.** Let $z_0 \in \mathbb{C}$, $R > 0$, and suppose f is holomorphic on $D_R(z_0)$. Then the Taylor series for f centred at z_0 converges to $f(z)$ for all $z \in D_R(z_0)$, and the convergence is uniform on $\overline{D}_r(z_0)$ for all $0 \leq r < R$.

Proof. The proof uses the Generalized Cauchy Integral Formula with an appropriate choice of contour, as shown in the diagram. Fix $r \in (0, R)$. By hypothesis, f is holo-



morphic inside and on the contour $C_\rho(z_0)$, where $r < \rho < R$. So for any $z \in \overline{D}_r(z_0)$, the Cauchy Integral Formula states that

$$f(z) = \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(w)}{w - z}\, dw.$$

Consider $w \in C_\rho(z_0)$, and rewrite

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0} \cdot \frac{1}{1-\frac{z-z_0}{w-z_0}},$$

and since $|z-z_0| \le r < \rho = |w-z_0|$, we have that $\left|\frac{z-z_0}{w-z_0}\right| < 1$, so the geometric series expansion

$$\frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{j=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^j,$$

is valid. Hence

$$\frac{1}{w-z} = \frac{1}{w-z_0} \sum_{j=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^j = \sum_{j=0}^{\infty} \frac{(z-z_0)^j}{(w-z_0)^{j+1}}.$$

Rearranging the expression for the partial sum

$$\sum_{j=0}^{n} \left(\frac{z-z_0}{w-z_0}\right)^j = \frac{1}{1-\frac{z-z_0}{w-z_0}} - \frac{\left(\frac{z-z_0}{w-z_0}\right)^{n+1}}{1-\frac{z-z_0}{w-z_0}} = \sum_{j=0}^{\infty} \frac{(z-z_0)^j}{(w-z_0)^j} - \frac{\left(\frac{z-z_0}{w-z_0}\right)^{n+1}}{1-\frac{z-z_0}{w-z_0}},$$

we write the series as the sum of the partial sum and the remainder to see that

$$\frac{1}{w-z} = \frac{1}{w-z_0} \left(\sum_{j=0}^{n} \left(\frac{z-z_0}{w-z_0}\right)^j + \frac{\left(\frac{z-z_0}{w-z_0}\right)^{n+1}}{1-\frac{z-z_0}{w-z_0}} \right) = \sum_{j=0}^{n} \frac{(z-z_0)^j}{(w-z_0)^{j+1}} + \frac{\left(\frac{z-z_0}{w-z_0}\right)^{n+1}}{w-z}.$$

This is only a finite sum, so we substitute it into the Cauchy Integral Formula, integrate term-by-term, and use the Generalized Cauchy Integral Formula to see that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(w)}{w-z}\, dw \\
&= \frac{1}{2\pi i} \int_{C_\rho(z_0)} f(w) \left(\sum_{j=0}^{n} \frac{(z-z_0)^j}{(w-z_0)^{j+1}} + \frac{\left(\frac{z-z_0}{w-z_0}\right)^{n+1}}{w-z} \right) dw \\
&= \sum_{j=0}^{n} \frac{(z-z_0)^j}{2\pi i} \int_{C_\rho(z_0)} \frac{f(w)}{(w-z_0)^{j+1}}\, dw + \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(w)}{w-z} \left(\frac{z-z_0}{w-z_0}\right)^{n+1} dw \\
&= \sum_{j=0}^{n} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j + R_n(z),
\end{aligned}$$

where

$$R_n(z) := \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(w)}{w-z} \left(\frac{z-z_0}{w-z_0}\right)^{n+1} dw.$$

So, in order to prove uniform convergence of the Taylor series, we only have to show that we can make $|R_n(z)|$ small for large n, uniformly in $z \in \overline{D}_R(z_0)$.

Now, we have that $|z-z_0| \le r$, and, for $w \in C_\rho(z_0)$, $|w-z_0| = \rho$. So, by the reverse triangle inequality, we have that

$$|w-z| = |(w-z_0)-(z-z_0)| \ge |w-z_0| - |z-z_0| \ge \rho - r,$$

whence $\frac{1}{|w-z|} \le \frac{1}{\rho-r}$. Since f is continuous, there exists $M > 0$ such that $|f(w)| \le M$ for all such $w \in C_\rho(z_0)$. Hence,

$$\left| \frac{f(w)}{w-z} \left(\frac{z-z_0}{w-z_0}\right)^{n+1} \right| = \frac{|f(w)|}{|w-z|} \left|\frac{z-z_0}{w-z_0}\right|^{n+1} \le \frac{M}{\rho-r} \left(\frac{r}{\rho}\right)^{n+1}.$$

Therefore since $\ell(C_\rho(z_0)) = 2\pi\rho$, by lemma 3.2.9 we have that

$$|R_n(z)| = \frac{1}{2\pi}\left|\int_{C_\rho(z_0)} \frac{f(w)}{w-z}\left(\frac{z-z_0}{w-z_0}\right)^{n+1} dw\right| \le \frac{2\pi\rho M}{2\pi(\rho - r)}\left(\frac{r}{\rho}\right)^{n+1}.$$

The right-hand side does not depend on z and can be made as small as desired by taking n large, since $r/\rho < 1$. □

Remark 4.3.3. Notice that this result implies that the Taylor series of f centred at z_0 will converge to $f(z)$ everywhere inside the largest open disc centred at z_0, on which f is holomorphic.

Definition 4.3.4. Let $U \subseteq \mathbb{C}$ be open, and $f\colon U \to \mathbb{C}$. Then we say that f is *analytic* if at every point $z \in U$, f can be expressed as a convergent power series.

Theorem 4.3.5. Let $U \subseteq \mathbb{C}$ be open, and $f\colon U \to \mathbb{C}$ be holomorphic. Then f is analytic.

Proof. This is nothing more than a restatement of theorem 4.3.2. □

Example 4.3.6. Let us compute the Taylor series for the function Log around $z_0 = 1$. The derivatives of the principal branch of the logarithm are:

$$\frac{d^j \operatorname{Log}(z)}{dz^j} = (-1)^{j+1}(j-1)!\frac{1}{z^j}.$$

Evaluating these at $z = 1$, we can construct the Taylor series:

$$\operatorname{Log}(z) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(j-1)!}{j!}(z-1)^j = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(z-1)^j,$$

which is valid on $D_1(1)$, the largest open disc centred at $z = 1$ on which $\operatorname{Log}(z)$ is holomorphic.

Example 4.3.7. Similarly, let us compute the Taylor series for the function $1/(1-z)$ around $z_0 = 0$. The derivatives are

$$\frac{d^j}{dz^j}\frac{1}{1-z} = \frac{j!}{(1-z)^{j+1}},$$

which we evaluate at $z = 0$ to see that the Taylor series is given by the geometric series

$$\frac{1}{1-z} = \sum_{j=0}^{\infty} \frac{j!}{j!} z^j = \sum_{j=0}^{\infty} z^j,$$

which is valid for on $D_1(0)$. This is of course familiar as the usual series representation for the function $1/(1-z)$ centred at the origin, although we have derived it through quite a different route. That this gives us the same series expansion is not a coincidence: we will see that series representations for holomorphic functions are unique: they are all essentially Taylor series.

Exercise 4.3.8. Verify the following familiar Taylor series expansions, centred at 0:

$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!};$$

$$\cos(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!}; \text{ and}$$

$$\sin(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}.$$

A very useful property of Taylor series is that we may differentiate them term-by-term. In general this is non-obvious, since differentiation and convergence of infinite series are both limiting processes, which in general do not commute.

Proposition 4.3.9. Let $z_0 \in \mathbb{C}$, $R > 0$, and f be holomorphic on $D_R(z_0)$. Then for $z \in D_R(z_0)$,

$$f'(z) = \sum_{j=0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!}(z - z_0)^j,$$

i.e. the Taylor series for f' is found by differentiating the Taylor series for f term-by-term.

Proof. By corollary 3.5.4, we know that the derivative f' of f is itself holomorphic on $D_R(z_0)$, and therefore has a Taylor series centred at z_0, given by

$$f'(z) = \sum_{j=0}^{\infty} \frac{(f')^{(j)}(z_0)}{j!}(z - z_0)^j = \sum_{j=0}^{\infty} \frac{f^{(j+1)}(z_0)}{j!}(z - z_0)^j. \qquad \square$$

Lemma 4.3.10. Let $z_0 \in \mathbb{C}$, $R > 0$, $\alpha, \beta \in \mathbb{C}$, and f, g be holomorphic on $D_R(z_0)$. Then

(i) the Taylor series for $\alpha f + \beta g$ centred at z_0, valid on $D_R(z_0)$, is the series

$$\sum_{j=0}^{\infty} \frac{\alpha f^{(j)}(z_0) + \beta g^{(j)}(z_0)}{j!}(z - z_0)^j;$$

and

(ii) the Taylor series for fg centred at z_0, valid on $D_R(z_0)$, is the series

$$\sum_{j=0}^{\infty} \frac{1}{j!} \left(\sum_{k=0}^{j} \binom{j}{k} f^{(k)}(z_0) g^{(j-k)}(z_0) \right) (z - z_0)^j.$$

Proof. Exercise. The generalized Leibniz formula

$$(fg)^{(j)}(z_0) = \sum_{k=0}^{j} \binom{j}{k} f^{(k)}(z_0) g^{(j-k)}(z_0)$$

might be useful. $\square$

Remark 4.3.11. Thus we can add and multiply Taylor series. The multiplication of two infinite series is a slightly delicate matter—it suffices for our purposes to observe that it is possible in this context, in exactly the way one would expect: the coefficient of $(z - z_0)^j$ in the product is the sum of all pairs of coefficients of $(z - z_0)^k$ and $(z - z_0)^{k-j}$, for $k = 0, \ldots, j$, from the two series being multiplied.

We have seen that a convergent power series about a point z_0 is holomorphic. It therefore has a Taylor series centred at z_0. We now prove that a power series *is* the Taylor series of the function it describes. This may not sound useful, but it is, because it allows us to compute the Taylor series of a function just by finding some power series expansion of the function, since this result tells us that the two are the same.

Theorem 4.3.12 (Uniqueness of Taylor series)**.** Let $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ be a convergent power series with radius of convergence R. Then the Taylor series of f centred at z_0 is the power series $\sum_{j=0}^{\infty} a_j (z - z_0)^j$, which is valid on $D_R(z_0)$.

Proof. By theorem 4.2.6 f is holomorphic on $D_R(z_0)$, so by theorem 4.3.2 f has a Taylor series centred at z_0. Let Γ be a loop inside $D_R(z_0)$, with z_0 in its interior. We use the

Generalized Cauchy Integral Formula (3.4) to express the coefficients in the Taylor series as

$$\frac{f^{(j)}(z_0)}{j!} = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{j+1}}\,dz.$$

There exists $0 < r < R$ such that Γ lies inside $D_r(z_0)$, so the power series converges uniformly on $D_r(z_0)$, and hence on Γ. Therefore lemma 4.1.22 implies that

$$\begin{aligned}\frac{f^{(j)}(z_0)}{j!} = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{j+1}}\,dz &= \frac{1}{2\pi i}\int_\Gamma \sum_{k=0}^{\infty} a_k \frac{(z-z_0)^k}{(z-z_0)^{j+1}}\,dz \\ &= \sum_{k=0}^{\infty} \frac{1}{2\pi i}\int_\Gamma a_k \frac{(z-z_0)^k}{(z-z_0)^{j+1}}\,dz \\ &= \sum_{k=0}^{\infty} \frac{a_k}{2\pi i}\int_\Gamma (z-z_0)^{k-j-1}\,dz.\end{aligned}$$

But we know from the Generalized Cauchy Integral Formula that

$$\int_\Gamma (z-z_0)^{k-j-1}\,dz = \begin{cases} 2\pi i & j = k, \\ 0 & \text{otherwise.}\end{cases}$$

Hence the only contribution to the jth coefficient is the term $k = j$, which contributes precisely $\frac{a_j}{2\pi i}2\pi i = a_j$. Thus the jth term of the Taylor series is a_j, which is exactly to say that the power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ is its own Taylor series centred at the point z_0. ☐

Example 4.3.13. Let us compute the Taylor series of the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

centred at 0. The series will converge on $D_1(0)$, but not on any larger disc, since f is not holomorphic at $z = 1$ and $z = 2$. The naive solution to this problem would be to take derivatives, evaluate them at the origin, and build the Taylor series this way. However from our discussion above, it is enough to exhibit any power series which converges to this function in the specified region. We use partial fractions to rewrite the function as a sum of simple fractions, and use geometric series:

$$\begin{aligned}\frac{1}{(z-1)(z-2)} = \frac{1}{1-z} - \frac{1}{2-z} = \frac{1}{1-z} - \frac{1}{2}\cdot\frac{1}{1-(z/2)} &= \sum_{j=0}^{\infty} z^j - \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j \\ &= \sum_{j=0}^{\infty}\left(1 - \frac{1}{2^{j+1}}\right) z^j,\end{aligned}$$

where the first expansion is valid on $D_1(0)$, and the second is valid on $D_2(0)$, so the resulting expansion for f is valid on $D_1(0) \cap D_2(0) = D_1(0)$.

Example 4.3.14. Let us consider the same function $f(z)$ as in the previous example, but let us find its Taylor series centred at $z_0 = -1$. The closest point to -1 at which the function fails to be holomorphic is the point $z = 1$, which is a distance 2 away, hence the Taylor series will converge in the disc $D_2(-1)$. To calculate the Taylor series centred at $z_0 = -1$, we artificially introduce terms of the form $z - (-1) = z + 1$ and make appropriate corrections. The function can be written as

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{((z+1)-2)((z+1)-3)} = \frac{1}{2-(z+1)} - \frac{1}{3-(z+1)},$$

where each term can be related to a geometric series:

$$\frac{1}{2-(z+1)} = \frac{1}{2}\cdot\frac{1}{1-(z+1)/2} = \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z+1}{2}\right)^j = \sum_{j=0}^{\infty}\frac{(z+1)^j}{2^{j+1}},$$

which is valid as long as $|z+1| < 2$; and

$$\frac{1}{3-(z+1)} = \frac{1}{3}\cdot\frac{1}{1-(z+1)/3} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{z+1}{3}\right)^j = \sum_{j=0}^{\infty}\frac{(z+1)^j}{3^{j+1}},$$

which is valid as long as $|z+1| < 3$.

Combining these expressions we get that

$$f(z) = \sum_{j=0}^{\infty}\frac{(z+1)^j}{2^{j+1}} - \sum_{j=0}^{\infty}\frac{(z+1)^j}{3^{j+1}} = \sum_{j=0}^{\infty}\left(\frac{1}{2^{j+1}} - \frac{1}{3^{j+1}}\right)(z+1)^j,$$

which is valid whenever $|z+1| < 2$, i.e. on $D_2(-1)$.

