Orbifolds and the magic theorems

6.1 Orbifolds

6.1.1 Introduction Our definition of orbifold will be a bit “19th century” in the sense that it will rely on intuition to some extent. We start with the plane or sphere with a pattern having wallpaper or spherical group symmetry. From that construct the orbifold: an object such that an ant walking around on the orbifold cannot tell they are not in fact walking over the original pattern.

There is a map natural from the plane or sphere to the fundamental domain that takes each point to the point in the fundamental domain to which it relates. This map is not always continuous and replacing the fundamental domain by the orbifold rectifies that. We saw examples of this originally in Chapter 0.

6.1.2 Back to geometry In the last chapter we studied topology and two spaces were considered identical if one could be deformed into another. Now and henceforth, we return to geometry where distances and angles matter.

We imagine the orbifolds for wallpaper groups to be made out of paper, and we can bend the paper but not crease or tear it. We can bend a sheet of paper in space into a cylinder of cone, but not into the surface of a sphere. For those who have studied some differential geometry, the point is that bending is an isometry.

We saw in §1.3.15 that the orbifold for *o* is a torus. There is a problem because in fact one cannot bend a piece of paper into a torus in $\mathbb{R}^3$ without creasing. One solution to this is to use the fourth dimension. Alternatively, just imaging smoothly zipping together opposite sides of a rectangle: if one knows about “manifolds” this is a perfectly good way to define an “abstract surface”.

Similarly we must imagine our spherical group orbifolds as zipped together from pieces of unit sphere. This is harder to imagine because you cannot bend a piece of the unit sphere the way you can bend paper.

6.1.3 Construction of the orbifold Mirror lines are always on the boundary of a fundamental domain and in the orbifold the only change is that we take the boundary to be reflecting. Gyrations also have to be on the boundary, and we resolve them by rolling the orbifold into a cone about the gyration, thus creating a singular “cone point” in the interior of the orbifold:

imagine the point of an idealised ice-cream cone.

Miracles manifest themselves in the need to identify two lengths of boundary of the fundamental domain with a "twist", thus creating an embedded Möbius band in the orbifold. Finally, and this appears for us only for the group o, a wandering causes identifications leading to a torus.

6.1.4 Signatures and orbifolds The orbifold notation for a wallpaper or spherical group also defines its orbifold. We already have discussed how signatures without numbers define topological surfaces. The additional information is that each blue number denotes a cone point of that order in the orbifold, and red numbers following a red star denote that the boundary curve from the star (which in fact always follows a straight line or great circle in the orbifold) has "corner points". A red k corresponds to an internal angle of π/k.

6.1.5 Decoding the signature Consider, for example, the wallpaper group $3{*}3$. Stripping away the numbers gives us the topology of the orbifold, in this case we have just $*$ or a disc with a boundary. The 3 tells us that there is a cone point of order 3, so geometrically we have a cone rather than a disc. The 3 tells us that the reflecting boundary has a corner point with internal angle $\pi/3$.

Or consider the spherical group 532. Stripping away the numbers leaves us with nothing, which is the signature of a sphere. The sphere has three cone points, one each of orders 2,3 and 5.

It might be an interesting exercise to take a fundamental domain for (say) 333 and try and understand how that becomes an orbifold that is topologically a sphere. I have never found an entirely convincing argument.

6.2 Magic Theorems and Orbifold Euler characteristics

Our aim now is to understand "where the magic theorems come from". We will start with the case of spherical symmetry.

In §5.2 we discussed the spherical symmetry group $\times$ which is the 2-element group $\{\pm\mathbb{I}\}$ and its orbifold, the projective plane. There is a 2-to-1 map that takes the sphere S^2 to $\mathbb{R}P_2$. We argued in 5.2.6 that if we take a map on the sphere that is invariant under $\times$ such as that generated by a cube, then one gets a corresponding map on $\mathbb{R}P_2$ with exactly half the number of vertices, edges and faces. It is exactly one half the number because the group $\times$ has two elements.

