Spherical symmetry groups

3.1 Identifying spherical symmetry groups

3.1.1 Scenario We are interested in patterns or objects with symmetries given by the group $O(3)$ of 3×3 orthogonal matrices, which is the group of symmetries generated by rotations about an axis through the origin and reflections in planes through the origin.

We will often think in terms of patterns drawn on the surface $x^2 + y^2 + z^2 = 1$ of the unit sphere. In that case, reflections reflect in *great circles* (circles whose centre is at the origin).

We will also consider solids such as the octahedron and the dodecahedron. If we think of an octahedron centred at the origin, we can project radially to obtain a pattern on the sphere: the vertices become points on the sphere (at the North pole, the South pole and four round the equator), the edges become arcs of great circles. The eight faces become three-sided, regular “spherical triangles”. On the sphere, four of these faces meet at a point, and so all the angles in our spherical triangle are all right angles!

3.1.2 Symmetries As we will see, all direct isometries of the sphere are rotations about some axis through the origin. But as for the plane, there are two sorts of indirect isometry:

- A *reflection* reflects the sphere in a great circle.
- A *(spherical) glide* is the result of composing a reflection with a rotation about the normal to the plane of reflection. For example, reflecting the earth in the plane through the equator and then rotating the earth by some amount around the North and South poles is a glide.

You should discover for a cube that there are three types of mirrors which reflect in the following planes planes through the origin.

- Normal to a line joining the midpoints of opposite faces.
- Normal to a line joining opposite vertices.
- Normal to a line joining the midpoints of opposite edges.

You should observe that four mirrors meet at the midpoints of faces, three at vertices and two at midpoints of edges. The cube has kaleidoscopic symmetry with signature $*432$.

A fundamental region is the radial projection onto the sphere of a triangle with vertices the centre of a face, the centre of an edge of that face, and a vertex belonging to that edge. (So it is defined by a flag.)

3.1.3 Orbifold Classification The basic reference here is the account in “Symmetry of Things” classifying the finite subgroups of $O(3)$ according to the “Magic Theorem” that states in this case that the sum of the costs is equal to

$$2 - \frac{2}{|G|}$$

(where $|G|$ is the number of elements in the subgroup G, which is denoted by g in the book).

Every possible signature with cost less than two gives rise to a symmetry group with just the exception that the cases MN and $*MN$ (where M, N are natural numbers) occur only when $M = N$.

I present here how I like to “organise” the resulting possibilities, which is a little different from Conway’s.

3.1.4 Seven groups for which there is no fixed axis

Full group	Rotation subgroup	Notes
$*332$ $(\#G = 24)$	332 $(\#G = 12)$	Symmetries of tetrahedron
$*432$ $(\#G = 48)$	432 $(\#G = 24)$	Symmetries of cube or octahedron
$*532$ $(\#G = 120)$	532 $(\#G = 60)$	Symmetries of icosahedron or dodecahedron
$3{*}2$ $(\#G = 24)$	This is the one group not fixing an axis that is not the rotational or full symmetry group of a platonic solid. It is a subgroup of the full symmetry group of a cube and its rotational subgroup is 332.	

3.1.5 Seven families that fix an axis Here, “fixing an axis” means that, for example, the line joining the North and South poles is sent to itself although it may be that the North and South poles are swapped over (e.g. by a reflection in the equator).

The trivial subgroup has no non-identity elements and hence the signature is the empty set. There is one family that fixes an axis and relies on a miracle:

$$N\times.$$

Here and following, N is an arbitrary natural number that may be omitted if $N = 1$.

Beyond that, there is a distinction as to whether the ends of the axis (the “North and South poles”) are kaleidoscope points or gyrations. And for each of those cases there is a “basic example” to which one can add either a reflection in the equator or gyrations of order two on the equator.