Proposition 4.3.9 tells us that the Taylor series for f' is obtained by termwise differentiation of the Taylor series for f. This is very useful, as illustrated by the next example.

Example 4.3.15. Let us compute the Taylor series of the function

$$f(z) = \frac{1}{(z-1)(z-2)^2}$$

centred at the point $z_0 = 0$. Since 1 is the closest point to 0 at which f is not holomorphic, this series expansion will be valid in the disc $D_1(0)$. Using partial fractions, and example 4.3.13, we see that

$$f(z) = \frac{1}{(z-1)(z-2)^2} = \frac{1}{z-1} - \frac{1}{z-2} + \frac{1}{(z-2)^2} = \sum_{j=0}^{\infty}\left(-1 + \frac{1}{2^{j+1}}\right)z^j + \frac{1}{(z-2)^2},$$

which is valid on $D_1(0)$. For the remaining term, we notice that, since we can differentiate convergent power series termwise,

$$\frac{1}{(z-2)^2} = \frac{d}{dz}\left(\frac{-1}{z-2}\right) = \frac{d}{dz}\left(\frac{1}{2-z}\right) = \frac{d}{dz}\left(\sum_{j=0}^{\infty}\frac{z^j}{2^{j+1}}\right) = \sum_{j=1}^{\infty}\frac{jz^{j-1}}{2^{j+1}},$$

which is valid wherever the differentiated series was valid, i.e. on $D_2(0)$. Combining, we get

$$f(z) = \sum_{j=0}^{\infty}\left(-1 + \frac{1}{2^{j+1}} + \frac{j+1}{2^{j+2}}\right)z^j,$$

which is valid on $D_1(0)$.

Exercise 4.3.16. Find the Taylor series centred at the point z_0 of each of the following functions f, and determine the set on which the expansion is valid:

(i) $f(z) = \cosh(z)$, $z_0 = 0$;
(ii) $f(z) = \sinh(z)$, $z_0 = 0$;
(iii) $f(z) = 1/(1-z)$, $z_0 = i$;
(iv) $f(z) = \mathrm{Log}(1-z)$, $z_0 = 0$;
(v) $f(z) = 1/(1+z)$, $z_0 = 0$;
(vi) $f(z) = z/(1-z)^2$, $z_0 = 0$;
(vii) $f(z) = z^3\sin(3z)$, $z_0 = 0$; and
(viii) $f(z) = (1+z)/(1-z)$, $z_0 = i$.

Exercise 4.3.17. Define $f(z) = \sum_{j=0}^{\infty} \frac{j^3}{3^j} z^j$. Compute the contour integrals of the following functions g around the contour $C_1(0)$:

(i) $g(z) = f(z)/z^4$;
(ii) $g(z) = \exp(z)f(z)$; and
(iii) $g(z) = f(z)\sin(z)/z^2$.

4.4. Laurent series. If a function fails to be holomorphic at a point $z_0 \in \mathbb{C}$, then we cannot expect it to have a Taylor series expansion centred at z_0, since that would be holomorphic at z_0. There is, however, a generalization of the idea of a Taylor series, which does indeed give us a series expansion centred at z_0, although it is not quite a power series.

Definition 4.4.1. Let $z_0 \in \mathbb{C}$, and $\ldots, a_{-1}, a_0, a_1, \ldots$ be a doubly-infinite sequence of complex numbers. A *Laurent series* centred at z_0 is a series of the form

$$\sum_{j=0}^{\infty} a_j(z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j} = \sum_{j=-\infty}^{\infty} a_j(z-z_0)^j.$$

We say the Laurent series *converges* if each of the two series on the left-hand side converges.

Remark 4.4.2. Thus a Laurent series $\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j$ converges if both the series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ and the series $\sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}$ converge. The first is just a power series, and therefore has a radius of convergence R. The second can be regarded as a power series in the variable $w = (z-z_0)^{-1}$, which similarly has a radius of convergence $r > 0$, in terms of the variable w. Thus the Laurent series converges for values of $z \in \mathbb{C}$ such that both $|z-z_0| < R$, and $|(z-z_0)^{-1}| < r$, i.e. such that $r < |z-z_0| < R$.

Definition 4.4.3. Let $z_0 \in \mathbb{C}$, and $r, R \in [0,\infty) \cup \{\infty\}$. Then the (open) *annulus* of radii r and R centred at z_0 is the set

$$A_{r,R}(z_0) = \{\, z \in \mathbb{C} : r < |z-z_0| < R \,\},$$

and the corresponding closed annulus is the set

$$\overline{A}_{r,R}(z_0) = \{\, z \in \mathbb{C} : r \leq |z-z_0| \leq R \,\}.$$

Thus a Laurent series converges on an annulus.

Theorem 4.4.4 (Laurent Series)**.** Let $z_0 \in \mathbb{C}$, $0 \leq r < R \leq \infty$, and f be holomorphic on $A_{r,R}(z_0)$. Then f can be expressed as a Laurent series centred at z_0 which converges on $A_{r,R}(z_0)$, uniformly on $\overline{A}_{r_1,R_1}(z_0)$ where $r < r_1 \leq R_1 < R$. Moreover, the coefficients of the Laurent series are given by

$$a_j = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-z_0)^{j+1}}\, dz,$$

for any loop Γ lying inside $A_{r,R}(z_0)$ and containing z_0 in its interior.

Proof. Fix $r < r_1 \leq R_1 < R$ and a point $z \in \overline{A}_{r_1,R_1}(z_0)$. In order to follow the logic of the proof, it will be convenient to keep figure 6 in mind. The left-hand picture shows the annuli $A_{r_1,R_1}(z_0) \subseteq A_{r,R}(z_0)$, and the contour Γ. The right-hand picture shows the equivalent contours Γ_1 and Γ_2, circles with radii ρ_1 and ρ_2 satisfying the inequalities $r < \rho_1 < r_1$ and $R_1 < \rho_2 < R$.

Consider the closed contour Γ, starting and ending at the point P in the figure, and defined as follows: follow Γ_2 all the way around until reaching P again, then go to Q via the "bridge" between the two circles, then all the way along Γ_1 until reaching Q again (in the negative direction), then back to P along the "bridge." This contour encircles

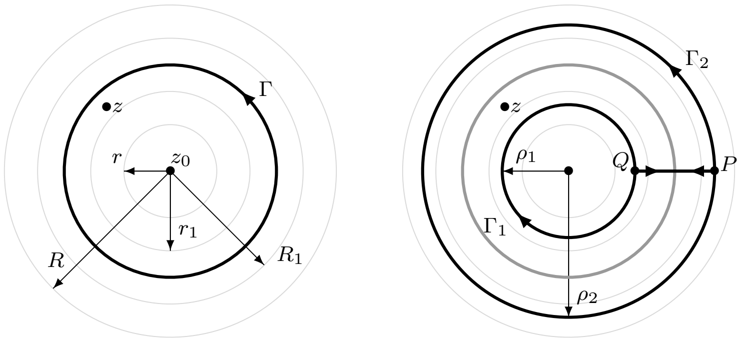


FIGURE 6. Contours Γ, Γ_1 and Γ_2.

the point z once in the positive sense, hence by the Cauchy Integral Formula we have that

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}\,dw.$$

On the other hand, because the "bridge" is traversed twice in opposite directions, their contribution to the integral cancels and we are left with

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(w)}{w-z}\,dw.$$

(To be pedantic, such a contour is not a loop, because the "bridge" precisely consists of a set of points of self-intersection. As stated, therefore, we are unable to apply the Cauchy Integral Formula. However, we may circumvent this problem by considering for a parameter $\varepsilon > 0$ a loop which does not quite traverse the same set of points twice, but passes along two parallel lines a distance ε apart, in opposite directions. We then use a continuity argument, letting $\varepsilon \to 0$, to recover the above results.)

We now consider each integral separately.

The integral along Γ_2 can be treated as we did the similar integral in the proof of the Taylor series theorem, and we simply quote the result:

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(w)}{w-z}\,dw = \sum_{j=0}^{\infty} a_j(z-z_0)^j,$$

where

$$a_j = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(w)}{(w-z_0)^{j+1}}\,dw. \tag{4.3}$$

Moreover the series converges uniformly in the closed disc $\overline{D}_{R_1}(z_0)$, as before.

The integral along Γ_1 can be treated along similar lines, except that because $|z-z_0| \geq r_1 > \rho_1 = |w-z_0|$ for $w \in \Gamma_1$, we must expand the integrand differently. Nonetheless, the idea is very much the same as was done for the Taylor series. We start by rewriting $1/(w-z)$ appropriately, and using a geometric series expansion, since $\left|\frac{w-z_0}{z-z_0}\right| < 1$, to see that

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = -\frac{1}{z-z_0}\frac{1}{1-\frac{w-z_0}{z-z_0}}$$

$$= -\frac{1}{z-z_0}\left(\sum_{j=0}^{n}\left(\frac{w-z_0}{z-z_0}\right)^j + \frac{\left(\frac{w-z_0}{z-z_0}\right)^{n+1}}{1-\frac{w-z_0}{z-z_0}}\right)$$

$$= -\sum_{j=0}^{n}\frac{(w-z_0)^j}{(z-z_0)^{j+1}} + \frac{1}{w-z}\frac{(w-z_0)^{n+1}}{(z-z_0)^{n+1}}.$$

Hence

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{w-z}\,dw = -\sum_{j=0}^{n}\frac{1}{2\pi i}\frac{1}{(z-z_0)^{j+1}}\int_{\Gamma_1}\frac{f(w)}{(w-z_0)^{-j}}\,dw + \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{w-z}\cdot\frac{(w-z_0)^{n+1}}{(z-z_0)^{n+1}}\,dw$$

$$= \sum_{j=1}^{n+1} a_{-j}(z-z_0)^{-j} + S_n(z),$$

where

$$a_{-j} = -\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{(w-z_0)^{-j+1}}\,dw, \tag{4.4}$$

and where

$$S_n(z) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{w-z}\cdot\frac{(w-z_0)^{n+1}}{(z-z_0)^{n+1}}\,dw.$$

Now, recall that $|z-z_0| \geq r_1$, and for $w \in \Gamma_1$ we have that $|w-z_0| = \rho_1$, so by the reverse triangle inequality, that $|w-z| \geq |z-z_0| - |w-z_0| \geq r_1 - \rho_1$. Furthermore, since f is continuous, there exists $M > 0$ such that $|f(w)| \leq M$ for all $w \in \Gamma_1$. Therefore using lemma 3.2.9 and the above inequalities,

$$|S_n(z)| \leq \frac{M\rho_1}{r_1-\rho_1}\left(\frac{\rho_1}{r_1}\right)^{n+1},$$

which is independent of z and, because $\rho_1 < r_1$, can be made arbitrarily small by choosing n large. Hence $S_n(z) \to 0$ as $n \to \infty$ uniformly on $\{\, z \in \mathbb{C} : |z-z_0| \geq r_1 \,\}$, and

$$\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)}{w-z}\,dw = \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j},$$

where the a_{-j} are given by (4.4). In summary, we have that proved that $f(z)$ is equal to the Laurent series

$$f(z) = \sum_{j=-\infty}^{\infty} a_j(z-z_0)^j,$$

everywhere on $A_{r,R}(z_0)$ and uniformly on any closed sub-annulus, where the coefficients a_j are given by (4.3) for $j \geq 0$ and by (4.4) for $j < 0$.

We are almost done, except that in the statement of the theorem the coefficients a_j are given by contour integrals along Γ and what we have shown is that they are given by contour integrals along Γ_1 or Γ_2. But notice that the integrand in (4.3) is holomorphic in the domain bounded by the contours Γ and Γ_2; and similarly for the integrand in (4.4) in the region bounded by the contours Γ and Γ_1. Therefore we can deform the contours Γ_1 and Γ_2 to $-\Gamma$ and Γ respectively, recalling that Γ_1 was negatively-oriented, to get that

$$a_{-j} = -\frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(w)}{(w-z_0)^{-j+1}}\,dw = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{(w-z_0)^{-j+1}}\,dw, \text{ and}$$
$$a_j = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(w)}{(w-z_0)^{j+1}}\,dw = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(w)}{(w-z_0)^{j+1}}\,dw. \qquad \square$$

Remark 4.4.5. Notice that this result generalizes the result about Taylor series for functions holomorphic in the disc. Indeed, if f were holomorphic on $D_R(z_0)$, then by the Cauchy Integral Theorem and the above formula for a_j, it would follow that $a_{-j} = 0$ for $j = 1, 2, \ldots$, and hence that the Laurent series is the Taylor series.

Example 4.4.6. Let us consider again the function $f(z) = \frac{1}{(z-1)(z-2)}$ but consider the expansion centred at $z_0 = 1$. Then f is not holomophic at z_0, so it cannot be holomorphic in any disc centred at z_0. But it is holomorphic on the punctured disc $D'_1(1)$, which is an annulus of the form $A_{0,1}(1)$. So we should be able to find a Laurent expansion.

We can write

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{(z-1)((z-1)-1)} = -\frac{1}{z-1}\cdot\frac{1}{1-(z-1)} = -\frac{1}{z-1}\sum_{j=0}^{\infty}(z-1)^j$$
$$= \sum_{j=-1}^{\infty}(-1)(z-1)^j,$$

which is valid on $A_{0,1}(1)$.

Theorem 4.4.7 (Uniqueness of Laurent series)**.** Let $z_0 \in \mathbb{C}$, and $0 \le r < R \le \infty$, and suppose the series

$$\sum_{j=-\infty}^{\infty} c_j(z-z_0)^j$$

converges on the annulus $A_{r,R}(z_0)$. Then there is a function f which is holomorphic on $A_{r,R}(z_0)$ with Laurent series centred at z_0 given by

$$f(z) = \sum_{j=-\infty}^{\infty} c_j(z-z_0)^j.$$

Proof. The series $\sum_{j=0}^{\infty} c_j(z-z_0)^j$ defines a holomorphic function f_1 on $D_R(z_0)$. The series $\sum_{j=1}^{\infty} c_{-j}(z-z_0)^{-j}$ defines a holomorphic function f_2 on the complement of $D_r(z_0)$, since it is the composition of the two holomorphic functions $z \mapsto \frac{1}{z-z_0}$ and $w \mapsto \sum_{j=1}^{\infty} c_{-j}w^j$. Hence the sum $f_1 + f_2$ is holomorphic on the set of values on which both f_1 and f_2 are holomorphic, i.e. $z \in \mathbb{C}$ such that $r < |z-z_0| < R$, which is exactly the annulus $A_{r,R}(z_0)$.

