Symmetry and Geometry

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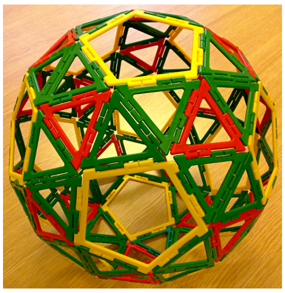
Preamble

0.1 Introduction

It was love at first sight. The first time I saw the snub dodecahedron I fell madly in love with it. I had heard of it before but while creating this course I realised that it could be made using some plastic pieces designed for schools.



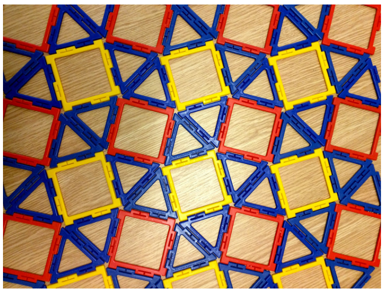


Figure 1: Snub things: dodecahedron (left); square tessellation (right)

The picture does not do it justice; there are better images online but nothing beats handling an object to appreciate its structure. We will get to understand symmetric objects in space in this course and you will soon be able to explain why this object exhibits 532 symmetry.

Something else I had not fully appreciated before was tilings of the plane by regular polygons and the way one can think of them as "infinite radius" versions of polyhedra. On the right above is the "snub square" tessellation which is a "flat" cousin of the snub dodecahedron. As a pattern in the plane, its symmetries provide two examples of "wallpaper groups": one case where only symmetries preserving colours are considered and one where we imagine all the colours to be the same. Colourings of patterns will also be a topic for us and by the end of the course you should also be able to explain why this provides an example of a 2-coloured wallpaper group of type $4{*}2/442$ (and why it is a 2-colouring despite seeming to involve three colours).

This year, I have become even more interested in such tilings and "Archimedean solids" such and they will feature rather more. I hope you will join me in working to understand them better.

What I am mainly trying to say in this introduction is that it was both instructive and fun preparing this course, that I have learned a lot from students taking it in previous years and I hope it will interesting and rewarding to study.

0.1.1 Problem (making a start) What symmetries can you find in the snub square tesselation in the two cases?

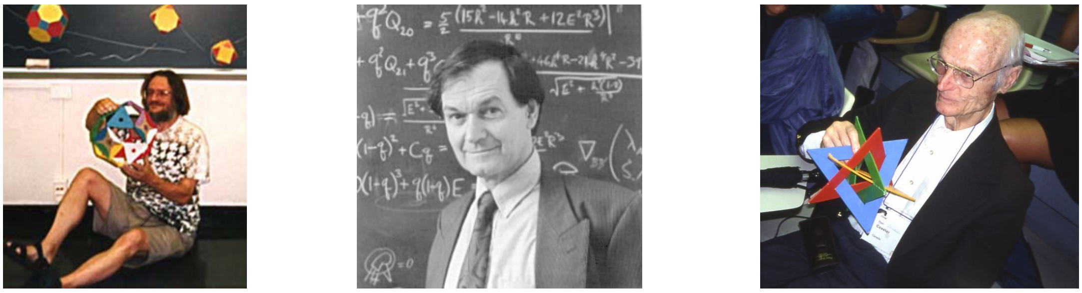


Figure 2: Conway, Penrose and Coxeter (left to right)

I would like to mention a few mathematicians who have influenced me or the content of the course. When I was an undergraduate I was lectured algebra by John Horton Conway (the inventor of “Life”); one of the few things I still have from lectures in those days is from his course on algebra, “everything you need to remember” summarised on two sides of a piece of paper and including a poem. I recall him saying that a group is not just some soulless, algebraic thing: if it is interesting it is the symmetries of something. This course was inspired by a book written by Conway (and collaborators) and takes a similar view in emphasising the visual side of things and the interplay between algebra and geometry.

As a postgraduate mathematician I was influenced a great deal by Roger Penrose whose whole approach to mathematics is geometrical. I was interested too by the relationship between some mathematical ideas and the art of Escher, an interplay he was much involved in. We will look at some of Escher’s more mathematical pictures later in the course.

Siobhan Roberts, who visited the School recently, wrote a biography of Conway (see the reading list) and also one of Coxeter, a mathematician I did not meet but who brought geometry back into the mathematical mainstream when the fashion was very much for abstraction.

I should mention William (“Bill”) Thurston too, a mathematician I never met. With Conway, he developed and promoted the “orbifold” approach to geometric symmetry that we will use. He also won the Fields Medal for realising that geometry provides a way in to understanding the classification of 3-manifolds: the analogues of surfaces one dimension up. Conway attributes a commandment to Thurston that we will be heeding: ”thou shalt know no geometrical group save by understanding its orbifold”.

0.2 Patterns on a line

Are the two patterns above equivalent as far as their symmetries are concerned? (We are of course imagining them continuing indefinitely in both directions.) Are there other fundamentally different repeating patterns in the line? The answer to the first question is easily seen to be "no" (one of the patterns has reflection symmetries in vertical lines and the other does not) but to answer the second question, we need first to think about the symmetries of an undecorated line and what "fundamentally different" might mean. While the pictures show "beads", I am imagining everything as one dimensional. A better model would be thinking of an infinitely long and thin coloured, straight rod.

0.2.1 Definition The *Euclidean group in one dimension* $E(1)$ consists of two types of symmetry of the line $\mathbb{R}$: the translations $T_a : x \mapsto x + a$ and the reflections given by $M_a : x \mapsto a - x$. In each case $a \in \mathbb{R}$. Note that T_a and M_a are *distance preserving* in that they preserve distances between pairs of points. Note that T_0 is the identity.

0.2.2 Exercise (BS) What point does M_a reflect in?

0.2.3 Exercise (BM) What happens when you compose translations and reflections with themselves and each other? Complete the following multiplication table:

$\circ$	T_b	M_b
T_a	T_{a+b}	$??$
M_a	$??$	$??$

Note that the entry already completed in the table essentially tells us that the translations for a subgroup of $E(1)$.

0.2.4 Proposition The map p from $E(1)$ to the 2-element group $\{\pm 1\}$ which sends translations to 1 and reflections to -1 is a group homomorphism. The translations are the kernel and thus form a normal subgroup of $E(1)$.

Proof. This essentially follows from the "multiplication table" constructed in Exercise 0.2.3 □

0.2.5 Definition (a little vague) A *discrete subgroup* of a group G is a subgroup that does not contain elements arbitrarily close to the identity 1_G. (This is vague because it is not clear what we mean by "close", but this should be unambiguous in practice for us. In this case, it just means that there are not arbitrarily small translations in the subgroup.)

0.2.6 Exercise (BS) Find an example of a non-discrete (or should that be "indiscrete") proper subgroup of $E(1)$.

0.2.7 Henceforth in this discussion, G is a discrete subgroup of $E(1)$. Restricting the group homomorphism of Proposition 0.2.4 to the subgroup G, we obtain a group homomorphism

$$G \to \{\pm 1\}.$$

The image of the composition is either the trivial subgroup (in which case G contains only translations) or the whole group in which case there are reflections also.

0.2.8 Proposition Every nontrivial discrete subgroup of translations is of the form $\{T_{ka} \mid k \in \mathbb{Z}\}$ for some fixed $a > 0$. We call such a discrete subgroup a 1-dimensional *lattice of translations*.

Proof. Given such a subgroup G, choose $a > 0$ as small as possible such that $T_a \in G$. Then the existence of elements of G not of the given form leads to a contradiction. (Exercise). □

0.2.9 Exercise (BM) Complete the proof of the above proposition. Where in the proof is the condition of discreteness used?

0.2.10 Definition By a (symmetric) *pattern in* $\mathbb{R}$ we mean one that is invariant under the action of a discrete subgroup of $E(1)$ that includes a lattice of translations. (The last condition is limiting ourselves to repeating patterns.)

0.2.11 If we take either of our patterns and stretch the underlying line uniformly by some factor, we have a pattern which is to all intents and purposes "the same". Our notion of equivalence of patterns is that if a pattern can be smoothly deformed into another while keeping the same symmetry group, we regard them as equivalent. This idea of "smooth isotopy" will not be defined any more exactly.

This means that we do not lose generality by assuming that the smallest translation in G is of magnitude 1.

0.2.12 Theorem Every symmetric pattern has its group G of symmetries one of the following two possibilities: the first is $T_{\mathbb{Z}}$ the lattice of translations $\{T_k \mid k \in \mathbb{Z}\}$ and the second is D_∞, the *infinite dihedral group* consisting of $T_{\mathbb{Z}}$ together with the reflections $\{M_k \mid k \in \mathbb{Z}\}$.

Proof. A sketch is as follows.

1. If G contains no reflections, then we know it must be $T_{\mathbb{Z}}$ from previously.

2. If G contains a reflection, consider first its subgroup of translations. (The translations do form a subgroup of G since they are the intersection of $G \subseteq E(1)$ with the subgroup of translations in $E(1)$.) From above, shrinking or expanding as necessary, we know this must be $T_{\mathbb{Z}}$.

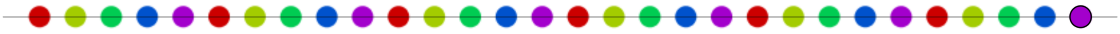
3. Now choose the origin so that M_0 is in G. Deduce by messing around with products that M_k is in G whenever $k \in \mathbb{Z}$.

4. Now show that there can be no further reflections in G. (Hint: the composition of two reflections in G has to be one of the translations in G.)

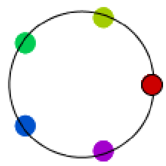
□

0.2.13 Exercise (NM) Complete the proof.

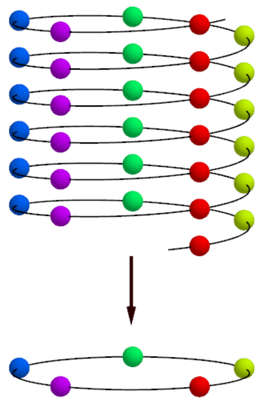
0.2.14 The translation group $T_{\mathbb{Z}}$



0.2.15 The pattern above has only translation symmetries. The whole pattern is generated by repeating  over and over again. We call this little section D a *fundamental domain*: it is a closed, connected region that contains one point from every orbit of $T_{\mathbb{Z}}$ and contains only one such point except possibly where the point in D is on its boundary.



0.2.16 We have a map from the whole line $\mathbb{R}$ with its coloured pattern to D which maps each point on the line to the element of D in its equivalence class. The trouble is that this map is not continuous. We can fix this by connecting the ends of D together to make a circle as above on the right. (I have changed the spacing of the beads for convenience.)



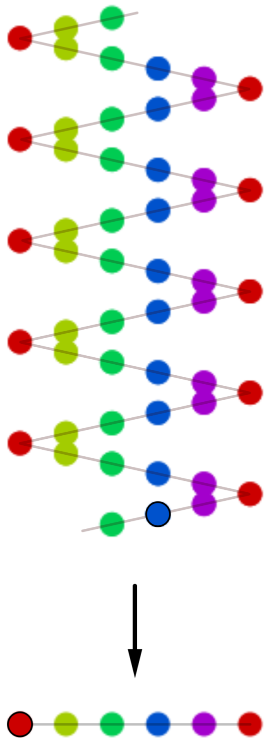
0.2.17 On the right, the real line with its pattern has been wound into a helix and the map that sends each point to its colour in the circle is projection vertically downwards and it is continuous. (Of course, the helix is meant to continue infinitely in both directions.) The circle is our first example of an *orbifold*. It completely defines the pattern in the sense that a 1-dimensional creature walking around on the orbifold would see exactly the same sequence of colours as if it was walking around on the original line. The whole pattern can be recovered from the orbifold by taking its "universal cover", and this idea may be familiar to some of you from Topology. We will think more about it later.

0.2.18 The infinite dihedral group D_∞

0.2.19 The case of D_∞ is rather different. There are two different sorts of red ball: those surrounded by green ones and those surrounded by violet ones. Therefore no symmetry of the pattern can take a red ball of one type to one of the other type. But for the other colours, every ball of that colour in the pattern is "the same". Correspondingly, there are two different types of reflection symmetry, centred in the two different types of red ball. When we talk about reflection points being "the same" or "different", the distinction we are making is whether there is a symmetry taking one to the other.

0.2.20 Exercise (BS) Choose a blue ball somewhere in the middle of the picture. What symmetries of the pattern takes this ball to each of the other pictured blue balls?

0.2.21 Exercise (NM) Set the scale and origin so that the red balls are at integer and half-integer points. Then the translation symmetries are by integer amounts. Show that if M is a reflection in a red ball and T_k is a translation by k then $T_k\, M\, T_k^{-1}$ is a reflection in a red ball distance k away. (In which direction?) So all the reflections about integer points are conjugate in the group D_∞, as are all those at half-integer points.



0.2.22 A fundamental domain for D_∞ looks like  and adjoining regions can be obtained by reflecting in the end-points. The orbifold is a closed interval *which we define to have reflecting ends*.

We already have a continuous map from $\mathbb{R}$ with its patterns to D that folds it up in a zigzag. (Hence the name "orbifold": it is the folded orbits of the symmetry group.) Given the orbifold, one can unwrap it to obtain the pattern.

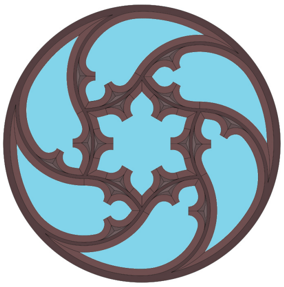
Again, a 1-dimensional creature crawling around on the orbifold, reflecting off the mirrors at the ends, sees exactly what it would crawling around on the original pattern.

0.2.23 Conclusion Note how our analysis reveals strong limitations in possible symmetries: one can have no reflectional symmetries, or two different types. But one cannot have just one type or more than two.

A final thought about this is that we classified the possible symmetry groups and then found their orbifolds. But if we accept that reflections in the pattern give rise to mirror boundaries in the orbifold then what other possible orbifolds are there? If we assume they are compact and connected, then a closed interval with reflecting boundaries or a circle seem to be the only possibilities. We might be able to deduce the completeness of our classification of groups from that. That is what we will seek to do in two dimensions.

0.2.24 Exercise (SN) If we allowed non-compact (but still closed) orbifolds, we would also have the whole real line $\mathbb{R}$ and a semi-infinite closed interval like $[0, \infty)$. To what subgroups of $E(1)$ do these correspond?

0.3 Patterns on a disc



0.3.1 Now we turn to patterns in a closed disc $K = \{(x, y) \mid x^2 + y^2 \leq 1\}$. The matrices

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

rotate the disc by an angle θ anticlockwise about the origin and reflect the plane in the line through the origin making an angle of $\theta/2$ with the positive x-axis respectively.

0.3.2 Definition The *orthogonal group* $O(2) = \{R_\theta\} \cup \{M_\theta\}$. The *special orthogonal group* $SO(2)$ is the subgroup consisting of all the rotations. The rotations are the kernel of the determinant homomorphism $\det : O(2) \to \{\pm 1\}$.

0.3.3 One can think of $SO(2)$ as being a circle with θ as an angular coordinate. The reflections form another separate circle. Because $O(2)$ is compact, a discrete subgroup turns out to be the same thing as a finite subgroup. We define a symmetric pattern on the disc as one which is invariant under a finite (and hence discrete) subgroup of $O(2)$.

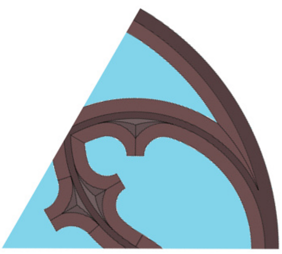
0.3.4 Proposition The finite nontrivial subgroups of $SO(2)$ are the groups

$$C_n = \{R_{2k\pi/n} \mid 0 \leq k < n\}, \qquad n \in \mathbb{Z}, \quad n \geq 2.$$

Proof. The same idea as for the classification of translation subgroups of $E(1)$ works. See Proposition 0.2.8. □

0.3.5 Exercise (NS) Complete the proof.

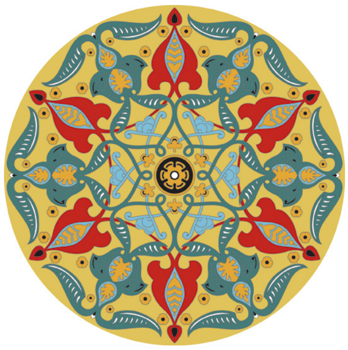
Rotational symmetry



0.3.6 The gothic tracery window at the top of this discussion has C_6 symmetry (by which we mean a cyclic group of rotations of order six). The obvious choice for its fundamental domain is a sector such as that above on the left, but the map from the disc to that would not be continuous. The fix this time is to wrap the sector into a cone, as above on the right. The original curved boundary is not reflecting because the pattern does not extend beyond: it is simply the boundary of our universe.

A 2-dimensional creature crawling around the orbifold experiences the same pattern as if they were on the original disc, and a special sort of unwinding recovers the disc from the orbifold.

Dihedral Symmetry



0.3.7 The Islamic-style decoration above has 6-fold rotational symmetry but, providing you don't look too hard, it also has reflectional symmetries. Its symmetry group is D_6, the rotational and reflectional symmetries of a regular hexagon.

0.3.8 In general, we write D_n for the symmetries of a regular n-gon when $n \geq 3$ and set D_2 to be the 4-element group of symmetries of an oblong (the "mattress group" - why so named?) and D_1 to be the group containing the identity and a single reflection.

0.3.9 Proposition A finite subgroup G of $O(2)$ has either no reflections or exactly as many reflections as rotations.

Proof. Consider the restriction of the determinant map to give a homomorphism $G \longrightarrow \{\pm 1\}$. □

0.3.10 Exercise (BS) Complete the proof. Remember that in a homomorphism of finite groups, every element in the image has the same number of preimages.

0.3.11 Theorem Every finite subgroup of $O(2)$ is either a cyclic group of rotations C_n (including the case $n = 1$ of the trivial subgroup) or a dihedral group D_n, $n \geq 1$.

Proof. This is broadly analogous to the proof of Proposition 0.2.12. Consider the subgroup of rotations. Then consider the consequences of having a reflection. □

0.3.12 Exercise (NM) Complete the proof.

0.3.13 Reflecting boundaries The orbifold is a sector bounded by the two adjacent reflection lines as on the right above. But the straight edge boundaries here should be taken as reflecting, just as for the earlier example of D_∞. For that reason they are marked in red, which is our preferred colour for mirror lines.

0.3.14 Slogans These phenomena that we have just seen will be central for us as we explore 2-dimensional geometric patterns.

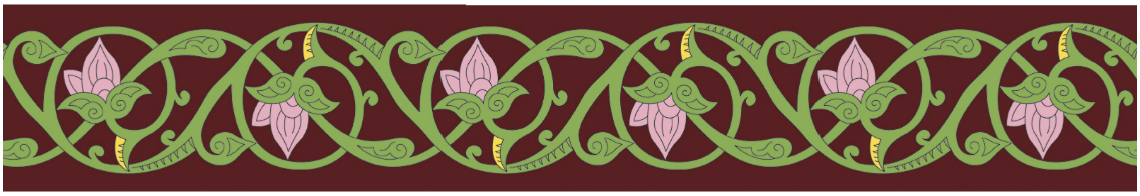
1. Reflection lines (which we sometimes call "mirrors") lead to reflecting boundaries in the orbifold.

2. A meeting of reflection lines leads to a "corner" in the reflecting boundary.

3. At a meeting point of mirrors, one always has rotational symmetry. We will call a rotational symmetry whose centre is not on a mirror a *gyration*. These lead to *cone points* in the orbifold.

0.3.15 Where are we headed? First of all, we are going to familiarise ourselves with symmetries of the Euclidean plane and explore the zoo of 17 different "wallpaper groups". But to understand their orbifolds and, in the end, see why there are just 17 we are going to need to learn quite a lot about Euler characteristics. (If you remember the famous formula $V - E + F = 2$ for polyhedra, that "2" at the end is the Euler characteristic of the sphere.)

We will need to understand and classify all possible surfaces including ones with holes in. To get a flavour of this, look at the "frieze pattern" below. (And yes, we will classify these as well.) To make its orbifold, take a length of the strip equal to the distance between a flower and an upside-down one, then fasten the ends together with a twist to make a Möbius band. (You really need a longer, thinner version of the picture to do this comfortably in three dimensional space.) If the pattern is printed on both sides of the paper, it will join up exactly right. We are going to have to understand surfaces including things like Möbius bands, toruses, Klein bottles and projective planes that may also have holes with reflecting boundaries, corners in their boundaries and cone points.



Wallpaper groups

1.1 The orbifold wallpaper shop

Our slogan is “ examples first and theory after”. This section is a crash course in associating a plane pattern with one of the 17 different possible symmetry groups. We will understand how and why all this works in a few weeks. A repeating pattern is one with translation symmetries in two independent directions. The symmetry group of such a pattern is often called a *wallpaper group*[1]

1.1.1 Types of symmetry As we will see, there are four types of symmetry of the plane:

- translation by a given vector;
- rotation by a given angle about a given point;
- reflection in a given line;
- glide reflection in a given line by a given non-zero amount.

The one of these that may be unfamiliar is the last, which we will often just call a glide. It is the composition of a reflection in a given line in the plane followed by a translation by a given non-zero distance in the direction parallel to the line. So, for example,

$$(x, y) \mapsto (1 - x, y + 2)$$

is a glide with underlying line $x = 1/2$.

Of the symmetries, the first two types are *direct* whereas the last two are *indirect*, meaning that in the latter case elements of the pattern are mapped to a mirror image of themselves. The identity can be regarded as either a translation by a zero vector or a rotation by zero angle.

1.1.2 Order of precedence In our analysis, we will look for symmetry features in a particular order and for this reason we have some nomenclature.

1. A *mirror* is a line of reflection symmetry and a *kaleidoscope* is an intersecting system of mirrors. A *kaleidoscope point* of order k is a point where k mirrors meet.

[1] There is an added technicality that there should not be arbitrarily small translations or rotations.

2. A *gyration* of order k is a point *not* on a mirror line about which there is rotational symmetry of order k.

3. A *miracle* is a glide symmetry of the pattern which is not "done with mirrors". (See discussion below.)

4. A *wandering* is a pair of independent translation symmetries in a pattern containing none of the three previous features.

We will be lax with our terminology and use e.g. "gyration" to refer to both the particular symmetry of the pattern and also the point in the pattern which is the centre of rotation.

If k mirrors meet at a point in a pattern, then there is k-fold rotational symmetry about that point. We don't count this as a gyration because the rotational symmetry comes from the mirrors.

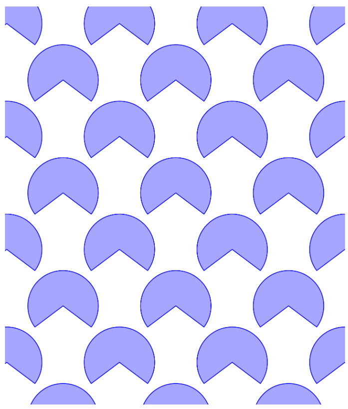
Wanderings are not difficult: if there are no mirrors, gyrations or miracles then for it to be a repeating pattern there must be a wandering.

1.1.3 We count features once We will be counting features only once. We will explain this eventually in terms of counting the features of the orbifold but for now, we will try and understand it intuitively.

Gyrations are easy: two gyrations in the pattern are "the same" feature if there is a symmetry of the pattern taking one centre of symmetry to the other. For two gyrations to be the same, they have to have the same order, but two gyrations of the same degree may be different. (For example, if one is green and the other is yellow they are certainly different.)

Mirrors are a bit more subtle although generally intuition works. We will discuss this more below.