Therefore we end up with

$$\chi_{\mathbb{R}P_2} = \frac{\chi_{S^2}}{2} = 1, \qquad \text{where the "2" is the size of the group } \times.$$

Under sufficiently "nice" n-to-1 maps, the Euler characteristic always behaves in this fashion, although proving that requires some better definitions and some work.

6.2.1 The problem As we saw even in the prologue, the map from a space with group acting to the orbifold is often not so decent. Consider the example we saw of the islamic pattern on a disk with D_6 symmetry. The map from the disc to its orbifold is 12-to-1 at most points but only 6-to-1 on the reflecting boundaries and 1-1 on the corner point.

And this leads to bad outcomes: the orbifold for the 120-element group $*532$ is a single spherical triangle which is topologically a disk and so has Euler characteristic 1. That's a long way from $2/120$.

6.2.2 The solution As all children know, if the game isn't working for you, then you change the rules. But you often have to give as well as take in this process.

- It might get us somewhere if the "Euler characteristic gets divided by the size of the group" game worked.
- But it might be enough if it just worked for these rather specific sorts of groups acting on the plane and sphere. In other words, we are imagining our orbifolds as being made out of flat space, pieces of the unit sphere or, in a later chapter the hyperbolic plane.

So let's see if we can invent a "better" Euler characteristic for this situation.

6.2.3 Exercise (NM) Consider the spherical symmetry group $*222$. How many elements does it have? What are they? Also, how many elements does $3{*}2$ have?

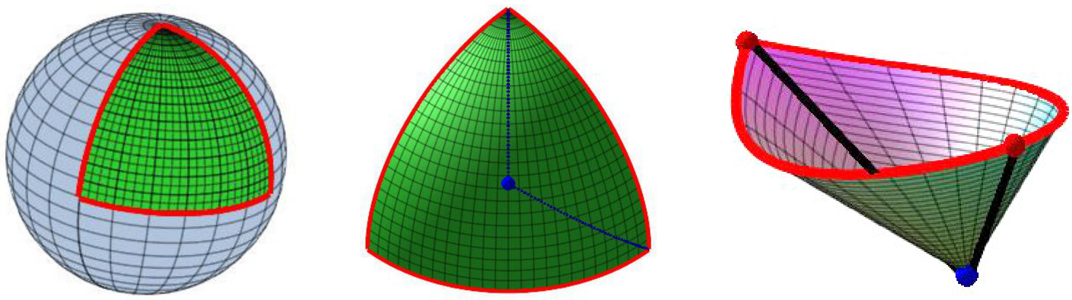


Figure 6.1: The orbifold for $*222$ (left); with a gyration point at the centre that exists in $3{*}2$ (middle). Fundamental region wrapped into cone to make the orbifold for $3{*}2$ (right).

6.2.4 Investigation Let's look closely at the relationship between symmetric maps on the sphere and maps on the corresponding orbifold. We will focus on the example of $3{*}2$.

Starting with $*222$, the orbifold for that is one quarter of the Northern hemisphere, a spherical triangle with three right angles and reflecting boundaries. (See the left-hand picture in Figure 6.1.)

In $3{*}2$ there is an additional 3-fold gyration at the centre of the spherical triangle. A fundamental region is now one third of the triangle, as delineated by the dotted blue lines in the centre picture of Figure 6.1.

As usual, we should identify the two blue dotted lines so that the orbifold is now a sort of cone as shown in the right-hand picture. *But that picture is misleading* because all the surface apart from the "conical singularity" at the gyration point should be curved like a piece of a sphere. (But that is hard to make a picture of, specially since the object cannot be embedded in $\mathbb{R}^3$.)

I have drawn a map on the orbifold too. There is a red vertex at the corner/kaleidoscope point and I have added one also half way round the boundary. I am taking the two pieces of reflecting boundary joining the red vertices as edges. I have a blue vertex at the gyration point and two black edges joining the blue vertex to each of the red ones.