To prove that the Laurent series of this holomorphic function $f = f_1 + f_2$ is indeed the series

$$\sum_{j=-\infty}^{\infty} c_j(z-z_0)^j,$$

we simply integrate term-by-term as we did with the corresponding result for Taylor series, theorem 4.3.12. $\square$

Just as with Taylor series, we now know that if we have expressed a function in terms of a series of the form of a Laurent series centred at a point z_0, then that is indeed the Laurent series of that function centred at that point.

Example 4.4.8. Let us compute the Laurent series of the rational function $f(z) = (z^2 - 2z + 3)/(z - 2)$ in the region $\mathbb{C} \setminus \overline{D}_1(1) = A_{1,\infty}(1)$. Let us first rewrite the numerator as a power series in $(z - 1)$:

$$z^2 - 2z + 3 = (z-1)^2 + 2.$$

Now we do the same with the denominator:

$$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1} \cdot \frac{1}{1 - 1/(z-1)} = \frac{1}{z-1} \sum_{j=0}^{\infty} \frac{1}{(z-1)^j} = \sum_{j=0}^{\infty} \frac{1}{(z-1)^{j+1}},$$

which is valid for $|z - 1| > 1$. Putting the two series together, we have that

$$f(z) = \frac{z^2 - 2z + 3}{z - 2} = \left((z-1)^2 + 2\right) \sum_{j=0}^{\infty} \frac{1}{(z-1)^{j+1}} = (z-1) + 1 + \sum_{j=0}^{\infty} \frac{3}{(z-1)^{j+1}}.$$

By the uniqueness of the Laurent series, this is *the* Laurent series for the function in the specified region.

Example 4.4.9. Consider the function $f(z) = 1/(z-1)(z-2)$. Let us find its Laurent expansions in the three different annuli centred at 0: $A_{0,1}(0)$, $A_{1,2}(0)$, and $A_{2,\infty}(0)$. Decomposing the function into partial fractions, we have that

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{1-z} - \frac{1}{2-z} = \sum_{j=0}^{\infty} z^j - \frac{1}{2} \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j,$$

which is valid on $D_1(0)$. Therefore on $A_{0,1}(0)$ we have that

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty} \left(1 - \frac{1}{2^{j+1}}\right) z^j.$$

On $A_{1,2}(0)$, the first of the geometric series above is not valid, but the second one is. Because on $A_{1,2}(0)$, $|z| > 1$, this means that $|1/z| < 1$, so we should try and use a geometric series in $1/z$:

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1 - (1/z)} = -\frac{1}{z} \sum_{j=0}^{\infty} \left(\frac{1}{z}\right)^j = \sum_{j=0}^{\infty} \frac{-1}{z^{j+1}},$$

which is valid on $\mathbb{C} \setminus \overline{D}_1(0)$. Therefore on $A_{1,2}(0)$ we have that

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty} \frac{-1}{z^{j+1}} + \sum_{j=0}^{\infty} \frac{-1}{2^{j+1}} z^j.$$

Finally on $A_{2,\infty}(0)$, we have that $|z| > 2$, so that we will have to find another series converging to $1/(z-2)$ in this region. Again, since now $|2/z| < 1$ we should try to use a geometric series in $2/z$:

$$-\frac{1}{2-z} = \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1 - (2/z)} = \frac{1}{z} \sum_{j=0}^{\infty} \left(\frac{2}{z}\right)^j = \sum_{j=0}^{\infty} \frac{2^j}{z^{j+1}}$$

which is valid on $\mathbb{C} \setminus \overline{D}_2(0)$. Therefore on $A_{2,\infty}(0)$ we have that

$$\frac{1}{(z-1)(z-2)} = \sum_{j=0}^{\infty} \left(-1 + 2^j\right) \frac{1}{z^{j+1}}.$$

Again by the uniqueness of the Laurent series, we know that these are *the* Laurent series for the function in the specified regions.

Exercise 4.4.10. Find the Laurent series for the function $f(z) = 1/(z + z^2)$ which is valid on the following domains:

(i) $A_{0,1}(0)$;
(ii) $A_{1,\infty}(0)$;
(iii) $A_{0,1}(-1)$; and
(iv) $A_{1,\infty}(-1)$.

Exercise 4.4.11. Find the Laurent series for the function $f(z) = z/(z+1)(z-2)$ which is valid on the following domains:

(i) $A_{0,1}(0)$;
(ii) $A_{1,2}(0)$; and
(iii) $A_{2,\infty}(0)$.

Exercise 4.4.12. Find the Laurent series for the function $f(z) = (z + 1)/z(z - 4)^3$ which is valid on the domain $A_{0,4}(4)$.

Exercise 4.4.13. Find the Laurent series of the following functions f on the domain $A_{0,\infty}(0)$:

(i) $f(z) = \sin(2z)/z^3$; and
(ii) $f(z) = z^2 \cos(1/3z)$.

4.5. Zeros and singularities.

Definition 4.5.1. Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in \mathbb{C}$, and $f : D \to \mathbb{C}$ be a function. We say z_0 is a *singularity* of f if f is not holomorphic at z_0. We say that a singularity z_0 of f is *isolated* if there exists $R > 0$ such that f is holomorphic on the punctured disc $D'_R(z_0)$ centred at z_0.

Example 4.5.2. The function $f(z) = \frac{1}{z-z_0}$ has an isolated singularity at z_0. On the other hand, the principal branch of the logarithm, Log, has non-isolated singularities at all non-positive real numbers.

The singularities of rational functions occur at the zeros of the denominator, so we first discuss zeros of holomorphic functions.

Definition 4.5.3. Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, and f be holomorphic on U. Then z_0 is a *zero* of f if $f(z_0) = 0$. We say that z_0 is a *zero of finite order* if there exists a positive integer m such that

$$f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0, \text{ but } f^{(m)}(z_0) \neq 0.$$

In this case we say that z_0 is a *zero of order m*. A zero of order one is also known as a *simple zero*. A zero z_0 of f is *isolated* if there exists $R > 0$ such that $f(z) \neq 0$ for $z \in D'_R(z_0)$.

Proposition 4.5.4. Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, and f be holomorphic on U, with a zero of finite order at z_0. Then z_0 is isolated.

Proof. Suppose z_0 is of order $m \geq 1$. Since f is holomorphic at z_0, there is $R > 0$ such that the Taylor series centred at z_0 converges to $f(z)$ on $D_R(z_0)$. But by assumption the coefficients for $j = 0, \ldots, m - 1$ are all zero, so we have that

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j = \sum_{j=m}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j = (z - z_0)^m g(z),$$

where

$$g(z) = \sum_{j=0}^{\infty} \frac{f^{(j+m)}(z_0)}{(j + m)!}(z - z_0)^j$$

is holomorphic on $D_R(z_0)$ and, by assumption, $g(z_0) = f^{(m)}(z_0)/m! \neq 0$. Since g is holomorphic at z_0, it is continuous at z_0, therefore there exists $\varepsilon > 0$, without loss of generality $\varepsilon < R$, such that $g(z) \neq 0$ for all $z \in D_\varepsilon(z_0)$. Since $(z - z_0)^m \neq 0$ on the punctured disc $D'_\varepsilon(z_0)$, we see that $f(z) = (z - z_0)^m g(z) \neq 0$ on $D'_\varepsilon(z_0)$, which shows that z_0 is an isolated zero of f. $\square$

The following easy consequence establishes a surprising fact about holomorphic functions.

Corollary 4.5.5. Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of z_0, f be holomorphic on U, and such that $f(z_n) = 0$ for a sequence of distinct points $z_n \in U$ which converge to z_0. Then f is identically zero on some disc centred at z_0.

Proof. Since f is holomorphic at z_0, it is continuous at z_0, and hence $f(z_0) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} 0 = 0$, since $z_n \to z_0$. Thus z_0 is a zero of f. Since $z_n \to z_0$ and $f(z_n) = 0$, we see that z_0 is *not* a zero of finite order, by the contrapositive of proposition 4.5.4. Thus there does not exist a positive integer m for which $f^{(m)}(z_0) \neq 0$. Then the Taylor expansion for f centred at z_0, which converges to f on some disc centred at z_0, has zeros for all its terms. Thus f is identically zero on the disc on which this Taylor expansion converges. $\square$

Corollary 4.5.6. Let $z_0 \in \mathbb{C}$ be a singularity of a rational function $f = P/Q$. Then z_0 is isolated.

Proof. The rational function P/Q is holomorphic everywhere except at the zeros of Q. Since Q is a polynomial, it is holomorphic, and—since it is evidently not identically zero, otherwise f would not be defined at all—its zeros are therefore isolated. Hence z_0 is isolated. $\square$

We now classify the isolated singularities of a holomorphic function.

Definition 4.5.7. Let $z_0 \in \mathbb{C}$ be an isolated singularity of a function f which is holomorphic on $D'_R(z_0)$ for some $R > 0$. Then f has a Laurent expansion centred at z_0 that is valid on $A_{0,R}(z_0)$. Suppose

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$$

is the Laurent expansion centred at z_0, valid on $A_{0,R}(z_0)$. Then

(i) z_0 is a *removable singularity* of f if $a_j = 0$ for all $j < 0$, i.e.

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j;$$

(ii) z_0 is a *pole of order* m of f, for a positive integer m, if $a_j = 0$ for $j < -m$ and $a_{-m} \neq 0$, i.e.

$$f(z) = \sum_{j=-m}^{\infty} a_j (z - z_0)^j;$$

and

(iii) z_0 is an *essential singularity* if $a_j \neq 0$ for infinitely many negative values of j.

Evidently one of these three cases occurs. A pole of order 1 is also known as a *simple pole.*

The name "removable" singularity is inspired by the following fact.

Theorem 4.5.8. Let $z_0 \in \mathbb{C}$ be a removable singularity of a function f which is holomorphic on $D'_R(z_0)$ for some $R > 0$. Then $f(z_0)$ can be (re-)defined so that f is holomorphic at z_0.

Proof. By definition we have a Laurent expansion centred at z_0 given by

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

which is valid on $D'_R(z_0)$. The function $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ is holomorphic on $D_R(z_0)$ (i.e. at z_0 too), since it is a power series, and agrees with f everywhere except possibly at z_0. So if we (re-)define $f(z_0) = a_0$, which is the value of this power series at the point z_0, then f is holomorphic, as required. $\square$

Example 4.5.9. The function $f \colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} \exp(z) & \text{for } z \neq 0, \\ 26 & \text{at } z = 0; \end{cases}$$

has a removable singularity at 0.

Example 4.5.10. The function $\frac{\sin(z)}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and has Laurent expansion centred at 0 and valid for $|z| > 0$ given by

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots ; \tag{4.5}$$

so it has a removable singularity at 0.

For quotients of holomorphic functions, the zeros of the denominator and the zeros of the numerator cancel in the following way.

Lemma 4.5.11. Let f, g be holomorphic at z_0, where z_0 is a zero of g of order m. Then

(i) if z_0 is not a zero of f, then f/g has a pole of order m at z_0; and
(ii) if z_0 is a zero of order k of f, then f/g has a pole of order $m-k$ at z_0 if $m > k$, and has a removable singularity at z_0 otherwise.

Proof. By assumption $g(z) = (z-z_0)^m q(z)$ for some holomorphic function q defined on a disc around z_0, satisfying $q(z_0) \neq 0$. If $f(z_0) \neq 0$, then f/q is a holomorphic function defined on a disc around z_0 satisfying $(f/q)(z_0) \neq 0$, and

$$\frac{f(z)}{g(z)} = (z-z_0)^{-m} \frac{f(z)}{q(z)},$$

whence we see that f/g has a pole of order m at z_0.

If f has a zero of order k at z_0, then $f(z) = (z-z_0)^k p(z)$, for some holomorphic function p defined on a disc around z_0, satisfying $p(z_0) \neq 0$. Therefore

$$\frac{f(z)}{g(z)} = \frac{(z-z_0)^k p(z)}{(z-z_0)^m q(z)} = \frac{1}{(z-z_0)^{m-k}} \frac{p(z)}{q(z)},$$

where p/q is non-zero at z_0 and holomorphic in a neighbourhood of z_0, hence the result follows. $\square$

4.6. Analytic continuation.

Definition 4.6.1. Let $D \subseteq \tilde{D} \subseteq \mathbb{C}$ be domains, and $f \colon D \to \mathbb{C}$ be holomorphic. We say that a holomorphic function $F \colon \tilde{D} \to \mathbb{C}$ is an *analytic continuation* of f if $F(z) = f(z)$ for $z \in D$.

Example 4.6.2. The function $\sum_{j=0}^{\infty} z^j$ is well-defined and holomorphic on $D_1(0)$, where it converges to $1/(1-z)$, which is holomorphic on $\mathbb{C} \setminus \{1\}$. The function $1/(1-z)$ is therefore an analytic continuation of the function $\sum_{j=0}^{\infty} z^j$.

Example 4.6.3. Suppose $D_1, D_2 \subseteq \mathbb{C}$ are domains such that $D_1 \cap D_2 \neq \emptyset$, and $f_1 \colon D_1 \to \mathbb{C}$ and $f_2 \colon D_2 \to \mathbb{C}$ are holomorphic, and such that $f_1(z) = f_2(z)$ for all $z \in D_1 \cap D_2$. Then the function $F \colon D_1 \cup D_2 \to \mathbb{C}$ defined by

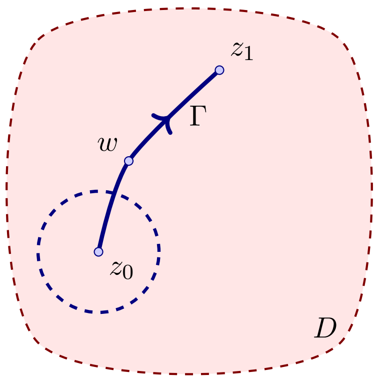
$$F(z) = \begin{cases} f_1(z) & z \in D_1, \\ f_2(z) & z \in D_2, \end{cases}$$

is an analytic continuation of f_1 and f_2 on $D_1 \cup D_2$. By assumption F is well-defined, and it is holomorphic because F' equals f_1' or f_2' in a neighbourhood of each point $z \in D_1 \cup D_2$.

Such a "gluing" technique works very easily, but the assumption that the two functions agree in $D_1 \cap D_2$ seems to be quite a strong assumption. Using series expansions it is often possible to show that two functions agree in a certain disc within a domain; a remarkable fact is that this suffices, as we now see.

Theorem 4.6.4 (Identity Theorem)**.** Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$, f be holomorphic on D and such that $f(z) = 0$ for all $z \in D_R(z_0)$, for some $R > 0$. Then $f(z) = 0$ for all $z \in D$.