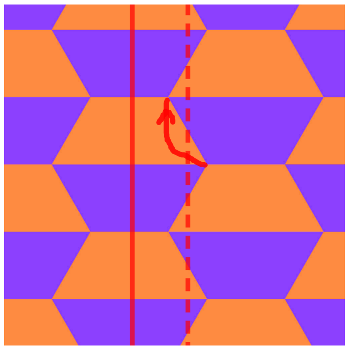
1.1.4 Miracles If a repeating pattern has mirrors then combining reflection in the mirror with a translation will produce a glide symmetry (unless the translation is perpendicular to the mirror). These are *not* miracles because these glides are done with mirrors.



The pattern on the right is a good example. You should see that it has vertical mirror lines. All the mirror lines are "the same". And if we were to reflect in one of these mirrors and then translate upwards by the distance between two of the slices of pie, we would have a glide symmetry, bu it is not a miracle because we did it with the aid of mirrors.

Draw a vertical line half way between two adjacent mirror lines. You should see that the pattern has a glide symmetry formed by reflecting in that line combined with a vertical translation by half the distance used previously. This glide cannot be obtained by combining our known mirrors with translations and so it is a real miracle. Mirrors and miracles alternate across the pattern. Note that the translation part of the miracle is exactly half of the smallest vertical translation symmetry; in fact, the square of the miracle is exactly that translation. For the purposes of classification, this pattern contains one mirror and one miracle.

1.1.5 When is it a miracle? It can be tricky to spot glide symmetries, and once one has one, it can be tricky to decide if it is a miracle.



- If the axis of the glide is a mirror then it is NOT a miracle, BUT the converse is NOT true.
- If you take a small area of the pattern and you can connect it to the area which is its image under the glide by a curve which does no cross a mirror, then it is a miracle.
- Generally for wallpaper groups the cost (see below) will tell you if you need a miracle.

The image on the right above shows a pattern with the same symmetry as the one in the previous paragraph. It is marked up: the solid red line shows the mirror and the dotted red line the axis of the miracle. The curved red arrow demonstrates that one can join a point in the pattern to its image under the glide without crossing a mirror.

1.1.6 Signature At the Orbifold Wallpaper Shop, wallpaper is priced according to its *signature*, which records the symmetry features. The costs appear in Table 1.1.

1.1.7 Conway's "Magic Theorem" The signatures of wallpaper groups are those that cost exactly $\$2$.

Proof. We will understand why the Magic Theorem is true later. □

For now, Conway's theorem tells us exactly which signatures are possible for wallpaper groups. A reference list is provided in Table 1.2.

If you are wondering what exactly constitutes a signature, it is a non-empty set of symbols consisting of:

- Zero or more blue circles, followed by
- Zero or more (blue) natural numbers (all greater than 1), followed by
- Zero or more (red) "stars" each independently followed by zero or more (red) natural numbers (each greater than 1), followed by
- zero or more (red) 'x's.

For the examples we will consider, the ordering of the sets of blue numbers and red numbers is unimportant, so for example $*442$ is the same signature as $*244$. And note that the use of red and blue numbers is redundant, although it can help to keep ones head straight.

1.2 The field guide to wallpaper groups

Kaleidoscopes

1.2.1 First look for mirror lines in your pattern (and it helps to mark some of them on a paper copy, or put a transparency over the top and mark on that to save spoiling it). If there are

Feature	Notation	Cost	Example
Kaleidoscope (a mirror or a system of intersecting mirrors)	$*$ for a single mirror or $*ab\ldots c$ for coincident mirrors where $a, b, \ldots, c$ correspond to points where that number of mirrors meet	$\$1$ for the $*$ and $\$\frac{n-1}{2n}$ for a red number n, etc	$**$ (two independent mirrors) costs $\$2$ and $*632$ costs $1+\frac{5}{12}+\frac{2}{6}+\frac{1}{4}=\$2$
Gyration (centre of rotation not on a mirror)	a where $a \geq 2$ is the order of the rotational symmetry	$\$\frac{n-1}{n}$ for a blue number n, etc	442 costs $\frac{3}{4}+\frac{3}{4}+\frac{1}{2}=\$2$
Miracle (a glide not generated by other symmetries)	$\times$	$\$1$	$22\times$ costs $\frac{1}{2}+\frac{1}{2}+1=\$2$
Wandering (Two independent translations not generated by other symmetries)	$\mathbf{o}$	$\$2$	Can only appear on its own as $\mathbf{o}$.

Table 1.1: The cost of wallpaper features

	Kaleidoscopic
$*632$	Fundamental domain is a 30-60-90 right-angled triangle (1/12 of a regular hexagon).
$*442$	Fundamental domain is a 45-45-90 right-angled triangle (1/8 of a square).
$*333$	Fundamental domain is an equilateral triangle.
$*2222$	Fundamental domain is a rectangle.
$**$	Two different types of mirror. The two types must be parallel to each other. It is a "kaleidoscope with the mirrors meeting at infinity".)
	Gyratory
632	Gyrations of orders $2, 3, 6$ arranged at the vertices of a 30-60-90 triangle. (Rarely seen in the wild.)
442	Gyrations of orders $2, 4, 4$ arranged at the vertices of a 45-45-90 triangle.
333	Gyrations of orders 3 (three different types) arranged at the vertices of an equilateral triangle.
2222	Gyrations of orders 2 (four different types) arranged at the vertices of a parallelogram. (NB: the corresponding kaleidoscope has to be rectangular).
	Gyroscopic (kaleidoscopic with central gyrations)
$3*3$	Gyration of order 3 at centre of an equilateral triangle kaleidoscope.
$2*22$	Gyration of order 2 at centre of a rectangular (possibly square) kaleidoscope.
$4*2$	Gyration of order 4 at centre of a square kaleidoscope.
$22*$	Mirrors all in the same direction and of the same type with two different types of order-2 gyrations between adjacent ones
	Miraculous (and trivial)
$*\times$	Mirrors all in the same direction and of the same type and a miracle. The glide line of the miracle is half way between adjacent mirrors.
$\times\times$	Two miracles of different types. The glide lines of the two have to be parallel.
$22\times$	As the previous case except there are two types of order-2 gyrations half way between adjacent glide lines.
o	A "wandering": there are no mirrors, gyrations or miracles – just a lattice of translations.

Table 1.2: List of the 17 wallpaper groups

mirrors in more than one direction, then find a polygon, as small as you can and bounded by mirrors. The possibilities are:

- a rectangle bounded by four mirrors;
- an equilateral triangle bounded by three mirrors;
- a right-angled isosceles triangle (45-45-90 — one eighth of square) bounded by three mirrors;
- a 30-60-90 right-angled triangle (one twelfth of a regular hexagon) bounded by three mirrors.

There is also the degenerate case of $**$ where there are two different sorts mirror (with all mirrors in the same direction).

If you are in the rectangular or equilateral triangle cases, don't forget to check for gyrations at the centre of the polygon, the existence of which means you have one of the "gyroscopic" cases: see discussion below. Also beware of gyrations half way between parallel mirrors in the degenerate case (which leads to $22*$).

1.2.2 In eg $*333$, the three segments of mirror bounding the triangle are different. There is no symmetry taking a point on one side to a point on one of the other sides. Furthermore, given a point on a mirror in the pattern there is a symmetry taking that point to one on the edges of the triangle.

On the other hand, adding a 3-fold gyration point in the centre makes all three mirror segments and all three kaleidoscope points equivalent and so we should count them only once. Hence the result is $3*3$ (and not $3*333$, which cannot be right because it is too expensive to buy).

1.2.3 In terms of thinking about combinations of features whose costs add up to $\$2$, it is worth noting that a red number costs half as much as the corresponding blue number. So, for example, since $*442$ costs exactly $\$2$ then so must $4*2$. The gyroscopic cases all come from this process from kaleidoscopes.

The gyratory groups

We next come to *gyrations* which are rotational symmetries that are not products of reflections.

1.2.4 By arithmetic, if $*abc$ costs $\$2$ then so does abc. Therefore the following signatures cost $\$2$ and so define wallpaper groups:

$$632, \quad 442, \quad 333, \quad 2222.$$

1.2.5 But more is true because the direct isometry (rotations and translations, remember) subgroups of the corresponding kaleidoscope do give examples of these wallpaper groups, with the gyration points being at the kaleidoscope points.

In fact, the gyrations have to be arranged in exactly the same way as the corresponding kaleidoscope points in every case except 2222 where the four gyrations points can be in an arbitrary parallelogram and not just in a rectangle.

1.2.6 There is one other group in the “true blue” category (ie which has only direct isometries), which is where there are no symmetries but the lattice of translations. This has signature **o** but do not diagnose that until you have checked for miracles below.

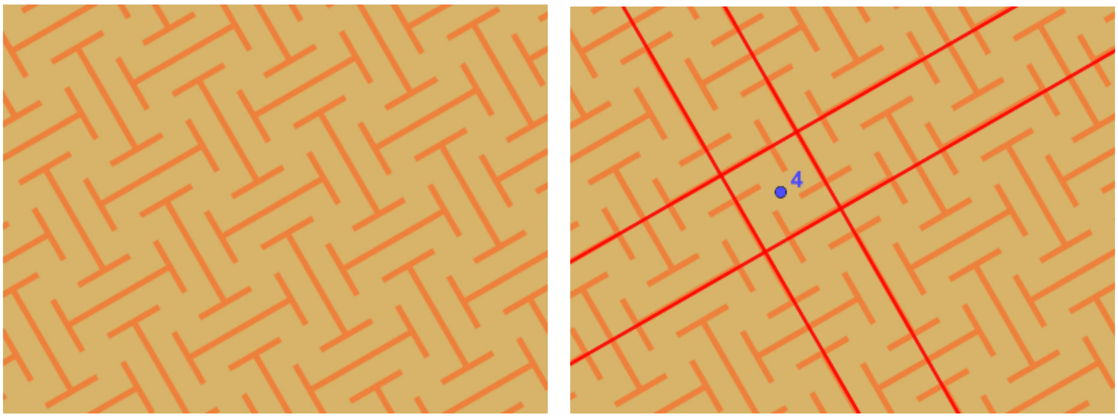


Figure 1.1: Marking up a pattern

Gyroscopic groups

1.2.7 Etymology “Gyroscopic” is a cross of “kaleidoscopic” and “gyratory”. The idea is that you have a kaleidoscope with a gyration in the centre of what would be its fundamental domain. You have to watch out for these. There are four possibilities listed in Table 1.2.

Miracles and odds and ends

1.2.8 Miracles The final thing you need to check for are *miracles*. A miracle is a glide reflection symmetry that does not come from mirrors and rotations and we denote it by $\times$. A give away is if you find a piece of pattern and its mirror image without there being a mirror dividing the two. This leads to the following signatures: $*\times$, $\times\times$, $22\times$. In all these, all mirror lines and glide lines denoted in the signature are parallel to each other.

1.2.9 Wanderings And if there are no mirrors, gyrations or miracles, then it must just be a *wandering*: a lattice of translations: *o* which should probably be in the “true blue” category.

1.2.10 Marking up We “mark up” a pattern to indicate its symmetry features and hence justify the identification of its signatures. We indicate mirrors by solid red lines and gyrations by blue points labelled with the degree. We can mark the axes of miracles by dashed red lines (my preference) and/or by a red, usually curly, arrow linking a piece of pattern to its mirror image.

1.3 Isometries of the plane

By a symmetry of the plane we mean an *isometry*: a distance preserving bijection $\mathbb{R}^2 \to \mathbb{R}^2$. Obvious examples include: translation $T_{\mathbf{a}}$ by a vector displacement $\mathbf{a}$; rotation $R_{p,\theta}$ by an angle θ anticlockwise about the point p and reflection M_l in a line l (not necessarily through the origin). The first two are called *direct isometries* and the final one is *indirect* since it sends shapes or patterns to their mirror images.

1.3.1 Definition The *Euclidean group* $E(2)$ is the group of all transformations of the plane of the form

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{a}$$

where $A \in O(2)$ (i.e. A is a 2×2 orthogonal matrix) and $\mathbf{a}$ is a vector in $\mathbb{R}^2$. An element of $E(2)$ is called an *isometry*. It is a *direct isometry* if A is a rotation ($\det A = 1$) and *indirect* if A is a reflection ($\det A = -1$).

Note that when $A = 1$ an isometry is a translation and when $\mathbf{b} = \mathbf{0}$ an isometry is an orthogonal linear transformation.

1.3.2 Exercise (BS) Write $(A, \mathbf{a})$ and $(B, \mathbf{b})$ for two elements of $E(2)$ as above. Compute the product of group elements $(A, \mathbf{a})(B, \mathbf{b})$ and express it in this notation. Hence find a formula for the inverse $(A, \mathbf{a})^{-1}$.

1.3.3 Proposition The map $f : E(2) \to O(2)$ defined by $f : (A, \mathbf{a}) \mapsto A$ is a group homomorphism. Its kernel is the normal subgroup $T \cong \mathbb{R}^2$ consisting of all translations.

Proof. The fact that it is a homomorphism follows from the previous exercise. Its kernel is clearly $\{(1, \mathbf{a}) \mid \mathbf{a} \in \mathbb{R}^2\}$, which is all translations of the plane. Kernels always are normal subgroups. □

1.3.4 Proposition We will write $R_{p,\theta}$ for rotation anticlockwise by θ about a point p. Then for all p, θ it is the case that $R_{p,\theta} \in E(2)$.

Proof. Let p have position vector $\mathbf{p}$ with respect to the origin and let

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

the orthogonal matrix that rotates by θ about the origin. Now write $\mathbf{x} = \mathbf{p} + \mathbf{y}$ and observe that

$$R_{P,\theta}(\mathbf{x}) = \mathbf{p} + A(\mathbf{y}) = \mathbf{p} + A(\mathbf{x} - \mathbf{p}) = A\mathbf{x} + (\mathbf{p} - A\mathbf{p})$$

which is of the required form. □

1.3.5 Exercise (SN) Show that given points p, q in the plane

$$R_{p,\theta} = T\, R_{q,\theta}\, T^{-1}$$

where T is the translation that takes q to p. Thus we observe that rotations by θ about different points are conjugate in $E(2)$. Explain this formula by imagining what happens to the plane under the sequence of three transformations. (By this I mean, think of the composition as being “shift the plane, rotate about q then shift back“.)

1.3.6 Proposition Every direct isometry of the plane is either a translation (including the zero translation) or rotation about some point.

Proof. Follows immediately from the above. □

1.3.7 Exercise (NL) Consider the following compositions and understand what you can about them:

1. The composition of two reflections through lines that meet at a point p;
2. The composition of two reflections in parallel lines;
3. The composition of reflection in a line and a translation perpendicular to that line;

1.3.8 Definition A *glide reflection* (just "glide" to its friends) is the composition of reflection in a line and a non-zero translation parallel to that line.

1.3.9 Proposition Every indirect isometry is either a reflection in a line or a glide along a line.

Proof. Consider an indirect isometry $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ where A is a reflection in a line through the origin Decompose $\mathbf{b} = \mathbf{p} + \mathbf{q}$ parallel and perpendicular to that line respectively. Identify $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{q}$ and hence the original transformation. □

Lattices

1.3.10 Definition A (2-dimensional) *lattice* in $\mathbb{R}^2$ is a subgroup of the translations in $E(2)$ of the form

$$\{k\mathbf{u} + l\mathbf{v} \mid k, l \in \mathbb{Z}\}$$

where $\mathbf{u}, \mathbf{v}$ are a pair of linearly independent vectors. We call $\mathbf{u}, \mathbf{v}$ *generators* of the lattice.

1.3.11 Proposition Let L be a discrete subgroup of the translations in $E(2)$ containing translations in two independent directions. Then L is a lattice.

Proof. For the proof, choose a non-zero translation $\mathbf{u}$ of minimum size in L and another non-zero translation $\mathbf{v}$ of minimum size subject to its not being a scalar multiple of $\mathbf{u}$. Clearly, all translations by elements of $\{k\mathbf{u} + l\mathbf{v} \mid k, l \in \mathbb{Z}\}$ are in L. Now show that no other translations can be in L. □

1.3.12 Exercise (MN) Complete the above proof.

1.3.13 Warning It is tempting and often useful when one has a lattice to draw a lattice of parallelograms based on $\mathbf{u}, \mathbf{v}$ but one must remember that the lattice is just the points in the picture, not the lines. In particular, different shaped parallelograms can generate the same lattice as for example in Fig 1.2.

1.3.14 Definition A *wallpaper group* is a discrete subgroup of $E(2)$ whose translations form a lattice.

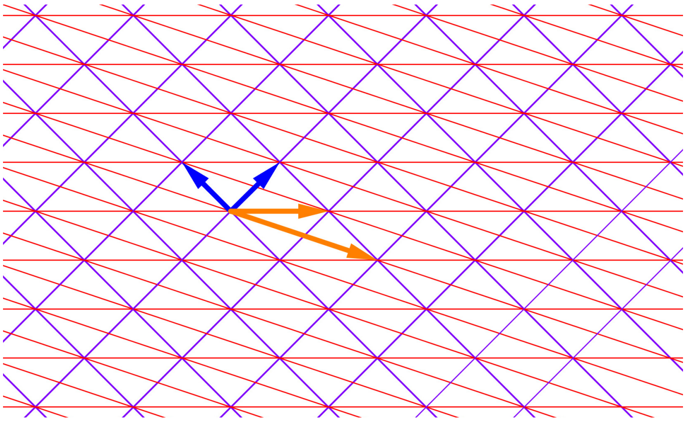
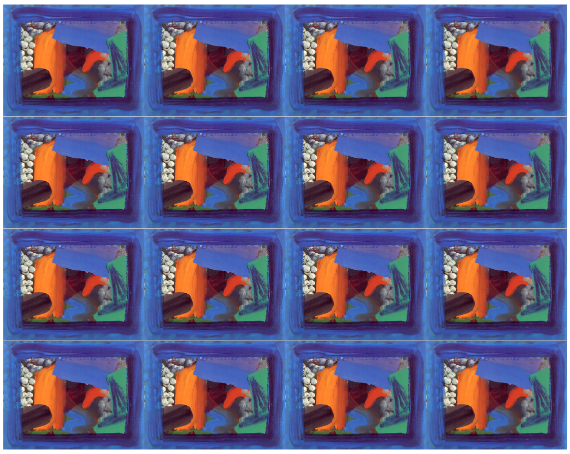


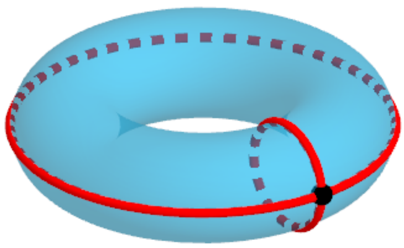
Figure 1.2: Two generating sets for the same lattice

1.3.15 A diversion: a wallpaper orbifold Take any lattice and into each panel put a copy of your favourite picture (not too symmetric please – we do not want extra symmetries). It does not have to be a rectangular lattice like mine below. Imagine it continued over the whole plane.



The symmetries of this are just a lattice of translations, the wallpaper group with signature **o**.

A fundamental domain for this pattern is (or rather, can be chosen to be) one copy of the picture, as below, left.



There is a map from the whole of $\mathbb{R}^2$ covered with this pattern to the fundamental domain but as in previous situations we have seen, it is not continuous. In the picture above left I have added some borders. To make the map continuous we need to zip together the two horizontal edges (or, more formally, *identify* them) to make a tube. And we also need to zip together the two vertical edges (which our previous operation has turned into circles).

The result is a *torus* as shown above on the right. The two zipped pairs of edges have become the two red circles and the vertices of the original confining rectangle have become a single point. You might (indeed you should) worry that we would have to distort Howard Hodgkin's painting as we carry out the second zipping above to produce the pictured torus. The answer to this is that the torus we really have as the orbifold is a mathematical construct where (essentially) we just take the rectangular painting and identify the edges so that, for example, if a 2-dimensional creature crawls off the right edge they magically appear on the left edge (at the same height). Find a simulation of the old computer game "Asteroids" online or install the "Torus Games" app from `www.geometrygames.org/TorusGames` to get a feeling for this.

So the picture of the torus correctly conveys the "topology" of the situation but not quite the correct "geometry". To get a proper "flat" torus in Euclidean space, we have to use four dimensions.

One point of this diversion is to see that we will need to understand the geometry and topology of surfaces quite well in order to understand wallpaper groups via their orbifolds.

1.3.16 Classification of lattices To return to our understanding of the Euclidean group, we need to classify lattices in the plane.

1.3.17 Definition

1. A lattice is *rectangular* if it has generators that are perpendicular.
2. A lattice is *rhombic* if it has generators of equal length.
3. A lattice is *square* if it has perpendicular generators of equal length.
4. A lattice is *hexagonal* if it has equal length generators at an angle of 60 degrees to each other.
5. We will call our lattice *general* or *generic* if it is none of the above.

In each case, it is not whether the generators one has been given (or first notices) that have to obey the conditions. The question is whether such generators exist.

1.3.18 Exercise (BM) Draw examples of all the above. Make sure your example fits only one category, don't e.g. make your rectangular lattice square. (You will need these below.)

1.3.19 Proposition The *symmetry group of a lattice*, is the subgroup of $O(2)$ consisting of elements that preserve the lattice. The symmetry groups are as follows.

Hexagonal D_6

Square D_4

Rhombic but not square or hexagonal D_2

Rectangular but not square D_2

General C_2

A two element group of rotations (half turn and the identity) is denoted C_2.

1.3.20 Exercise (NM) For each type of lattice, indicate the symmetries on an example diagram.

1.3.21 Exercise (NM) Draw a diagram that illustrates how some of the categories of lattices are special cases of others. (Perhaps five blobs for the types and arrows to indicate which are special cases of other.)

1.3.22 Exercise (NM) For each class of lattice, draw a picture of the corresponding lattice of parallelograms (which will specialise to be squares rhombuses, etc except for the general lattice) and determine the wallpaper group signature for that pattern.

Note that this is different from the symmetry group of the lattice as discussed above. (The symmetry group above is the local group of the wallpaper group at on eof the vertices.)

Three definitions and three theorems

Throughout, G is a wallpaper group. Restricting the group homomorphism $f : E(2) \longrightarrow O(2)$ to G we obtain a group homomorphism $G \longrightarrow O(2)$.

1.3.23 Definition The image of $G \to O(2)$ is called the *point group* of G and is often denoted by P.

So P is obtained as follows: write every element of G in the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$. Then take all the "$A$"s that arise – they form a subgroup P of $O(2)$.

1.3.24 Images in P The images of elements of a wallpaper group in the point group are as follows.

- Translations map to the identity $\mathbb{I}$ in P.
- A rotation by θ about a point maps to a rotation by θ about the origin.
- A reflection in a line l and a glide based on the line l both map to a reflection about the line through the origin parallel to l.

The point group of a wallpaper group G is a sort of reduced picture of the group and it often allows us to deduce things about G itself.

1.3.25 Examples

- In the wallpaper group $*\times$ every element that is not a translation is a reflection or glide and the lines of all these are parallel. Thus P is the two element group consisting of the identity and the reflection in the line through the origin parallel to all the mirrors and glide lines.

- In the group $22\times$ we have order two gyrations that give rise to a half-turn in P and the miracles that one finds are all glides along parallel lines giving rise to a single reflection in P. Also of course translations give rise to the identity in P. But these three elements of P do not form a group: the product of a half-turn and a reflection in $O(2)$ is a reflection in a line perpendicular to that of the first reflection. We deduce then that this new reflection must also be in P.

  This reflection in the point group must come from some symmetries in the pattern: it cannot mirrors so it must be glides and so there must also be glides along lines perpendicular to the ones we first identified in the features. In fact, we always have a choice when identifying the glide lines for the miracle in $22\times$ and those choices have perpendicular glide lines.