6.2.5 Consider the map on the whole sphere generated by our orbifold map. In other words, think of the map as defining a symmetric pattern on the whole sphere with $3{*}2$ symmetry. You might like to try and sketch a bit of this to get the idea.

1. The orbifold map has two faces. The fundamental region that we made the orbifold from is duplicated #$3{*}2 = 24$ times on the sphere. Therefore the resulting map has $F = 48$.

2. Each black edge on the orbifold also leads to 24 on the whole sphere and so that contributes 48 edges on the whole sphere. But each red edge generates only 12 edges on the whole sphere because the reflection in that edge fixes it. (Or, to put it another way, on the sphere, the resulting edges are each shared by two copies of the fundamental region.) Overall therefore we have $E = 72$.

3. The red vertex in the foreground similarly leads to only 12 vertices on the whole sphere and the red vertex at the corner point only 6. The gyration point leads to only 8 vertices on the whole sphere because there is a 3-element cyclic subgroup of rotations that fix it. Therefore $V = 26$.

4. Doing the sum, $\chi = 2 = 26 - 72 + 48 = 2$, as it must be.

6.2.6 The orbifold Now, thinking just about the orbifold, we will change how we count things so as to reflect the calculations above. Count the blue vertex as $1/3$ of a vertex; the red corner vertex as $1/4$ of a vertex; the other red vertex as $1/2$ a vertex; the red edges as half an edge each and count the black edges and the faces normally. One gets

$$V = 13/12, \quad E = 3, \quad F = 2 \quad \text{and } V - E + F = \frac{1}{12} = \frac{2}{24} = \frac{2}{\#3{*}2}.$$

This calculation is the motivation for the following definition.

6.2.7 Definition Let X be an orbifold for a (spherical or wallpaper) symmetry group. Consider a map on X such that in addition to the normal requirements for computing Euler characteristics for a surface with boundary we have a vertex at each gyration point and a vertex at each corner (kaleidoscope) point on the boundary.

The *orbifold Euler characteristic* χ_o of X is defined by $\chi_o = V - E + F$ where we count vertices, edges and faces normally except that:

- Edges on a reflecting boundary count as $1/2$ an edge;
- Vertices at an order n kaleidoscope point (i.e. a corner point in a reflecting boundary) count as $1/(2n)$ of a vertex;
- Other vertices on the boundary count as $1/2$ of a vertex;
- Vertices at gyration points of order n count as $1/n$ of a vertex.

By construction, we have the following theorem.

6.2.8 Theorem Let G be a spherical symmetry group. Then for its orbifold we have

$$\chi_o = \frac{2}{\#G}.$$

And now the following theorem proves the Magic Theorem for the spherical case.

6.2.9 Theorem For a spherical symmetry group G,

$$\chi_o = 2 - \text{the cost in dollars of the signature.}$$

Proof. Consider building the orbifold from a sphere. Since a sphere has no boundaries or gyration points, its orbifold Euler characteristic is the same as its standard Euler characteristic: $\chi_o(S^2) = \chi(S^2) = 2$.

1. In building the orbifold, $*$ and $\times$ will, as normal cost one unit of Euler characteristic.

2. Draw a map on the orbifold, respecting the conditions required for it to compute χ_o. We start by imagining computing the standard Euler characteristic χ.

3. Now changing the weighting of all boundary edges and vertices (including any corner points) to $1/2$ has no effect on computing $V - E + F$ because each boundary component has exactly as many edges as vertices. So imagine that we have made this change. The edges and non-kaleidoscope vertices on the boundary are now counting as they do in the definition of χ_o.

4. The introduction of an "n" gyration point means that one vertex now counts for $1/n$ of a vertex, thus reducing $V - E + F$ by $(n-1)/n$.