Proof. Suppose for a contradiction that there is a point $z_1 \in D$ with $f(z_1) \neq 0$. Let Γ be a curve in D connecting z_0 to z_1.



As we move from z_0 to z_1 along Γ we must meet a point $w \in \Gamma$ such that

(i) $f(z) = 0$ for all z preceding w along Γ, and
(ii) there exist points $z \in \Gamma$ arbitrarily close to w such that $f(z) \neq 0$.

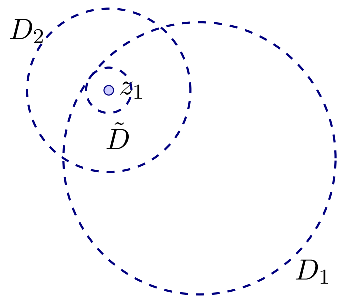
Then by (i), all derivatives of f vanish at points z preceding w along Γ and by continuity again, also at w itself. But then the Taylor expansion of f centred at w is identically zero, violating (ii). $\square$

Corollary 4.6.5. Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$, $f, g \colon D \to \mathbb{C}$ be holomorphic and such that $f(z) = g(z)$ for all $z \in D_R(z_0)$, for some $R > 0$. Then $f(z) = g(z)$ for all $z \in D$.

Proof. Apply the Identity Theorem to the holomorphic function $f - g$. $\square$

We can now consider analytic continuation via Taylor series.

Example 4.6.6. Consider a power series converging on a disc D_1 to a holomorphic function $f_1 \colon D_1 \to \mathbb{C}$.



Pick a point $z_1 \in D_1$ and consider the Taylor series of f_1 centred at z_1. Then this will converge (to f_1) on the largest open disc $\tilde{D}$ centred at z_1 and contained inside D_1. However it may happen that the Taylor series converges in a larger disc $D_2 \supseteq \tilde{D}$ to a holomorphic function $f_2 \colon D_2 \to \mathbb{C}$. We would like to glue them together, as in our first example of analytic continuation, to a holomorphic function $F \colon D_1 \cup D_2 \to \mathbb{C}$. For this we would need f_1 and f_2 to agree on all of $D_1 \cap D_2$. *A priori* this is not true, but by construction, f_1 and f_2 agree on $\tilde{D}$, and the Identity Theorem implies that they do indeed agree on $D_1 \cap D_2$, as required.

Corollary 4.6.7. Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$, and f be holomorphic on D and such that $f(z_n) = 0$ for a sequence of distinct points $z_n \in D$ which converge to z_0. Then $f(z) = 0$ for all $z \in D$.

Proof. By corollary 4.5.5 f is identically zero on some disc around z_0, and hence the Identity Theorem implies that f is identically zero on the whole domain D. ☐

Corollary 4.6.8. Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$, and f, g be holomorphic on D and such that $f(z_n) = g(z_n)$ for a sequence of distinct points $z_n \in D$ which converge to z_0. Then $f(z) = g(z)$ for all $z \in D$.

Proof. Apply the preceding result to the holomorphic function $f - g$. ☐

Example 4.6.9. Consider the two holomorphic functions $f, g \colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \cos^2(z) + \sin^2(z), \quad \text{and} \quad g(z) = 1.$$

Since we know that $\cos^2 x + \sin^2 x = 1$ for $x \in \mathbb{R}$, we know that $f = g$ on the real axis. Since this contains convergent sequences, e.g. $1/n \to 0$, they agree on their common domain, which is the whole complex plane. Thus we have another proof that

$$\cos^2(z) + \sin^2(z) = 1$$

for all $z \in \mathbb{C}$.

5. The residue calculus and its applications

We now begin a wonderful section of the course, which sees us apply the material we have been developing in some most unexpected ways.

5.1. The Cauchy Residue Theorem. The following seemingly innocuous calculation will have many ramifications. We simply evaluate the contour integral of a function f, where f is holomorphic on the interior of the contour, *except at an isolated singularity* z_0.

Theorem 5.1.1. Let $z_0 \in \mathbb{C}$, f be holomorphic on the punctured disc $D'_R(z_0)$ for some $R > 0$, with an isolated singularity at z_0, and Γ be a loop inside $D'_R(z_0)$, with z_0 in its interior. Then

$$\int_\Gamma f(z)\,dz = 2\pi i a_{-1},$$

where a_{-1} is the coefficient of the $(z-z_0)^{-1}$ term in the Laurent expansion of f centred at z_0 valid on $D'_R(z_0)$,

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j.$$

Proof. Since the Laurent expansion converges uniformly on closed subannuli of the punctured disc, it converges uniformly on Γ, therefore we may integrate the Laurent series term-by-term, to get

$$\int_\Gamma f(z)\,dz = \int_\Gamma \left(\sum_{j=-\infty}^{\infty} a_j (z-z_0)^j \right) dz = \sum_{j=-\infty}^{\infty} a_j \int_\Gamma (z-z_0)^j\,dz.$$

From the Generalized Cauchy Integral Formula, or by deforming the contour to a circle of radius $r < R$, we see that

$$\int_\Gamma (z-z_0)^j\,dz = \begin{cases} 2\pi i & j=-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus only the $j=-1$ term contributes anything to the sum, and so

$$\int_\Gamma f(z)\,dz = 2\pi i a_{-1}.$$

□

Since the coefficient a_{-1} has a special role in this theory, we give it a name.

Definition 5.1.2. Let $z_0 \in \mathbb{C}$, and f be holomorphic on the punctured disc $D'_R(z_0)$, for some $R > 0$, with an isolated singularity at z_0. Then the *residue of f at z_0*, $\operatorname{Res}(f, z_0)$, is

$$\operatorname{Res}(f, z_0) = a_{-1},$$

where the Laurent series of f valid on $D'_R(z_0)$ is

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j.$$

Example 5.1.3. Consider the function $f(z) = z\exp(1/z)$. This function has an essential singularity at the origin and is holomorphic everywhere else. The residue can be computed from the Laurent series centred at 0:

$$z\exp(1/z) = z\sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{z^j} = \sum_{j=0}^{\infty} \frac{1}{j!}\frac{1}{z^{j-1}} = z + 1 + \frac{1}{2z} + \cdots,$$

whence $\operatorname{Res}(f, 0) = \frac{1}{2}$.

It is not always necessary to compute the Laurent expansion to evaluate the residue at a point.

Lemma 5.1.4. Let $z_0 \in \mathbb{C}$, and f be holomorphic on the punctured disc $D'_R(z_0)$ for some $R > 0$, with a removable singularity at z_0. Then $\operatorname{Res}(f, z_0) = 0$.

Proof. This just follows by definition, since a singularity being removable means that there are no negative powers of $(z - z_0)$ in the Laurent expansion centred at z_0, in particular $\operatorname{Res}(f, z_0) = a_{-1} = 0$. □

Lemma 5.1.5. Let $z_0 \in \mathbb{C}$, and f be holomorphic on the punctured disc $D'_R(z_0)$ for some $R > 0$, with a pole of order m at z_0. Then

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)). \tag{5.1}$$

Proof. By definition the Laurent series for f valid on $D'_R(z_0)$ is of the form

$$f(z) = \sum_{j=-m}^{\infty} a_j (z - z_0)^j.$$

Multiplying this by $(z - z_0)^m$, we obtain

$$(z - z_0)^m f(z) = \sum_{j=-m}^{\infty} a_j (z - z_0)^{j+m} = \sum_{j=0}^{\infty} a_{j-m} (z - z_0)^j.$$

Taking $m - 1$ derivatives term-by-term, we have that

$$\frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right) = \sum_{j=m-1}^{\infty} a_{j-m} \frac{j!}{(j - (m-2))!} (z - z_0)^{j-(m-1)},$$

and thus by continuity,

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right) = (m-1)! a_{m-1-m} = (m-1)! \mathrm{Res}\,(f, z_0). \qquad \square$$

Example 5.1.6. The function $f(z) = \frac{\exp(z)}{z(z+1)}$ has simple poles at $z = 0$ and $z = -1$; therefore,

$$\mathrm{Res}\,(f, 0) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{\exp(z)}{z+1} = 1; \text{ and}$$
$$\mathrm{Res}\,(f, -1) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{\exp(z)}{z} = -\frac{1}{e}.$$

Lemma 5.1.7. Let $z_0 \in \mathbb{C}$, and g, h be holomorphic on $D'_R(z_0)$, for some $R > 0$, such that h has a simple zero at z_0, while $g(z_0) \neq 0$. Then defining $f = g/h$, we have that

$$\mathrm{Res}\,(f, z_0) = \frac{g(z_0)}{h'(z_0)}. \tag{5.2}$$

Proof. By lemma 4.5.11, since z_0 is a simple zero of h, it is a simple pole of f. Applying (5.1) with $m = 1$, we see that

$$\mathrm{Res}\,(f, z_0) = \lim_{z \to z_0} ((z - z_0) f(z)) = \lim_{z \to z_0} \left((z - z_0) \frac{g(z)}{h(z)} \right) = \lim_{z \to z_0} \frac{g(z)}{\frac{h(z) - h(z_0)}{z - z_0}} = \frac{g(z_0)}{h'(z_0)}. \qquad \square$$

Example 5.1.8. Consider $f(z) = \cot(z)$. Since $\cot(z) = \cos(z)/\sin(z)$, f has singularities at the zeros of the sine function: $z = n\pi$ for integers n. These zeros are simple because $\sin'(n\pi) = \cos(n\pi) = (-1)^n \neq 0$. Therefore by (5.2), for all integers n,

$$\mathrm{Res}\,(f, n\pi) = \frac{\cos(n\pi)}{\sin'(n\pi)} = \frac{\cos(n\pi)}{\cos(n\pi)} = 1.$$

This result will be crucial for the applications concerning infinite series later on in this section.

Example 5.1.9. Consider the function

$$f(z) = \frac{\cos(z)}{z^2 (z - \pi)^3}.$$

By lemma 4.5.11, this function has a pole of order 2 at the origin and a pole of order 3 at $z = \pi$. Therefore (5.1) implies that

$$\mathrm{Res}\,(f, 0) = \lim_{z \to 0} \frac{d}{dz} \left(z^2 f(z)\right) = \lim_{z \to 0} \frac{d}{dz} \left(\frac{\cos(z)}{(z - \pi)^3} \right) = \lim_{z \to 0} \left(\frac{-\sin(z)}{(z - \pi)^3} - \frac{3\cos(z)}{(z - \pi)^4} \right) = -\frac{3}{\pi^4},$$

and

$$\begin{aligned}\operatorname{Res}(f,\pi) = \lim_{z\to\pi}\frac{1}{2!}\frac{d^2}{dz^2}\left((z-\pi)^3 f(z)\right) &= \lim_{z\to\pi}\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{\cos(z)}{z^2}\right)\\ &= \lim_{z\to\pi}\frac{1}{2}\left(\frac{6\cos(z)}{z^4}+\frac{4\sin(z)}{z^3}-\frac{\cos(z)}{z^2}\right)\\ &= -\frac{6-\pi^2}{2\pi^4}.\end{aligned}$$

Exercise 5.1.10. Determine all the isolated singularities of the following functions f, and compute the residue at each singularity:

(i) $f(z) = \exp(3z)/(z-2)$;
(ii) $f(z) = (z+1)/(z^2-3z+2)$;
(iii) $f(z) = (\cos(z))/z^2$;
(iv) $f(z) = ((z-1)/(z+1))^3$;
(v) $f(z) = \exp(z)/z(z+1)^3$; and
(vi) $f(z) = (z-1)/\sin(z)$.

We are now ready to state the main result of this section, which concerns the formula for the integral of a function f which is holomorphic on a positively-oriented loop Γ and has only a finite number of isolated singularities $\{z_1,\ldots,z_k\}$ in the interior of the loop.

Theorem 5.1.11 (Cauchy Residue Theorem)**.** Let Γ be a loop, and f be holomorphic inside and on Γ except for finitely many isolated singularities $z_1,\ldots,z_k$ in the interior of Γ. Then

$$\int_\Gamma f(z)\,dz = 2\pi i\sum_{j=1}^{k}\operatorname{Res}(f,z_j).$$

Proof. We may deform the contour Γ into k loops Γ_j, each one having precisely one isolated singularity z_j of f in its interior. Therefore by theorem 5.1.1

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{k}\int_{\Gamma_j} f(z)\,dz = \sum_{j=1}^{k}2\pi i\operatorname{Res}(f,z_j) = 2\pi i\sum_{j=1}^{k}\operatorname{Res}(f,z_j). \qquad \square$$

Example 5.1.12. Let us compute the integral

$$\int_{C_2(0)}\frac{1-2z}{z(z-1)(z-3)}\,dz.$$

By lemma 4.5.11, the integrand $f(z)=\frac{1-2z}{z(z-1)(z-3)}$ has simple poles at $z=0$, $z=1$ and $z=3$, of which only $z=0$ and $z=1$ lie in the interior of the contour. Thus by the residue theorem,

$$\int_{C_2(0)}\frac{1-2z}{z(z-1)(z-3)}\,dz = 2\pi i\left(\operatorname{Res}(f,0)+\operatorname{Res}(f,1)\right).$$

We use (5.1) to see that

$$\begin{aligned}\operatorname{Res}(f,0) &= \lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{(1-2z)}{(z-1)(z-3)} = \frac{1}{3}, \text{ and}\\ \operatorname{Res}(f,1) &= \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{(1-2z)}{z(z-3)} = \frac{1}{2};\end{aligned}$$

so

$$\int_{C_2(0)}\frac{1-2z}{z(z-1)(z-3)}\,dz = 2\pi i\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{5\pi i}{3}.$$

Exercise 5.1.13. Evaluate, using the Cauchy Residue Theorem, the following contour integrals of the given function f around the given contour $C_R(0)$:

(i) $f(z) = \sin(z)/(z^2 - 4)$, $R = 5$;
(ii) $f(z) = \exp(z)/z(z-2)^3$, $R = 3$;
(iii) $f(z) = \tan(z)$, $R = 2\pi$;
(iv) $f(z) = 1/z^2 \sin(z)$, $R = 1$;
(v) $f(z) = (3z+2)/(z^4+1)$, $R = 3$; and
(vi) $f(z) = 1/(z^2+z+1)$, $R = 8$;

5.2. The Argument Principle and Rouché's Theorem.

Definition 5.2.1. Let $D \subseteq \mathbb{C}$ be a domain. A function f is *meromorphic on D* if for all $z \in D$, either f has a pole of some finite order at z or f is holomorphic at z.

Lemma 5.2.2. Let $D \subseteq \mathbb{C}$ be a domain, Γ be a loop in D, and f be meromorphic on D, and not identically zero. Then f has a finite number of poles and zeros on the interior of Γ.