1.3.26 Exercise (NS) Use the point group to deduce that if a wallpaper group is such that there are points p, q in the pattern about which there is a rotational symmetry of orders 2 and 3 respectively, then somewhere in the pattern there is a point about which there is rotational symmetry of order 6.

1.3.27 Exercise (NM) Let g, g' be glides in a wallpaper group with the lines of the two glides being orthogonal. What can you deduce about their product gg' by considering images in the point group.

1.3.28 Exercise (NM) Take three wallpaper groups of different character and identify the point group for each.

1.3.29 Definition Let p be a point in $\mathbb{R}^2$. The *local group* at p is the subgroup of G that fixes p. (In group action terms, it is the stabilizer of C.)

Choosing the origin to be at the point we are interested in, the local group can be regarded as a subgroup of $O(2)$.

1.3.30 Exercise (NS) Explain why the local group for a wallpaper groups is trivial except at points at gyrations or on mirrors.

1.3.31 Theorem – the "crystallographic restriction" Rotations appearing in a wallpaper group have order $2, 3, 4$ or 6.

Proof. This follows from the Magic Theorem because there are no signatures with cost \$ 2 containing numbers other than $2, 3, 4, 6$. Alternatively (and non-examinably), we can argue as follows.

Translations and rotations in the group must take centres of n-fold rotational symmetry to centres of n-fold rotational symmetry, and so if there is one centre of rotation then there are lots of them. So let p, q be two centres of rotational symmetry as close together as possible. If $n > 6$, rotate q about p by one n-th of a turn to obtain another centre of n-fold rotational symmetry q'. This is closer to q than p was, giving a contradiction.

For $n = 5$, we find that q' is further from q than p is. But rotating p about q' we do obtain a centre of rotation q'' that gives the contradiction. $\square$

1.3.32 Exercise (BM) Draw some pictures to illustrate the above proof.

1.3.33 Corollary Nontrivial local groups can only be D_1, D_2, D_3, D_4, D_6 or their rotational subgroups.

1.3.34 Theorem The point group P of G is contained in the symmetry group of the lattice L of translations in G. (In other words, if $A \in P$ and $\mathbf{x} \mapsto \mathbf{x} + \mathbf{b}$ is a translation symmetry of our pattern then so is $\mathbf{x} \mapsto \mathbf{x} + A\mathbf{b}$

Proof. Let $A \in P$. Then there exists $\mathbf{a}$ such that $(A, \mathbf{a}) \in G$. Let $\mathbf{t}$ be a vector in the lattice L so that $(1, \mathbf{t}) \in L$. Now

$$(A, \mathbf{a})(1, \mathbf{t})(A, \mathbf{a})^{-1} = (1, A\mathbf{t})$$

and so $A\mathbf{t} \in L$. □

1.3.35 Corollary Nontrivial point groups can only be D_1, D_2, D_3, D_4, D_6 or their rotational subgroups.

1.3.36 Exercise (NM) Prove the Corollary. Bear in mind that the theorem says that the point group is a subgroup of the symmetry group of the lattice, not necessarily the whole thing!

1.3.37 Exercise (NL) Which point groups can occur with which lattice types?

1.3.38 Note By the way, if we believe it is clear that there are no lattices with rotational symmetry of order 5 or greater than 6, then the last theorem provides an alternative proof of the crystallographic restriction. But the contradiction argument we gave is appealingly free of technology.

1.3.39 Definition A *cell* in a wallpaper pattern is a parallelogram whose edges form two translations that generate the lattice of translations. (It is thus a fundamental domain for the subgroup of the wallpaper group consisting of all its translations.)

1.3.40 Theorem Fix a cell C in a wallpaper pattern. The point group P acts on C. Fixing a fundamental domain $F \subseteq C$, the images $\{p \cdot F \mid p \in P\}$ "tile" C with $|P|$ fundamental domains.

1.3.41 Corollary The size of the point group is equal to the area of a cell divided by the area of a fundamental domain.

Proof. (not examinable) We will not give a full proof of the theorem: we will just define the action of P. So let $A \in P$ and suppose $\mathbf{y} \in C$. Then by definition there exists $\mathbf{b}$ such that $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ is in the group. Now let $\mathbf{z}$ be the unique point in C that is obtained by a translation symmetry from $A\mathbf{y} + \mathbf{b}$. Then define $A \cdot \mathbf{y} = \mathbf{z}$; this is well defined because different choices of representing $A \in P$ by an element of the wallpaper group differ by a translation. □

1.3.42 Reference The Wikipedia wallpaper groups page `https://en.wikipedia.org/wiki/Wallpaper_group` gives ways (not unique) of tiling a cell with fundamental domains.

Tiling the plane

2.1 Polygons

A *polygon* is a chain of line segments (*sides* or sometimes *edges* joined at *vertices* or *corners*) which form a closed loop. We assume this is in the plane unless we state otherwise. A polygon is *simple* if it does not intersect itself, in which case it has an interior which is a region bounded by line segments. We will use the word “polygon” for both the chain of segments and for its interior, according to context. A simple polygon is *convex* if any line that meets its interior does so in a single connected segment.

We refer to a polygon with n sides as an n-gon or by a familiar name such as “triangle”, “quadrilateral”, “pentagon”, etc.

We are most interested in polygons with some symmetry. For example, we know that a quadrilateral has D_4 symmetry if it is a square and D_2 symmetry if it is an oblong. On the other hand, a generic quadrilateral has trivial symmetry group.

2.1.1 Definition A *flag* of a polygon is a triple (v, e, f) where v is a vertex, e is the centre of an edge attached to that vertex and f is the polygon’s centre.

In general, to geometers a “flag” is a 0-dimensional object, a 1-dimensional object, and so on up to a k-dimensional object with each object contained in the next.

2.1.2 Definition A *regular* polygon is one where the symmetry group acts transitively on its flags. (Thus, given vertices v_1, v_2 attached to edges e_1, e_2 respectively, there is a symmetry of the figure taking v_1 to v_2 and e_1 to e_2.)

The convex, regular polygons are exactly all the *regular n-gons* (the equilateral triangle, square, regular pentagon, etc). Regular polygons and polyhedra are often denoted by their Schläfli symbol. The regular, convex n-gon has Schläfli symbol $\{n\}$.

2.1.3 For interest There are also non-convex regular *star polygons* (see `http://mathworld.wolfram.com/StarPolygon.html`). These are denoted $\{\frac{n}{m}\}$ where $n \geq 5$ and $2 \leq m < n/2$. If the gcd of m and n is not 1 then $\{\frac{n}{m}\}$ is not a star polygon but a *compound*.

2.1.4 Definition A polygon is *vertex transitive* (“v-transitive” for short) if its symmetry group acts transitively on its vertices. It is *edge transitive* (“e-transitive” for short) if its symmetry group acts transitively on its edges.

2.1.5 Problem (NL) A triangle is v-regular if and only if it is equilateral. Similarly for e-regular.

Consider now convex quadrilaterals. A generic quadrilateral has no non-trivial symmetries. Special, more symmetric quadrilaterals include rectangles, rhombuses, parallelograms, isosceles trapeziums, kites and squares. In each case what is the symmetry group and which cases are e-transitive and which are v-transitive.

2.2 Honeycombs

In this chapter we will consider only tilings that are "edge to edge", meaning that where two polygons meet, they do so only at a vertex or they share a complete common edge. Unless otherwise stated, tilings are by regular polygons.

2.2.1 Definition A *regular tiling* or *honeycomb* in the plane is a tiling of the plane by identical regular polygons. It is easy to see that there are exactly three possibilities:

- The *quadrille*: a tiling by squares with four meeting at each vertex. Symmetry group $*442$. Schäfli symbol $\{4, 4\}$.
- The *deltille*: a tiling by equilateral triangles with six meeting at each vertex. Symmetry group $*632$. Schäfli symbol $\{3, 6\}$.
- The *hextille*: a tiling by regular hexagons with three meeting at each vertex. Symmetry group $*632$. Schäfli symbol $\{6, 3\}$.

In each case, flags in each polygon in the tiling are fundamental domains for the symmetry group and so given two flags in two polygons in the tiling, there is a unique symmetry taking one to the other.

The *Schäfli symbol* $\{p, q\}$ indicates that the tiling is by regular p-gons and q are meeting at each vertex.

2.2.2 Duality / reciprocation A trick with tilings by regular polygons is to take the the centres of the tiles to be vertices in a new "dual" tiling. Two vertices are joined by an edge in the new tiling if and only if the corresponding polygons in the original tiling had a common edge. This

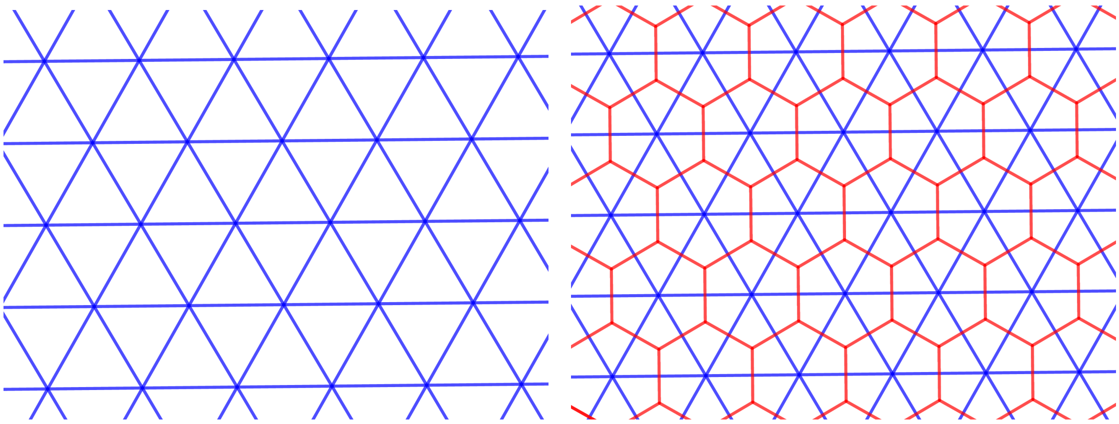


Figure 2.1: Deltille $\{3, 6\}$ and with the dual hexatille $\{6, 3\}$.

duality always reverses the numbers in the Schäfli symbol: as we see above, the detille and hextille are dual and the dual of the quadrille $\{4, 4\}$ is itself. Draw the picture and check!

2.3 Archimedean tilings

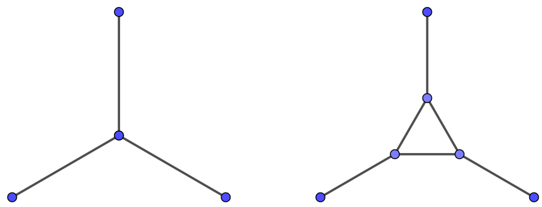


Figure 2.2: Truncating a vertex in a hextille.

2.3.1 Truncation If one takes a vertex in the $\{p, q\}$ honeycomb, one can replace the vertices by little regular q-gons, centred at the original vertex as illustrated for $q = 3$ in Figure 2.2. The resulting tiling consists in the hextille case of equilateral triangles and 12-gons.

If you slowly increase the size of the equilateral triangles, at some point the 12-gons become regular and we have a new tiling of the plane by regular polygons, called the *truncated hextille*. See Figure 2.3 below.

2.3.2 Exercise (NS) What do you get by truncating the quadrille and by truncating the deltille?

2.3.3 Vertex transitive tilings The truncated hextille has exactly the same symmetries as the original hextille: it's $*632$. The tiling now has two sorts of tiles and also two sorts of edges (those separating two 12-gons and those separating a 12-gon from a triangle). But it has only one type of vertex. More precisely, given two vertices there is a symmetry of the pattern taking one to the other. We say that the tiling is *vertex transitive* of *v-transitive* for short. The word *uniform* is aalso often used.

Tilings that are v-transitive but not regular are called *Archimedean or semiregular*.

2.3.1 The Wythoff construction

2.3.1 The Wythoff construction constructs most of the v-transitive tilings of the plane by regular polygons; it constructs all those with kaleidoscopic symmetry group of the form $G = *2pq$.

To be more precise, for each such group we find all tilings by regular polygons on which the group acts by symmetries v-transitively. We are most interested in the case where G is the full symmetry group of the tiling but sometimes we get cases where G is a proper subgroup of the full symmetry group in which case we say the tiling is *relative* rather than absolute. We will see examples soon.

For tilings in the plane, the only possibilities are $*632$ and $*442$ but we proceed with a general p and q because it turns out we can use other values in the spherical and hyperbolic cases.

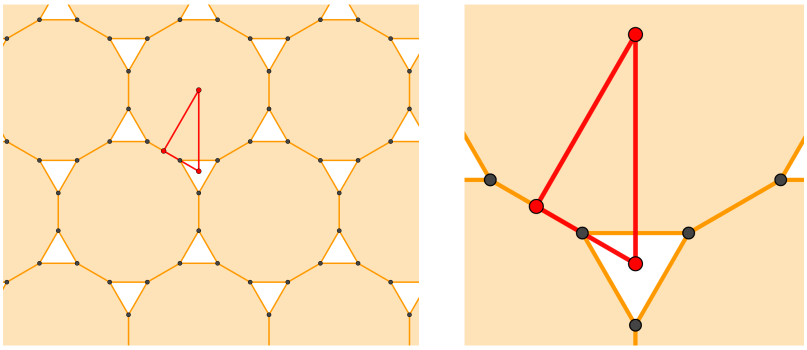
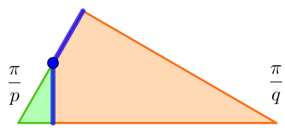


Figure 2.3: Truncated hextille with fundamental domain.

2.3.2 Wythoff's idea In figure 2.3 we see a fundamental domain in the truncated hextille and an expanded picture. On the right we see that picture in a different orientation. The base angles of the triangle are π/p and π/q where in this example we have $p = 3$ and $q = 6$. Note the following facts.



- There is one and only one vertex of the tiling in the orbifold. It is the blue point. (This follows from the vertex transitivity.)

- The blue half-edges meet the reflecting boundary of the orbifold orthogonally. (If they did not then the edge would change direction at the boundary.)

- The blue vertex lies on the angle bisector of the opposite angle and consequently the two half edges are the same length. (Thus the two half-edges generate a regular orange 12-gon when kaleidoscoped about the right-hand vertex.)

- The lower half-edge generates a green equilateral triangle when kaleidoscoped about the left-hand vertex.

2.3.3 Face code We define the *face code* of a vertex-transitive tiling to be the list of the number of sides of each *n*-gon surrounding a vertex. It is defined only up to cyclic permutations and reversal. For the truncated hextille the face code is $(12)^2(3)$ because at each vertex we encounter a 12-gon then a 12-gon then a 3-gon as we go round a vertex. The face code can be seen from the orbifold picture by considering the result of reflecting the orbifold in its shortest side.

2.3.4 Wythoff's construction Wythoff observed that in general if one is to have $*2pq$ acting vertex transitively on a tiling by regular polygons, that for essentially the same reasons sketched above for the truncated hextille there are exactly seven possibilities.

- The blue tiling vertex can be at one of the three corners of the orbifold with an edge joining it perpendicularly to the opposite side. (Note that in the case of the lower two vertices, the edge is one of the edges of the original orbifold.)

- It can be at the point on one of the three sides where the angle bisector of the opposite angle meets it. There are two blue edges joining the vertex orthogonally to the two other sides.

- Finally, the blue vertex can be at the meeting of the three angle bisectors (the “incentre” of the triangle), with a blue edge joining it orthogonally to each of the sides.

2.3.5 The $*632$ regular and Archimedean tilings Table 2.1 shows the seven results of the Wythoff construction for $*632$. It produces two honeycombs (regular tilings) that we know about and four v-regular “Archimedean” tilings. A case of note is the “truncated deltille”. A little thought reveals that truncating a deltille leads to a hextille and so not to a new v-regular tiling. But since it comes from the Wythoff construction we see more: the resulting hextille will be two coloured (orange and green) and the $*632$ symmetry group of the coloured pattern is a subgroup of the symmetry group of the whole pattern (a “larger” $*632$) which still acts transitively on the vertices.

2.3.6 The $*442$ regular and Archimedean tilings The case $*442$ is less productive: one obtains only the quadrille and the truncated quadrille (squares and octagons with face code $(8)^2(4)$). This is because the symmetry of the fundamental domain leads to two pairs of possibilities becoming identical and the others lead to relative cases.

2.3.7 The non-Wythoff v-regular tilings There are three non-Wythoff Archimedean tilings which we will discuss later: the snub quadrille, the snub hextille and the isosnub quadrille. See Table 2.2. In all then we have 11 v-regular tilings, 3 of which are regular.

2.3.8 Catalan tilings One can take the dual of the 8 Archimedean tilings and obtain 8 rather attractive *Catalan tilings*. These are not tilings by regular polygons. The same symmetry group acts transitively on the faces of the Catalan tiling because they correspond to vertices in the original tiling. The Catalan tiling will have different types of vertex, but because the faces of the original tiling were regular polygons, the vertices of the Catalan tiling are regular in the sense that the emerging edges are equally spaced around a vertex. Each edge of the Catalan tiling perpendicularly bisects and edge of the original tiling. (Why?)

You can find pictures of the 8 Catalan tilings at `https://en.wikipedia.org/wiki/List_of_convex_uniform_tilings`. The “Cairo tiling” in particular seems particularly attractive.

2.3.9 Exercise () Consider the Wythoff construction of the “truncated deltille”. What colouring of the hextille does that give you?

2.3.10 Exercise () Consider the Wythoff construction for $*442$. There are three “relative cases” which give rise to colourings of tilings. Identify the colourings.

2.3.11 Exercise () What are the symmetry groups of the three non-Wythoffian Archimedean tilings?

2.3.12 Exercise () One of the Archimedean tilings is “chiral” meaning that its mirror image is not equivalent to the original under direct isometry. Which one?

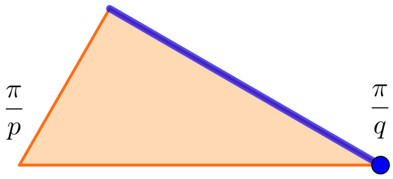
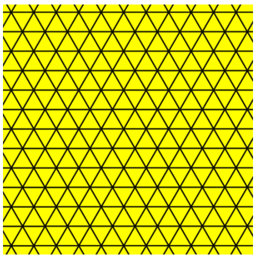
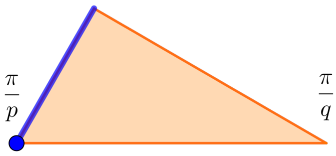
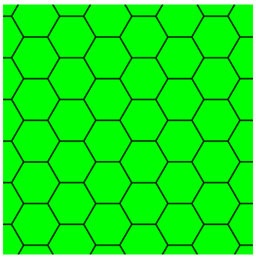
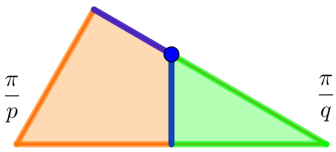
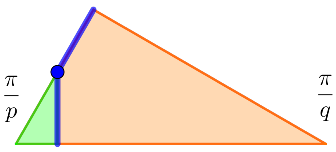
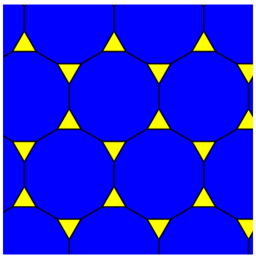
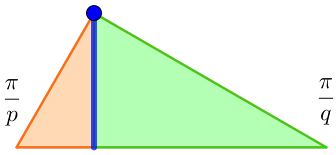
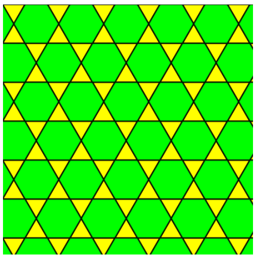
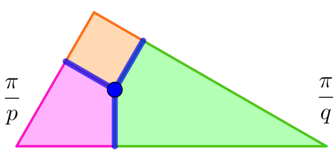
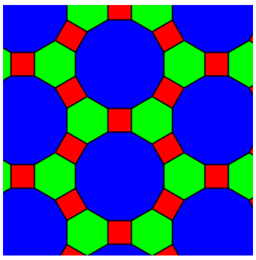
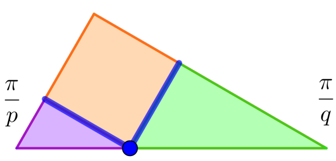
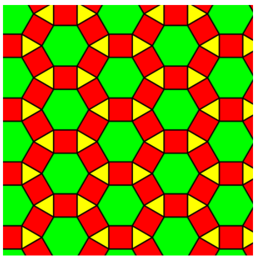
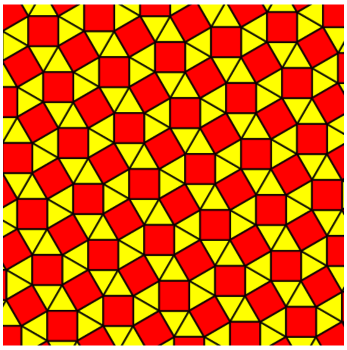
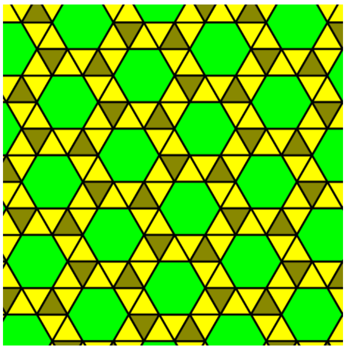
Construction	**Face code / Name**	**The** $*632$ **case**
	3^6 Deltille	
	6^3 Hextille	
	$(6)^2 6$ Truncated deltille	= coloured hextille Relative
	$(12)^2 3$ Truncated hextille	
	$(3)(6)(3)(6)$ Hexadeltille	
	$(4)(6)(12)$ Truncated hexadeltille	
	$(4)(3)(4)(6)$ Rhombihexadeltille	

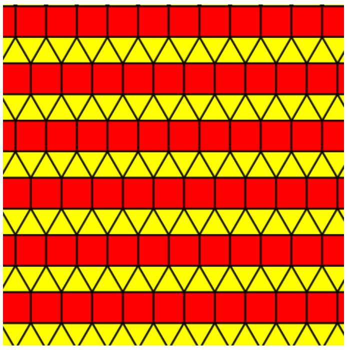
Table 2.1: The Wythoff construction (tiling images from Wikipedia)



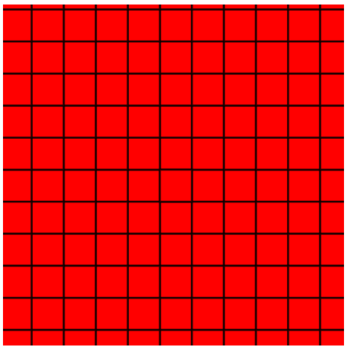
Snub quadrille



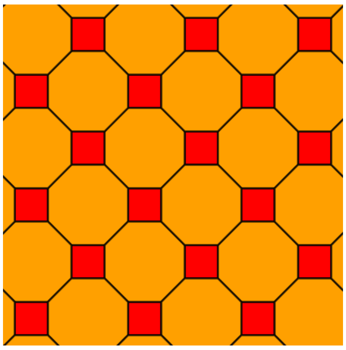
Snub hextille



Isosnub quadrille



Quadrille



Truncated quadrille

On the left are the two Wythoff $*442$ cases. Of the other 5, two are duplicates because of symmetry and the other three are relative: they correspond to colourings of tilings already in the list.

Table 2.2: Two Wythoff $*442$ and three non-Wythoff v-regular tilings (images from Wikipedia)

2.3.13 Exercise () Of the Archimedean tilings, one is also edge-regular. Which one, and what feature of the Wythoff construction corresponds to this property?