5. The introduction of a corner (kaleidoscope) point of order n on the boundary changes a vertex that is currently counting as $1/2$ to one that counts as $1/(2n)$ thus reducing $V - E + F$ by $(n-1)/(2n)$.

6. Thus χ_o for the orbifold is exactly 2 minus the accumulated costs of the features in the signature.

□

6.2.10 The big view So there we have it: the "Magic Theorem" for the group G spherical case is just combining the fact that χ_o is by construction equal to $2/\#G$ with calculating what χ_o has to be if you consider constructing it feature by feature (as per the signature of G) from a sphere. Magic indeed!

The 17 Wallpaper groups

6.2.11 Introduction So what of the original case we studied of patterns in the plane? The plane itself is not compact and so does not have an Euler characteristic in the sense we have been discussing.

6.2.12 Theorem The orbifold of a wallpaper group has orbifold Euler characteristic zero.

Proof. What follows is a bit sketchy, but reasonably convincing, I believe. Draw a map on the orbifold (as you would in the the sphere case) and let V, E, F be the numbers that appear in calculating χ_o. Consider the corresponding map on the plane. Cut the map off so as to include only regions lying entirely inside a circle of radius $R > 0$.

Let N denote the number of complete fundamental regions within the cut-off. Then

$$V_R \approx NV, \quad E_R \approx NE, \quad F_R \approx NF$$

where the "R" subscripts indicate the vertex, edge and face counts for the whole map cut off at radius R.

The errors we are making in this approximation is caused by the pieces crossing over the circle of radius R, and so will be proportional to R. Thus for some c independent of R we have

$$(V_R - NV) - (E_R - NE) + (F_R - NF) < cR$$

and using $V_R - E_R + F_R = 1$ from Euler's theorem and observing that N grows as R^2, we have a contradiction unless we have $\chi_o = V - E + F = 0$. □

Notes on orbifolds

6.2.13 Orbifolds with no reflecting boundaries and gyrations In the plane case we have o (a torus) and $\times\times$, which are the two possible "flat compact worlds". In the sphere case we have the projective plane $\times$. The defining feature of these examples is that the action of the group is "free", meaning that every element except the identity moves every point, or equivalently that the stabilizer of every point is trivial. It's points having non-trivial stabilizers that gives rise to reflecting boundaries and cone points.

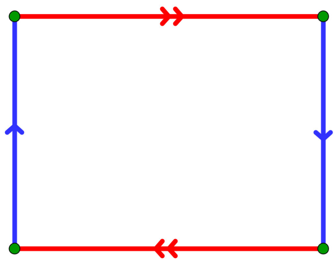


Figure 6.2: A projective plane or not? Are you a topologist or a geometer?

6.2.14 Topologists and the projective plane Here is something I only appreciated when preparing this course. Topologists will tell you that if you take a piece of paper and identify each pair of opposite edges with each other *with direction reversed* as in the figure then the result is a projective plane. You might think therefore that because you have made the projective plane out of paper (i.e. if you have studied differential geometry it has a flat $K = 0$ metric) that it should be the orbifold for a wallpaper group.

The reason it does not is that when you identify, the four corners get identified in pairs, and each of the two resulting points in the projective plane is surrounded by just two right-angles of paper. So what we really have, at least if you are a geometer is a projective plane with two order-2 "cone points' points, which is the orbifold of the wallpaper pattern $22\times$.

If you identify to get the torus or Klein bottle, all four corners come together in a single point which is surrounded by four right angles of paper and so there is no angle deficit.

6.2.15 Exercise (NM) For the projective plane, take the map as in the figure and compute the Euler characteristic to prove that the topologists are (in their way) correct. Compute also the orbifold Euler characteristic, assuming that we have the two cone points.

Check also that if you change the indicated direction of just one edge so that you have a Klein bottle, you get the correct Euler characteristic for that also.