Nature of "N& S poles"	**Additions**	**Signature**
Gyration	None	NN
Gyration	Eq. Refl.	$N*$
Gyration	Eq. Rots.	$22N$
Kaleidoscope	None	$*NN$
Kaleidoscope	Eq. Refl.	$*22N$
Kaleidoscope	Eq. Rots.	$2*N$

3.1.6 Notes Note that the North and South poles are separate gyration/kaleidoscope points for the two "basic" examples NN and $*NN$ respectively. In fact, the corresponding group in these cases is exactly the rotational or full symmetries of a regular N-gon.

For the other cases where one has in the signature a reflection or glide in the equator or an order-2 gyration centred on the equator, those symmetries interchange the North pole and South poles and so there is only one 'N' in the signature.

3.2 $O(3)$ and $SO(3)$

3.2.1 Definition The *orthogonal group* $O(n)$ consists of all $n \times n$ matrices A satisfying $A^T A = \mathbb{I}$. This is equivalent to asking that A preserves dot products (and hence lengths and angles). Taking determinants, we deduce that $\det(A) = \pm 1$. In fact, $\det : O(n) \to \{\pm 1\}$ is a group homomorphism and its kernel is the *special orthogonal group* $SO(n)$.

3.2.2 Elements of $O(3)$ Let $A \in SO(3)$. Then the characteristic polynomial is equal to pone when $\lambda = 0$ and tends to minus infinity as $\lambda- > +\infty$. Thus A has a positive eigenvalue. Since A preserves length the eigenvalue is $\lambda = 1$.

Taking the eigenvector to be in the direction of the positive z-axis, A must act as a rotation in the orthogonal (x, y) plane. Thus every non-identity element is a rotation about a fixed axis and in a basis chosen so that the axis of rotation is the z-axis it has matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, an element $A \in O(3)$ with $\det A = -1$ hand must have -1 as an eigenvalue.

Taking the eigenvector in this case to be the z-axis, A must act as a rotation in the orthogonal (x, y) plane (because if it was a reflection we would have $\det A = 1$). If that rotation is zero, we have a reflection in a *great circle* (the intersection of the sphere with a plane through the origin), otherwise they are that composed with a rotation about the normal: what we have called a *(spherical) glide*. With the original eigenvector being the z-axis again, in matrix form we have

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

3.2.3 Summary We have understood the elements of $O(3)$. If they are "direct" then they are rotations about an axis through the origin or the identity (which we think of as a rotation by zero). If they are "indirect" they are reflections in great circles or "glides".

3.2.4 Warning A spherical glide is determined by a unit vector $\mathbf{n}$ which is normal to the plane of reflection and an angle $\theta \in (0, 2\pi)$ which is the angle rotated about $\mathbf{n}$, measured clockwise as one looks along $\mathbf{n}$. In general therefore $(\mathbf{n}, \theta)$ and $(-\mathbf{n}, -\theta)$ are the same spherical glide.

More than that however, the spherical glide determined by $(\mathbf{n}, \pi)$ is given by minus the identity (or, if you like, the "antipodal map") for all $\mathbf{n}$.

3.3 Polyhedra

3.3.1 Definition A polyhedron is a collection of plane polygons in space, with each edge being common to exactly two polygons. A convex one has an interior which is a region of space. We may use polyhedron to refer to the solid body or to its surface.

The most familiar examples are the five *Platonic Solids*, which are the convex polyhedrons whose symmetry group acts transitively on their flags[1].

To understand the possibilities for regular, convex polyhedra, the only possibilities are that 3,4 or 5 triangles, 3 squares or 3 pentagons meet at a vertex: two does not make a polygon and large numbers do not fit properly round a vertex.

The Euler characteristic gets us a long way. Projecting out from centre of a convex polyhedron, the vertices and edges yield a graph on the sphere with the edges being arcs of great circles. Recall (from PPS) that for such a graph on a sphere that divides the surface into F "connected and simply connected" regions and having E edges and V vertices we have Euler's famous formula

$$V - E + F = 2.$$

In a cube, for example, we have $V = 8$, $E = 12$, $V = 6$.

3.3.2 Exercise (BS) Check the formula also for the regular tetrahedron and octahedron.

3.3.3 Definition The *vertex figure* of a polyhedron is the plane polygon whose vertices are the midpoints of all the edges that are incident with a chosen vertex.