2.3.14 Exercise () For each of the seven Wythoff cases, consider the corresponding Catalan tiling. What is the shape of the tile? Which cases give rise to Catalan tilings by Rhombuses and why?

Spherical symmetry groups

3.1 Identifying spherical symmetry groups

3.1.1 Scenario We are interested in patterns or objects with symmetries given by the group $O(3)$ of 3×3 orthogonal matrices, which is the group of symmetries generated by rotations about an axis through the origin and reflections in planes through the origin.

We will often think in terms of patterns drawn on the surface $x^2 + y^2 + z^2 = 1$ of the unit sphere. In that case, reflections reflect in *great circles* (circles whose centre is at the origin).

We will also consider solids such as the octahedron and the dodecahedron. If we think of an octahedron centred at the origin, we can project radially to obtain a pattern on the sphere: the vertices become points on the sphere (at the North pole, the South pole and four round the equator), the edges become arcs of great circles. The eight faces become three-sided, regular "spherical triangles". On the sphere, four of these faces meet at a point, and so all the angles in our spherical triangle are all right angles!

3.1.2 Symmetries As we will see, all direct isometries of the sphere are rotations about some axis through the origin. But as for the plane, there are two sorts of indirect isometry:

- A *reflection* reflects the sphere in a great circle.
- A *(spherical) glide* is the result of composing a reflection with a rotation about the normal to the plane of reflection. For example, reflecting the earth in the plane through the equator and then rotating the earth by some amount around the North and South poles is a glide.

You should discover for a cube that there are three types of mirrors which reflect in the following planes planes through the origin.

- Normal to a line joining the midpoints of opposite faces.
- Normal to a line joining opposite vertices.
- Normal to a line joining the midpoints of opposite edges.

You should observe that four mirrors meet at the midpoints of faces, three at vertices and two at midpoints of edges. The cube has kaleidoscopic symmetry with signature $*432$.

A fundamental region is the radial projection onto the sphere of a triangle with vertices the centre of a face, the centre of an edge of that face, and a vertex belonging to that edge. (So it is defined by a flag.)

3.1.3 Orbifold Classification The basic reference here is the account in “Symmetry of Things” classifying the finite subgroups of $O(3)$ according to the “Magic Theorem” that states in this case that the sum of the costs is equal to

$$2 - \frac{2}{|G|}$$

(where $|G|$ is the number of elements in the subgroup G, which is denoted by g in the book).

Every possible signature with cost less than two gives rise to a symmetry group with just the exception that the cases MN and $*MN$ (where M, N are natural numbers) occur only when $M = N$.

I present here how I like to “organise” the resulting possibilities, which is a little different from Conway’s.

3.1.4 Seven groups for which there is no fixed axis

Full group	Rotation subgroup	Notes
$*332$ $(\#G = 24)$	332 $(\#G = 12)$	Symmetries of tetrahedron
$*432$ $(\#G = 48)$	432 $(\#G = 24)$	Symmetries of cube or octahedron
$*532$ $(\#G = 120)$	532 $(\#G = 60)$	Symmetries of icosahedron or dodecahedron
$3{*}2$ $(\#G = 24)$	This is the one group not fixing an axis that is not the rotational or full symmetry group of a platonic solid. It is a subgroup of the full symmetry group of a cube and its rotational subgroup is 332.	

3.1.5 Seven families that fix an axis Here, “fixing an axis” means that, for example, the line joining the North and South poles is sent to itself although it may be that the North and South poles are swapped over (e.g. by a reflection in the equator).

The trivial subgroup has no non-identity elements and hence the signature is the empty set. There is one family that fixes an axis and relies on a miracle:

$$N\times.$$

Here and following, N is an arbitrary natural number that may be omitted if $N = 1$.

Beyond that, there is a distinction as to whether the ends of the axis (the “North and South poles”) are kaleidoscope points or gyrations. And for each of those cases there is a “basic example” to which one can add either a reflection in the equator or gyrations of order two on the equator.

Nature of "N& S poles"	**Additions**	**Signature**
Gyration	None	NN
Gyration	Eq. Refl.	$N*$
Gyration	Eq. Rots.	$22N$
Kaleidoscope	None	$*NN$
Kaleidoscope	Eq. Refl.	$*22N$
Kaleidoscope	Eq. Rots.	$2*N$

3.1.6 Notes Note that the North and South poles are separate gyration/kaleidoscope points for the two "basic" examples NN and $*NN$ respectively. In fact, the corresponding group in these cases is exactly the rotational or full symmetries of a regular N-gon.

For the other cases where one has in the signature a reflection or glide in the equator or an order-2 gyration centred on the equator, those symmetries interchange the North pole and South poles and so there is only one 'N' in the signature.

3.2 $O(3)$ and $SO(3)$

3.2.1 Definition The *orthogonal group* $O(n)$ consists of all $n \times n$ matrices A satisfying $A^T A = \mathbb{I}$. This is equivalent to asking that A preserves dot products (and hence lengths and angles). Taking determinants, we deduce that $\det(A) = \pm 1$. In fact, $\det : O(n) \to \{\pm 1\}$ is a group homomorphism and its kernel is the *special orthogonal group* $SO(n)$.

3.2.2 Elements of $O(3)$ Let $A \in SO(3)$. Then the characteristic polynomial is equal to pone when $\lambda = 0$ and tends to minus infinity as $\lambda - > +\infty$. Thus A has a positive eigenvalue. Since A preserves length the eigenvalue is $\lambda = 1$.

Taking the eigenvector to be in the direction of the positive z-axis, A must act as a rotation in the orthogonal (x, y) plane. Thus every non-identity element is a rotation about a fixed axis and in a basis chosen so that the axis of rotation is the z-axis it has matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, an element $A \in O(3)$ with $\det A = -1$ hand must have -1 as an eigenvalue.

Taking the eigenvector in this case to be the z-axis, A must act as a rotation in the orthogonal (x, y) plane (because if it was a reflection we would have $\det A = 1$). If that rotation is zero, we have a reflection in a *great circle* (the intersection of the sphere with a plane through the origin), otherwise they are that composed with a rotation about the normal: what we have called a *(spherical) glide*. With the original eigenvector being the z-axis again, in matrix form we have

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

3.2.3 Summary We have understood the elements of $O(3)$. If they are “direct” then they are rotations about an axis through the origin or the identity (which we think of as a rotation by zero). If they are “indirect” they are reflections in great circles or “glides”.

3.2.4 Warning A spherical glide is determined by a unit vector $\mathbf{n}$ which is normal to the plane of reflection and an angle $\theta \in (0, 2\pi)$ which is the angle rotated about $\mathbf{n}$, measured clockwise as one looks along $\mathbf{n}$. In general therefore $(\mathbf{n}, \theta)$ and $(-\mathbf{n}, -\theta)$ are the same spherical glide.

More than that however, the spherical glide determined by $(\mathbf{n}, \pi)$ is given by minus the identity (or, if you like, the “antipodal map”) for all $\mathbf{n}$.

3.3 Polyhedra

3.3.1 Definition A polyhedron is a collection of plane polygons in space, with each edge being common to exactly two polygons. A convex one has an interior which is a region of space. We may use polyhedron to refer to the solid body or to its surface.

The most familiar examples are the five *Platonic Solids*, which are the convex polyhedrons whose symmetry group acts transitively on their flags[1].

To understand the possibilities for regular, convex polyhedra, the only possibilities are that 3,4 or 5 triangles, 3 squares or 3 pentagons meet at a vertex: two does not make a polygon and large numbers do not fit properly round a vertex.

The Euler characteristic gets us a long way. Projecting out from centre of a convex polyhedron, the vertices and edges yield a graph on the sphere with the edges being arcs of great circles. Recall (from PPS) that for such a graph on a sphere that divides the surface into F "connected and simply connected" regions and having E edges and V vertices we have Euler's famous formula

$$V - E + F = 2.$$

In a cube, for example, we have $V = 8$, $E = 12$, $V = 6$.

3.3.2 Exercise (BS) Check the formula also for the regular tetrahedron and octahedron.

3.3.3 Definition The *vertex figure* of a polyhedron is the plane polygon whose vertices are the midpoints of all the edges that are incident with a chosen vertex.

3.3.4 Example In a regular polyhedron, all the vertex figures are the same. For example, the vertex figure of a cube is the equilateral triangle $\{3\}$.

3.3.5 Definition The *Schläfli symbol of a regular, convex polyhedron* is $\{p, q\}$ where the faces are regular p-gons (so have Schläfli symbol $\{p\}$) and the vertex figures are regular q-gons $\{q\}$ (and so q of the p-gons are incident at each vertex).

3.3.6 Example The Schläfli symbol of the cube is $\{4, 3\}$.

3.3.7 Exercise (NL) Suppose we would like to build a convex, regular polyhedron with three pentagonal faces meeting at each vertex. Show that it must have twelve faces. (Hint: if it has F faces then they have $5F$ edges between them and so the polyhedron must have $5F/2$ edges. Now consider also vertices and apply Euler's Theorem. Now carry out a similar analysis for squares and triangles.

3.3.8 Conclusion So the only possible convex, regular polyhedrons are the so-called *platonic solids*:

- The tetrahedron $\{3, 3\}$;
- The cube $\{4, 3\}$;
- The octahedron $\{3, 4\}$;
- The icosahedron $\{3, 5\}$;

[1] Recall a flag is a triple (v, e, f) consisting of a vertex, edge and face with the vertex being on the edge and the edge being on the face.

- The dodecahedron $\{5, 3\}$.

The duality we studied for tilings also works here. The face centres of a cube form the vertices of an octahedron and *vice versa*. Similarly for the dodecahedron and icosahedron. This duality is reflected in the fact that the Schläfli symbols are reversed.

We should worry briefly about existence because the previous analysis only tells you, for example, that a $\{5, 3\}$ does not contradict Euler's Theorem and that if it exists then it has 12 faces.

3.3.9 Exercise (BM) Give a formula for the vertices in $\mathbb{R}^3$ for a cube, regular tetrahedron and regular octahedron centred at the origin. (For the tetrahedron, you might find it helpful to observe that it is possible to divide the vertices of a cube into two sets of four in such a way that each set forms the vertices of a regular tetrahedron.)

3.3.10 Exercise (NL) To construct an icosahedron, let $\phi = (1 + \sqrt{5})/2 \approx 1.618034$ be the "golden ratio", so that ϕ is the larger root of $x^2 - x - 1 = 0$. Then $\phi^2 = 1 + \phi$ and $1/\phi = \phi - 1$. These formulae are very helpful in simplifying expressions in ϕ.

The vertices of the four "golden rectangles"

$$(\pm 1, \pm\phi, 0), \qquad (\pm\phi, 0, \pm 1), \qquad (0, \pm 1, \pm\phi)$$

in $\mathbb{R}^3$ are the vertices of an icosahedron (and the shorter sides are some of its edges). Make sense of this by (a) identifying the five "closest neighbours" that each vertex has and (b) identifying the vertices of the 20 equilateral triangles that are its faces.

By the way, the three rectangles are an example of the famous Borromean rings configuration: while the three are inextricably linked together, if any one is removed the other two fall apart.

3.3.11 Exercise (NM) Find the 20 face centres of the icosahedron, and scaling them if it makes the results neater, give a formula along the lines of that in 3.3.3.10 for the 20 vertices of a dodecahedron.

3.3.12 Exercise (CM) There are five subsets of the vertices of a dodecahedron that form cubes. Can you identify them in this picture?

3.3.13 Exercise (NM) Find a general calculation that tells us how many vertices, edges and faces the platonic solid $\{p, q\}$ has in terms of p, q.

3.3.14 Exercise (NM) (Descartes' Theorem) If, say, four equilateral triangles meet at a vertex of a regular polyhedron, we say that the "(angle) deficit" at that vertex is $360 - (4 \times 60) = 120$ degrees. For a regular, convex polyhedron, show that the total vertex deficit (i.e. the sum of the deficit at all the vertices) is 720 degrees. Do this with a general calculation: do no just check all five cases.

3.4 Symmetry groups of convex regular polyhedra

3.4.1 Rotational symmetry groups These are discussed in Year 2 FPM. Each consists of rotations about all the vertices and about the centres of all the edges and faces.

- The rotational symmetry group of the tetrahedron is A_4 (size 12), the group of all even permutations of four objects. In this case the four objects can be taken to be the vertices (or indeed the faces).
- The rotational symmetry group of the cube and of the octahedron is S_4 (size 24), the group of permutations of four objects. The four objects can be taken to be the diagonals of the cube or the four pairs of opposite faces in the octahedron.
- The rotational symmetry group of the dodecahedron and icosahedron is A_5 (size 60), the group of all even permutations of five objects, which can be taken to be the five cubes in the earlier exercise.

3.4.2 Exercise (NL)

1. Find all the rotational symmetries of a cube. Identify how many are rotations about opposite vertices, how many about centres of opposite edges and how many about centres of opposite faces.
2. Do the same for another platonic solid.

3.4.3 Proposition A finite subgroup of $O(3)$ is either contained in $SO(3)$ (i.e. it is entirely rotations) or it has as many reflections as rotations.

Proof. Consider the homomorphism $\det : O(3) \longrightarrow \{\pm 1\}$. □

We need to understand the full (i.e. including indirect) symmetry groups of the platonic solids. Take, say, the cube. Choose a flag (ie. choose a face, an edge on that face and a vertex on that edge). The centre of the face, the centre of the edge and the vertex form a triangle (making one eighth of a square face) which we call a *flagstone*. (We may also think oof the gflagstone as the spherical triangle obtained by projecting this radially onto the sphere.)

A little thought and we see that the reflectional symmetry group of the cube acts transitively on the flagstones: given two faces of a cube, there is a rotational symmetry taking one to the other and given two flagstones in a face there is a symmetry fixing that face and taking one flagstone to the other. (And that symmetry will be a further rotation if the two flags have the same handedness and a reflection if one is a mirror image of the other.)

There are no symmetries that fix a chosen flagstone and so (by the Orbit-stabilizer theorem), once one has chosen a reference flagstone, there is a canonical bijection between the full symmetry group of the cube and the set of all 48 flagstones.

3.4.4 We can project the edges of the cube from its centre onto the sphere containing all its vertices. The edges become arcs of circles and the (projected) flagstones divide the sphere into 48 identical (provided we allow mirror images) “spherical triangles”. Thinking of the symmetry group acting on the sphere, one has a kaleidoscope type pattern with signature $*432$. See Joan Baez’s blog post at `https://johncarlosbaez.wordpress.com/2012/05/27/symmetry-and-the-fourth-dimension-part-2/`

3.4.5 Exercise (NM) Carry out the same analysis for the other four platonic solids.

3.4.6 Theorem The reflectional symmetry group of the cube and octahedron is isomorphic to $S_4 \times \mathbb{Z}_2$; of the icosahedron and dodecahedron it is $A_5 \times \mathbb{Z}_2$ and for the tetrahedron it is S_4.

Proof. For all except the tetrahedron $-\mathbb{I}$ is an element of the reflectional symmetry group and it is in the centre of the group (i.e. it commutes with all other symmetries). Consequently, the full symmetry group is a product of the rotational subgroup and the two-element subgroup generated by $-\mathbb{I}$. See **??** for conditions for a group to be a product.)

For the tetrahedron every permutation of the vertices can be realised by the full symmetry group (whereas only the even permutations arise from rotational symmetries). Thus the full group is S_4. $\square$

3.4.7 Exercise (NS) Find all the planes of reflectional symmetry for a cube.

Find a glide symmetry of a cube. Find one where the plane of reflection for the glide is not a plane of reflectional symmetry for the cube.

Note that the glide you found is *not* a miracle: every path on the sphere between a point and its image under your glide crosses at least one plane of reflection.

Classify the reflectional symmetries of a cube. How many are reflections and how many are glides? Describe the glides.

Friezes, Archimedean and Catalan solids

4.1 Frieze groups

4.1.1 Definition Let F be the subgroup of $E(2)$ that fixes an infinite, constant width, closed strip.

4.1.2 Proposition Taking the strip to by $-1 \leq y \leq 1$ in the (x, y)-plane, F consists of the following types of symmetry.

- Translations in the x direction
- Rotations by π about points on the x-axis.
- Reflections in lines perpendicular to the x-axis.
- Reflections in the x-axis.
- Glides along the x axis.

4.1.3 Definition A *frieze group* is a discrete subgroup of F containing a translation.

4.1.4 Theorem There are seven different types of frieze pattern. They correspond to the seven infinite families of subgroups of $O(3)$ that fix an axis.

The argument here is just that if we take N periods of a frieze pattern we can wrap that around the equator of a sphere and the frieze symmetries become a subgroup of $O(3)$ fixing the axis joining the poles.

4.1.5 Notation Our signature for a frieze group is obtained from the corresponding subgroup of $O(3)$ by replacing instances of N by an infinity symbol of the same colour. The blue and red infinity symbols have a cost of $\$1$ and $\$0.5$ respectively, in which case all frieze patterns cost $\$2$ (which seems reasonable given that they are groups of symmetries of the plane.)

4.1.6 The frieze groups

Vert Ref Lines?	Additions	Signature	Footprints	LaTeX example
No	Miracle	$\infty\times$	walk	pbpbpbpbpbp
No	None	$\infty\infty$	hop	LLLLLLLLLL
No	Horiz. Refl.	$\infty*$	jump	EEEEEEEEEE
No	Rots.	22∞	dizzy hop	NNNNNNNNNN
Yes	None	$*\infty\infty$	sidle	MMMMMMMMMM
Yes	Horiz. Refl.	$*22\infty$	dizzy jump	XXXXXXXXXX
Yes	Rots.	$2*\infty$	dizzy sidle	MWMWMWM

OK – I cheated with the last one and used rotated ‘M’s for the ‘W’s because real ‘w’s have slanting sides in all the fonts I could access.

For the “footprints” classification, see the image on Learn or search online.

4.1.7 Exercise (NL) For each frieze group, define a periodic “function” $\mathbb{R} \longrightarrow [-1,1]$ whose graph has that frieze symmetry. Don’t be frightened of discontinuous or piecewise-defined examples. The inverted commas are because in some cases your function may need to be of the form $y^2 = ??$ in order to accommodate the symmetries in the group.

How does this relate to the usual analysis of “odd” and “even” periodic functions? You might want to allow the idea of functions being odd or even about points other than just the origin.

4.1.8 Exercise (CS) If you were a wallpaper manufacturer wanting to produce rolls of wallpaper that could be “hung” to produce a given wallpaper pattern, how do you go about this? You need to find a way of cutting the pattern into horizontal strips of fixed width so that all the strips have the same repeating pattern (which would in fact then have the symmetry of a frieze pattern).

4.1.9 Exercise (NL) Show that every element of $O(3)$ that fixes the z-axis is of the form

$$\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \text{where } A \in O(2).$$

Four possible types of matrix arise according to the plus/minus sign and whether or not $A \in SO(2)$. Identify how these correspond with the different frieze transformations listed in 4.1.2.

The identification is the one that takes a number of periods of a frieze pattern and wraps it round the equator of the sphere.

4.2 Archimedean and Catalan solids

4.2.1 Introduction We can talk about vertex transitive polyhedra just as we can for tilings. The five platonic solids are v-transitive because they are flag transitive. The trick of truncation can be used on platonic solids also: imagine sawing off the corners of a cube with planes perpendicular to lines joining opposite vertices. If you do this just right the cut faces will be equilateral triangles and the original square faces will be reduced to regular octagons.

4.2.2 The Wythoff construction The Wythoff construction allows us to construct all the v-transitive convex polyhedra with symmetry group $*2pq$. It works precisely as the plane version does for tilings, although one has to imagine the fundamental triangle as being a spherical triangle on the sphere; it is a fundamental domain for the group. Note from Table 4.1 how the naming completely parallels the tiling case.

4.2.3 Archimedean solids The v-transitive convex polyhedra comprise the following.

- five platonic solids (which are flag-regular)
- Thirteen *Archimedean solids* made up of
  - Five with symmetry $*532$ (from the Wythoff construction)
  - Five with symmetry $*432$ (from the Wythoff construction)
  - The truncated tetrahedron (symmetry group $*332$).
  - The snub cube and snub dodecahedron
- *Prisms* (face code $(4)(4)(N)$ for $N \geq 3$) and *Antiprisms* (face code $(3)(3)(3)(N)$ for $N \geq 3$)

Conway's "The symmetries of things" gives a (very) generalised Wythoff construction that, among other things, proves that this list is complete.

4.2.4 Example How can one calculate how many faces of different types a Wythoffian Archimedean solid has? Take the rhomboocosidodecahedron for example. The group has size 120 and so the sphere is tiled by 120 Wyhoff triangles. The vertex is on an edge of the fundamental domain and so counts for two fundamental domains. Thus it has $V = 60$. From the face code, each vertex is on a single triangle but triangles have three vertices. So overall there are $60/3 = 20$ triangles. Similarly there are $60/5 = 12$ pentagons and $2(60/4) = 30$ squares. Thus $F = 62$. Euler's formula then tells us that there are 120 edges. (We can double check that last figure e.g. by noticing that the Wythoff triangle contains two half-edges.)

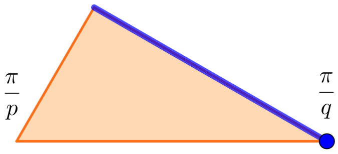
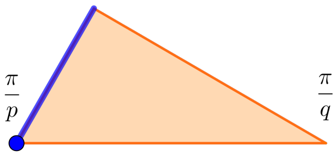
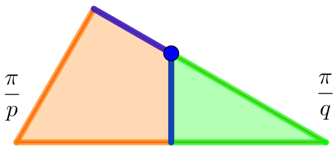
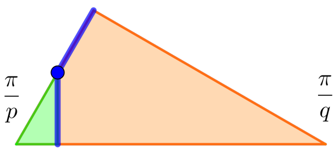
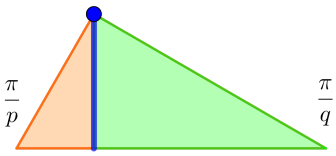
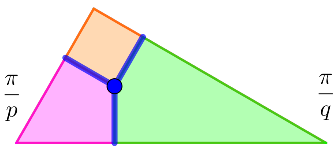
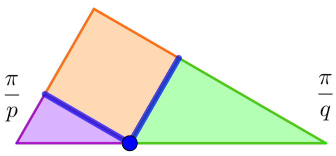
Construction	$*235$	$*234$
	3^5 Icosahedron	3^4 Octahedron
	5^3 Dodecahedron	4^3 Cube
	$(6)^2 5$ Truncated icosahedron	$(6)^2 4$ Truncated octahedron
	$(10)^2 3$ Truncated dodecahedron	$(8)^2 3$ Truncated cube
	$(3)(5)(3)(5)$ Icosidodecahedron	$(3)(4)(3)(4)$ Cuboctahedron
	$(4)(6)(10)$ Truncated icosidodecahedron	$(4)(6)(8)$ Truncated cuboctahedron
	$(4)(3)(4)(5)$ Rhombicosidodecahedron	$(4)(3)(4)(4)$ Rhombicuboctahedron

Table 4.1: The Wythoff construction for $*235$ and $*235$

4.2.5 Catalan solids Just as for tilings, one can take the dual of the Archimedean solids. These *Catalan solids* have the same symmetry group as the solid from which they are derived, and the symmetry group acts transitively on the faces. They have two or three different types of vertex (corresponding to the different types of face of the corresponding Archimedean solid) but each vertex is regular in the sense that points unit distance along the emerging edges form a regular polygon.

4.2.6 Exercise (NS) How many faces of what sort does a truncated icosidodecahedron have? And how many vertices and edges?

