Mathematics 3 **Honours Analysis** **2018/19**

Workshop 1 – Revision from Fundamentals of Pure Mathematics

NAME:

The purpose of this workshop activity is to revisit some of the ideas from Fundamentals of Pure Mathematics which will be crucial in Honours Analysis. Please feel free to confer and ask for help, but try to give individual and honest answers to the questions in your own words, without referring to your notes or books. This is in order to get a sense of how much of the material you've internalised. The work will be collected at the end of the workshop and passed back to you next week for feedback purposes only. It does **not** *count towards the assessment of the course.*

1. Let $A_\alpha \subseteq X$ for all $\alpha \in \mathcal{A}$. Then $\bigcap_{\alpha\in\mathcal{A}}(X \setminus A_\alpha) = \ldots$?

2. A map $f : A \to B$ is injective iff ...?

3. If $A \subseteq \mathbb{R}$ is bounded, what is meant by its **supremum**?

4. Give the definition of **convergence** of the sequence (a_n) to a number $a \in \mathbb{R}$. (Use ϵ's and N's.) Give the definition of a **Cauchy sequence** (a_n). Intuitively, how do these definitions differ?

5. State the Bolzano–Weierstrass theorem.

6. True or false? (Give reasons.)
(i) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence (a_n) is convergent;
(ii) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n \to 0$ as $n \to \infty$;
(iii) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ converges.

7. $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if p satisfies $\ldots$?

8. Let $a_n > 0$ and suppose that $\sum_{n=1}^{\infty} a_n$ converges. Is it true that $\sum_{n=1}^{\infty} a_n^2$ converges? If so, why, if not, why not?

9. What is the sum of the finite geometric series $\sum_{n=1}^{N} r^n$? (If you can't remember, then work it out!) Deduce the range of $r \in \mathbb{R}$ for which the infinite geometric series $\sum_{n=1}^{\infty} r^n$ converges.

10. State the **ratio test** and explain how it is related to geometric series.

11. Let $f(x) = x^2$ when x is rational and $f(x) = 0$ when x is irrational. Discuss the continuity and differentiability of f.

12. State the intermediate value theorem and the extreme value theorem for a continuous function $f : [0, 1] \to \mathbb{R}$. Why do these theorems have the names that they do?

13. State carefully the mean value theorem for a function $f : [0, 1] \to \mathbb{R}$. Why is it called the "mean value" theorem?

14. A real number is **algebraic** if it satisfies some polynomial equation with integer coefficients. Why is the set of algebraic numbers countable?

15. Let $f : (0,1) \to \mathbb{R}$ be a function and let $a \in (0,1)$. Match each statement in Group A with a statement from Group B which means the same thing.

Group A:

(i) $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(ii) $\forall \epsilon > 0, \forall \delta > 0$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(iii) $\exists \epsilon > 0$ such that $\forall \delta > 0$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(iv) $\exists \epsilon > 0$ and $\exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(v) $\forall \delta > 0, \exists \epsilon > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(vi) $\exists \delta > 0$ such that $\forall \epsilon > 0$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

Group B:

(a) f is continuous at a.

(b) f is bounded on $(0,1)$.

(c) f is constant on $(0,1)$

(d) There is some neighbourhood of a on which f is bounded.

(e) There is some neighbourhood of a on which f is constant.

Mathematics 3 **Honours Analysis**

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The purpose of this workshop activity is to revisit some of the ideas from Fundamentals of Pure Mathematics which will be crucial in Honours Analysis. Please feel free to confer and ask for help, but try to give individual and honest answers to the questions in your own words, without referring to your notes or books. This is in order to get a sense of how much of the material you've internalised. The work will be collected at the end of the workshop and passed back to you next week for feedback purposes only. It does **not** *count towards the assessment of the course.*

1. Let $A_\alpha \subseteq X$ for all $\alpha \in \mathcal{A}$. Then $\bigcap_{\alpha\in\mathcal{A}}(X \setminus A_\alpha) = \ldots$?

   Solution: $\bigcap_{\alpha\in\mathcal{A}}(X \setminus A_\alpha) = X \setminus \bigcup_{\alpha\in\mathcal{A}} A_\alpha$.

2. A map $f : A \to B$ is injective iff . . . ?

   Solution: $f(a_1) = f(a_2)$ implies $a_1 = a_2$.

3. If $A \subseteq \mathbb{R}$ is bounded, what is meant by its **supremum**?

   Solution: Its supremum is its least upper bound, i.e. the unique real number M such that (i) $a \leq M$ for all $a \in A$ and (ii) if M' is any real number such that $a \leq M'$ for all $a \in A$, then $M \leq M'$.

4. Give the definition of **convergence** of the sequence (a_n) to a number $a \in \mathbb{R}$. (Use ϵ's and N's.) Give the definition of a **Cauchy sequence** (a_n). Intuitively, how do these definitions differ?

   Solution: a_n converges to a if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - a| < \epsilon$. (a_n) is Cauchy if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|a_m - a_n| < \epsilon$. The former says that the numbers a_n are getting closer and closer to the number a as n gets bigger while the latter says that the numbers a_n are getting closer and closer to each other as n gets bigger.