Proof. Suppose for a contradiction that there are infinitely many poles z_n in the interior of Γ. Then since the interior of Γ is bounded, the sequence z_n is a bounded sequence, hence the Bolzano-Weierstrass Theorem implies that there is a convergent subsequence, $z_{n_k} \to z$, for some $z \in \mathbb{C}$, say, as $k \to \infty$. Then z lies either in the interior of Γ or on Γ, and is a point such that there are poles of f arbitrarily close to z. Then f is not holomorphic at z, and moreover z is not an isolated singularity of f. But this is a contradiction, since f is meromorphic inside and on Γ, thus f must either be holomorphic at z or have a pole at z. But a pole is, by definition, an *isolated* singularity. Thus z satisfies neither condition.

Applying this argument to the function $1/f$, we see similarly that f has at most finitely many zeros in the interior of Γ. □

Definition 5.2.3. Let Γ be a loop, and f be meromorphic on the interior of Γ, with zeros $w_1, \dots, w_l$ and poles $z_1, \dots, z_k$ in the interior of Γ. Then $N_0(f)$, the number of zeros of f inside Γ, counted with multiplicity, and $N_\infty(f)$, number of poles of f inside Γ, counted with multiplicity, are defined as

$$N_0(f) = \sum_{j=1}^{l} \text{order of } w_j, \quad \text{and} \quad N_\infty(f) = \sum_{j=1}^{k} \text{order of } z_j.$$

Example 5.2.4. Consider the rational function

$$f(z) = \frac{(z-8)z^3}{(z-5)^4(z+2)^2(z-1)^5}$$

and the contour $C_4(0)$. Then the only zero of f inside $C_4(0)$ is at $w = 0$, whence

$$N_0(f) = (\text{order of } w) = 3,$$

and the only poles inside $C_4(0)$ are at $z_1 = -2$ and $z_2 = 1$, whence

$$N_\infty(f) = (\text{order of } z_1) + (\text{order of } z_2) = 2 + 5 = 7.$$

Theorem 5.2.5 (The Argument Principle)**.** Let Γ be a loop in $\mathbb{C}$, and f be meromorphic on the interior of Γ, and holomorphic and non-zero on Γ. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz = N_0(f) - N_\infty(f).$$

Proof. Define a meromorphic function G on Γ and its interior by $G(z) = f'(z)/f(z)$. We will locate the singularities of G and compute the residues there. Notice that $G(z)$ is holomorphic on Γ because f is holomorphic and nonzero there. Inside Γ, $G(z)$ is meromorphic, with singularities whenever f has a zero or a pole. If z_0 is a zero of f of order m in the interior of Γ, for a positive integer m, then

$$f(z) = (z - z_0)^m h(z),$$

where $h(z)$ is holomorphic and nonzero at z_0. Then

$$G(z) = \frac{f'(z)}{f(z)} = \frac{m(z - z_0)^{m-1} h(z) + (z - z_0)^m h'(z)}{(z - z_0)^m h(z)} = \frac{m}{z - z_0} + \frac{h'(z)}{h(z)}.$$

Since h'/h is holomorphic at z_0, we see that $G(z)$ has a simple pole at z_0 with residue m. On the other hand, if z_∞ is a pole of f of order k in the interior of Γ, for a positive integer k, then

$$f(z) = \frac{H(z)}{(z - z_\infty)^k},$$

where H is holomorphic and nonzero at z_∞. Then in a punctured neighbourhood of z_∞,

$$G(z) = \frac{f'(z)}{f(z)} = \frac{(z - z_\infty)^k H'(z) - H(z) k (z - z_\infty)^{k-1}}{(z - z_\infty)^{2k} H(z)/(z - z_\infty)^k} = \frac{-k}{z - z_\infty} + \frac{H'(z)}{H(z)}.$$

Since H'/H is holomorphic at z_∞, $G(z)$ has a simple pole at z_∞ with residue $-k$. Thus the residues of G are precisely the orders of the zeros of f and -1 times the orders of the poles of f. The Cauchy Residue Theorem and the definitions of the numbers $N_0(f)$ and $N_\infty(f)$ therefore imply that

$$\frac{1}{2\pi i} \int_\Gamma G(z)\, dz = N_0(f) - N_\infty(f). \qquad \square$$

Corollary 5.2.6. Let Γ be a loop in $\mathbb{C}$, and f be holomorphic inside and on Γ, and non-zero on Γ. Then

$$\frac{1}{2\pi i} \int_\Gamma \frac{f'(z)}{f(z)}\, dz = N_0(f).$$

Proof. This follows immediately, since by assumption there are no poles inside Γ. $\square$

Theorem 5.2.7 (Rouché's Theorem)**.** Let Γ be a loop, and f, g be holomorphic on and inside Γ, such that for all $z \in \Gamma$,

$$|f(z) - g(z)| < |f(z)|.$$

Then $N_0(f) = N_0(g)$.

Proof. Since the inequality in the hypothesis is strict, both f and g are non-zero on Γ. Therefore $F = g/f$ is holomorphic and non-zero on Γ, and meromorphic on the interior of Γ, with poles at the zeros of f which are not cancelled by zeros of g. Since we are counting both zeros and poles with multiplicity, we see that

$$N_0(F) - N_\infty(F) = N_0(g) - N_0(f).$$

Dividing the inequality in the assmption by $|f(z)|$, we have that on Γ, $|1 - F(z)| < 1$, which says that the image of Γ under the function F lies in the open disc $D_1(1)$. So the principal value Log of the logarithm function is holomorphic on the image of Γ under F. By the Argument Principle and the Fundamental Theorem of Calculus, then,

$$N_0(g) - N_0(f) = N_0(F) - N_\infty(F) = \frac{1}{2\pi i} \int_\Gamma \frac{F'(z)}{F(z)}\, dz = \frac{1}{2\pi i} \int_\Gamma \frac{d}{dz} \operatorname{Log}(F(z))\, dz = 0. \quad \square$$

Example 5.2.8. Consider the polynomial $g(z) = z^5 + 3z + 1$. Rouché's Theorem allows us to deduce in what regions the zeros of this polynomial lie. We claim that all five zeros of g lie inside the disc $D_2(0)$. To show this, we consider the contour $C_2(0)$, and let $f(z) = z^5$, which clearly has five zeros, counted with multiplicity, in $\operatorname{Int}(C_2(0))$. Now, for $z \in C_2(0)$ we have that $|f(z) - g(z)| = |-3z - 1| \le 3|z| + 1 = 7$, whereas

$|f(z)| = |z^5| = |z|^5 = 32$. Hence by Rouché's Theorem $g(z)$ also has five zeros in $\mathrm{Int}(C_2(0))$.

Now, instead take $f(z) = 3z$ and consider $C_1(0)$. Then on $C_1(0)$, $|f(z)| = 3$, whereas $|f(z) - g(z)| = |-z^5 - 1| \leq |z|^5 + 1 = 2$, so that there is only one zero of $g(z)$ inside the unit disc. Thus there are four zeros in the annulus $A_{1,2}(0)$.

Example 5.2.9. Similarly we can show that $g(z) = z + 3 + 2\exp(z)$ has precisely one zero in the left half-plane. However, since the left half-plane is not bounded, we cannot simply apply Rouché's Theorem without modification. So we consider a large bounded subset with the hope that it is large enough to be able to say something about the whole left half-plane. Let us consider the semi-disc of radius R in the left half-plane, centred at 0, for R sufficiently large and let Γ_R be the contour consisting of two regular components: the straight line from $-iR$ to iR along the imaginary axis and the semicircle parametrized by $\gamma(t) = R\exp(it)$ for $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. The function $f(z) = z + 3$ has exactly one zero in the left half-plane, at $z = -3$. For $z = x + iy \in \Gamma_R$, we have that $|f(z) - g(z)| = |-2\exp(z)| = 2e^x \leq 2e^0 = 2$, while $|f(z)|$ is bounded below on Γ_R by

$$|f(z)| = |z + 3| \geq \begin{cases} 3 & z = iy, \\ |z| - 3 = R - 3 & |z| = R. \end{cases}$$

Therefore for all $R > 5$, $|f(z) - g(z)| < |f(z)|$ for all $z \in \Gamma_R$ and thus by Rouché's Theorem, $g(z)$ has precisely one zero inside Γ_R. Since this holds for arbitrarily large R, $g(z)$ has precisely one zero in the left-hand plane.

Exercise 5.2.10. Use Rouché's Theorem to show that the polynomial $z^6 + 4z^2 - 1$ has exactly two zeros in the disc $D_1(0)$.

Exercise 5.2.11. Prove that the equation $z^3 + 9z + 27 = 0$ has no roots in the disc $D_2(0)$.

Exercise 5.2.12. Prove that all the roots of the equation $z^6 - 5z^2 + 10 = 0$ lie in the annulus $A_{1,2}(0)$.

Exercise 5.2.13. Find the number of roots of the equation $6z^4 + z^3 - 2z^2 + z - 1 = 0$ in the disc $D_1(0)$.

Exercise 5.2.14. Prove that the equation $z = 2 - \exp(-z)$ has exactly one root in the right half-plane.

Theorem 5.2.15 (Fundamental Theorem of Algebra)**.** Let $g(z) = a_n z^n + \cdots + a_1 z + a_0$ be a polynomial of degree n, for $a_0, \ldots, a_n \in \mathbb{C}$. Then g has n zeros, counted with multiplicity.

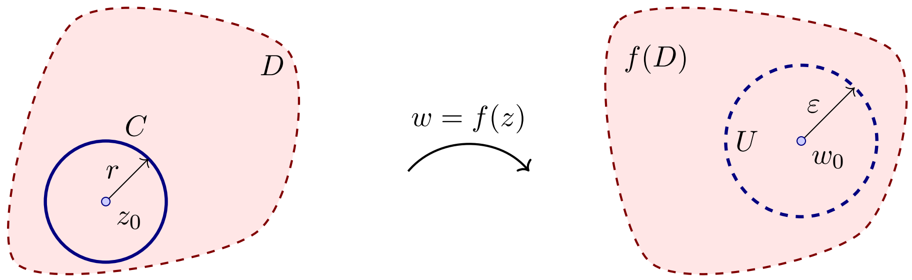
Proof. Since $a_n \neq 0$ by definition, we may divide through by a_n and assume, without loss of generality, that $a_n = 1$. In that case, we take $f(z) = z^n$ which clearly has n zeros (counted with multiplicity) at $z = 0$. Let $R > 1$ be such that $R > |a_{n-1}| + \cdots + |a_1| + |a_0|$. Then we see that, for $z \in C_R(0)$,

$$\begin{aligned} |f(z) - g(z)| \leq |a_{n-1}|R^{n-1} + \cdots + |a_1|R + |a_0| &\leq R^{n-1}\left(|a_{n-1}| + \cdots + \frac{|a_1|}{R^{n-2}} + \frac{|a_0|}{R^{n-1}}\right) \\ &\leq R^{n-1}\left(|a_{n-1}| + \cdots + |a_1| + |a_0|\right) \\ &< R^{n-1}R \\ &= R^n \\ &= |f(z)|. \end{aligned}$$

Hence by Rouché's theorem g has as many zeros as f inside $C_R(0)$ for all R satisfying the above inequality. Since this holds for arbitrarily large R, we see that g has precisely n zeros counted with multiplicity. ∎

Theorem 5.2.16 (Open Mapping Theorem)**.** Let $D \subseteq \mathbb{C}$ be a domain, and suppose f is non-constant and holomorphic on D. Then the image of D under f, $f(D) = \{\, f(z) : z \in D \,\}$, is an open subset of $\mathbb{C}$.

Proof. Let $w_0 \in f(D)$, so there exists $z_0 \in D$ such that $f(z_0) = w_0$. The function $F\colon D \to \mathbb{C}$ defined by $F(z) = f(z) - w_0$ is non-constant and holomorphic on D, and has a zero at z_0. The zeros of such a function are isolated, since otherwise by corollary 4.6.8, the function would be identically zero. Hence there exists $r > 0$ such that $\overline{D}_r(z_0) \subseteq D$ and F is nonzero on the boundary circle $C_r(z_0)$.



Let $\varepsilon = \min_{z \in C_r(z_0)} |F(z)|$, which is positive, by continuity of F. We claim that the open disc $U = D_\varepsilon(w_0)$ is contained in $f(D)$. For $w \in U$, define $G\colon D \to \mathbb{C}$ by $G(z) = f(z) - w$. For all $z \in D$,

$$|F(z) - G(z)| = |(f(z) - w_0) - (f(z) - w)| = |w - w_0| < \varepsilon \leq |F(z)|,$$

whence by Rouché's Theorem, G has the same number of zeros inside $C_r(z_0)$ as F, but F has a zero at z_0, hence G has at least one zero inside $C_r(z_0)$. Therefore there exists some z inside $C_r(z_0)$ with $f(z) = w$ and hence $w \in f(D)$. □

We can now give another proof of the following result.

Corollary 5.2.17 (Maximum Modulus Principle)**.** Let $D \subseteq \mathbb{C}$ be a domain, and f be holomorphic and non-constant on D. Then $|f(z)|$ does not attain a maximum on D.

Proof. Suppose for a contradiction that $|f(z)|$ attains its maximum at $z_0 \in D$. Then for some $\varepsilon > 0$, the open disc $U = D_\varepsilon(z_0)$ is contained in D and for all $z \in U$, $|f(z_0)| \geq |f(z)|$. By the Open Mapping theorem, $f(U)$ is open, so there is some $\delta > 0$ such that $V = D_\delta(f(z_0))$ is contained inside $f(U)$. But we can find $w \in V$ with $|w| > |f(z_0)|$ and since $V \subset f(U)$, there is some $z \in U$ with $w = f(z)$ and hence $|f(z)| > |f(z_0)|$, which is a contradiction. □

Corollary 5.2.18. Let $D \subseteq \mathbb{C}$ be a domain, and suppose f is holomorphic on D and such that any of the values $\mathrm{Re}(f(z))$, $\mathrm{Im}(f(z))$, $|f(z)|$, or $\mathrm{Arg}(f(z))$ is constant. Then f is constant.

Proof. Suppose that $\mathrm{Re}(f(z)) = c$ for all $z \in D$, but for a contradiction that f is not constant. Choose $z_0 \in D$ and $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subseteq D$. This is an open set, so $f(D_\varepsilon(z_0))$ is an open subset of $\mathbb{C}$, by the Open Mapping Theorem. So there exists $\delta > 0$ such that $D_\delta(f(z_0)) \subseteq f(D_\varepsilon(z_0)) \subseteq f(D)$. But the complex numbers $f(z_0) + \delta/2$ and $f(z_0) - \delta/2$ have different real parts, and lie in $D_\delta(f(z_0))$, and therefore lie in the image of D under f. This is a contradiction.