4.2.7 Exercise (NL) Investigate the Wythoff construction for $*332$. Of the seven possibilities, you should discover that the tetrahedron and truncated tetrahedron appear twice owing to symmetry. The remaining three lead to “coloured versions” of Archimedean solids with a larger symmetry group where $*332$ is the symmetry group of the coloured object which still acts transitively on its vertices. Identify the three coloured objects.

Conway refers to these as “relative Archimedean solids”.

4.2.8 Exercise (NM) Which cases of the Wythoff construction results in polyhedra that are also edge-transitive? Hence identify the edge-transitive Archimedian solids.

What characterises the Catalan solid dual to an edge-regular Archimedean solid?

4.2.9 Exercise (NM) For each case of the Wythoff construction, draw on the Wythoff triangle the edges of the faces of the corresponding Catalan solid.

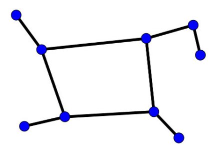
Euler's formula and topology of surfaces

5.0.1 Introduction In this chapter we will be considering surfaces, which is necessary background to understanding orbifolds and where the "magic theorems" come from.

We will be doing *topology* rather than *geometry*. In topology we study features of things which are unchanged by a continuous deformation and so to a topologist the top half of a sphere (including the equator), the closed unit disc in the plane, a (closed) square in the plane and half of the closed square rolled in to a cone are all the same thing. To a geometer, the first is a (positively) curved surface whereas the others are flat and the last two feature singular points such as corners on a boundary or a "cone point".

So for this Chapter, a disc is the same as a triangle and a football the same as a rugby ball.

5.1 A quick guide to Euler characteristics



5.1.1 Definition By a *map* on a surface we mean a connected graph such that the regions into which the surface is divided are connected and simply connected. (The final condition means that any loop in a region can be continuously shrunk down to a point.)

5.1.2 Euler's theorem Euler's famous theorem says that for a map on a sphere that divides it into F regions and has V vertices and E edges we have

$$V - E + F = 2.$$

The "2" on the right-hand side is the *Euler characteristic of the sphere* and we will denote Euler characteristics generally by the Greek letter χ ("chi").

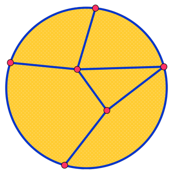
This is proved in Year 1 PPS - see Liebeck's book if you have forgotten. We will also sketch a proof below.

5.1.3 Example Imagine the little map (above-right) drawn on a sphere. We have $V = 9$, $E = 9$ and $F = 2$, where the two regions are the inside of the quadrilateral and the rest of the sphere outside of it.

It is convenient to allow "dangling" edges and vertices as in this picture, which are not part of a border of any region. Observe that if one picks the dangling "cherries", including their stalk, then one removes exactly one vertex and one edge each time leaving $V - E + F$ unchanged. So dangling pieces can always be removed.

5.1.4 Exercise (BM) Find a map on a sphere which violates the condition of its regions being simply connected and such that $V - E + F \neq 2$.

5.1.5 Definition The Euler characteristic for a map on a surface with boundary is defined in the same way as for one without, except there is an extra requirement that one has edges joined by vertices running round every boundary.



5.1.6 Exercise (BS) Compute χ using the map in the closed disc on the right. (Note that here $F = 5$, the disk including boundary is our whole surface, there is no "region outside".)

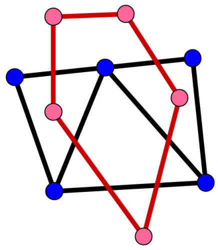
5.1.7 Theorem For the closed disk, $\chi = 1$.

Proof. Imagine the disc as drawn on the surface of the sphere. Then calculating χ for the sphere is the same as calculating it for the disc, except that one counts one extra region (the exterior of the disc). $\square$

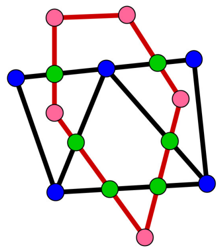
5.1.8 Vague Definition For us, a *surface* is a 2-dimensional object, closed (so that if, like the disk, it has a boundary, then the boundary is included) and "compact" (which in the case of surfaces in $\mathbb{R}^n$ means that they do not go off to infinity).

5.1.9 Theorem Given a compact surface, the Euler characteristic $\chi = V - E + F$ is independent of the choice of map used to calculate it.

A (non-examinable) sketch proof for the case without boundary goes as follows. Suppose you have two maps M and N on a surface such as the black and red ones to the right. Deform them a little if necessary so that the vertices of each do not lie on edges or vertices of the other, and so that where edges of one meet edges of the other they cross transversely (meaning that the cross one another cleanly rather than, say, meeting tangentially and veering off again).



Now, wherever an edge of G meets an edge of G' add that point as a vertex to both graphs. These extra vertices are the green ones in the lower picture. Note that adding such a vertex divides an existing edge of G in two and so leaves χ unchanged. And the same applies for χ'. Now consider the combined graph H (all 16 vertices 10 regions and 24 edges in the lower picture). We claim that the Euler characteristic of H is equal to that of G and equal to that of G' and so $\chi = \chi'$.



To see this, start with the black/blue graph G with the added green vertices. Choose a region in G and add in all the red/pink edges and vertices within it. Since a disk $\chi = 1$, this does not change χ for G. Repeat until you arrive at H.

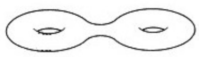
5.1.10 Exercise (BS) Recall that in Chapter 1 we constructed a torus by taking a rectangle and zipping the edges together. Think of the rectangle as being a single "face" for our map and think of the edges of the rectangle as being edges of our map and the corners being vertices. As one identifies the edges to obtain a torus, some of the edges and corners get identified so that the resulting map on the torus has fewer edges and vertices than we started with.

Decide how many vertices ad edges the map on the torus has and deduce that for a torus $\chi = 0$.

5.1.11 Exercise (NS) For a 1-dimensional object, a map divides the object into intervals (which are edges) and the Euler characteristic is just $V - E$. Draw a graph on the circle and deduce that $\chi = 0$ and draw a graph on a closed interval and deduce that $\chi = 1$.

5.1.12 Exercise (NM) Compute χ for (the surface of) a cylinder (including its two circular boundaries but without caps on the end) and for the Möbius band including its single circular boundary.

5.1.13 Definition A *connected sum* of two surfaces Σ_1, Σ_2 is the surface that results if one punches a hole in both surfaces and connects the boundaries of the resulting holes together with a tube. It is denoted $\Sigma_1\#\Sigma_2$. A connected sum of two toruses is shown on the left. Taking a connected sum with a torus is called "adding a handle" to a surface. The picture on the right is a "torus with a handle" or a "sphere with two handles".



5.1.14 Theorem The Euler characteristic of a connected sum is given by

$$\chi_{\Sigma_1\#\Sigma_2} = \chi_{\Sigma_1} + \chi_{\Sigma_2} - 2.$$

In particular, adding a handle to a surface reduces its Euler characteristic by two.

5.1.15 Exercise (NM) Prove the above theorem. (Try cutting a triangular face out of each surface and connecting them together with a triangular prism.)

5.2 The projective plane

5.2.1 The group $N\times$ has a glide g which is reflection in the equator followed by a rotation about the poles of π/N. Thus g^2 is a rotation about the poles by $2\pi/N$ and so generally there is a gyration point at the poles. (There is only one type of gyration because g swaps the North and South poles over.)

In the particular case $N = 1$ there is no gyration and $1\times = \times$ is the 2-element subgroup of $O(3)$ containing just $\pm\mathbb{I}$.

5.2.2 The orbifold of $\times$ can be thought of in several different ways. Firstly, a topologist would say it is just "the sphere with opposite points (i.e. $\mathbf{x}$ and $-\mathbf{x}$) identified".

Alternatively we could try and find a fundamental domain for $\times$. If we take the open Northern hemisphere, then each point is in a different orbit of the group. If we include the equator however, then opposite points on it have to be identified: you have to imagine that if you walk South over the equator you magically reappear at the diametrically opposite point heading North.

If we are just trying to get a grip on the topology, we could flatten the closed Northern hemisphere out into a disk, keeping the rule about walking off the edge and appearing at the opposite point.

Finally, if you are an algebraic geometer you would identify the orbifold with the set of all 1-dimensional subspaces of $\mathbb{R}^3$ and know it as *the projective plane* $\mathbb{R}P_2$.

5.2.3 Imagine that before identifying the opposite points on the sphere you cut two identically sized discs centred on the poles out of the sphere leaving just a strip around the equator. The identification will turn that strip into a Möbius band, as you can probably convince yourself (see exercise below).

Now the identification identifies the two discs you cut out, and so the Möbius band is the result of punching an open disc out of $\mathbb{R}P_2$ leaving a single boundary curve. Reversing this, we see that you can also construct $\mathbb{R}P_2$ by taking a Möbius band and a disk and "zipping together" their boundary curves. Of course, you cannot do this in $\mathbb{R}^3$ without some self-intersection unpleasantness, but that can be avoided by dodging into the fourth dimension where necessary.

5.2.4 Exercise (NS) Cut out a long thin strip of paper and imagine that it is wrapped around the equator of a sphere. Observe that "identifying opposite points" on the sphere can be realised by wrapping the strip twice round a Möbius band.

5.2.5 There is a canonical map p from the sphere S^2 to $\mathbb{R}P_2$ which takes the point on the sphere to the point in $\mathbb{R}P_2$ that it defines. Each point in $\mathbb{R}P_2$ has two inverse images in S^2. Notice that this does *not* mean that S^2 is somehow to separate copies of $\mathbb{R}P_2$: it cannot be since S^2 is connected. Rather it means that S^2 is what is called a "bundle" over $\mathbb{R}P_2$. The situation is rather like that illustrated in the picture in 0.2.17 except that there there are an infinite number of inverse images whereas here if you go round twice, you are back where you started.

We discuss this example here for three reasons. Firstly it is additional motivation toward the fact that we need to understand surfaces better in order to understand orbifolds. Secondly $\mathbb{R}P_2$ will be an important to us in that discussion. Thirdly, it's fun.

5.2.6 Theorem The Euler characteristic of the projective plane $\mathbb{R}P_2$ is $\chi = 1$.

Proof. Consider a map on the sphere (such as that corresponding to a cube) which has the property that it is unchanged under $\mathbf{x} \mapsto -\mathbf{x}$. This defines a map on $\mathbb{R}P_2$ with half as many regions, edges and vertices. $\square$

5.2.7 Making surfaces by identifying edges of a rectangle We observed earlier how to identify the edges of a square (or rectangle) in pairs to make a torus.

If we take a rectangle and just identify one pair of opposite sides (in the same direction) we obtain a cylinder (without end caps). If we identify them in opposite directions we get a Möbius band.

The projective plane comes from identifying both pairs of opposite sides, taking opposite orientation in each case, and there is one final possibility: identify one pair of opposite edges in corresponding directions and one pair reverse: the result is a Klein bottle.

5.3 The classification of surfaces

5.3.1 Aim Our aim is to classify all possible compact surfaces (allowing boundaries). Our strategy will be to construct surfaces by performing operations on a sphere. We will then argue that every surface is equivalent to one of those we have constructed.

5.3.2 Definition A surface is *orientable* if it is possible to continuously choose over the whole of it which direction of rotation in the surface we regard as "positive".

A sphere is orientable: choose for example, rotations to be positive if they are anticlockwise as viewed from outside the sphere. A Möbius band is not orientable: a clock face transported once round the band returns to its starting point as a mirror image. (You may say that that the clock face has come back on the "other side" of the Möbius band. But when we think of surfaces we are imaging the surface to be the whole universe: it does not have "two sides" any more than our three-dimensional world has two sides.)

For a surface in $\mathbb{R}^3$ that does not cross itself, a continuous choice of unit normal over the surface is possible if and only if the surface is orientable.

5.3.3 Equivalent condition If a surface contains a Möbius band, then it clearly cannot be orientable. The converse is true also: if a surface is not orientable there is a path in the surface such that if you carry a clock along it, when you return to the start you have a mirror image clock. A thin band around this path will be a Möbius band. (Or it will if the path does not cross itself. We will no worry about that issue here.)

5.3.4 We now consider four operations that we can use to construct a surface from a sphere.

Punching holes

5.3.5 Definition By "punching a hole" in a surface we mean removing something that is topologically an open disc, leaving the surface with a new boundary curve.

5.3.6 Theorem Punching out a hole reduces χ by 1.

To see this, simply choose a map where the hole to be punched out is the interior of a region.

5.3.7 Notation Our notation for surfaces uses symbols to denote operations performed on a sphere to obtain the surface. We denote the sphere with k holes punched out by $*^k$. It has $\chi = 2 - k$. Note that for topological purposes it does not matter what the configuration or shape of the holes is. Provided they do not overlap, they can be resized and moved about without changing the surface topologically.

5.3.8 Exercise (NS) A closed disc is $*$ and a cylinder is $**$ or $*^2$. Check that this gives the same Euler characteristic as you found in 5.1.12.

Adding crosscaps

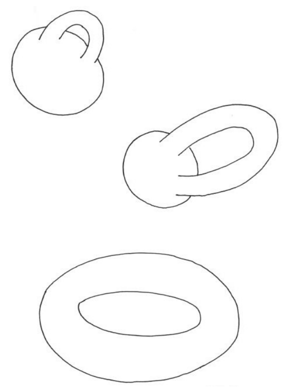
5.3.9 Definition By *adding a crosscap* to a surface we mean the process of cutting out a disc and then "filling the hole" by zipping the boundary of a Möbius band to it. We observed in §5.2 that cutting a disc out of a projective plane leaves a Möbius band, and so one could describe "adding a crosscap" as being taking the connected sum with a projective plane. We will write $\times$ for a crosscap. Thus $\times^m$ denotes the sphere with m crosscaps.

5.3.10 Theorem Adding a crosscap reduces χ by 1. The resulting surface is not orientable, irrespective of whether the original surface was.

Proof. The Euler characteristic follows from the formula for a connected sum and our knowledge that $\chi = 1$ for a projective plane. □

5.3.11 Exercise (NM) Explain why a Möbius band is the surface $*\times$. (The notation means a sphere with a hole punched out and also a crosscap added.)

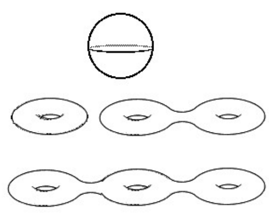
Adding handles (standard and cross)

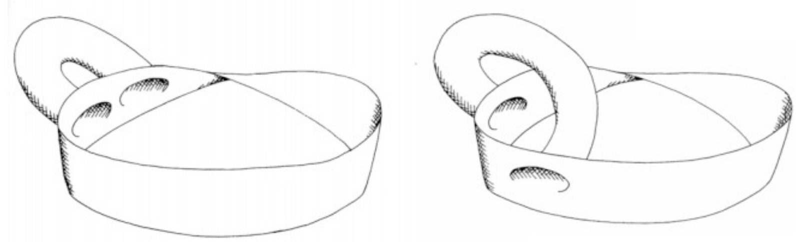


On the right at the top, a handle has been added to a sphere. Expanding the handle and shrinking the sphere it becomes clear we have a torus.

Adding more handles to a sphere, we obtain the sequence of surfaces in the bottom picture on the right. We will write $\mathbf{o}$ for a handle and so describe those surfaces as $\mathbf{o}^g$ where $g = 0, 1, 2, 3$ in the picture but can be any natural number. Since adding a handle decreases χ by 2, the surface we are calling $\mathbf{o}^g$ has $\chi = 2 - 2g$.

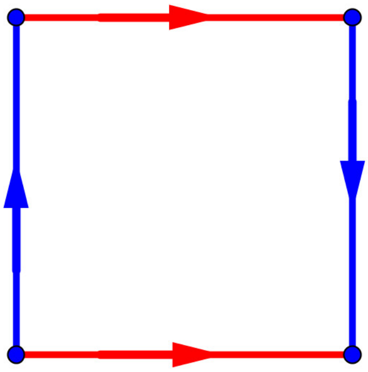
Adding handles as we have on the right maintains orientability. The surfaces pictured have an inside and an outside and the outward facing normal can be used to orient them.

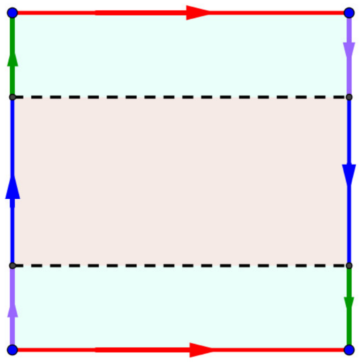




5.3.12 The pictures above show the addition of a handle (left) and "cross handle" (right) to a Möbius band. In the cross handle the other end of the connecting tube is connected to the "other side" of the surface.

5.3.13 The Klein bottle If we attach a cross handle to a sphere the result is a "Klein bottle": the example on the left was unfortunately manufactured at a 3-dimensional factory and so the handle has to pass through the sphere. In four dimensions one could avoid that. In general, adding a cross handle to a surface is the same as taking a connected sum with a Klein bottle.





5.3.14 Exercise (NS) We calculated $\chi = 0$ for the Klein bottle earlier because our argument about attaching handles a surface did not depend on whether we were talking about a proper handle or a cross handle. Check that by checking how many vertices and edges there are after the identification in the left-hand picture above.

5.3.15 Proposition A Klein bottle can be cut by a single closed curve into two Möbius bands.
Proof. To prove this, consider the picture on the right above. The two dotted lines together constitute a single closed curve on the Klein bottle that divides it into two and does not intersect itself.

Before identifying, cut along the dotted lines. The central beige area can now be identified by the blue arrows to form a Möbius band. Now identify the two pale blue strips along the red edge. Identifying green with green and lilac with lilac creates another Möbius band. □

Drawing these things together we have the following two results.

5.3.16 Proposition A Klein bottle is the connected sum of two projective planes. (So a Klein bottle is the surface $\times\times$.)

5.3.17 Proposition Adding a cross handle to a surface is the same as adding two crosscaps.

5.3.18 Exercise (NM) Explain how the two propositions just above follow from the previous discussions.

Crosscaps and handles

5.3.19 Proposition In the presence of a crosscap, a cross handle is the same thing as a handle. Thus two crosscaps are the same thing as a handle, provided there is another crosscap somewhere.

Proof. Look at the "handle on a Möbius band" picture above. If you fix one point of attachment and then take the other point of attachment of the handle and move it all the way round the band until it becomes close to the fixed one again, the handle becomes a cross handle! But the spare crosscap contains a Möbius band, so you can walk around that. □

5.3.20 Definition We say a surface is *tidy* if it is equivalent to one obtained by punching out discs, and adding crosscaps, handles and cross handles to a sphere.

5.3.21 Theorem Every tidy surface is equivalent to one that is of the form

$$*^a\mathbf{o}^b \text{ (where } a, b \geq 0) \quad \text{or} \quad *^a\times^c \text{ (where } a \geq 0, c \geq 1).$$

The Euler characteristics are $2 - a - 2b$ and $2 - a - c$ respectively. All these forms are distinct surfaces except for the obvious case where $b = c = 0$.

Proof. The first part follows from the above. The uniqueness follows because the number of boundary curves is equal to a. Then the Euler characteristic determines the surface except where we need to decide whether we have n handles or $2n$ crosscaps. But the former is orientable and the latter is not. □

5.3.22 Exercise (NL) Expand the above proof by adding some details.

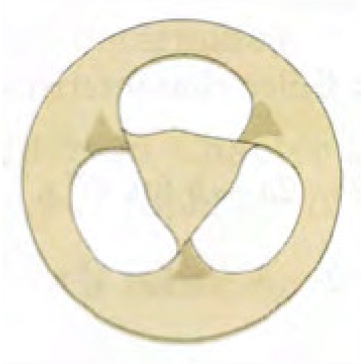
5.3.23 The classification of surfaces This fundamental result in geometry and topology states that the previous theorem is in fact a classification of all (connected, compact) surfaces. To see that, one needs to show that every surface is equivalent to a tidy one.

The key definition is of a "trianglarisable surface" meaning one can cut it into "triangles" that "zip together" along edges. it is non-trivial, but one can show that any "reasonable" surface is triangularisable.

The idea then is to imagine your surface arriving as a self-assembly kit from IKEA. You tip all the triangles with their zips onto the floor. Pick up a triangle. In your hand you now have a tidy surface which is a sphere with one punched hole. Now find another triangle that fastens to that one and add it. You still have a tidy surface – in fact its topology has not changed. Now keep adding triangles: the key is to show that each addition you make keeps the surface tidy. (After you finish you probably find some spare pieces on the floor of course.)

This argument is Conway's ZIP ("zero-irrelevancy proof"). You can find a more complete account of it on Learn under "Resources".

5.3.24 Exercise (NL) Consider the surface on the right (taken from the Conway book). What is its signature? To calculate this you need to count how many boundary components there are, find its Euler characteristic (perhaps by making a paper model) and check if it is orientable.



5.3.25 Surfaces and geometric groups Of course, the point of studying surfaces is to understand orbifolds and you will not be surprised to discover that all the wallpaper and spherical groups we have studied that have no numbers in their signature, the orbifold is the surface given by the same notation. The orbifold is made out of Euclidean space (flat) material for the wallpaper groups and out of "surface of a sphere" (constant positive curvature) material for the spherical groups.

For miracles, if one has a curve joining a point to its mirror image in the pattern which does not cross a mirror, then thickening that to a little strip gives us a Möbius band in the orbifold and so a crosscap.

5.3.26 Exercise (CM) For the four wallpaper groups and two spherical groups with no numbers in their signature, understand how the orbifold can be considered to be the corresponding surface.

Orbifolds and the magic theorems

6.1 Orbifolds

6.1.1 Introduction Our definition of orbifold will be a bit “19th century” in the sense that it will rely on intuition to some extent. We start with the plane or sphere with a pattern having wallpaper or spherical group symmetry. From that construct the orbifold: an object such that an ant walking around on the orbifold cannot tell they are not in fact walking over the original pattern.

There is a map natural from the plane or sphere to the fundamental domain that takes each point to the point in the fundamental domain to which it relates. This map is not always continuous and replacing the fundamental domain by the orbifold rectifies that. We saw examples of this originally in Chapter 0.

6.1.2 Back to geometry In the last chapter we studied topology and two spaces were considered identical if one could be deformed into another. Now and henceforth, we return to geometry where distances and angles matter.

We imagine the orbifolds for wallpaper groups to be made out of paper, and we can bend the paper but not crease or tear it. We can bend a sheet of paper in space into a cylinder of cone, but not into the surface of a sphere. For those who have studied some differential geometry, the point is that bending is an isometry.

We saw in §1.3.15 that the orbifold for *o* is a torus. There is a problem because in fact one cannot bend a piece of paper into a torus in $\mathbb{R}^3$ without creasing. One solution to this is to use the fourth dimension. Alternatively, just imaging smoothly zipping together opposite sides of a rectangle: if one knows about “manifolds” this is a perfectly good way to define an “abstract surface”.

Similarly we must imagine our spherical group orbifolds as zipped together from pieces of unit sphere. This is harder to imagine because you cannot bend a piece of the unit sphere the way you can bend paper.

6.1.3 Construction of the orbifold Mirror lines are always on the boundary of a fundamental domain and in the orbifold the only change is that we take the boundary to be reflecting. Gyrations also have to be on the boundary, and we resolve them by rolling the orbifold into a cone about the gyration, thus creating a singular “cone point” in the interior of the orbifold:

imagine the point of an idealised ice-cream cone.

Miracles manifest themselves in the need to identify two lengths of boundary of the fundamental domain with a "twist", thus creating an embedded Möbius band in the orbifold. Finally, and this appears for us only for the group o, a wandering causes identifications leading to a torus.

6.1.4 Signatures and orbifolds The orbifold notation for a wallpaper or spherical group also defines its orbifold. We already have discussed how signatures without numbers define topological surfaces. The additional information is that each blue number denotes a cone point of that order in the orbifold, and red numbers following a red star denote that the boundary curve from the star (which in fact always follows a straight line or great circle in the orbifold) has "corner points". A red k corresponds to an internal angle of π/k.