6.2.16 Kaleidoscopes Here the orbifolds are polygons in the plane case and spherical triangles and similar in the spherical case. This includes the interesting case of *NN* where the orbifold is a spherical 2-gon, with vertices at the North and South poles and two lines of longitude as edges. (The shape is also called a "lune".)

6.2.17 Exercise () Sketch the orbifold for $2{*}22$ and define a map on it and use it to compute the orbifold Euler characteristic and check that it is as you would expect.

6.2.18 Exercise (NM) Relate the orbifold of $**$ (a cylinder) and of $*\times$ (a Möbius band) to fundamental regions in an example pattern.

6.2.19 Final thoughts Has what we have done "proved" that our classification of wallpaper groups and finite subgroups of $O(3)$ is correct and complete?

One issue is that of existence: it is clear that for a wallpaper or spherical symmetry group to exist there must be an orbifold with the correct orbifold Euler characteristic. But if we have an orbifold with a possible value of the orbifold Euler characteristic, does that mean that a group exists and is unique? As we saw, some possible signatures such as *MN* with $M \neq N$ do not result in spherical symmetry groups. But all those groups we have listed do exist because we can find patterns with those symmetries. Uniqueness (a careful discussion of which depends on having a good definition of "unique") one can in fact establish case by case.

I would probably argue that what we have done so far explains and makes sense of the classification, connects it with lots of important ideas and establishes a great notation. But it is probably not the sort of proof you would want to stake your reputation on if the final results were not well-established. Where are the gaps? What do you think?

6.3 Isohedral tilings and Heesch types

6.3.1 Isohedral tilings and Catalan things

6.3.1 The problem A topic of long-standing interest is tessellating the plane (or indeed the sphere) with identical (not necessarily regular polygon) tiles. More precisely, we will consider tilings where the symmetry group of the tiling is a wallpaper group that acts transitively on the tiles. We call such things "isohedral tilings".

If the symmetry group includes reflections, then one will need two types of tiles, the types being mirror images of each other if you want in practice to cover your bathroom wall.

6.3.2 Catalan tilings/solids We are already familiar with one class of isohedral tilings of the plane or sphere, the Catalan tilings and Catalan solids. These are the duals of Archimedean (i.e. vertex regular but not regular) tilings and convex polyhedra. They are isohedral since the dual of a vertex transitive object is face transitive.

6.3.3 Wythoff construction for Catalan objects It is instructive to consider what the vertices and edges of the Catalan object look like in the Wythoff construction. The rule is simple:

- The vertices of the Catalan object are the vertices of the Wythoff triangle that are the centres of polygons if one constructs the Archimedean object.
- The edges of the Catalan object are the edges of the Wythoff triangle that meet an edge of the Archimedean object perpendicularly.

The Wythoff triangles (omitting the two of the seven cases that lead to honeycombs and regular polyhedra) can be seen in Table 6.1.

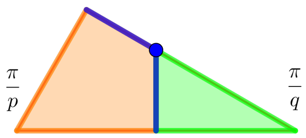
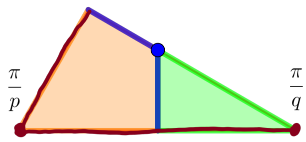
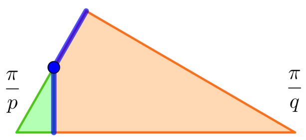
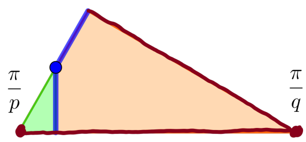
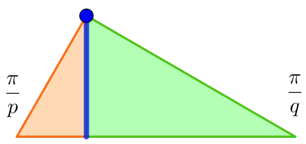
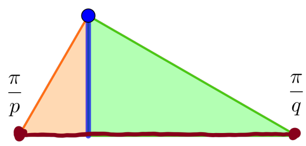
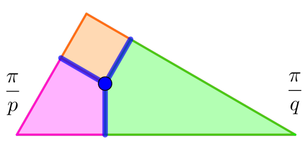
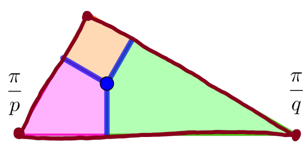
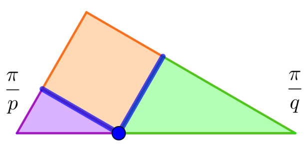
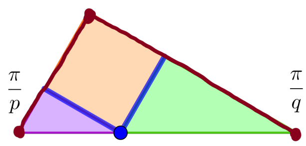
Archimedean	**Catalan**	**Notes**
		Catalan face is an icosoles triangle made of two Wythoff triangles.
		Catalan face is an icosoles triangle made of two Wythoff triangles.
		Archimedean and Catalan have only one type of edge. So these are edge-transitive. Catalan face is a rhombus (four Wythoff triangles).
		Catalan face is the Wythoff triangle.
		Catalan face is a kite (two Wythoff triangles).

