Mathematics 3 **Honours Analysis** **2018/19**

Workshop 6 – Examples of Metric Spaces (full version)

In this workshop we shall meet a variety of examples of metric spaces. In most – but not all – cases the subtlety involved in checking that a given function is a metric lies in verification of the triangle inequality. You'll notice that there are several different metrics called d_1, several called d_2 and several called d_∞, depending on what the underlying set X is. Why do we use these notations?

1. Show that both

$$d_1(x,y) := \sum_{i=1}^{n} |x_i - y_i| \quad \text{and} \quad d_\infty(x,y) := \max_{1\le i\le n} |x_i - y_i|$$

define metrics on $\mathbb{R}^n$. **Remark.** So it is quite possible for a given set to have many distinct metrics defined on it.

2. Show that

$$d_1(f,g) := \int_0^1 |f-g|$$

defines a metric on the class $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$.

3. Show that

$$d_2(f,g) := \left(\int_0^1 |f-g|^2\right)^{1/2}$$

defines a metric on the class $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$. (**Hint:** Recall that in the last workshop we proved that if $F, G : [0,1] \to \mathbb{R}$ are continuous functions, then we have the Cauchy–Schwarz inequality $|\int_0^1 FG| \le \left(\int_0^1 F^2\right)^{1/2} \left(\int_0^1 G^2\right)^{1/2}$ which is analogous to the Cauchy–Schwarz inequality $|\mathbf{x}\cdot\mathbf{y}| \le |\mathbf{x}||\mathbf{y}|$ for euclidean space $\mathbb{R}^n$ – which we used in lectures to establish the triangle inequality for $\mathbb{R}^n$ with the usual metric. So use the Cauchy–Schwarz inequality for integrals to deduce $\left(\int |F+G|^2\right)^{1/2} \le \left(\int |F|^2\right)^{1/2} + \left(\int |G|^2\right)^{1/2}$.)

4. Let $\mathcal{R}$ denote the vector space of Riemann-integrable functions $f : \mathbb{R} \to \mathbb{R}$. For $f, g \in \mathcal{R}$ let

$$d_1(f,g) := \int |f-g|.$$

Does d_1 define a metric on $\mathcal{R}$? (If you don't know what a vector space is, don't worry.)

5. Which of the follwing are metrics on $\mathbb{R}$?

(i) $d(x,y) = \sin|x-y|$

(ii) $d(x,y) = |\sin(x-y)|$

(iii) $d(x,y) = \log(1+|x-y|)$

(iv) $d(x,y) = |x-y|^2$

(v) $d(x,y) = |x-y|^{1/2}$.

6. a) On the same picture, sketch the unit balls $B(0,1)$ in $\mathbb{R}^2$ with respect to each of the metrics d_1, d_2 (i.e. the usual metric) and d_∞. Also sketch $B(0,2)$ for d_1. [$B(0,2)$ is the ball centred at 0 with radius2.]
b) Show that for $\mathbb{R}^n$ with the metrics d_1, d_2 and d_∞ we have

$$d_\infty(x,y) \le d_2(x,y) \le d_1(x,y) \le n d_\infty(x,y).$$

What does this have to do with part a)?
c) Show that $d_1(x,y) \le \sqrt{n} d_2(x,y)$ and that $d_2(x,y) \le \sqrt{n} d_\infty(x,y)$.

7. a) Let $f : \mathbb{R} \to \mathbb{R}$. What conditions on f ensure that $d(x,y) = |f(x)-f(y)|$ defines a metric on $\mathbb{R}$?
b) (Harder.) Let $g : [0,\infty) \to \mathbb{R}$. What conditions on g ensure that $\rho(x,y) = g(|x-y|)$ defines a metric on $\mathbb{R}$?

8. Let X be the set of strings of 0's and 1's of length 2^{1000}. For a pair of strings consider the two quantities

(i) the number of entries in which the two strings differ;

(ii) 2^{-j} where j is the first entry in which two strings differ (taken to be 0 if the two strings are identical);

does either define a metric on X?

9. Consider a graph whose vertices are the students of the University of Edinburgh and whose edges (of length 1) link each pair of students who have

shaken hands. Define $d(x, y)$ to be the length of the shortest path in the graph which joins student x to student y. (Of course we set $d(x, x) = 0$.) Show that d defines a metric on the set of students of the University of Edinburgh. (We assume that the University of Edinburgh is sufficiently sociable so that each pair of students is joined by a path of *some* finite length.)

Assessment task to be handed in on Wednesday 31 October of Week 7: Questions 6 and 7(a).

Some supplementary questions:

A. Let ℓ^1 be the set of all absolutely convergent series of real numbers, that is,

$$\ell^1 = \{(x_n)_{n=1}^{\infty} \ : \ x_n \in \mathbb{R}, \ \sum_{n=1}^{\infty} |x_n| < \infty\}.$$

Show that for (x_n) and $(y_n) \in \ell^1$, $\sum_{n=1}^{\infty} |x_n - y_n|$ converges and that

$$d((x_n), (y_n)) = \sum_{n=1}^{\infty} |x_n - y_n|$$

defines a metric on ℓ^1.