The other cases are similar. □

Exercise 5.2.19 (Schwarz's Lemma)**.** Let f be holomorphic on $D_1(0)$, and satisfy $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D_1(0)$. Prove that $|f(z)| \leq |z|$, using the following steps.

(i) Define $F(z) = f(z)/z$ for $z \in D_1'(0)$, and $F(0) = f'(0)$. Show that F is holomorphic on $D_1(0)$.
(ii) Let $z \in D_1(0)$ be non-zero, and suppose $|z| < r < 1$. Use the Maximum Modulus Principle to show that
$$|F(z)| \leq \max_{w \in C_r(0)} \left| \frac{f(w)}{w} \right| = \max_{w \in C_r(0)} \frac{|f(w)|}{r} \leq \frac{1}{r}.$$
(iii) Letting $r \to 1$ in part (ii), deduce that $|f(z)| \leq |z|$ for all $z \in D_1(0)$.

Exercise 5.2.20. Let f satisfy the hypothesis of exercise 5.2.19.

(i) Show that if there exists a non-zero point $z_0 \in D_1(0)$ such that $|f(z_0)| = |z_0|$, then $f(z) = e^{i\theta} z$ for some constant $\theta \in \mathbb{R}$.
(ii) Show that if $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ for some constant $\theta \in \mathbb{R}$.

Exercise 5.2.21. Let h be holomorphic on $\mathbb{C}$ and satisfy $|h(z)| < 1$ for all z such that $|z| = 1$. Prove that the equation $h(z) = z$ has exactly one solution, counting multiplicity, in $D_1(0)$. (Such a solution is a *fixed point* of h.)

5.3. Trigonometric integrals. We now start applying the residue calculus, using it to evaluate real integrals. Integrals of the form
$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$
for a rational function R can often be evaluated by considering a contour integral of an appropriate function around the unit circle centred at 0. Recall that for $z = e^{i\theta}$ on the circle of radius 1 centred at the origin, we have that
$$\cos\theta = \mathrm{Re}(z) = \frac{z + \overline{z}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \text{and} \quad \sin\theta = \mathrm{Im}(z) = \frac{z - \overline{z}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$
Defining $f(z) = \frac{1}{iz} R\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)$, we therefore have that
$$f(\exp(i\theta)) = \frac{1}{i\exp(i\theta)} R(\cos\theta, \sin\theta).$$
Consider the contour $C_1(0)$, which is parametrized by $\gamma\colon [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(\theta) = \exp(i\theta)$, so
$$\begin{aligned}\int_\Gamma f(z)\, dz = \int_0^{2\pi} f(\gamma(\theta))\gamma'(\theta)\, d\theta &= \int_0^{2\pi} f(\exp(i\theta)) i \exp(i\theta)\, d\theta \\ &= \int_0^{2\pi} \frac{1}{i\exp(i\theta)} R(\cos\theta, \sin\theta)\, i\exp(i\theta)\, d\theta \\ &= \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta.\end{aligned}$$
In other words, with a cunning choice of f, we reduce this real integral to a contour integral, which we of course do not have to evaluate by hand—we have the Cauchy Residue Theorem to tell us what the integral equals.

This motivates the choice of integrand in the following examples, which might otherwise be somewhat mysterious.

Example 5.3.1. Consider the integral
$$I = \int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta.$$

Define

$$f(z) = \frac{1}{iz}\frac{\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^2}{5+4\frac{1}{2}\left(z+\frac{1}{z}\right)} = \frac{i}{4}\frac{(z^2-1)^2}{z^2(2z^2+5z+2)} = \frac{i}{8}\frac{(z^2-1)^2}{z^2(z+\frac{1}{2})(z+2)},$$

which has a double pole at $z=0$ and single poles at $z=-\frac{1}{2}$ and $z=-2$. Of these, only the poles at $z=0$ and $z=-\frac{1}{2}$ lie inside $D_1(0)$, so

$$I = \int_{C_1(0)} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f,0) + \operatorname{Res}(f,1/2)\right).$$

Let us compute the residues. By (5.1), we have

$$\operatorname{Res}(f,0) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{i}{8}\frac{(z^2-1)^2}{(z+\frac{1}{2})(z+2)}\right) = \frac{-5i}{16}.$$

The pole at $z=-\frac{1}{2}$ is simple, so its residue is even simpler to compute:

$$\operatorname{Res}(f,1/2) = \lim_{z\to-\frac{1}{2}}\left(\frac{i}{8}\frac{(z^2-1)^2}{z^2(z+2)}\right) = \frac{3i}{16}.$$

Therefore, the integral becomes

$$I = 2\pi i\left(\frac{-5i}{16}+\frac{3i}{16}\right) = \frac{\pi}{4}.$$

Note that, as we would expect, our answer is indeed real.

Example 5.3.2. Consider the integral

$$I = \int_0^{\pi}\frac{1}{2-\cos\theta}\,d\theta.$$

This time the integral is only over $[0,\pi]$, so we cannot immediately use the residue theorem. However in this case we notice that because $\cos(2\pi-\theta)=\cos\theta$, we have that

$$\int_{\pi}^{2\pi}\frac{1}{2-\cos\theta}\,d\theta = -\int_{\pi}^{0}\frac{1}{2-\cos\theta}\,d\theta = \int_0^{\pi}\frac{1}{2-\cos\theta}\,d\theta.$$

Therefore

$$I = \frac{1}{2}\int_0^{2\pi}\frac{1}{2-\cos\theta}\,d\theta.$$

So, we define

$$f(z) = \frac{1}{iz}\frac{1}{2-\frac{1}{2}(z+\frac{1}{z})} = \frac{2i}{z^2-4z+1} = \frac{2i}{(z-2+\sqrt{3})(z-2-\sqrt{3})},$$

which has two simple poles at $z=2\pm\sqrt{3}$. Only $z=2-\sqrt{3}$ lies inside the unit disc $D_1(0)$, hence we have that

$$\begin{aligned} I = \frac{1}{2}\int_0^{2\pi}\frac{1}{2-\cos\theta}\,d\theta = \frac{1}{2}\int_{C_1(0)} f(z)\,dz = \frac{2\pi i}{2}\operatorname{Res}\left(f,2-\sqrt{3}\right) &= \pi i \lim_{z\to 2-\sqrt{3}}\frac{2i}{z-2-\sqrt{3}} \\ &= \frac{\pi}{\sqrt{3}}. \end{aligned}$$

Exercise 5.3.3. Evaluate the following real integrals:

(i) $\int_0^{2\pi}\frac{1}{2+\sin\theta}\,d\theta$;
(ii) $\int_0^{\pi}\frac{8}{5+2\cos\theta}\,d\theta$;
(iii) $\int_{-\pi}^{\pi}\frac{1}{1+\sin^2\theta}\,d\theta$; and
(iv) $\int_0^{\pi}\frac{1}{(3+2\cos\theta)^2}\,d\theta$.

5.4. Improper integrals.

Definition 5.4.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Then we define the *improper integrals*

$$\begin{aligned}\int_0^\infty f(x)\,dx &= \lim_{R\to\infty} \int_0^R f(x)\,dx;\\ \int_{-\infty}^0 f(x)\,dx &= \lim_{r\to-\infty} \int_r^0 f(x)\,dx; \text{ and}\\ \int_{-\infty}^\infty f(x)\,dx &= \lim_{\substack{R\to\infty\\ r\to-\infty}} \int_r^R f(x)\,dx;\end{aligned}$$

whenever the appropriate limits exist. Note that the limits are taken independently in the final improper integral; however, if the improper integral $\int_{-\infty}^\infty f(x)\,dx$ exists, then it is equal to

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{\rho\to\infty} \int_{-\rho}^\rho f(x)\,dx.$$

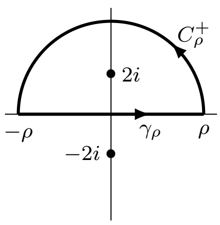
The latter expression might exist even when the improper integral, defined by taking the upper and lower limits independently, does not. In this case we define the *Cauchy principal value* of the integral by

$$\text{p. v.}\int_{-\infty}^\infty f(x)\,dx := \lim_{\rho\to\infty} \int_{-\rho}^\rho f(x)\,dx.$$

Example 5.4.2. Consider the improper integral

$$I = \text{p. v.}\int_{-\infty}^\infty \frac{1}{x^2+4}\,dx = \lim_{\rho\to\infty} \int_{-\rho}^\rho \frac{1}{x^2+4}\,dx.$$

The integral on the right-hand side for fixed ρ can be interpreted as the contour integral of the function $f(z) = 1/(z^2+4)$ along the contour γ_ρ which is the straight line segment on the real axis from $-\rho$ to ρ. In order to use the residue theorem we need to *close the contour*; that is, we must define a closed contour along which we can apply the residue theorem. Of course, in so doing we are introducing a further integral, and the success of the method depends on whether the extra integral is computable. We will see that in this case, the extra integral, if chosen judiciously, vanishes.



Consider the semicircular contour C_ρ^+ in the upper half plane, parametrized by $\gamma(t) = \rho\exp(it)$, for $t \in [0,\pi]$. Let Γ_ρ be the composition of both contours: it is a closed contour as shown in the figure. Then

$$\int_{\Gamma_\rho} \frac{1}{z^2+4}\,dz = \int_{\gamma_\rho} \frac{1}{z^2+4}\,dz + \int_{C_\rho^+} \frac{1}{z^2+4}\,dz,$$

where the integral on the left-hand side can be evaluated by the residue theorem. The integral we want is on the right-hand side, but there is an additional term. We will now argue that the integral along C_ρ^+ tends to 0 as $\rho \to \infty$.

For $\rho > 2$, by the reverse triangle inequality, we have on C_ρ^+ that $|z^2+4| \geq |z^2| - 4 = |z|^2 - 4 = \rho^2 - 4$, whence

$$\frac{1}{|z^2+4|} \leq \frac{1}{\rho^2-4}.$$

Thus by lemma 3.2.9,

$$\left| \int_{C_\rho^+} \frac{1}{z^2+4}\, dz \right| \leq \ell(C_\rho^+) \frac{1}{\rho^2-4} = \frac{\pi\rho}{\rho^2-4} \to 0 \quad \text{as } \rho \to \infty. \tag{5.3}$$

The function f has simple poles at $z = \pm 2i$, of which only $z = 2i$ lies inside the closed contour Γ_ρ, for $\rho > 2$. From (5.1) we have that

$$\operatorname{Res}(f, 2i) = \lim_{z \to 2i} \left(\frac{1}{z+2i} \right) = \frac{1}{4i},$$

hence

$$\begin{aligned}
\text{p. v.} \int_{-\infty}^{\infty} \frac{1}{x^2+4}\, dx = \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \frac{1}{x^2+4}\, dx &= \lim_{\rho\to\infty} \left(\int_{\Gamma_\rho} \frac{1}{z^2+4}\, dz - \int_{C_\rho^+} \frac{1}{z^2+4}\, dz \right) \\
&= \lim_{\rho\to\infty} \int_{\Gamma_\rho} \frac{1}{z^2+4}\, dz - \lim_{\rho\to\infty} \int_{C_\rho^+} \frac{1}{z^2+4}\, dz \\
&= 2\pi i \operatorname{Res}(f, 2i) - 0 \\
&= \frac{\pi}{2}.
\end{aligned}$$

The approach in this example will apply equally to any rational function $R = P/Q$ on the real line, where Q is non-zero, and $\deg(Q) \geq \deg(P) + 2$. The higher power in the denominator ensures that the integral around the semicircular arc used to close the contour tends to zero, as above.

Example 5.4.3. Consider the following integral

$$I = \text{p. v.} \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx.$$

Using the same approach, we let γ_ρ denote the straight line segment on the real axis from $-\rho$ to ρ, C_ρ^+ denote the semicircular contour in the upper half plane parametrized by $\gamma(t) = \rho \exp(it)$ for $t \in [0, \pi]$, and Γ_ρ denote the composition of the two contours.

Let $f(z) = z^2/(z^2+1)^2$, which is holomorphic except for poles of order 2 at the points $z = \pm i$, of which only $z = i$ lies in the interior of Γ_ρ for $\rho > 1$. We use (5.1) to see that

$$\operatorname{Res}(f, i) = \lim_{z \to i} \frac{d}{dz} \left(\frac{z^2}{(z+i)^2} \right) = \lim_{z \to i} \left(\frac{2iz}{(z+i)^3} \right) = \frac{-i}{4},$$

hence, for $\rho > 1$, we have by the Cauchy Residue Theorem that

$$\int_{\Gamma_\rho} f(z)\, dz = 2\pi i \frac{-i}{4} = \frac{\pi}{2}.$$

For $z \in C_\rho^+$, we have that, by the reverse triangle inequality,

$$|f(z)| = \left| \frac{z^2}{(z^2+1)} \right| = \frac{|z|^2}{|z^2+1|^2} \leq \frac{\rho^2}{(\rho^2-1)^2},$$

thus, by lemma 3.2.9,

$$\left| \int_{C_\rho^+} f(z)\, dz \right| \leq \ell(C_\rho^+) \frac{\rho^2}{(\rho^2-1)^2} = \frac{\pi\rho^3}{(\rho^2-1)^2} \to 0 \quad \text{as } \rho \to \infty.$$

Therefore, since

$$\int_{\Gamma_\rho} f(z)\,dz = \int_{\gamma_\rho} f(z)\,dz + \int_{C_\rho^+} f(z)\,dz,$$

for all ρ, we have that

$$\text{p. v.}\int_{\infty}^{\infty} f(x)\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx = \lim_{\rho\to\infty}\int_{\Gamma_\rho} f(z)\,dz - \lim_{\rho\to\infty}\int_{C_\rho^+} f(z)\,dz = \frac{\pi}{2}.$$

Exercise 5.4.4. Evaluate the following integrals:

(i) $\int_0^\infty \frac{x^2+1}{x^4+1}\,dx$;
(ii) $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$;
(iii) $\int_0^\infty \frac{1}{x^3+1}\,dx$ (consider the contour which is the boundary of the sector $\{\, z = re^{i\theta} : 0 \le \theta \le 2\pi/3, \text{ and } 0 \le r \le \rho \,\}$);
(iv) $\text{p. v.}\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx$;
(v) $\text{p. v.}\int_{-\infty}^\infty \frac{x^2}{(x^2+9)^2}\,dx$;
(vi) $\text{p. v.}\int_{-\infty}^\infty \frac{1}{(x^2+1)(x^2+4)}\,dx$; and
(vii) $\text{p. v.}\int_{-\infty}^\infty \frac{x}{(x^2+4x+13)^2}\,dx$.