6.1.5 Decoding the signature Consider, for example, the wallpaper group $3{*}3$. Stripping away the numbers gives us the topology of the orbifold, in this case we have just $*$ or a disc with a boundary. The 3 tells us that there is a cone point of order 3, so geometrically we have a cone rather than a disc. The 3 tells us that the reflecting boundary has a corner point with internal angle $\pi/3$.

Or consider the spherical group 532. Stripping away the numbers leaves us with nothing, which is the signature of a sphere. The sphere has three cone points, one each of orders 2,3 and 5.

It might be an interesting exercise to take a fundamental domain for (say) 333 and try and understand how that becomes an orbifold that is topologically a sphere. I have never found an entirely convincing argument.

6.2 Magic Theorems and Orbifold Euler characteristics

Our aim now is to understand "where the magic theorems come from". We will start with the case of spherical symmetry.

In §5.2 we discussed the spherical symmetry group $\times$ which is the 2-element group $\{\pm\mathbb{I}\}$ and its orbifold, the projective plane. There is a 2-to-1 map that takes the sphere S^2 to $\mathbb{R}P_2$. We argued in 5.2.6 that if we take a map on the sphere that is invariant under $\times$ such as that generated by a cube, then one gets a corresponding map on $\mathbb{R}P_2$ with exactly half the number of vertices, edges and faces. It is exactly one half the number because the group $\times$ has two elements.

Therefore we end up with

$$\chi_{\mathbb{R}P_2} = \frac{\chi_{S^2}}{2} = 1, \qquad \text{where the "2" is the size of the group } \times.$$

Under sufficiently "nice" n-to-1 maps, the Euler characteristic always behaves in this fashion, although proving that requires some better definitions and some work.

6.2.1 The problem As we saw even in the prologue, the map from a space with group acting to the orbifold is often not so decent. Consider the example we saw of the islamic pattern on a disk with D_6 symmetry. The map from the disc to its orbifold is 12-to-1 at most points but only 6-to-1 on the reflecting boundaries and 1-1 on the corner point.

And this leads to bad outcomes: the orbifold for the 120-element group $*532$ is a single spherical triangle which is topologically a disk and so has Euler characteristic 1. That's a long way from $2/120$.

6.2.2 The solution As all children know, if the game isn't working for you, then you change the rules. But you often have to give as well as take in this process.

- It might get us somewhere if the "Euler characteristic gets divided by the size of the group" game worked.
- But it might be enough if it just worked for these rather specific sorts of groups acting on the plane and sphere. In other words, we are imagining our orbifolds as being made out of flat space, pieces of the unit sphere or, in a later chapter the hyperbolic plane.

So let's see if we can invent a "better" Euler characteristic for this situation.

6.2.3 Exercise (NM) Consider the spherical symmetry group $*222$. How many elements does it have? What are they? Also, how many elements does $3{*}2$ have?

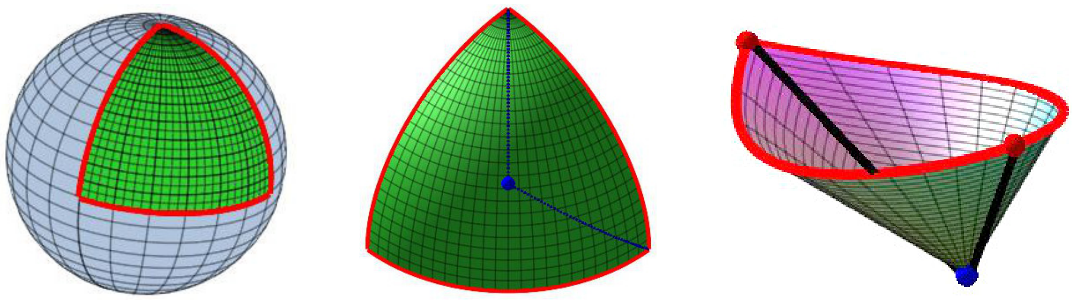


Figure 6.1: The orbifold for $*222$ (left); with a gyration point at the centre that exists in $3{*}2$ (middle). Fundamental region wrapped into cone to make the orbifold for $3{*}2$ (right).

6.2.4 Investigation Let's look closely at the relationship between symmetric maps on the sphere and maps on the corresponding orbifold. We will focus on the example of $3{*}2$.

Starting with $*222$, the orbifold for that is one quarter of the Northern hemisphere, a spherical triangle with three right angles and reflecting boundaries. (See the left-hand picture in Figure 6.1.)

In $3{*}2$ there is an additional 3-fold gyration at the centre of the spherical triangle. A fundamental region is now one third of the triangle, as delineated by the dotted blue lines in the centre picture of Figure 6.1.

As usual, we should identify the two blue dotted lines so that the orbifold is now a sort of cone as shown in the right-hand picture. *But that picture is misleading* because all the surface apart from the "conical singularity" at the gyration point should be curved like a piece of a sphere. (But that is hard to make a picture of, specially since the object cannot be embedded in $\mathbb{R}^3$.)

I have drawn a map on the orbifold too. There is a red vertex at the corner/kaleidoscope point and I have added one also half way round the boundary. I am taking the two pieces of reflecting boundary joining the red vertices as edges. I have a blue vertex at the gyration point and two black edges joining the blue vertex to each of the red ones.

6.2.5 Consider the map on the whole sphere generated by our orbifold map. In other words, think of the map as defining a symmetric pattern on the whole sphere with $3{*}2$ symmetry. You might like to try and sketch a bit of this to get the idea.

1. The orbifold map has two faces. The fundamental region that we made the orbifold from is duplicated #$3{*}2 = 24$ times on the sphere. Therefore the resulting map has $F = 48$.

2. Each black edge on the orbifold also leads to 24 on the whole sphere and so that contributes 48 edges on the whole sphere. But each red edge generates only 12 edges on the whole sphere because the reflection in that edge fixes it. (Or, to put it another way, on the sphere, the resulting edges are each shared by two copies of the fundamental region.) Overall therefore we have $E = 72$.

3. The red vertex in the foreground similarly leads to only 12 vertices on the whole sphere and the red vertex at the corner point only 6. The gyration point leads to only 8 vertices on the whole sphere because there is a 3-element cyclic subgroup of rotations that fix it. Therefore $V = 26$.

4. Doing the sum, $\chi = 2 = 26 - 72 + 48 = 2$, as it must be.

6.2.6 The orbifold Now, thinking just about the orbifold, we will change how we count things so as to reflect the calculations above. Count the blue vertex as $1/3$ of a vertex; the red corner vertex as $1/4$ of a vertex; the other red vertex as $1/2$ a vertex; the red edges as half an edge each and count the black edges and the faces normally. One gets

$$V = 13/12, \quad E = 3, \quad F = 2 \quad \text{and } V - E + F = \frac{1}{12} = \frac{2}{24} = \frac{2}{\#3{*}2}.$$

This calculation is the motivation for the following definition.

6.2.7 Definition Let X be an orbifold for a (spherical or wallpaper) symmetry group. Consider a map on X such that in addition to the normal requirements for computing Euler characteristics for a surface with boundary we have a vertex at each gyration point and a vertex at each corner (kaleidoscope) point on the boundary.

The *orbifold Euler characteristic* χ_o of X is defined by $\chi_o = V - E + F$ where we count vertices, edges and faces normally except that:

- Edges on a reflecting boundary count as $1/2$ an edge;

- Vertices at an order n kaleidoscope point (i.e. a corner point in a reflecting boundary) count as $1/(2n)$ of a vertex;

- Other vertices on the boundary count as $1/2$ of a vertex;

- Vertices at gyration points of order n count as $1/n$ of a vertex.

By construction, we have the following theorem.

6.2.8 Theorem Let G be a spherical symmetry group. Then for its orbifold we have

$$\chi_o = \frac{2}{\#G}.$$

And now the following theorem proves the Magic Theorem for the spherical case.

6.2.9 Theorem For a spherical symmetry group G,

$$\chi_o = 2 - \text{the cost in dollars of the signature.}$$

Proof. Consider building the orbifold from a sphere. Since a sphere has no boundaries or gyration points, its orbifold Euler characteristic is the same as its standard Euler characteristic: $\chi_o(S^2) = \chi(S^2) = 2$.

1. In building the orbifold, $*$ and $\times$ will, as normal cost one unit of Euler characteristic.

2. Draw a map on the orbifold, respecting the conditions required for it to compute χ_o. We start by imagining computing the standard Euler characteristic χ.

3. Now changing the weighting of all boundary edges and vertices (including any corner points) to $1/2$ has no effect on computing $V-E+F$ because each boundary component has exactly as many edges as vertices. So imagine that we have made this change. The edges and non-kaleidoscope vertices on the boundary are now counting as they do in the definition of χ_o.

4. The introduction of an "n" gyration point means that one vertex now counts for $1/n$ of a vertex, thus reducing $V - E + F$ by $(n-1)/n$.

5. The introduction of a corner (kaleidoscope) point of order n on the boundary changes a vertex that is currently counting as $1/2$ to one that counts as $1/(2n)$ thus reducing $V - E + F$ by $(n-1)/(2n)$.

6. Thus χ_o for the orbifold is exactly 2 minus the accumulated costs of the features in the signature.

□

6.2.10 The big view So there we have it: the "Magic Theorem" for the group G spherical case is just combining the fact that χ_o is by construction equal to $2/\#G$ with calculating what χ_o has to be if you consider constructing it feature by feature (as per the signature of G) from a sphere. Magic indeed!

The 17 Wallpaper groups

6.2.11 Introduction So what of the original case we studied of patterns in the plane? The plane itself is not compact and so does not have an Euler characteristic in the sense we have been discussing.

6.2.12 Theorem The orbifold of a wallpaper group has orbifold Euler characteristic zero.

Proof. What follows is a bit sketchy, but reasonably convincing, I believe. Draw a map on the orbifold (as you would in the the sphere case) and let V, E, F be the numbers that appear in calculating χ_o. Consider the corresponding map on the plane. Cut the map off so as to include only regions lying entirely inside a circle of radius $R > 0$.

Let N denote the number of complete fundamental regions within the cut-off. Then

$$V_R \approx NV, \quad E_R \approx NE, \quad F_R \approx NF$$

where the "R" subscripts indicate the vertex, edge and face counts for the whole map cut off at radius R.

The errors we are making in this approximation is caused by the pieces crossing over the circle of radius R, and so will be proportional to R. Thus for some c independent of R we have

$$(V_R - NV) - (E_R - NE) + (F_R - NF) < cR$$

and using $V_R - E_R + F_R = 1$ from Euler's theorem and observing that N grows as R^2, we have a contradiction unless we have $\chi_o = V - E + F = 0$. □

Notes on orbifolds

6.2.13 Orbifolds with no reflecting boundaries and gyrations In the plane case we have o (a torus) and $\times\times$, which are the two possible "flat compact worlds". In the sphere case we have the projective plane $\times$. The defining feature of these examples is that the action of the group is "free", meaning that every element except the identity moves every point, or equivalently that the stabilizer of every point is trivial. It's points having non-trivial stabilizers that gives rise to reflecting boundaries and cone points.

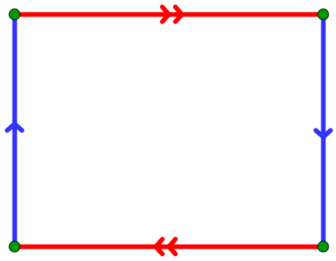


Figure 6.2: A projective plane or not? Are you a topologist or a geometer?

6.2.14 Topologists and the projective plane Here is something I only appreciated when preparing this course. Topologists will tell you that if you take a piece of paper and identify each pair of opposite edges with each other *with direction reversed* as in the figure then the result is a projective plane. You might think therefore that because you have made the projective plane out of paper (i.e. if you have studied differential geometry it has a flat $K = 0$ metric) that it should be the orbifold for a wallpaper group.

The reason it does not is that when you identify, the four corners get identified in pairs, and each of the two resulting points in the projective plane is surrounded by just two right-angles of paper. So what we really have, at least if you are a geometer is a projective plane with two order-2 “cone points’ points, which is the orbifold of the wallpaper pattern $22\times$.

If you identify to get the torus or Klein bottle, all four corners come together in a single point which is surrounded by four right angles of paper and so there is no angle deficit.

6.2.15 Exercise (NM) For the projective plane, take the map as in the figure and compute the Euler characteristic to prove that the topologists are (in their way) correct. Compute also the orbifold Euler characteristic, assuming that we have the two cone points.

Check also that if you change the indicated direction of just one edge so that you have a Klein bottle, you get the correct Euler characteristic for that also.

6.2.16 Kaleidoscopes Here the orbifolds are polygons in the plane case and spherical triangles and similar in the spherical case. This includes the interesting case of NN where the orbifold is a spherical 2-gon, with vertices at the North and South poles and two lines of longitude as edges. (The shape is also called a “lune”.)

6.2.17 Exercise () Sketch the orbifold for $2{*}22$ and define a map on it and use it to compute the orbifold Euler characteristic and check that it is as you would expect.

6.2.18 Exercise (NM) Relate the orbifold of $**$ (a cylinder) and of $*\times$ (a Möbius band) to fundamental regions in an example pattern.

6.2.19 Final thoughts Has what we have done “proved” that our classification of wallpaper groups and finite subgroups of $O(3)$ is correct and complete?

One issue is that of existence: it is clear that for a wallpaper or spherical symmetry group to exist there must be an orbifold with the correct orbifold Euler characteristic. But if we have an orbifold with a possible value of the orbifold Euler characteristic, does that mean that a group exists and is unique? As we saw, some possible signatures such as MN with $M \neq N$ do not result in spherical symmetry groups. But all those groups we have listed do exist because we can find patterns with those symmetries. Uniqueness (a careful discussion of which depends on having a good definition of ”unique”) one can in fact establish case by case.

I would probably argue that what we have done so far explains and makes sense of the classification, connects it with lots of important ideas and establishes a great notation. But it is probably not the sort of proof you would want to stake your reputation on if the final results were not well-established. Where are the gaps? What do you think?

6.3 Isohedral tilings and Heesch types

6.3.1 Isohedral tilings and Catalan things

6.3.1 The problem A topic of long-standing interest is tessellating the plane (or indeed the sphere) with identical (not necessarily regular polygon) tiles. More precisely, we will consider tilings where the symmetry group of the tiling is a wallpaper group that acts transitively on the tiles. We call such things “isohedral tilings”.

If the symmetry group includes reflections, then one will need two types of tiles, the types being mirror images of each other if you want in practice to cover your bathroom wall.

6.3.2 Catalan tilings/solids We are already familiar with one class of isohedral tilings of the plane or sphere, the Catalan tilings and Catalan solids. These are the duals of Archimedean (i.e. vertex regular but not regular) tilings and convex polyhedra. They are isohedral since the dual of a vertex transitive object is face transitive.

6.3.3 Wythoff construction for Catalan objects It is instructive to consider what the vertices and edges of the Catalan object look like in the Wythoff construction. The rule is simple:

- The vertices of the Catalan object are the vertices of the Wythoff triangle that are the centres of polygons if one constructs the Archimedean object.
- The edges of the Catalan object are the edges of the Wythoff triangle that meet an edge of the Archimedean object perpendicularly.

The Wythoff triangles (omitting the two of the seven cases that lead to honeycombs and regular polyhedra) can be seen in Table 6.1.

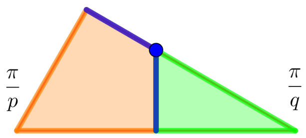
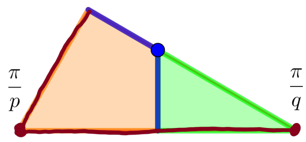
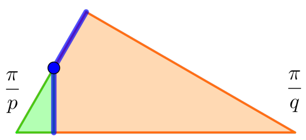
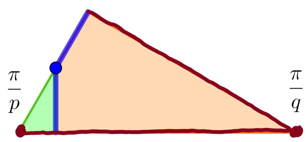
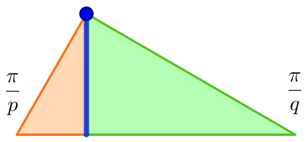
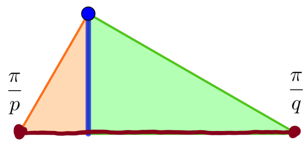
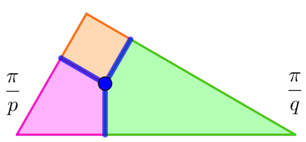
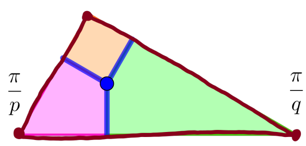
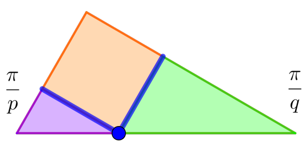
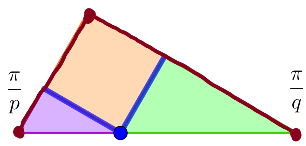
Archimedean	**Catalan**	**Notes**
		Catalan face is an icosoles triangle made of two Wythoff triangles.
		Catalan face is an icosoles triangle made of two Wythoff triangles.
		Archimedean and Catalan have only one type of edge. So these are edge-transitive. Catalan face is a rhombus (four Wythoff triangles).
		Catalan face is the Wythoff triangle.
		Catalan face is a kite (two Wythoff triangles).

Table 6.1: Wythoff for Catalan. The vertices and edges of the Catalan object are in deep red.

6.3.2 Heesch types

6.3.1 The problem Given a wallpaper pattern, classify the ways we can tile the plane with tiles (not necessarily polygons) with each tile being a fundamental domain?

Clearly for kaleidoscopic groups, there is only one option as all mirror lines must be tile boundaries and so there is one and only one such tiling. But for other groups there are more possibilities.

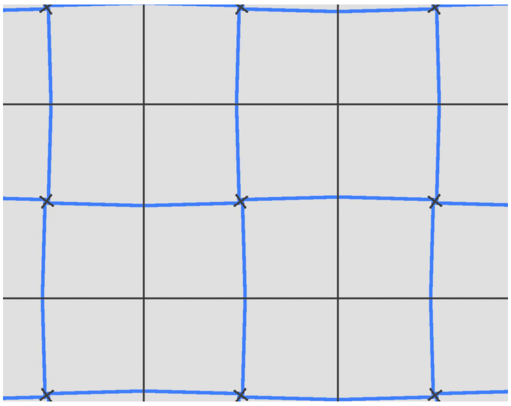
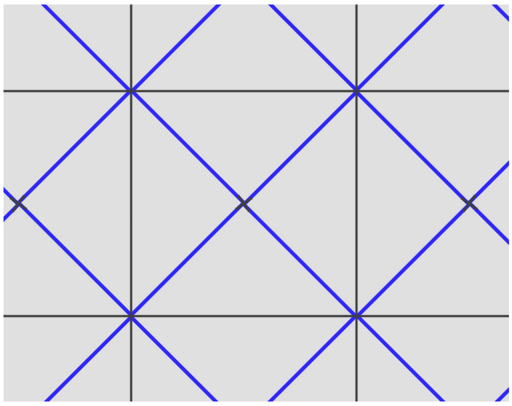
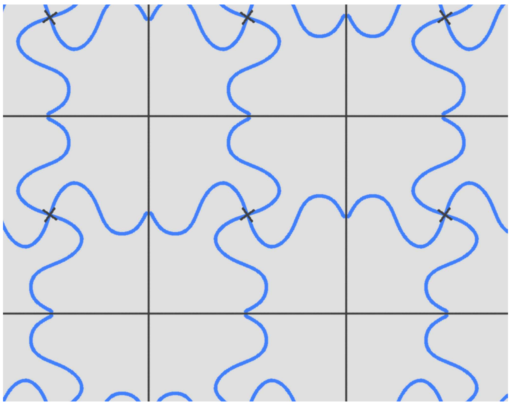
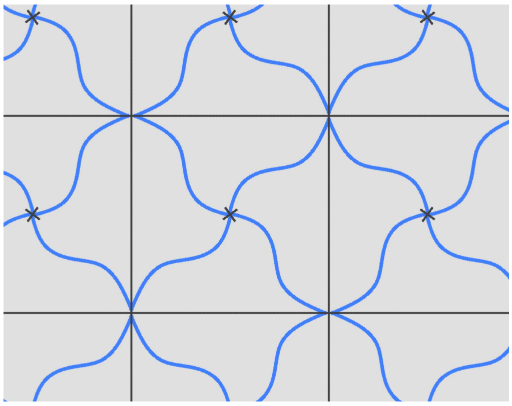
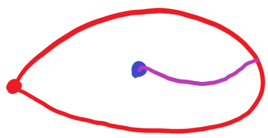
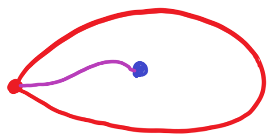
Heesch type 1	**Heesch type 2**
	
	
	
A walk around a tile: Gyration - edge - point where 4 tiles meet - mirror - kaleidoscope point - mirror - point where 4 tiles meet - edge - Gyration	**A walk around a tile:** Gyration - edge - kaleidoscope point - mirror - kaleidoscope point - edge - gyration

Table 6.2: Heesch types for $4{*}2$. (The tile boundaries are the black mirror lines and the blue pieces.)

6.3.2 Example Consider for example $4{*}2$. The top row shows the two obvious ways of dividing a square surrounded by mirrors (in black) into four fundamental regions: joint the

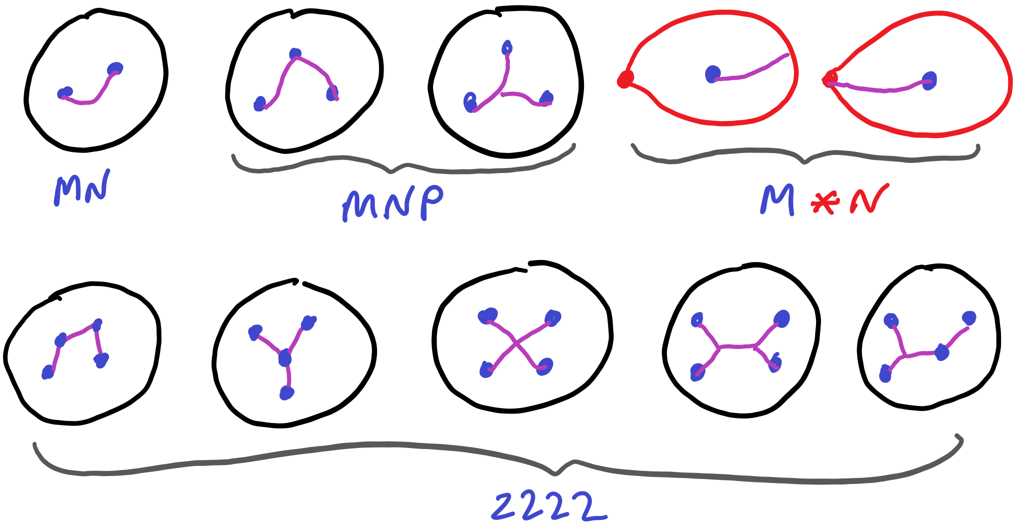


Figure 6.3: Heesch types for some gyratory and gyroscopic groups. (Note that the left-hand *MNP* case corresponds to 3 different Heesch types according to which of the gyrations is the one connected to both the others.)

gyration at the centre of the square by a straight line either to the centre of an edge or to the centre of a side. But as in the second row, it works equally well if we replace the straight lines by general curves, and in the first case the blue curve does not have to meet the edge at its endpoint.

The Type 1 cases on the left are qualitatively different from those on the right as shown by checking the order of features one encounters on walking round a tile.