Table 6.1: Wythoff for Catalan. The vertices and edges of the Catalan object are in deep red.

6.3.2 Heesch types

6.3.1 The problem Given a wallpaper pattern, classify the ways we can tile the plane with tiles (not necessarily polygons) with each tile being a fundamental domain?

Clearly for kaleidoscopic groups, there is only one option as all mirror lines must be tile boundaries and so there is one and only one such tiling. But for other groups there are more possibilities.

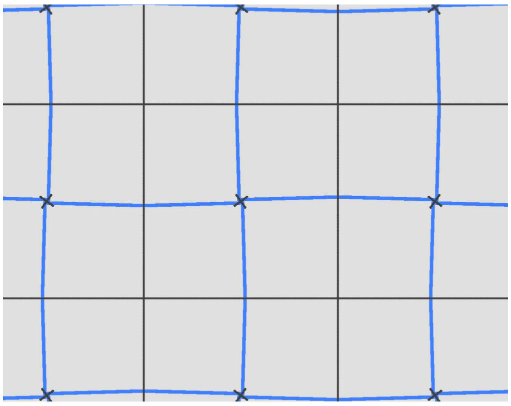
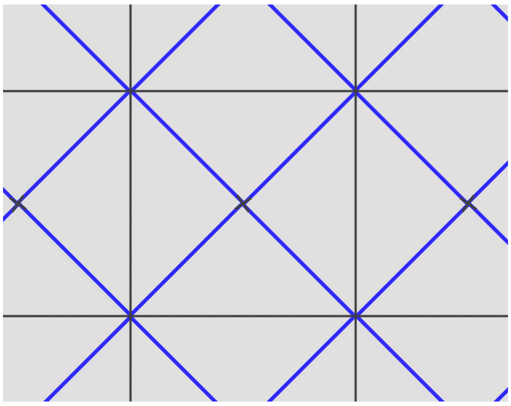
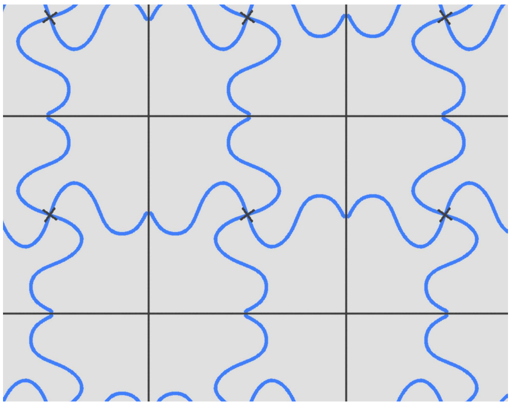
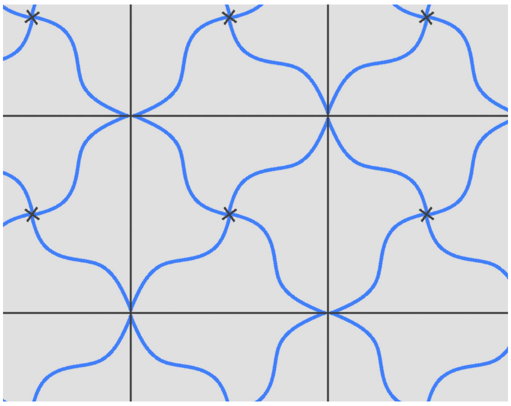
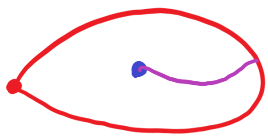
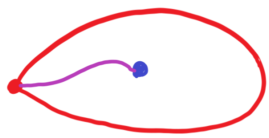
Heesch type 1	**Heesch type 2**
	