5. State the Bolzano–Weierstrass theorem.

   Solution: Every sequence of real numbers which is bounded must have a convergent subsequence.

6. True or false? (Give reasons.)
(i) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence (a_n) is convergent;
(ii) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $a_n \to 0$ as $n \to \infty$;
(iii) the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} |a_n|$ converges.

Solution:
(i) False, e.g. the sequence $a_n = 1$ for all n converges but the series diverges.
(ii) False: e.g. $a_n = 1/n$.
(iii) False: e.g. $a_n = (-1)^n/n$.

7. $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if p satisfies ...?

Solution: $p > 1$.

8. Let $a_n > 0$ and suppose that $\sum_{n=1}^{\infty} a_n$ converges. Is it true that $\sum_{n=1}^{\infty} a_n^2$ converges? If so, why, if not, why not?

Solution: Yes. Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Then the sequence (a_n) satisfies $a_n \to 0$. In particular, there exists an N such that $n \geq N$ implies $|a_n| < 1$. So for $n \geq N$ we have $a_n^2 < a_n$ and using the comparison test for series with non-negative terms we get the conclusion.

9. What is the sum of the finite geometric series $\sum_{n=1}^{N} r^n$? (If you can't remember, then work it out!) Deduce the range of $r \in \mathbb{R}$ for which the infinite geometric series $\sum_{n=1}^{\infty} r^n$ converges.

Solution: Let $S_N = \sum_{n=1}^{N} r^n$. Then $rS_N = \sum_{n=2}^{N+1} r^n$. So $rS_N - S_N = r^{N+1} - r$ and so $S_N = (r^{N+1} - r)/(r-1)$ provided $r \neq 1$. If $r = 1$ then the sum is N. So when $r = 1$ the infinite geometric series diverges, and for $r \neq 1$ it converges iff the sequence S_N converges, which is when $-1 < r < 1$.

10. State the **ratio test** and explain how it is related to geometric series.

Solution: If a sequence (a_n) satisfies $\lim_{n\to\infty} |a_{n+1}/a_n| = r < 1$, then the series $\sum_n a_n$ converges. Under the hypothesis one can show by induction that $|a_n| \leq As^n$ for sufficiently large n (where A is some constant and $r < s < 1$) and then one can use the convergence of the geometric series and the comparison test. If $r > 1$ the series diverges and if $r = 1$ it is inconclusive.

11. Let $f(x) = x^2$ when x is rational and $f(x) = 0$ when x is irrational. Discuss the continuity and differentiability of f.

Solution: f is discontinuous at all points except 0, where it is continuous since if $\epsilon > 0$, then $|x| < \sqrt{\epsilon}$ implies $|f(x) - f(0)| < \epsilon$. Therefore it cannot be differentiable at any point except possibly 0, where we have to apply the definition of derivative to be sure. Indeed, it is differentiable at 0 with derivative 0 since $\lim_{h\to 0} |(f(h) - f(0))/h - 0| \leq \lim_{h\to 0} |h| = 0$.

12. State the intermediate value theorem and the extreme value theorem for a continuous function $f : [0,1] \to \mathbb{R}$. Why do these theorems have the names that they do?

Solution: The IVT states that for every value c **intermediate** between $f(0)$ and $f(1)$ (i.e. such that $f(0) < c < f(1)$ or $f(1) < c < f(0)$) there is some $x \in [0,1]$ such that $f(x) = c$. The Extreme Value Theorem states that there are a and b in $[0,1]$ at which f takes on the **extreme values** m and M respectively where $m \leq f(x) \leq M$ for all $x \in [0,1]$. (Meaning that $f(a) = m$ and $f(b) = M$.)

13. State carefully the mean value theorem for a function $f : [0,1] \to \mathbb{R}$. Why is it called the "mean value" theorem?

Solution: Suppose f is continuous on $[0,1]$ and differentiable on $(0,1)$. Then there exists a $c \in (0,1)$ such that the derivative of f at c equals the slope of the chord joining $(0, f(0))$ to $(1, f(1))$, i.e. $f'(c) = (f(1) - f(0))/(1-0)$

14. A real number is **algebraic** if it satisfies some polynomial equation with integer coefficients. Why is the set of algebraic numbers countable?

Solution: We use repeatedly that a countable union of countable sets is countable. Let $\mathcal{A}$ be the set of all algebraic numbers. Then

$$\mathcal{A} = \bigcup_{n\in\mathbb{N}} \bigcup_{a_0\in\mathbb{Z}} \cdots \bigcup_{a_n\in\mathbb{Z}} S_{a_0,\ldots,a_n}$$

where $S_{a_0,\ldots,a_n}$ is the finite set $\{x \in \mathbb{R} : a_0 + a_1 x + \cdots + a_n x^n = 0\}$.

15. Let $f : (0, 1) \to \mathbb{R}$ be a function and let $a \in (0, 1)$. Match each statement in Group A with a statement from Group B which means the same thing.

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(iv) $\exists \epsilon > 0$ and $\exists \delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

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(c) f is constant on $(0, 1)$

(d) There is some neighbourhood of a on which f is bounded.

(e) There is some neighbourhood of a on which f is constant.

Solution:
(i) (a)
(ii) (c)
(iii) (b)
(iv) (d)
(v) (b)
(vi) (e).