6.3.3 Relation to the orbifold The orbifold for $4{*}2$ is a cone with a corner point on its reflecting boundary, as shown in the schematic orbifold pictures in Table 6.2. *In order to form a fundamental domain which can be laid flat on the plane, you have to "cut open" the orbifold from the gyration to the boundary.* There are just two distinct ways two do this, according as to whether one arrives on the boundary at the corner point or not.

Note that the walks around the boundary are deducible from the cut orbifold: for type 1, start at the gyration, walk along the top edge of the purple cut, arrive at the boundary at a point where four tiles will meet, turn left along the mirror, passing the kaleidoscope point and continue to visit the "four tiles meeting" point again and return to the gyration along the bottom edge of the purple cut.

6.3.4 General theory The Heesch classification of tilings by fundamental domains is thus a classification of distinct ways to make cuts in an orbifold so that the orbifold becomes topologically a disc. Additionally, there must be at least one cut emerging from every cone point and cuts can only begin or end at cone points, on a reflecting boundary (either at "ordinary" points or at corners) or by meeting other cuts.

The theory above gives the Heesch classification for all groups of the form *N*∗*M* since the same orbifold picture works for all these cases. As mentioned before, the theory is trivial for kaleidoscopic groups.

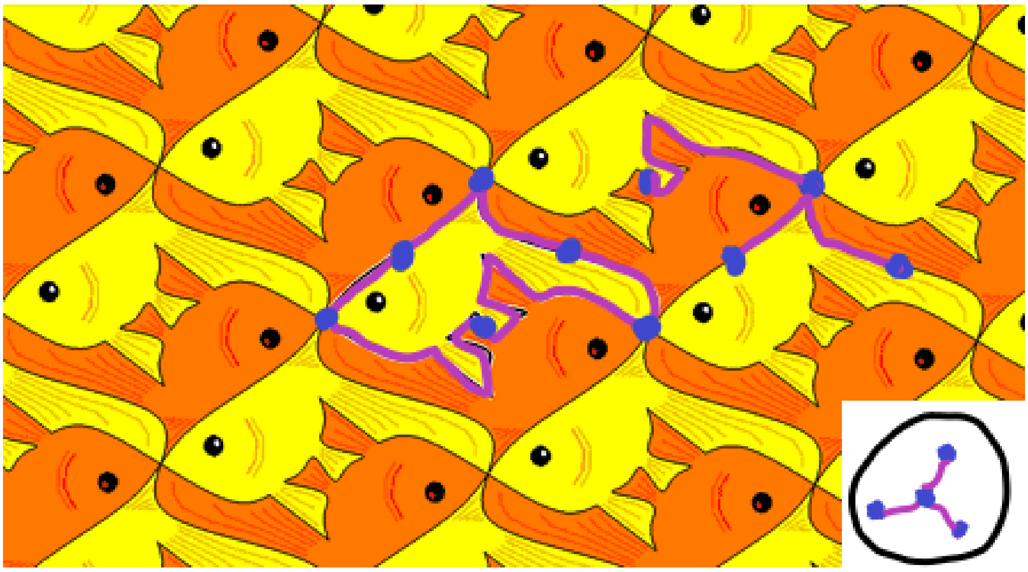


Figure 6.4: A Heesch tessellation with group 2222 (ignoring the colouring).

6.3.5 Gyratory groups We will concentrate on purely gyratory groups where the orbifold is a sphere with 2,3 or four gyrations. Topologically, we simply have to cut open the sphere so that it becomes a disk topologically. There are no reflecting boundaries in this case. The results are in Figure 6.3: there are only two basic cases for groups with three gyrations but five if there are four gyrations.

Note also that for the group *MNP*, in the left-hand case in Figure 6.3 there are three different Heesch types depending on which gyration is connected to both the others.

6.3.6 Example Figure 6.4 has symmetry 2222 ignoring the colours. Its Heesch type is drawn at the bottom left of the picture. One can identify the type by looking at the outlined tile near the centre and noticing that traversing its boundary one encounters gyrations ABACADA where A,B,C,D are the three gyrations defining the pattern. Alternatively, towards the top right one can see the pattern of orbifold cuts realised in the tessellation.

Presentations and colourings

7.1 Introduction to colourings

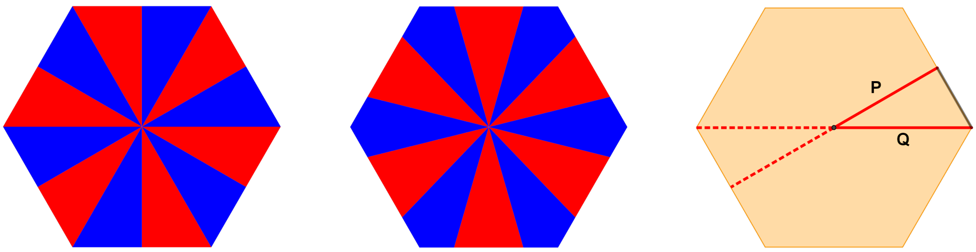


Figure 7.1: A 2-colouring (left); NOT a colouring (centre); Generators (right)

7.1.1 By a *colouring* of a pattern with initial symmetry group G, we mean the addition of colours in such a way that each element of g permutes the colours in a consistent way. The hexagon on the left in the figure, taken to have initial symmetry group D_6 as usual is 2-coloured: in this case, all the reflection symmetries swap red and blue, while all the rotation symmetries leave them fixed. Once the hexagon is coloured, the symmetry group has been reduced from D_6 to the subgroup C_6 of rotations.

The hexagon in the middle however is not 2-coloured: we will rule out the this trivial case where all the symmetries preserve the colours. To put it another way, the coloured pattern always has a symmetry group that is a proper subgroup of the original.

Our aim is to understand possible colourings. To do that we need a proper definition and some group theory.

7.1.2 Definition An n-colouring of a symmetry group G is a group homomorphism $c : G \longrightarrow S_n$ from G to the symmetric group S_n (where $n \geq 2$) of permutations of n colours such that $c(G)$ acts transitively on the colours. (The last condition means that given any two colours a, b there is a symmetry $g \in G$ such that $c(g)$ takes a to b.)

We will regard two colourings as the same if they differ by a permutation of the colours.

7.1.3 Commentary The map $c : G \longrightarrow S_n$ encapsulates the requirement that each $g \in G$ acts consistently on colours: if g takes one red point in the pattern to a blue point then it takes every red point to a blue point.

The fact that c is a group homomorphism is just a consistency requirement so that the permutation of colours associated with a symmetry gg' is equal to the permutation effected by g' followed by the permutation effected by g.

The transitivity requirement means that the hexagon in the centre is not 2-coloured. It also means that if we were to colour the rim of the hexagon on the left yellow, it remains a 2-colouring.

7.1.4 Theorem The group homomorphism of a colouring $c : G \longrightarrow S_n$ has kernel H where H is the normal subgroup of G that comprises all the symmetries that preserve the colours.

Proof. Should be clear from the previous discussion. □

7.1.5 Example The hexagon on the left above illustrates a 2-colouring. The homomorphism sends all rotations to the identity and all reflections to the non-trivial element of S_2 which exchanges the two colours. We will write it as (RB) using cycle notation.

The subgroup preserving the colours in this case is the cyclic group C_6 of rotational symmetries of the hexagon.

7.1.6 Notation We will sometimes refer to a colouring as in the above theorem as a G/H colouring. This often defines the colouring. For instance, there is only one C_6 subgroup of D_6 and so a D_6/C_6 colouring has to be the one where reflections swap the colours and rotations preserve them.

7.2 Colouring dihedral groups

7.2.1 Definition Let $h_1, \dots, h_k$ be elements of a group G. A *word* in $h_1, \dots, h_k$ is a product of elements of the group where each term in the product is one of the h_j or the inverse of one of the h_j.

Note here and hereafter: the "inverses" can be omitted for generators of finite order because if $h^n = 1$ then h^{-1} can be rewritten as = h^{n-1}.

7.2.2 Definition We say that the group G *is generated by* $h_1, \dots, h_k$ and write $G = \langle h_1, \dots, h_k \rangle$ if every element of G can be expressed as a word in $h_1, \dots, h_k$.

7.2.3 Proposition The dihedral group D_6 is generated by the two reflections P, Q in Figure 7.1.

7.2.4 Exercise (NM) Prove the proposition. (It may help to identify the symmetry PQ first.)

7.2.5 Exercise (NM) This continues from the previous proposition.

1. Show that $P^2 = Q^2 = (PQ)^6 = 1$

2. Deduce from part 1 (without geometric reasoning) that also $(QP)^6 = 1$.

3. Show that the symmetries

$$1, P, QP, PQP, QPQP, PQPQP, QPQPQP, PQPQPQP,$$
$$QPQPQPQP, PQPQPQPQP, QPQPQPQPQP, PQPQPQPQPQP$$

constitute the whole group D_6.

4. Show that elements of D_6 such as $QPQPQ$ that terminate in a "Q" are equal to one of the elements in part 3 using only the relations in part 1.

Thus we know everything about D_6 if we know it is generated by two elements satisfying the "relations" in part 1.

7.2.6 Definition Suppose that the group G is generated by $h_1, \dots, h_k$ and we have a finite number of *relations* $R_1, \dots, R_l$ which are equations relating the generators. We say that the generators and relations constitute a *presentation* of G and write $G = \langle h_1, \dots, h_k \mid R_1, \dots, R_l \rangle$ if two words in the generators are equal as elements of G if and only if one can be obtained from the other by repeated use of the relations.

7.2.7 Example A cyclic group of size n has a presentation $\langle a \mid a^n = 1 \rangle$.

7.2.8 Example Generalising what we saw in the previous example,

$$D_n = \langle P, Q \mid P^2 = Q^2 = (PQ)^n = 1 \rangle.$$

7.2.9 Exercise (NM) Generators and relations are not unique. For example, if we write $R = PQ$ in D_n then another presentation of D_n is

$$D_n = \langle P, R \mid P^2 = R^n = (PR)^2 = 1 \rangle.$$

Check you agree with this.

7.2.10 Exercise (NM) Consider a lattice L of translations (such as is contained in every wallpaper group). Let S, T be two translations that generate the lattice. Then we have $L = \langle S, T | STS^{-1}T^{-1} = 1 \rangle$. Explain the relation with a suitable picture.

7.2.11 Examples Consider the group $D_6 = \langle P, Q \mid P^2 = Q^2 = (PQ)^6 = 1 \rangle$. We will try and construct the 2-colouring at the top of this Chapter by means of a group homomorphism $c : D_6 \longrightarrow S_2$. We start by deciding, inspired by inspection of the pattern, that $c(P) = c(Q) = (RB)$ where R (red) and B (blue) define the colours and the parentheses denote the 2-cycle that permutes them.

Since every element of the group can be written in terms of P, Q this determines the value of c on every element by the group laws. (In fact, it says that $c(g) = (RB)$ if g can be written as a product of an odd number of generators and $c(g) = 1$ otherwise.)

For this to be consistent, we need to check that this respects the relations. Clearly $c(P)c(P) = c(Q)c(Q) = 1$. Also $c(P)c(Q) = 1$ and so $(c(P)c(Q))^6 = 1$. These things have to be true for c to be well defined. The theorem encapsulating this is stated after the next exercise.

7.2.12 Exercise (NM) Consider the group $D_n = \langle P, Q \mid P^2 = Q^2 = (PQ)^n = 1\rangle$. Consider the following three possible recipes for 2-colourings.

- $c(P) = c(Q) = (RB)$
- $c(P) = (RB), \quad c(Q) = 1$
- $c(P) = 1, \quad c(Q) = (RB)$

Determine for what values of n each of these exists and where it does sketch a representative example. (A good way of generating an example is to take a fundamental domain and colour it red and see how that pans out.) Why are we not including the possibility $c(P) = c(Q) = 1$?

7.2.13 Theorem Let $G = \langle h_1, \dots, h_k \mid R_1, \dots, R_l\rangle$ and let H be a group. Then there is a one to one correspondence between homomorphisms $G \longrightarrow H$ and maps f from the set of generators to H such that for each relation R_j, the equation defined by replacing each generator h_i by $f(h_i)$ is satisfied in H.

Proof. A sketch is to observe first that a group homomorphism clearly gives rise to such a map f on the generators. Secondly, given a map f defined for the generators, one can extend it uniquely to a map on the whole group. It is well-defined because of the condition imposed on the relations. Making this completely rigorous requires a better definition of the group defined by some generators and relations. □

7.2.14 Exercise (NM) Consider 3-colourings (with colours R,G,B) of the group D_n. Explain why for a 3-colouring $c : D_n \longrightarrow S_3$ we cannot have $c(P)$ or $c(Q)$ being a 3-cycle of the colours. Deduce that up to relabelling the colours the only possibility is that $c(P) = (RB)$ and $c(Q) = (GB)$.

Determine for what values of n this exists and where it does sketch a representative example.

7.3 Generators and relations for wallpaper and other groups

To understand colourings of wallpaper and spherical groups we need to obtain generators and relations. As usual, we will start with the results and justify them afterwards. What follows applies to the wallpaper, spherical and hyperbolic cases.

7.3.1 The general scheme As we will see below, each feature in the signature of a symmetry group gives rise to one "Greek" generator and zero or more "Latin" ones. The Greek generators tie everything together via the overall relation that the product of all the Greek generators is the identity.

Kaleidoscopes

7.3.2 The general case A kaleidoscope $*ab\dots c$ with n mirror meetings (or equivalently, n corner points on the corresponding orbifold boundary) corresponds to one Greek generator α

and $n + 1$ Latin ones $P, Q, \dots, T$. The relations are

$$\begin{aligned} P^2 = Q^2 = \cdots = S^2 = T^2 = 1 & \quad \text{Since these are reflections} \\ (PQ)^a = (QR)^b = \cdots = (ST)^c = 1 & \quad \text{Rotations around mirror meetings} \\ \alpha^{-1} P \alpha = T & \end{aligned}$$

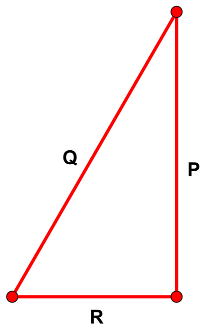
7.3.3 Kaleidoscope on its own If the symmetry of a pattern is a single kaleidoscope with no extra features $*ab\dots c$, then since there is only one Greek generator in the whole pattern, the overall relation becomes just $\alpha = 1$. Thus we have $P = T$ and the generators α and T can be omitted to get:

$$\begin{aligned} P^2 = Q^2 = \cdots = S^2 = 1 & \quad P, Q, \dots, S \text{ are reflections} \\ (PQ)^a = (QR)^b = \cdots = (SP)^c = 1 & \quad \text{Rotations around mirror meetings} \end{aligned}$$

7.3.4 Example Choose you favourite $*632$ pattern. Identify a fundamental region, which should be a triangle similar to that in the diagram. The full symmetry group is generated by P, Q, R with relations

$$P^2 = Q^2 = R^2 = (PQ)^6 = (QR)^3 = (RP)^2 = 1.$$

The last three relations tell us the order of rotational symmetry about the three vertices.



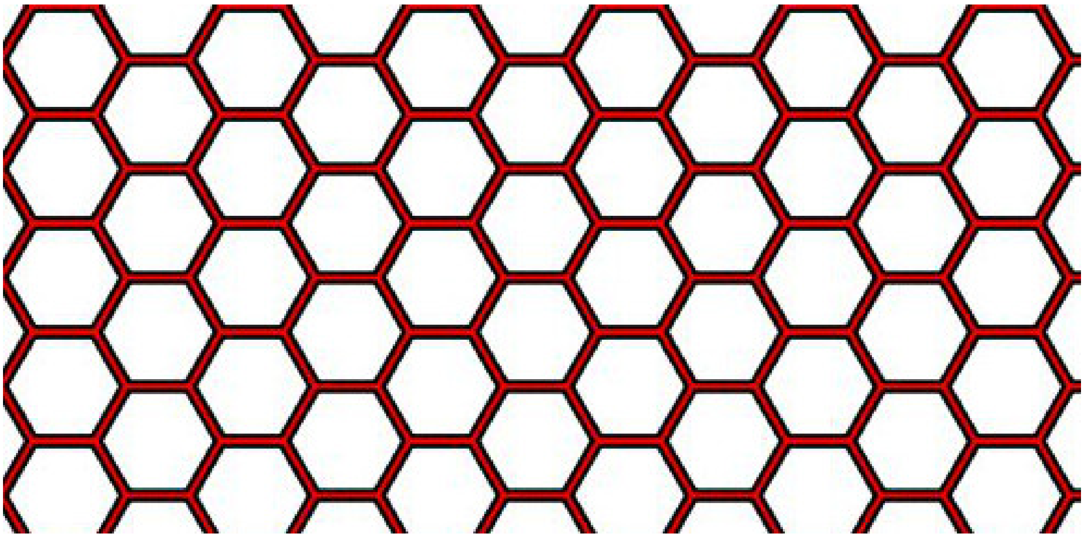
7.3.5 Exercise (NM) Let us consider 3-colourings of $*632$, calling our colours A, B, C. The first three relations tell us that each of P, Q, R must be colour transpositions or the identity. The identity $(PQ)^6 = 1$ tells us nothing since the size of S_3 is 6 and so every element to the 6th power is the identity.

Consider the consequences of $(QR)^3 = 1$. If one of Q, R acts by the identity then so must the other and then we cannot act transitively on the colours.

1. Suppose Q, R act by the same transposition: say $Q \mapsto (AB)$ and $R \mapsto (AB)$. For a transitive action, P must act by a different transposition, but then PR acts by a 3-cycle contradicting $(PR)^2 = 1$. Thus there is no colouring.

2. We can assume therefore that Q, R act by distinct transpositions: without loss of generality, we can assume $Q \mapsto (AB)$, $R \mapsto (BC)$. It is easy to check that the relations are satisfied if P acts by the identity or by (BC) but not otherwise. Thus we have the 3-colourings
$$P \mapsto 1, \qquad Q \mapsto (AB), \quad R \mapsto (BC).$$
and
$$P \mapsto (BC), \qquad Q \mapsto (AB), \quad R \mapsto (BC).$$

Illustrate these colourings on the hexagonal grid. You can start by choosing a fundamental region and hence identifying P, Q and R. Colour it with one of the colours and see how it works out.



7.3.6 Exercise (NM) Investigate the 2-colourings of $*632$ — you should find there are three. Draw examples.

7.3.7 Exercise (NL) Consider the wallpaper group $**$. Each $*$ gives a Greek and Latin generator, and since we have two features the product of the Greek generators must be 1.

$$\begin{gathered} P^2 = 1, \quad \alpha^{-1} P \alpha = P \\ Q^2 = 1, \quad \beta^{-1} Q \beta = Q \\ \alpha\beta = 1 \end{gathered}$$

Now, the more complicated relations on the right of the top two lines just say that the Greek generator commutes with its Latin companion. And since the product of the Greek generators is 1, we can eliminate one of them in favour of the other. We arrive at

$$P^2 = Q^2 = 1, \quad P\alpha = \alpha P, \quad Q\alpha = \alpha Q.$$

The generators P, Q correspond to two adjacent mirrors of different sorts and the generator α corresponds to a translation symmetry parallel to the lines of reflection.

Find the 2-colourings and 3-colourings of this. (There are seven 2-colourings but only two 3-colourings.) Give examples in all cases.

7.3.8 Exercise (NM) What 2-colourings and 3-colourings of the full symmetry group $*432$ of the cube are there?

The true blues

7.3.9 Gyrations A gyration, corresponding to n gives rise to a single Greek generator β and the single relation

$$\beta^n = 1$$

7.3.10 Exercise (CM) Consider the group 333. According to the rules, it has three generators satisfying

$$\alpha^3 = \beta^3 = \gamma^3 = \alpha\beta\gamma = 1.$$

Show that there are no 2-colourings of this group and determine the possible 3-colourings, giving examples. Take care to account fully for equivalence under relabelling of the colours.

7.3.11 Handles A handle (i.e. an **o**) gives rise to two one Greek and two Latin generators satisfying

$$X^{-1}Y^{-1}XY = \alpha.$$

In the case of the group **o** with just one handle, $\alpha = 1$ and so we simply have two generators satisfying

$$X^{-1}Y^{-1}XY = 1 \quad \text{or equivalently} \quad XY = YX.$$

The generators X and Y are then just two generators of the lattice of translations.

7.3.12 Exercise (NM) What can you say about 2-colourings and 3-colourings of **o**? Give examples.

Miracles

7.3.13 Miracles A miracle has one Latin and one Greek generator satisfying

$$Z^2 = \delta.$$

In essence, Z is the glide and δ the translation which is its square.

7.3.14 Exercise (NM) Consider $\times\times$. We have generators Y, Z and γ, δ satisfying

$$Y^2 = \gamma, \qquad Z^2 = \delta, \qquad \gamma\delta = 1.$$

We can simplify this: we do not need explicitly to mention γ and δ. The whole thing reduces to a group generated by Y, Z subject only to

$$Y^2Z^2 = 1.$$

Find the 2-colourings and 3-colourings of this.

7.3.15 Exercise (NM) Write down standard generators and relations for $*\times$ and simplify them. What can you say about 2-colourings and 3-colourings?

Gyroscopic groups

7.3.16 Gyroscopic groups are those with rotational symmetry about the centre of what would otherwise be a simple kaleidoscope. The colourings can be a little confusing!

7.3.17 Exercise (NM) Consider the group $3{*}3$. Following the rules, we have generators and relations as follows.

$$P^2 = Q^2 = 1; \quad (PQ)^3 = 1; \quad \alpha^{-1}P\alpha = Q$$
$$\beta^3 = 1$$
$$\alpha\beta = 1$$

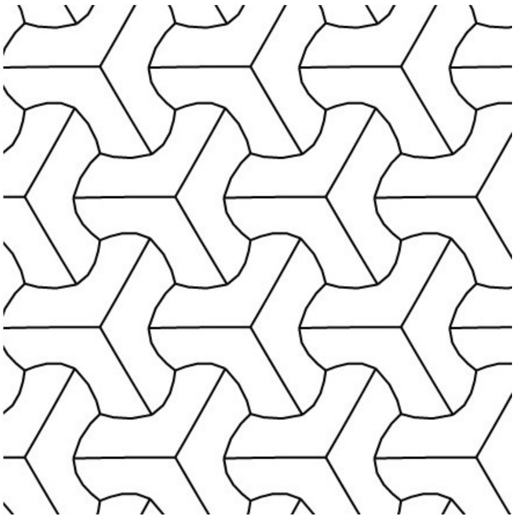
The last two lines imply that $\alpha = \beta^2$ and so we can eliminate α. We arrive at

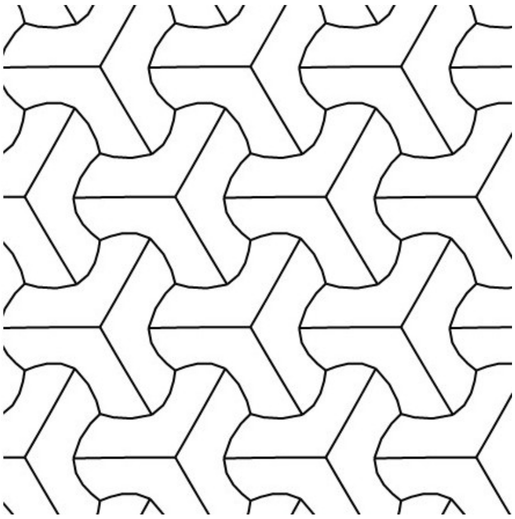
$$P^2 = Q^2 = 1, \quad (PQ)^3 = 1, \quad Q = \beta P \beta^2.$$

Next we note that $Q^2 = 1$ follows from $P^2 = 1$ and the final relation and so it can be dropped. Then we can eliminate Q altogether to arrive at

$$P^2 = 1, \quad \beta^3 = 1, \quad (P\beta P\beta^2)^3 = 1.$$

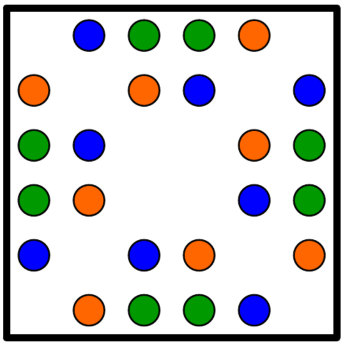
The generator β is a gyration and P is a reflection in a chosen side of the triangle surrounding β. Investigate the 2-colourings and 3-colourings. Draw some pictures!