	
	
A walk around a tile: Gyration - edge - point where 4 tiles meet - mirror - kaleidoscope point - mirror - point where 4 tiles meet - edge - Gyration	**A walk around a tile:** Gyration - edge - kaleidoscope point - mirror - kaleidoscope point - edge - gyration

Table 6.2: Heesch types for $4{*}2$. (The tile boundaries are the black mirror lines and the blue pieces.)

6.3.2 Example Consider for example $4{*}2$. The top row shows the two obvious ways of dividing a square surrounded by mirrors (in black) into four fundamental regions: joint the

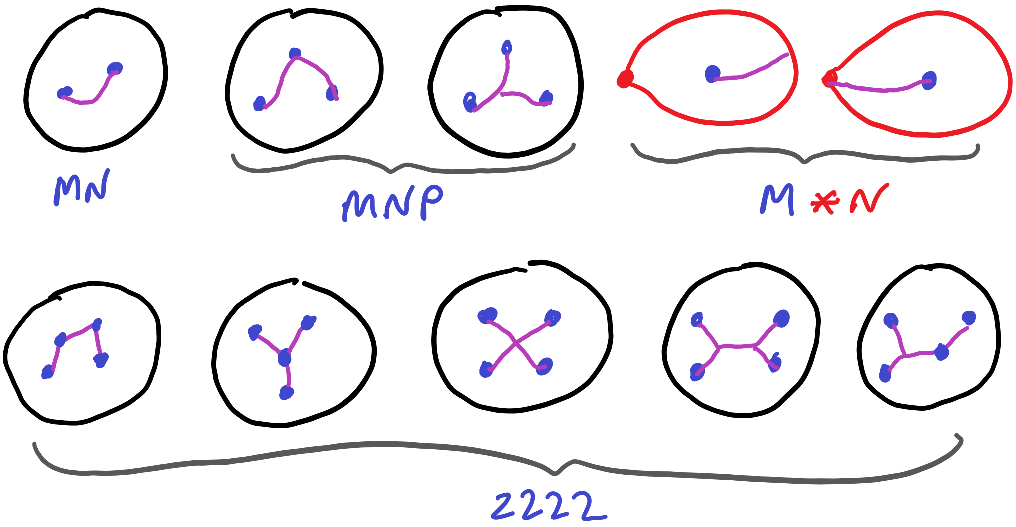


Figure 6.3: Heesch types for some gyratory and gyroscopic groups. (Note that the left-hand *MNP* case corresponds to 3 different Heesch types according to which of the gyrations is the one connected to both the others.)

gyration at the centre of the square by a straight line either to the centre of an edge or to the centre of a side. But as in the second row, it works equally well if we replace the straight lines by general curves, and in the first case the blue curve does not have to meet the edge at its endpoint.

The Type 1 cases on the left are qualitatively different from those on the right as shown by checking the order of features one encounters on walking round a tile.

6.3.3 Relation to the orbifold The orbifold for $4{*}2$ is a cone with a corner point on its reflecting boundary, as shown in the schematic orbifold pictures in Table 6.2. *In order to form a fundamental domain which can be laid flat on the plane, you have to "cut open" the orbifold from the gyration to the boundary.* There are just two distinct ways two do this, according as to whether one arrives on the boundary at the corner point or not.