We can evaluate improper integrals of trigonometric functions by regarding them as the improper integrals of real or imaginary parts of functions involving the complex exponential.

Example 5.4.5. Consider the integral

$$I = \text{p. v.}\int_{-\infty}^{\infty} \frac{\cos(3x)}{x^2+4}\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} \frac{\cos(3x)}{x^2+4}\,dx.$$

The integral is the real part of the complex integral

$$I_0 = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} \frac{\exp(i3x)}{x^2+4}\,dx.$$

As previously, for $\rho > 2$, let γ_ρ be the line segment on the real axis from $-\rho$ to ρ, and close this contour using the upper semicircle C_ρ^+. For $z \in C_\rho^+$ we have, by the reverse triangle inequality, that

$$\left|\frac{\exp(i3z)}{z^2+4}\right| = \frac{e^{-3\mathrm{Im}(z)}}{|z^2+4|} \le \frac{e^{-3\mathrm{Im}(z)}}{\rho^2-4} \le \frac{1}{\rho^2-4},$$

since $e^{-3\mathrm{Im}(z)}$ is bounded above by 1 in the upper half-plane. Therefore by lemma 3.2.9

$$\left|\int_{C_\rho^+} \frac{\exp(i3z)}{z^2+4}\,dz\right| \le \ell(C_\rho^+)\frac{1}{\rho^2-4} = \frac{\pi\rho}{\rho^2-4} \to 0 \quad \text{as } \rho\to\infty.$$

The function $f(z) = \exp(i3z)/(z^2+4)$ is holomorphic except for simple poles at $z = \pm 2i$, of which only $z = 2i$ lies in the interior of Γ_ρ, for $\rho > 2$. The residue is given by

$$\operatorname{Res}(f, 2i) = \lim_{z\to 2i}\left(\frac{\exp(i3z)}{z+2i}\right) = \frac{e^{-6}}{4i},$$

hence, for $\rho > 2$,

$$\int_{\Gamma_\rho} f(z)\,dz = 2\pi i\frac{e^{-6}}{4i} = \frac{\pi}{2e^6}.$$

So

$$I_0 = \lim_{\rho\to\infty}\int_{\rho}^{\rho} f(x)\,dx = \lim_{\rho\to\infty}\int_{\Gamma_\rho} f(z)\,dz - \lim_{\rho\to\infty}\int_{C_\rho^+} f(z)\,dz = \frac{\pi}{2e^6},$$

whence $I = \mathrm{Re}(I_0) = \frac{\pi}{2e^6}$.

Generalizing this approach allows us to compute integrals of the form

$$\text{p. v.}\int_{-\infty}^{\infty} R(x)\exp(iax)\,dx \tag{5.4}$$

where a is real and non-zero, given the following lemma, which tells us when the semi-circular contour—chosen carefully in the upper or lower half of the plane—tends to 0.

Lemma 5.4.6 (Jordan Lemma)**.** Let $R = P/Q$ be a rational function, where $\deg(Q) \geq \deg(P) + 1$, and $a \in \mathbb{R}$ be non-zero. Then

$$\lim_{\rho\to\infty}\int_{C_\rho^+} \exp(iaz)\frac{P(z)}{Q(z)}\,dz = 0 \quad \text{if } a > 0\text{, and}$$
$$\lim_{\rho\to\infty}\int_{C_\rho^-} \exp(iaz)\frac{P(z)}{Q(z)}\,dz = 0 \quad \text{if } a < 0,$$

where C_ρ^+ and C_ρ^- are the semicircular contours from ρ to $-\rho$ in the upper and lower half-plane respectively.

Proof. Omitted. □

Example 5.4.7. Consider the integral

$$I = \text{p. v.}\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\,dx.$$

This is the imaginary part of the integral

$$I_0 = \text{p. v.}\int_{-\infty}^{\infty} \frac{x\exp(ix)}{1+x^2}\,dx.$$

Conside the usual line segment γ_ρ on the real axis from $-\rho$ to ρ, and close it with the upper semicircle C_ρ^+ to create a closed contour Γ_ρ. Letting $f(z) = z\exp(iz)/(1+z^2)$, we see that f is holomorphic except for simple poles at $z = \pm i$, of which only $z = i$ lies in the interior of Γ_ρ, for $\rho > 1$. The residue is given by

$$\operatorname{Res}(f, i) = \lim_{z\to i}\left(\frac{z\exp(iz)}{z+i}\right) = \frac{ie^{-1}}{2i} = \frac{1}{2e}.$$

By the Jordan Lemma, $\int_{C_\rho^+} f(z)\,dz \to 0$ as $\rho \to \infty$, so

$$I_0 = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx = \lim_{\rho\to\infty}\int_{\Gamma_\rho} f(z)\,dz - \lim_{\rho\to\infty}\int_{C_\rho^+} f(z)\,dz = 2\pi i\frac{1}{2e} = \frac{i\pi}{e},$$

so $I = \operatorname{Im}(I_0) = \frac{\pi}{e}$.

Exercise 5.4.8. Evaluate the following integrals:

(i) $\text{p. v.}\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1}\,dx$;
(ii) $\int_0^{\infty} \frac{\cos x}{(x^2+1)^2}\,dx$;
(iii) $\text{p. v.}\int_{-\infty}^{\infty} \frac{x\sin 3x}{x^2+4}\,dx$;
(iv) $\text{p. v.}\int_{-\infty}^{\infty} \frac{\exp(3ix)}{x-2i}\,dx$;
(v) $\text{p. v.}\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\,dx$;
(vi) $\text{p. v.}\int_{-\infty}^{\infty} \frac{\exp(-2ix)}{x^2+4}\,dx$;
(vii) $\text{p. v.}\int_{-\infty}^{\infty} \frac{\cos(2x)}{x-3i}\,dx$; and
(viii) $\int_0^{\infty} \frac{x^3\sin(2x)}{(x^2+1)^2}\,dx$.

Example 5.4.9. Let $0 < a < 1$ and consider the improper integral

$$I = \text{p. v.}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} \frac{e^{ax}}{e^x+1}\,dx.$$

As usual consider the straight line segment γ_ρ on the real axis from $-\rho$ to ρ. However, if we try to close the contour using the upper or lower semicircle from ρ to $-\rho$, it is hard to estimate the contribution from the semicircle, even in the limit as $\rho \to \infty$, not to mention the fact that the integrand has an infinite number of poles in either the upper or lower half-planes due to the periodicity of the exponential function.

It is precisely the periodicity of the exponential function which suggests a way around this difficulty. Consider closing the contour into the rectangular loop Γ_ρ depicted in figure 7.

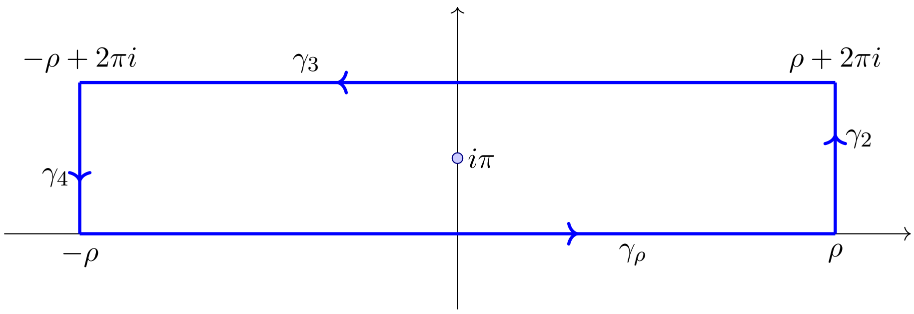


FIGURE 7. The rectangular loop $\Gamma = \gamma_\rho + \gamma_2 + \gamma_3 + \gamma_4$

There are two interesting properties of this rectangular loop: the first is that we will be able to show that the contributions from the vertical sides of the rectangle tend to 0 as $\rho \to \infty$, and the second is that the function $f(z) = \frac{\exp(az)}{\exp(z)+1}$ satisfies, by properties of the exponential,

$$f(z+2\pi i) = \frac{\exp(az + i2\pi a)}{\exp(z+2\pi i)+1} = \frac{\exp(i2\pi a)\exp(az)}{\exp(z)+1} = \exp(i2\pi a)f(z),$$

so that

$$\int_{\gamma_3} f(z)dz = \int_{-\gamma_\rho} f(z+2\pi i)\,dz = -\exp(i2\pi a)\int_{\gamma_\rho} f(z)dz,$$

where the sign change is due to the fact that γ_3 goes in the opposite direction to γ_ρ.

Consider γ_2, which we parametrize as $\gamma(t) = \rho + it$, for $t \in [0, 2\pi]$. Then, using the reverse triangle inequality, we have that

$$\begin{aligned}\left|\int_{\gamma_2} f(z)dz\right| = \left|\int_0^{2\pi} \frac{\exp(a\rho + ait)}{\exp(\rho+it)+1} i\,dt\right| &\le \int_0^{2\pi} \frac{e^{a\rho}}{|e^\rho e^{it}+1|}\,dt \\ &= \int_0^{2\pi} \frac{e^{-(1-a)\rho}}{|e^{it}+e^{-\rho}|}\,dt \\ &\le \int_0^{2\pi} \frac{e^{-(1-a)\rho}}{1-e^{-\rho}}\,dt \\ &= \frac{2\pi e^{-(1-a)\rho}}{1-e^{-\rho}} \to 0 \qquad \text{as } \rho\to\infty.\end{aligned}$$

A very similar argument shows that the integral along γ_4 also converges to 0 as $\rho \to \infty$.

Now, the function f is holomorphic on $\mathbb{C}$ except for simple poles at $z = i\pi + i2k\pi$ for integers k, of which only $z = i\pi$ lies insider Γ_ρ. We use (5.2) to see that

$$\operatorname{Res}(f, i\pi) = \frac{\exp(ai\pi)}{\exp(i\pi)} = -\exp(ia\pi).$$

So by the Cauchy Residue Theorem, the integral of $f(z)$ around the rectangular loop Γ_ρ, which is the composition of γ_ρ, γ_2, γ_3, γ_4, for $\rho > 4$ is given by

$$\int_{\Gamma_\rho} f(z)\,dz = 2\pi i \operatorname{Res}(f, i\pi) = -2\pi i \exp(i\pi a).$$

Putting this all together, we see that

$$-2\pi i \exp(i\pi a) = \lim_{\rho\to\infty} \int_{\Gamma_\rho} f(z)\,dz = \lim_{\rho\to\infty} (1-\exp(i2\pi a)) \int_{-\rho}^{\rho} f(x)\,dx = (1-\exp(i2\pi a))I,$$

hence, simplifying also with some algebra and trigonometric identities,

$$I = \frac{-2\pi i \exp(i\pi a)}{1-\exp(i2\pi a)} = \pi \csc(\pi a).$$

5.5. Improper integrals with poles. Consider the integral

$$\text{p. v.}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx.$$

Our approach to this would be to consider the function $f(z) = \frac{\exp(iz)}{z}$, and consider a suitable closed contour. However, notice that this function has a pole at $z = 0$, which is on the contour we would consider. We need to find a way of avoiding this.

Definition 5.5.1. Let $c \in (a, b) \subseteq \mathbb{R}$, and $f : [a, b] \setminus \{c\} \to \mathbb{R}$ be continuous. We define the *improper integrals* by

$$\int_a^c f(x)\,dx = \lim_{r\downarrow 0} \int_a^{c-r} f(x)\,dx;$$

$$\int_c^b f(x)\,dx = \lim_{s\downarrow 0} \int_{c+s}^{b} f(x)\,dx; \text{ and}$$

$$\int_a^b f(x)\,dx = \lim_{r\downarrow 0} \int_a^{c-r} f(x)\,dx + \lim_{s\downarrow 0} \int_{c+s}^{b} f(x)\,dx;$$

whenever the relevant limits exist. (The notation $r \downarrow 0$ indicates that $r \to 0$ through positive values only.) If the limits in the third case exist, then the result is equal to the *principal value* of the integral,

$$\text{p. v.}\int_a^b f(x)\,dx = \lim_{r\downarrow 0} \left(\int_a^{c-r} f(x)\,dx + \int_{c+r}^{b} f(x)\,dx \right).$$

Note that this value might exist even if the improper integral does not.

We can combine our two definitions of principal values in the following way.

Definition 5.5.2. Let $c \in \mathbb{R}$ and $f : \mathbb{R} \setminus \{c\} \to \mathbb{R}$ be continuous. Then we define the *principal value* of the integral by

$$\text{p. v.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{\rho\to\infty \\ r\downarrow 0}} \left(\int_{-\rho}^{c-r} f(x)\,dx + \int_{c+r}^{\rho} f(x)\,dx \right);$$

provided the two limits exist independently.

It turns out that principal value integrals of this type can often be evaluated using the residue theorem. The residue theorem applies to closed contours, so in computing a principal value integral we need to close the contour, not just ρ to $-\rho$ as in the previous section, but also $c - r$ to $c + r$. One way to do this is to consider the small semicircle $-C_r^+(c)$ of radius r around the singular point c, as in Figure 8 (we use the minus sign since this semicircle is traversed in the opposite direction to usual).

During our calculations, we shall be interested in the limit $\lim_{r\downarrow 0} \int_{-C_r^+(c)} f(z)\,dz$, so we record the following lemma.

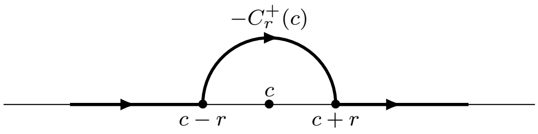


FIGURE 8. Closing the contour around a singularity.

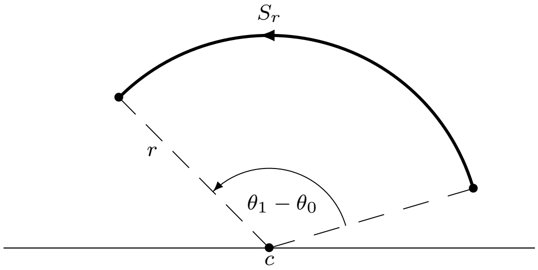


FIGURE 9. A small circular arc.