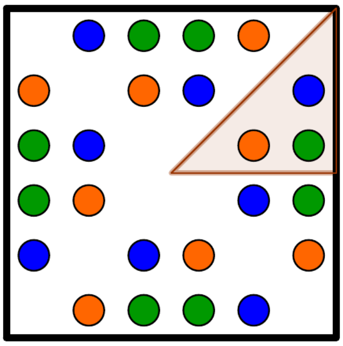


More theory of colourings

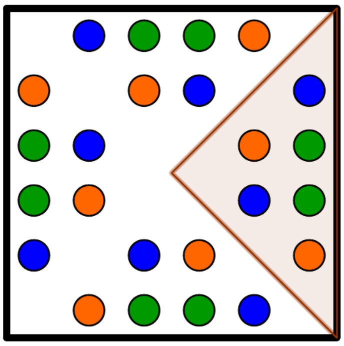
7.3.18 Example Just a reminder of the fact that not every part of a pattern needs to be coloured to have an *n*-colouring, and indeed there may be extra colours in the pattern. The little pattern on the right is a 2-colouring of D_4.



7.3.19 Example Neglecting colours in the pattern, the "full" symmetry group is D_4 and a fundamental domain is the little flagstone on the right. The coloured pattern has symmetry C_4 (the subgroup of rotations). So we have the surjective group homomorphism $c : D_4 \longrightarrow S_2$ with kernel C_4.
Note also that when colouring, you do not have to start by colouring a whole fundamental domain in one colour.



7.3.20 Example The coloured pattern has symmetry group C_4 which has a fundamental domain as in the picture on the right. The fundamental domain is twice the size of that for the uncoloured pattern.



7.3.21 The general case In general, given a colouring $c : G \longrightarrow S_n$ with kernel H (the symmetry group of the coloured pattern) and image $K \leq S_n$, the size of a fundamental domain for H is $\#K$ times that for G.

Furthermore, the orbifold for H can be obtained by "unfolding" the orbifold for G.

Thinking about the proof of the "Magic Theorems" this tells us also that the orbifold Euler characteristic of H must be $\#K$ times that of G. Of course, this only tells us anything in the case of spherical or hyperbolic geometry: in the Euclidean case both Euler characteristics must be zero.

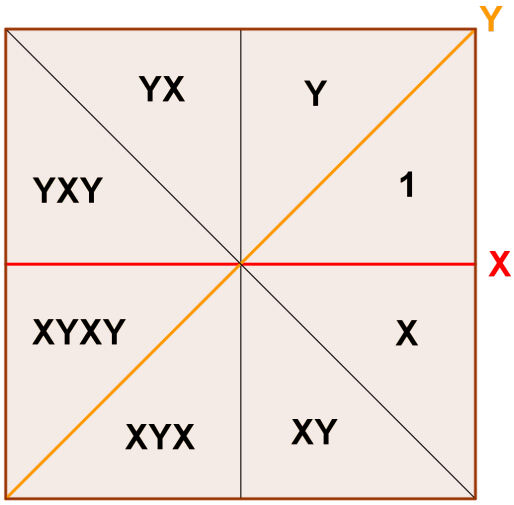
7.3.22 Restrictions on K Finally, note that for a colouring the image of $c : G \longrightarrow S_n$ the image K must be a subgroup of S_n that acts transitively on the n colours. For $n = 2$ the only possibility is $K = S_2$ but for $n = 3$ there are two possibilities: the image may be S_3 or just the subgroup of even permutations. So the orbifold for H may be 6 or 3 times the size of that for G.

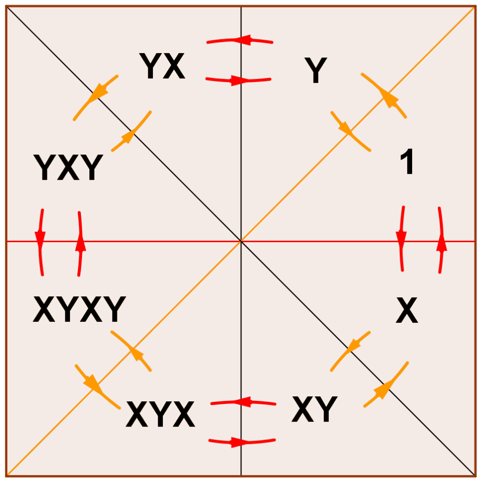
7.4 Understanding the presentations

It remains to convince ourselves that our recipes for the presentations of the groups are correct. We will try and do that here.

Dihedral groups revisited

7.4.1 Generators for D_4 In the left-hand figure, two generators X, Y for D_4 have been identified. Once we identify one "reference" fundamental domain as corresponding to $1 \in D_4$, then the other seven corresponding domains can be labelled by the unique group element that takes our reference domain to the given one.





Note that, for instance, the domain labelled YX is mapped by the action of X to the domain labelled $X(YX) = XYX$. Do not get confused by noticing, for example, that Y sends $XYXY$ to XYX: in fact $YXYXY = XYX$ (exercise).

7.4.2 Right multiplication In the right-hand picture, red arrows have been drawn indicating all the reflections conjugate in D_4 to the original X and the same with orange arrows for Y. Notice how the the red arrows correspond to multiplication *on the right* by X and similarly for the orange arrows and Y. (Take care again not to be confused by the fact that there are different ways of writing an element in terms of the generators.)

There are several things to notice about this: starting with a group element like YX, multiplying on the left by a generator often takes that element to one on the far side of the pattern. On the other hand, multiplying on the right by a generator takes you to an adjacent fundamental domain.

Notice that this makes it easy to write down a formula for the group element corresponding to a final domain in terms of the generators: simply find a path along the arrows to the domain and write down the generators you have used in order from left to right.

We sum up the situation in the following proposition.

7.4.3 Proposition Consider a generator $Q \in G$ of a group G. Let $g \in G$. Then

$$(gQg^{-1})g = gQ.$$

That is, multiplying g on the left by the conjugate of Q by g leads to the same result as multiplying g on the right by Q.

7.4.4 Definition In Figure 7.2, we have thrown away the geometric parts of the picture to leave just the *Cayley graph* for this presentation of D_4. This is a directed graph with one vertex for each group element and directed edges that tell us the result of multiplying each group element on the right by each of the generators.

7.4.5 Relations Every closed loop of arrows that one can find in the graph corresponds to a word in the generators that is equal to the identity. So, the trivial loops that occur from

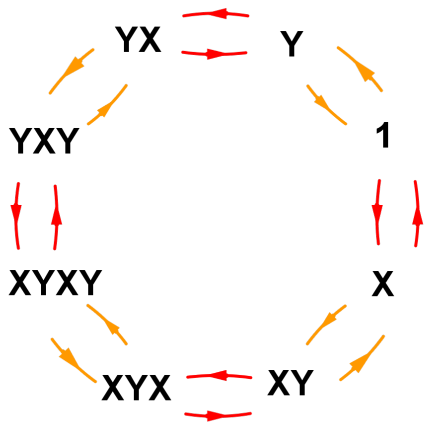


Figure 7.2: The Cayley graph of this presentation of D_4.

following a red or orange edge and then returning immediately by the same coloured loop to our starting point correspond to $X^2 = 1$ and $Y^2 = 1$. And the loop starting at 1 and travelling clockwise round the circle by loops of alternating colour to return to its starting place tells us that $(XY)^4 = XYXYXYXY = 1$.

This picture gives an alternative proof that the three relations above are sufficient to define the group. Consider a loop in the graph corresponding to a word that is equal to 1. We can eliminate any "backtracking" by using $X^2 = Y^2 = 1$ and reduce to a loop that travels k times round the circle. But that reduces the word to some power of $(XY)^4$.

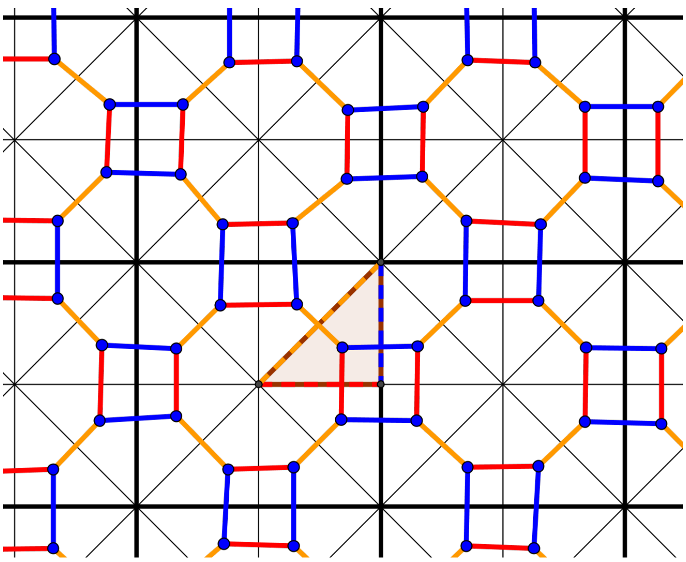


Figure 7.3: generators and relations for $*442$

7.4.6 Presentation of a kaleidoscope Figure 7.3 shows a square grid in black. Its symmetry group is $*442$, a kaleidoscope based on the shaded right-angled triangle near the centre. We take the reference fundamental domain to be the shaded one and the generators to be reflections in the boundary of that region: P (red), Q (gold) and R (blue). All the conjugate reflections have been added in the appropriate colour. Each edge in the picture should be thought of as two directed edges, one in each direction; this simplification is reasonable because reflections are of order two and so they are their own inverse.

Our simplified presentation for $*442$ is that P, Q, R satisfy

$$P^2 = Q^2 = R^2 = (PQ)^4 = (QR)^4 = (RP)^2 = 1.$$

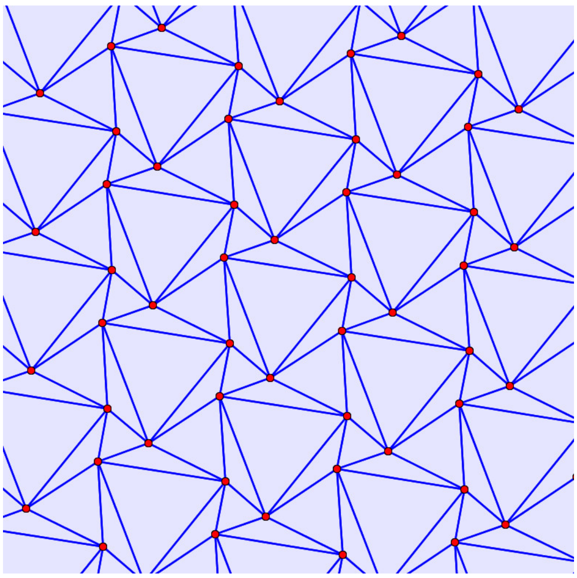
The fact that P, Q, R are order two is implicit in the picture. The remaining three relations correspond exactly to the three different sorts of tiles (two octagonal, one square) in the coloured tiling.

Now consider a closed loop of coloured edges in the graph, corresponding to a word in the generators which equals 1. We can reduce the loop by cutting out one polygon at a time, which corresponds to a simplification of the word using one of the known relations. Thus eventually one can reduce the loop to going round a single coloured tile and hence to nothing: correspondingly, we have shown that the word can be simplified to 1 using just the relations we have stated.

We can check the other kaleidoscopes similarly. The argument works for wallpaper, spherical and hyperbolic groups. The critical feature is all three underlying spaces are simply connected, and so every closed loop in them can be shrunk to a point.

7.4.7 Exercise (NM) Find the smallest translation symmetry of the square lattice of the example. Identify the image of the reference fundamental domain under it. Hence write down the translation in terms of the given generators.

7.4.8 Exercise (NL) Take two other kaleidoscopic groups, at least one of which is not a "wallpaper" example and construct the analogous tiling that proves the relations are complete.



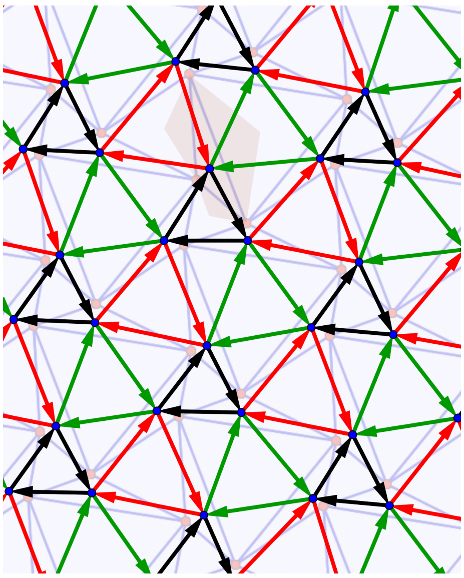


Figure 7.4: Presentation of 333

7.4.9 A gyratory example In Figure 7.4 we see a wallpaper with 333 symmetry. A fundamental domain can be taken to be a whole scalene triangle with one third of each adjacent equilateral one, as marked rather faintly on the right-hand picture.

The three generators α (green), β (red), γ (black) are one-third turns about the centres of large, medium and small equilateral triangles respectively. Travelling "backwards" along an edge corresponds to right multiplication by the inverse (which is also the square) of the corresponding generator.

We see the "tiles" in the graph correspond to the relations

$$\alpha^3 = \beta^3 = \gamma^3 = \alpha\beta\gamma = 1.$$

Thus the picture shows that 333 is generated by these generators with these relations.

7.5 Where do the tilings come from? (Not examinable)

7.5.1 Introduction For the wallpaper and spherical groups at least, one can construct tilings of the sort studied in the previous section and hence establish that the generators and relations completely describe the group and so justify our analysis of colourings.

But one should of course be curious as to where the formulae for writing down the generators and relations comes from. To do that we need to return to thinking about orbifolds.

7.5.2 Way back in §1.3.15 we saw how for the wallpaper group **o**, the orbifold, obtained by folding the orbits, is a torus. The picture in that section shows a torus with two red curves on it. Thinking of the black dot where they intersect as a reference point in a fundamental region, when you travel around one of the closed curves in the orbifold, in the unfolded picture it brings you back to an adjacent fundamental region. And if you go round the loop the other way, it takes you to a different, neighbouring fundamental domain.

In fact, the fundamental domains are in one to one correspondence with pairs of integers (m, n), representing a loop on the torus that travels m times round one of the loops and n times round the other. One loop on the torus can be deformed into another precisely when they have the same values of m and n. Perhaps surprisingly, it does not matter which order you go round the loops: see the video at `https://www.youtube.com/watch?v=nLcr-DWVEto`.

If you have done some topology, you will probably see (or know already) that what we are doing here is considering the "fundamental group" of the torus and the plane is its "universal cover".

7.5.3 The picture in Fig 7.5 shows the two closed curves on the torus (with a chosen direction) "lifted" to the whole plane. The result, denoting the yellow generator by X and the red one by Y is a tiling like the ones in the previous section. We can see that there is just one sort of "tile" which corresponds to the relation

$$XYX^{-1}Y^{-1} = 1.$$

Thus we see that the tiling with relations is something that is lifted from the orbifold to the plane.

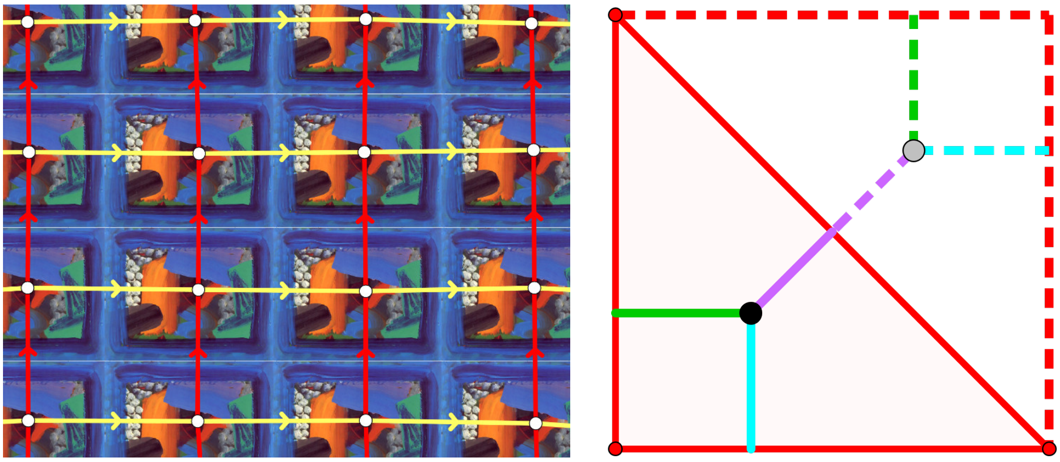


Figure 7.5: Generators for **o** and $*442$

7.5.4 On the right in Fig 7.5 there is a picture of the orbifold for $*442$ with a black dot marking a reference point. The three coloured segments joining the black dots to the sides are curves that one could travel along, bounce off the reflecting boundary and return to the reference point. That curve, lifted to the whole pattern (as shown only for the pink one) carries us to an adjoining fundamental domain. But repeating travel along the curve takes us back where we started. Those little segments therefore give us generators P, Q, R each squaring to the identity — and it is because they square to the identity that we do not need an arrow on them: each is equal to its inverse.

One sees also in the picture the beginnings of the three resulting relational tiles, each alternating between two of the colours. These give us the relations

$$(PQ)^4 = (QR)^4 = (RP)^2 = 1.$$

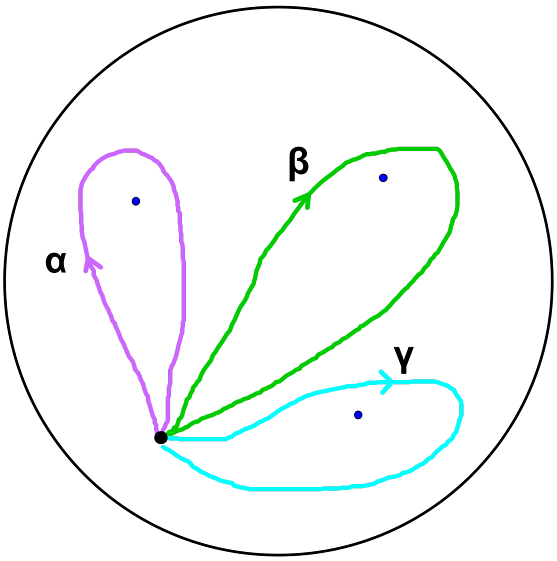
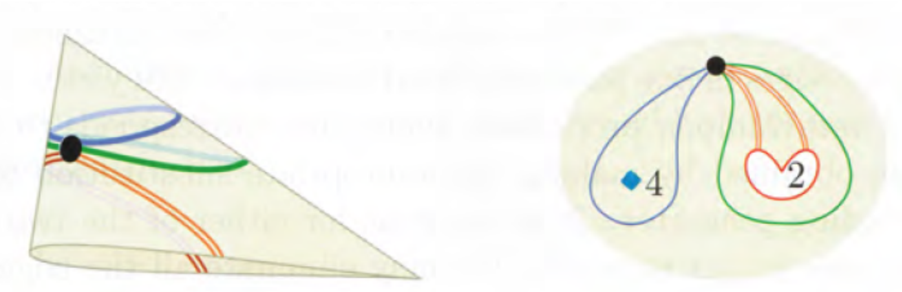


Figure 7.6: Generators for **333**

7.5.5 The Greek generators A Greek generator representing a gyration is clear: in the orbifold it is a curve from the base point winding round the cone point and returning to base. If the gyration has order 3 then we have a Greek generator α with $\alpha^3 = 1$. To reconstruct the pattern, the orbifold has to be "carved open" with cuts originating at cone points so that it can be "laid flat" on the plane.

Supposing now we consider the case of 333. The orbifold is a sphere with three cone points. The picture in the figure is a little symbolic and shows the three generators as loops surrounding one of the three cone points. You have to imagine the picture on the surface of a sphere. When you do that, you see that the curve $\alpha\beta\gamma$ surrounds a portion of the sphere (the "outside" of the three loops) with no symmetry feature inside it. Therefore that loop can be shrunk to nothing and so represents the identity. This accounts for the final relation in the presentation of 333 as

$$\alpha^3 = \beta^3 = \gamma^3 = \alpha\beta\gamma = 1.$$

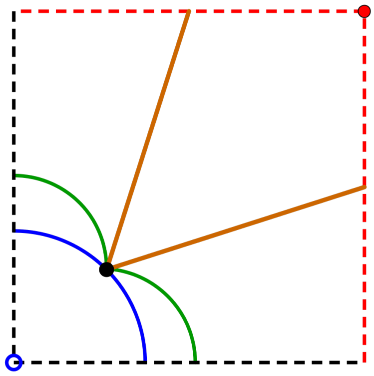


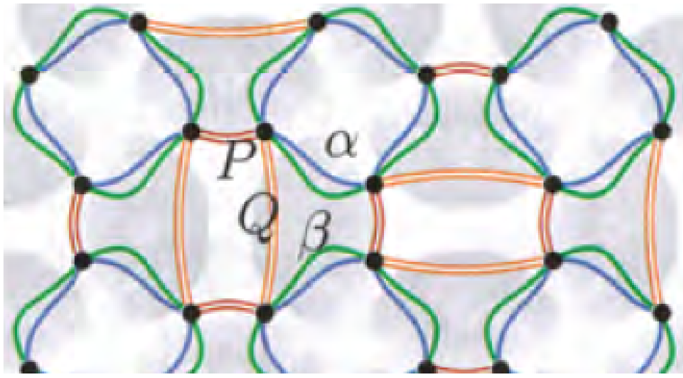
7.5.6 So, finally, the origin of the generators and relations is that one takes the orbifold, thought of as a sphere with added features, chooses a base point and throws a loop around each symmetry feature which one labels as a Greek generator. Within each loop (except in the case of the gyration where the only thing one has is that the correct power of the Greek generator is the identity) one has Latin generators corresponding to generators associated with that feature.

We conclude with an example. In the figure, the orbifold of $4{*}2$ is shown: it is a cone with a reflecting boundary, containing a single corner point. On the right is a symbolic picture of the orbifold where a sphere with a chosen black base point has a blue loop thrown around the cone point and a green one around the kaleidoscope. Suitably oriented, these will become generators α and β with $\alpha\beta = 1$. On the left, the same picture is drawn on the orbifold with the addition of two double red lines leading to the reflecting edge. These represent the reflection generators P, Q.

7.5.7 To generate the corresponding tiling on a pattern on the plane, the orbifold needs to be carved open by cut running from the cone point to the reflecting boundary, meeting the latter at a point on the boundary between the red generators on the side away from the corner point.

The result is shown on the left in the final figure. The fundamental region is one quarter of a square kaleidoscope — the red dashed edges are the reflecting boundary with the corner point on the top right and the black dashed lines are the cut that carved open the orbifold. The cone point is in the bottom-left corner.





On the right of the last figure, the generators appear on a $4{*}2$ pattern. With orientations consistently chosen one gets precisely the presentation produced by the rules we have been using:

$$P^2 = Q^2 = (PQ)^2 = 1, \quad Q = \alpha^{-1} P \alpha, \quad \beta^4 = 1, \quad \alpha\beta = 1.$$