3.3.4 Example In a regular polyhedron, all the vertex figures are the same. For example, the vertex figure of a cube is the equilateral triangle $\{3\}$.

3.3.5 Definition The *Schläfli symbol of a regular, convex polyhedron* is $\{p, q\}$ where the faces are regular p-gons (so have Schläfli symbol $\{p\}$) and the vertex figures are regular q-gons $\{q\}$ (and so q of the p-gons are incident at each vertex).

3.3.6 Example The Schläfli symbol of the cube is $\{4, 3\}$.

3.3.7 Exercise (NL) Suppose we would like to build a convex, regular polyhedron with three pentagonal faces meeting at each vertex. Show that it must have twelve faces. (Hint: if it has F faces then they have $5F$ edges between them and so the polyhedron must have $5F/2$ edges. Now consider also vertices and apply Euler's Theorem. Now carry out a similar analysis for squares and triangles.

3.3.8 Conclusion So the only possible convex, regular polyhedrons are the so-called *platonic solids*:

- The tetrahedron $\{3, 3\}$;
- The cube $\{4, 3\}$;
- The octahedron $\{3, 4\}$;
- The icosahedron $\{3, 5\}$;

[1] Recall a flag is a triple (v, e, f) consisting of a vertex, edge and face with the vertex being on the edge and the edge being on the face.

- The dodecahedron $\{5,3\}$.

The duality we studied for tilings also works here. The face centres of a cube form the vertices of an octahedron and *vice versa*. Similarly for the dodecahedron and icosahedron. This duality is reflected in the fact that the Schläfli symbols are reversed.

We should worry briefly about existence because the previous analysis only tells you, for example, that a $\{5,3\}$ does not contradict Euler's Theorem and that if it exists then it has 12 faces.

3.3.9 Exercise (BM) Give a formula for the vertices in $\mathbb{R}^3$ for a cube, regular tetrahedron and regular octahedron centred at the origin. (For the tetrahedron, you might find it helpful to observe that it is possible to divide the vertices of a cube into two sets of four in such a way that each set forms the vertices of a regular tetrahedron.)

3.3.10 Exercise (NL) To construct an icosahedron, let $\phi = (1+\sqrt{5})/2 \approx 1.618034$ be the "golden ratio", so that ϕ is the larger root of $x^2 - x - 1 = 0$. Then $\phi^2 = 1 + \phi$ and $1/\phi = \phi - 1$. These formulae are very helpful in simplifying expressions in ϕ.

The vertices of the four "golden rectangles"

$$(\pm 1, \pm\phi, 0), \qquad (\pm\phi, 0, \pm 1), \qquad (0, \pm 1, \pm\phi)$$

in $\mathbb{R}^3$ are the vertices of an icosahedron (and the shorter sides are some of its edges). Make sense of this by (a) identifying the five "closest neighbours" that each vertex has and (b) identifying the vertices of the 20 equilateral triangles that are its faces.

By the way, the three rectangles are an example of the famous Borromean rings configuration: while the three are inextricably linked together, if any one is removed the other two fall apart.

3.3.11 Exercise (NM) Find the 20 face centres of the icosahedron, and scaling them if it makes the results neater, give a formula along the lines of that in 3.3.3.10 for the 20 vertices of a dodecahedron.

3.3.12 Exercise (CM) There are five subsets of the vertices of a dodecahedron that form cubes. Can you identify them in this picture?

3.3.13 Exercise (NM) Find a general calculation that tells us how many vertices, edges and faces the platonic solid $\{p, q\}$ has in terms of p, q.

3.3.14 Exercise (NM) (Descartes' Theorem) If, say, four equilateral triangles meet at a vertex of a regular polyhedron, we say that the "(angle) deficit" at that vertex is $360 - (4 \times 60) = 120$ degrees. For a regular, convex polyhedron, show that the total vertex deficit (i.e. the sum of the deficit at all the vertices) is 720 degrees. Do this with a general calculation: do no just check all five cases.