Note that the walks around the boundary are deducible from the cut orbifold: for type 1, start at the gyration, walk along the top edge of the purple cut, arrive at the boundary at a point where four tiles will meet, turn left along the mirror, passing the kaleidoscope point and continue to visit the "four tiles meeting" point again and return to the gyration along the bottom edge of the purple cut.

6.3.4 General theory The Heesch classification of tilings by fundamental domains is thus a classification of distinct ways to make cuts in an orbifold so that the orbifold becomes topologically a disc. Additionally, there must be at least one cut emerging from every cone point and cuts can only begin or end at cone points, on a reflecting boundary (either at "ordinary" points or at corners) or by meeting other cuts.

The theory above gives the Heesch classification for all groups of the form *N*∗*M* since the same orbifold picture works for all these cases. As mentioned before, the theory is trivial for kaleidoscopic groups.

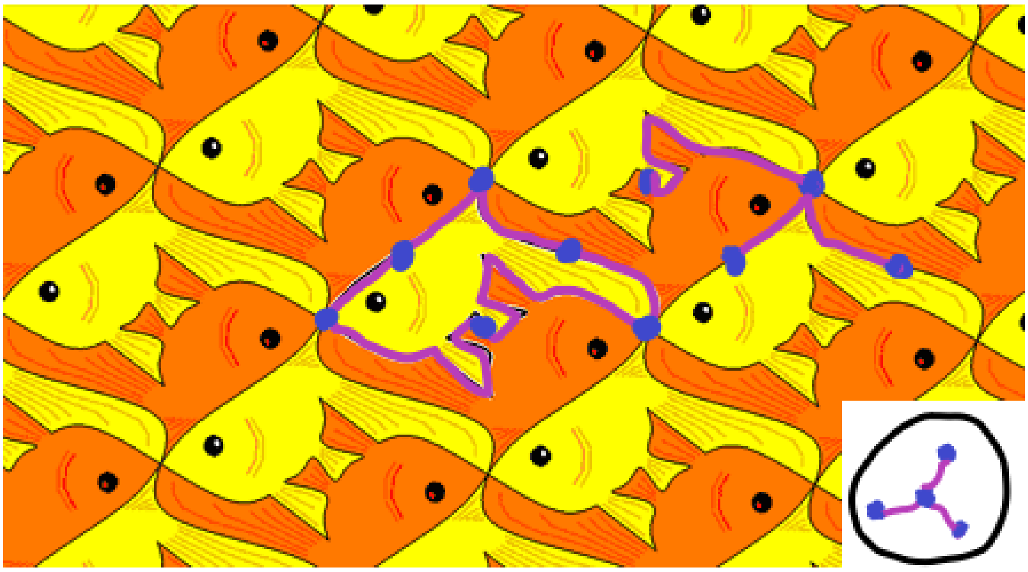


Figure 6.4: A Heesch tessellation with group 2222 (ignoring the colouring).

6.3.5 Gyratory groups We will concentrate on purely gyratory groups where the orbifold is a sphere with 2,3 or four gyrations. Topologically, we simply have to cut open the sphere so that it becomes a disk topologically. There are no reflecting boundaries in this case. The results are in Figure 6.3: there are only two basic cases for groups with three gyrations but five if there are four gyrations.

Note also that for the group *MNP*, in the left-hand case in Figure 6.3 there are three different Heesch types depending on which gyration is connected to both the others.

6.3.6 Example Figure 6.4 has symmetry 2222 ignoring the colours. Its Heesch type is drawn at the bottom left of the picture. One can identify the type by looking at the outlined tile near the centre and noticing that traversing its boundary one encounters gyrations ABACADA where A,B,C,D are the three gyrations defining the pattern. Alternatively, towards the top right one can see the pattern of orbifold cuts realised in the tessellation.