B. Let (X, d) be a metric space. Show that $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ defines a metric on X.

C. Does $\sigma(x, y) = \left(|x_1 - y_1|^{1/2} + |x_2 - y_2|^{1/2}\right)^2$ define a metric on $\mathbb{R}^2$?

D. Let X be a vector space with an inner product $\langle x, y\rangle$. Let

$$d(x, y) = \langle x - y, x - y\rangle^{1/2}.$$

Show that d defines a metric on X.

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1. Show that both

$$d_1(x,y) := \sum_{i=1}^{n} |x_i - y_i| \text{ and } d_\infty(x,y) := \max_{1\le i\le n} |x_i - y_i|$$

define metrics on $\mathbb{R}^n$. **Remark.** So it is quite possible for a given set to have many distinct metrics defined on it.

Solution: In both cases, only the triangle inequality is maybe not obvious. We have $|x_i - y_i| \le |x_i - z_i| + |z_i - y_i|$ for each i. We can sum this to obtain the triangle inequality for d_1; and for d_∞ we have $|x_i - y_i| \le |x_i - z_i| + |z_i - y_i| \le \max_i |x_i - z_i| + \max_i |z_i - y_i| = d_\infty(x,z) + d_\infty(z,y)$ so that $d_\infty(x,y) = \max_i |x_i - y_i| \le d_\infty(x,z) + d_\infty(z,y)$.

2. Show that

$$d_1(f,g) := \int_0^1 |f-g|$$

defines a metric on the class $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$.

3. Show that

$$d_2(f,g) := \left(\int_0^1 |f-g|^2\right)^{1/2}$$

defines a metric on the class $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$. (**Hint:** Recall that in the last workshop we proved that if $F, G : [0,1] \to \mathbb{R}$ are continuous functions, then we have the Cauchy–Schwarz inequality $|\int_0^1 FG| \le \left(\int_0^1 F^2\right)^{1/2} \left(\int_0^1 G^2\right)^{1/2}$ which is analogous to the Cauchy–Schwarz inequality $|\mathbf{x}\cdot\mathbf{y}| \le |\mathbf{x}||\mathbf{y}|$ for euclidean space $\mathbb{R}^n$ – which we used in lectures to establish the triangle inequality for $\mathbb{R}^n$ with the usual metric. So use the Cauchy–Schwarz inequality for integrals to deduce $\left(\int |F+G|^2\right)^{1/2} \leq \left(\int |F|^2\right)^{1/2} + \left(\int |G|^2\right)^{1/2}$.)

Solutions: In both cases we need to worry about whether $d(f,g) = 0$ implies $f = g$. If $d(f,g) = 0$ we have that $|f-g|$ (or $|f-g|^2$) has integral 0, so by last week's assignment, the nonnegative continuous function $|f-g|$ (or $|f-g|^2$) is identically 0; that is, $f = g$. The triangle inequality for d_1 follows by integrating the inequality $|f(s)-g(s)| \leq |f(s)-h(s)| + |h(s)-g(s)|$ on $[0,1]$. The triangle inequality for d_2 follows since we have

$$\int (F+G)^2 = \int F^2 + \int G^2 + 2\int FG \leq \left(\left(\int F^2\right)^{1/2} + \left(\int G^2\right)^{1/2}\right)^2$$

and from this we deduce that $d_2(f,g) \leq d_2(f,h) + d_2(h,g)$ upon setting $F = f - h$ and $G = h - g$.

4. Let $\mathcal{R}$ denote the vector space of Riemann-integrable functions $f : \mathbb{R} \to \mathbb{R}$. For $f, g \in \mathcal{R}$ let

$$d_1(f,g) := \int |f-g|.$$

Does d_1 define a metric on $\mathcal{R}$? (If you don't know what a vector space is, don't worry.)

Solution: No: the function $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$ is a step function, hence is in $\mathcal{R}$, is not the zero function, yet $d_1(f,0) = \int |f| = 0$.

5. Which of the follwing are metrics on $\mathbb{R}$?

(i) $d(x,y) = \sin|x-y|$

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(iii) $d(x,y) = \log(1+|x-y|)$

(iv) $d(x,y) = |x-y|^2$

(v) $d(x,y) = |x-y|^{1/2}$.

Solution:

(i) No, because $d(3\pi/2, 0) = -1 \not\geq 0$.

(ii) No, because $d(2\pi, 0) = 0$ but $2\pi \neq 0$.

(iii) This is positive, symmetric and $d(x, y) = 0$ implies $x = y$. So we only have to consider the triangle inequality. Do we have

$$\log(1 + |x - y|) \leq \log(1 + |x - z|) + \log(1 + |z - y|)?$$

Exponentiating both sides reduces this to

$$1 + |x - y| \leq (1 + |x - z|)(1 + |z - y|)?$$

and upon multiplying out the RHS and applying the triangle inequality for $\mathbb{R}$ we see that this is true, so d is a metric on $\mathbb{R}$.