3.4 Symmetry groups of convex regular polyhedra

3.4.1 Rotational symmetry groups These are discussed in Year 2 FPM. Each consists of rotations about all the vertices and about the centres of all the edges and faces.

- The rotational symmetry group of the tetrahedron is A_4 (size 12), the group of all even permutations of four objects. In this case the four objects can be taken to be the vertices (or indeed the faces).
- The rotational symmetry group of the cube and of the octahedron is S_4 (size 24), the group of permutations of four objects. The four objects can be taken to be the diagonals of the cube or the four pairs of opposite faces in the octahedron.
- The rotational symmetry group of the dodecahedron and icosahedron is A_5 (size 60), the group of all even permutations of five objects, which can be taken to be the five cubes in the earlier exercise.

3.4.2 Exercise (NL)

1. Find all the rotational symmetries of a cube. Identify how many are rotations about opposite vertices, how many about centres of opposite edges and how many about centres of opposite faces.
2. Do the same for another platonic solid.

3.4.3 Proposition A finite subgroup of $O(3)$ is either contained in $SO(3)$ (i.e. it is entirely rotations) or it has as many reflections as rotations.

Proof. Consider the homomorphism $\det : O(3) \longrightarrow \{\pm 1\}$. □

We need to understand the full (i.e. including indirect) symmetry groups of the platonic solids. Take, say, the cube. Choose a flag (ie. choose a face, an edge on that face and a vertex on that edge). The centre of the face, the centre of the edge and the vertex form a triangle (making one eighth of a square face) which we call a *flagstone*. (We may also think oof the gflagstone as the spherical triangle obtained by projecting this radially onto the sphere.)

A little thought and we see that the reflectional symmetry group of the cube acts transitively on the flagstones: given two faces of a cube, there is a rotational symmetry taking one to the other and given two flagstones in a face there is a symmetry fixing that face and taking one flagstone to the other. (And that symmetry will be a further rotation if the two flags have the same handedness and a reflection if one is a mirror image of the other.)

There are no symmetries that fix a chosen flagstone and so (by the Orbit-stabilizer theorem), once one has chosen a reference flagstone, there is a canonical bijection between the full symmetry group of the cube and the set of all 48 flagstones.

3.4.4 We can project the edges of the cube from its centre onto the sphere containing all its vertices. The edges become arcs of circles and the (projected) flagstones divide the sphere into 48 identical (provided we allow mirror images) “spherical triangles”. Thinking of the symmetry group acting on the sphere, one has a kaleidoscope type pattern with signature $*432$. See Joan Baez’s blog post at `https://johncarlosbaez.wordpress.com/2012/05/27/symmetry-and-the-fourth-dimension-part-2/`

3.4.5 Exercise (NM) Carry out the same analysis for the other four platonic solids.

3.4.6 Theorem The reflectional symmetry group of the cube and octahedron is isomorphic to $S_4 \times \mathbb{Z}_2$; of the icosahedron and dodecahedron it is $A_5 \times \mathbb{Z}_2$ and for the tetrahedron it is S_4.

Proof. For all except the tetrahedron $-\mathbb{I}$ is an element of the reflectional symmetry group and it is in the centre of the group (i.e. it commutes with all other symmetries). Consequently, the full symmetry group is a product of the rotational subgroup and the two-element subgroup generated by $-\mathbb{I}$. See **??** for conditions for a group to be a product.)

For the tetrahedron every permutation of the vertices can be realised by the full symmetry group (whereas only the even permutations arise from rotational symmetries). Thus the full group is S_4. $\square$

3.4.7 Exercise (NS) Find all the planes of reflectional symmetry for a cube.

Find a glide symmetry of a cube. Find one where the plane of reflection for the glide is not a plane of reflectional symmetry for the cube.

Note that the glide you found is *not* a miracle: every path on the sphere between a point and its image under your glide crosses at least one plane of reflection.

Classify the reflectional symmetries of a cube. How many are reflections and how many are glides? Describe the glides.