Lemma 5.5.3. Let $D \subseteq \mathbb{C}$ be a domain, $c \in D$, f be meromorphic on D, with a simple pole at $z = c$, and let S_r be the circular arc parametrized by $\gamma(\theta) = c + r\exp(i\theta)$ for $\theta \in [\theta_0, \theta_1]$ for some $0 \leq \theta_0 < \theta_1 \leq 2\pi$. Then

$$\lim_{r\downarrow 0} \int_{S_r} f(z)\,dz = i(\theta_1 - \theta_0)\mathrm{Res}\,(f, c)\,. \tag{5.5}$$

Proof. Since f has a simple pole at c, its Laurent expansion centred at c is valid on some punctured disc $D'_R(c)$, for $R > 0$, and has the form

$$f(z) = \frac{a_{-1}}{z-c} + \sum_{j=0}^{\infty} a_j(z-c)^j,$$

where $g(z) := \sum_{j=0}^{\infty} a_j(z-c)^j$ defines a holomorphic function on the disc $D_R(c)$. Now let $0 < r < R$ and consider the integral

$$\int_{S_r} f(z)\,dz = a_{-1}\int_{S_r} \frac{dz}{z-c} + \int_{S_r} g(z)\,dz.$$

Because g is holomorphic it is in particular bounded on $\overline{D}_{R/2}(c)$, i.e. there exists $M > 0$ such that $|g(z)| \leq M$ on $\overline{D}_{R/2}(c)$. Then lemma 3.2.8 implies that, for $r \leq R/2$

$$\left|\int_{S_r} g(z)\,dz\right| \leq M\ell(S_r) = M\pi r(\theta_1 - \theta_0) \to 0 \quad \text{as } r \downarrow 0.$$

On the other hand,

$$\int_{S_r} \frac{1}{z-c}\,dz = \int_{\theta_0}^{\theta_1} \frac{ri\exp(i\theta)}{r\exp(i\theta)}\,d\theta = i\int_{\theta_0}^{\theta_1} d\theta = i\,(\theta_1 - \theta_0)\,.$$

Therefore

$$\lim_{r\downarrow 0}\int_{S_r} f(z)\,dz = \lim_{r\downarrow 0}\int_{S_r} \frac{a_{-1}}{z-c} + g(z)\,dz = i\,(\theta_1-\theta_0)a_{-1} + 0 = i(\theta_1-\theta_0)\,\mathrm{Res}\,(f,c)\,. \quad \square$$

Example 5.5.4. We now consider the example with which we began this section,

$$I = \text{p.v.}\int_{\infty}^{\infty} \frac{\sin x}{x}\,dx,$$

regarding it, as suggested, as the imaginary part of the principal value integral

$$I_0 = \lim_{\substack{\rho\to\infty \\ r\downarrow 0}} \left(\int_{-\rho}^{-r} \frac{e^{ix}}{x}\,dx + \int_{r}^{\rho} \frac{e^{ix}}{x}\,dx \right).$$

This is a contour integral in the complex plane along the subset of the real axis consisting of the intervals $[-\rho,-r]$ and $[r,\rho]$. In order to use the residue theorem we must close this contour. We join ρ and $-\rho$ via the large semicircle C_ρ^+ of radius ρ in the upper half-plane. In order to join $-r$ and r we choose a small semicircle $-C_r^+$ also in the upper half-plane, travelled in the negative direction.

Because the function $f(z) = \exp(iz)/z$ is holomorphic on and inside the contour, the Cauchy Integral Theorem says that the contour integral vanishes. Splitting this contour integral into its different pieces, we have that

$$\left(\int_{-\rho}^{-r} + \int_{-C_r^+} + \int_{r}^{\rho} + \int_{C_\rho^+} \right) \frac{\exp(iz)}{z}\,dz = 0,$$

thus the limit as $\rho\to\infty$ and $r\downarrow 0$ is 0 too. By the Jordan Lemma, the integral along C_ρ^+ converges to 0 as $\rho\to\infty$. The function has a simple pole at $z=0$, so by (5.2), $\operatorname{Res}(f,0)=1$. So, using (5.5), and recalling that $-C_r^+$ is negatively oriented,

$$I_0 = -\lim_{r\downarrow 0} \int_{-C_r^+} \frac{\exp(iz)}{z}\,dz = \lim_{r\downarrow 0} \int_{C_r^+} \frac{\exp(iz)}{z}\,dz = i\pi \operatorname{Res}(f,0) = i\pi.$$

Therefore, we have that $I = \operatorname{Im}(I_0) = \pi$.

Exercise 5.5.5. Evaluate the following integrals:

(i) p. v.$\int_{-\infty}^{\infty} \frac{1}{x^3-1}\,dx$;
(ii) $\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,dx$ where $0<a<b$; and
(iii) $\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx$.

5.6. Infinite series. We can use contour integration to evaluate sums like $\sum_{n=1}^\infty \frac{1}{n^2}$.

Example 5.6.1. Consider the function $f(z) = \cot(\pi z)/z^2$. This is holomorphic on $\mathbb{C}$ except for poles at $z=n$ for all integers n. Consider a non-zero integer n. Then the pole at n is a simple pole, and by (5.2) the residue is given by

$$\operatorname{Res}(f, n\pi) = \frac{\cos(n\pi)/n^2}{\pi \sin'(n\pi)} = \frac{\cos(n\pi)/n^2}{\pi\cos(n\pi)} = \frac{1}{\pi n^2}.$$

The residue at $z=0$ can be calculated by computing the Laurent series centred at 0, which we do by using known expansions of the trigonometric functions, remembering that we only care about the coefficient of z^{-1}, so we can ignore higher order terms in z:

$$\begin{aligned} f(z) = \frac{\cos \pi z}{z^2 \sin \pi z} &= \frac{1}{z^2}\cdot \frac{1-(\pi z)^2/2!+\cdots}{\pi z - (\pi z)^3/3! + \cdots} \\ &= \frac{1}{\pi z^3}\cdot\frac{1-(\pi z)^2/2+\cdots}{1-((\pi z)^2/6-\cdots)} \\ &= \frac{1}{\pi z^3}\left(1-(\pi z)^2/2+\cdots\right)\left(1+\left((\pi z)^2/6-\cdots\right)+\left((\pi z)^2/6-\cdots\right)^2+\cdots\right) \\ &= \frac{1}{\pi z^3}\left(1+\left(\frac{1}{6}-\frac{1}{2}\right)\pi^2 z^2+\cdots\right), \end{aligned}$$

where we used the geometric expansion of the value $\frac{1}{1-((\pi z)^2/6-\cdots)}$ in terms of powers of $\left((\pi z)^2/6-\cdots\right)$, which is valid for small $|z|$. From this Laurent expansion we see that the coefficient of z^{-1}, i.e. $\operatorname{Res}(f,0)$, is $\frac{-\pi}{3}$.

Now consider the contour Γ_N, for N a positive integer, defined as the positively oriented square with vertices at $(N+\frac{1}{2})(1+i)$, $(N+\frac{1}{2})(-1+i)$, $(N+\frac{1}{2})(-1-i)$ and $(N+\frac{1}{2})(1-i)$, as shown in figure 10. Notice that the contour misses all the poles of f.

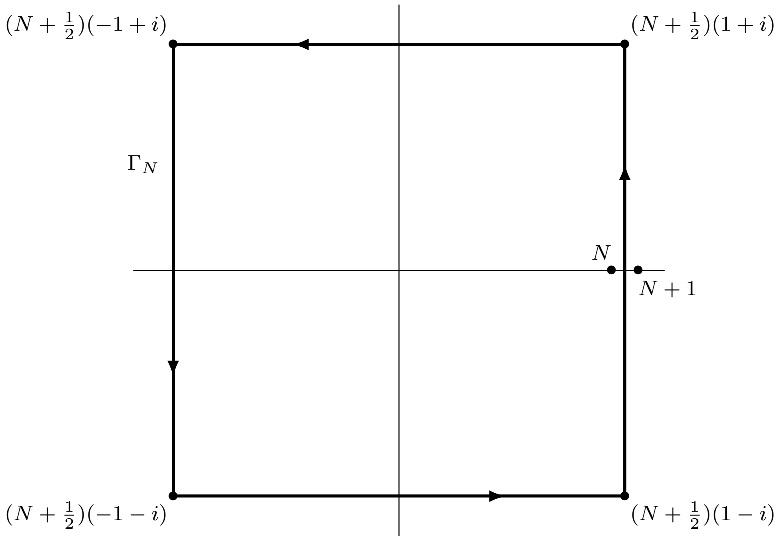


FIGURE 10. The contour Γ_N.

For fixed N, the poles at the integers $-N, -N+1, \dots, -1, 0, 1, \dots, N-1, N$ lie in the interior of Γ_N. Thus by the Cauchy Residue Theorem,

$$\begin{aligned}
\int_{\Gamma_N} f(z)\,dz = 2\pi i \sum_{n=-N}^{N} \operatorname{Res}(f,n) &= 2\pi i \left(\sum_{n=-N}^{-1} \operatorname{Res}(f,n) + \operatorname{Res}(f,0) + \sum_{n=1}^{N} \operatorname{Res}(f,n) \right) \\
&= 2\pi i \left(\sum_{n=-N}^{1} \frac{1}{\pi n^2} - \frac{\pi}{3} + \sum_{n=1}^{N} \frac{1}{\pi n^2} \right) \\
&= 2\pi i \left(2 \sum_{n=1}^{N} \frac{1}{n^2 \pi} - \frac{\pi}{3} \right) \\
&= 4i \left(\sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{6} \right).
\end{aligned}$$

Now, it can be shown that there exists $K > 0$ such that $|\cot(\pi z)| \le K$ for all $z \in \Gamma_N$, where K is independent of N. Thus, for $z \in \Gamma_N$, since $|z| \ge N + \frac{1}{2}$, we have that

$$|f(z)| \le \left| \frac{\cot n\pi}{z^2} \right| \le \frac{K}{(N+\frac{1}{2})^2} \le \frac{K}{N^2},$$

and hence, by lemma 3.2.9,

$$\left| \int_{\Gamma_N} f(z)\,dz \right| \le \ell(\Gamma_N) \frac{K}{N^2} = \frac{4(2N+1)K}{N^2} \to 0 \quad \text{as } N \to \infty.$$

Therefore

$$0 = \lim_{N\to\infty} \int_{\Gamma_N} f(z)\,dz = \lim_{N\to\infty} 4i \left(\sum_{n=1}^{N} \frac{1}{n^2} - \frac{\pi^2}{6} \right) = 4i \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{\pi^2}{6} \right),$$

which implies that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 5.6.2. (i) By considering the function $f(z) = \pi \cot(\pi z)/(z^2+1)$, evaluate
$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}.$$

(ii) By considering the function $f(z) = \pi \cot(\pi z)/(z-(1/2))^2$, evaluate
$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-(1/2))^2}.$$

Example 5.6.3. Consider the alternating sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. Let $f(z) = \csc(\pi z)/z^2$, which is holomorphic on $\mathbb{C}$ except for poles at $z = n$ for each integer n. For non-zero integers n, the pole is a simple pole, and (5.2) implies that
$$\operatorname{Res}(f, n) = \frac{1}{\pi \sin'(\pi n)} \frac{1}{n^2} = \frac{1}{\pi \cos(\pi n)} \frac{1}{n^2} = (-1)^n \frac{1}{\pi n^2}.$$
To find the residue at $z = 0$, we compute the Laurent expansion centred at 0:
$$\begin{aligned} f(z) = \frac{1}{z^2}\frac{1}{\sin(\pi z)} &= \frac{1}{\pi z^3} \cdot \frac{1}{1 - (\pi z)^2/3! + \cdots} \\ &= \frac{1}{\pi z^3} \cdot \frac{1}{(1 - ((\pi z)^2/6 - \cdots))} \\ &= \frac{1}{\pi z^3} \left(1 + ((\pi z)^2/3! + \cdots) + ((\pi z)^2/6 + \cdots)^2 + \cdots\right) \\ &= \frac{1}{\pi z^3} + \frac{\pi}{6z} + \cdots, \end{aligned}$$
whence $\operatorname{Res}(f, 0) = \pi/6$.

Now, consider the square contour Γ_N from the previous example. Since we know there exists $K > 0$ such that $|\cot(\pi z)| \le K$ for all $z \in \Gamma_N$, where K is independent of N, the trigonometric identity $\csc^2(\pi z) = 1 + \cot^2(\pi z)$, which follows by dividing the usual $\sin^2(\pi z) + \cos^2(\pi z) = 1$ by $\sin^2(\pi z)$, implies that $|\csc^2(\pi z)| \le 1 + K^2$ for all $z \in \Gamma_N$. Hence by exactly the same argument as in the preceding example, we have that $\int_{\Gamma_N} f(z)\, dz \to 0$ as $N \to \infty$.

Applying the Cauchy Residue Theorem, however, tells us that
$$\int_{\Gamma_N} f(z)\, dz = 2\pi i \sum_{n=-N}^{N} \operatorname{Res}(f, n) = 2\pi i \left(2 \sum_{n=1}^{N} (-1)^n \frac{1}{\pi n^2} + \frac{\pi}{6}\right),$$
so, taking the limit as $N \to \infty$, we have that
$$0 = 2\pi i \left(2 \sum_{n=1}^{\infty} (-1)^n \frac{1}{\pi n^2} + \frac{\pi}{6}\right) = 4i \left(\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} + \frac{\pi^2}{12}\right),$$
hence
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{12}.$$

Lemma 5.6.4. Let $0 \le k \le n$ be non-negative integers, and let $\binom{n}{k}$ be the usual binomial coefficient, and let Γ be a loop with 0 in its interior. Then
$$\binom{n}{k} = \frac{1}{2\pi i} \int_{\Gamma} \frac{(1+z)^n}{z^{k+1}}\, dz.$$

Proof. Exercise. □

Example 5.6.5. Consider the sum $S = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}$. We can substitute the integral representation for the binomial coefficient from lemma 5.6.4 to get the expression

$$S = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{(1+z)^{2n}}{z^{n+1}}\, dz \right) \frac{1}{5^n} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\Gamma} \frac{(1+z)^{2n}}{(5z)^n} \frac{1}{z}\, dz,$$

for any loop Γ around the origin, in particular for $C_1(0)$. Since for $z \in C_1(0)$ we have

$$\left| \frac{(1+z)^2}{5z} \right| \leq \frac{4}{5},$$

the Weierstrass M-test implies that the integrand converges uniformly on $C_1(0)$, so we can interchange the order of the summation and the integration to see that

$$S = \frac{1}{2\pi i} \int_{C_1(0)} \sum_{n=0}^{\infty} \frac{(1+z)^{2n}}{(5z)^n} \frac{1}{z}\, dz = \frac{5}{2\pi i} \int_{C_1(0)} \frac{1}{3z - 1 - z^2}\, dz,$$

which can now be calculated using the Cauchy Residue Theorem. The integrand $f(z) = \frac{1}{3z-1-z^2}$ has simple poles at $(3 \pm \sqrt{5})/2$, of which only $(3-\sqrt{5})/2$ lies inside the contour. Therefore,

$$S = 5\operatorname{Res}\left(f, (3-\sqrt{5})/2\right) = 5\frac{1}{\sqrt{5}} = \sqrt{5}.$$

Email address: `R.Gratwick@ed.ac.uk`