(iv) No, the triangle inequality fails. If we take $x = 0$, $z = 1$ and $y = 2$ then we have $d(x, y) = 4$ while $d(x, z) + d(z, y) = 2$.

(v) This is positive, symmetric and $d(x, y) = 0$ implies $x = y$. So we only have to consider the triangle inequality. Do we have

$$|x - y|^{1/2} \leq |x - z|^{1/2} + |z - y|^{1/2}?$$

Yes, since

$$|x - y| \leq |x - z| + |z - y| \leq (|x - z|^{1/2} + |z - y|^{1/2})^2$$

as is seen by multiplying out the last term.

6. a) On the same picture, sketch the unit balls $B(0, 1)$ in $\mathbb{R}^2$ with respect to each of the metrics d_1, d_2 (i.e. the usual metric) and d_∞. Also sketch $B(0, 2)$ for d_1. [$B(0, 2)$ is the ball centred at 0 with radius2.]
b) Show that for $\mathbb{R}^n$ with the metrics d_1, d_2 and d_∞ we have

$$d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq n d_\infty(x, y).$$

What does this have to do with part a)?
c) Show that $d_1(x, y) \leq \sqrt{n} d_2(x, y)$ and that $d_2(x, y) \leq \sqrt{n} d_\infty(x, y)$.

7. a) Let $f : \mathbb{R} \to \mathbb{R}$. What conditions on f ensure that $d(x, y) = |f(x) - f(y)|$ defines a metric on $\mathbb{R}$?
b) (Harder.) Let $g : [0, \infty) \to \mathbb{R}$. What conditions on g ensure that $\rho(x, y) = g(|x - y|)$ defines a metric on $\mathbb{R}$?

Solution: For g we require that it map into $[0, \infty)$ in order that the metric be nonnegative, and we require that $g(0) = 0$ and that 0 is the only value of t such that $g(t) = 0$ in order that $\rho(x, x) = 0$ and that $\rho(x, y) = 0$ implies $x = y$. Symmetry is obvious. The triangle inequality is more delicate: if g is increasing we have $\rho(x, y) = g(|x - y|) \le g(|x - z| + |z - y|)$ by the ordinary triangle inequality for $\mathbb{R}$. So if we had

$$g(p + q) \le g(p) + g(q) \text{ for all } p, q \ge 0$$

we'd then have

$$\rho(x, y) = g(|x-y|) \le g(|x-z|+|z-y|) \le g(|x-z|)+g(|z-y|) = \rho(x, z)+\rho(z, y).$$

Now if g is differentiable with continuous derivative which is nonnegative and decreasing, then we have

$$g(p + q) - g(p) = \int_0^q g'(t + p)dt \le \int_0^q g'(t)dt = g(q) - g(0) = g(q).$$

So a sufficient condition is that g map into $[0, \infty)$, $g(0) = 0$, g be strictly increasing and g' being continuous and decreasing. **Any reasonable argument is acceptable. The idea is to get you to think a bit...** Note that $g(x) = \sqrt{x}$ satisfies these conditions.

8. Let X be the set of strings of 0's and 1's of length 2^{1000}. For a pair of strings consider the two quantities

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(ii) 2^{-j} where j is the first entry in which two strings differ (taken to be 0 if the two strings are identical);

does either define a metric on X?

Solution: Both are metrics. Positivity, symmetry and $d(x, y) = 0$ implies $x = y$ are clear. Suppose x, y and z are strings and that x and z differ in j entries and that z and y differ in k entries. Then x and y can differ in at most $j + k$ entries. So we have a metric for item (i). Suppose now that the first entry where x and z differ is j and that the first entry where z and y differ is k. Then x and y have the same entries at least until the $\min\{j, k\}$'th one. So $d(x, y) \le 2^{-\min\{j,k\}} \le 2^{-j} + 2^{-k}$.

9. Consider a graph whose vertices are the students of the University of Edinburgh and whose edges (of length 1) link each pair of students who have shaken hands. Define $d(x, y)$ to be the length of the shortest path in the graph which joins student x to student y. (Of course we set $d(x, x) = 0$.) Show that d defines a metric on the set of students of the University of Edinburgh. (We assume that the University of Edinburgh is sufficiently sociable so that each pair of students is joined by a path of *some* finite length.)

Solution: The only thing to check is the triangle inequality. If x and z can be joined by a chain of acquaintances of length j and if z and y can be joined by a chain of acquaintances of length k, then mainfestly x and y can be joined by a chain of acquaintances of length at most $j + k$.

Some supplementary questions:

A. Let ℓ^1 be the set of all absolutely convergent series of real numbers, that is,

$$\ell^1 = \{(x_n)_{n=1}^{\infty} \ : \ x_n \in \mathbb{R}, \ \sum_{n=1}^{\infty} |x_n| < \infty\}.$$

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defines a metric on ℓ^1.

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