HONOURS COMPLEX VARIABLES WORKSHOP 1 SOLUTIONS
14$^{\text{th}}$ JANUARY 2019

Solutions to questions 6, 7, and 8 are due 11.00, Monday 21$^{\text{st}}$ January 2019.

The purpose of this workshop is to gain some dexterity in calculating with complex numbers. It should be mostly revision, but make sure you master these calculations, since this is the language of this course.

Question 1. Evaluate the following complex numbers, in the form $x + iy$:
(a) $(1 + 2i)^3$;
(b) $\frac{5}{-3+4i}$;
(c) $\left(\frac{2+i}{3-2i}\right)^2$; and
(d) $(1 + i)^n + (1 - i)^n$ for a non-negative integer n.

There is no trickery here—just do the algebra. The last one might benefit from testing one or two small cases first to see what happens.

Solution 1. (a) Since $(1 + 2i)^2 = 1 - 4 + 4i = -3 + 4i$, we have that $(1 + 2i)^3 = (1 + 2i)^2(1 + 2i) = (-3 + 4i)(1 + 2i) = -3 - 8 - 2i = -11 - 2i$.

(b) Multiplying the top and bottom of the fraction by the complex conjugate $-3 - 4i$ of the denominator, we have that

$$\frac{5}{-3+4i} = \frac{5}{-3+4i} \cdot \frac{-3-4i}{-3-4i} = \frac{-15-20i}{25} = \frac{-3}{5} - \frac{4}{5}i.$$

(c) We have that

$$\begin{aligned}\left(\frac{2+i}{3-2i}\right)^2 = \frac{(2+i)^2}{(3-2i)^2} = \frac{4-1+4i}{9-4-12i} = \frac{3+4i}{5-12i} = \frac{3+4i}{5-12i} \cdot \frac{5+12i}{5+12i} &= \frac{15-48+56i}{169} \\ &= -\frac{33}{169} + \frac{56}{169}i.\end{aligned}$$

(d) Notice that $(1 + i)^2 = 1 - 1 + 2i = 2i$, so $(1 + i)^4 = \left((1 + i)^2\right)^2 = (2i)^2 = -4$, and similarly $(1-i)^2 = -2i$, so $(1-i)^4 = (-2i)^2 = -4$. Thus $(1+i)^{4k} = (-4)^k = (1-i)^{4k}$ for any integer k. We also have that

$$\begin{aligned}(1 + i) + (1 - i) &= 2, \\ (1 + i)^2 + (1 - i)^2 &= 2i - 2i = 0, \text{ and} \\ (1 + i)^3 + (1 - i)^3 &= 2i(1 + i) - 2i(1 - i) = 2i\left((1 + i) - (1 - i)\right) = 2i(2i) = -4,\end{aligned}$$

so writing $n = 4k + l$, for some non-negative integer k and $l = 0, 1, 2, 3$, we have that

$$\begin{aligned}(1 + i)^n + (1 - i)^n = (1 + i)^{4k+l} + (1 - i)^{4k+l} &= (-4)^k(1 + i)^l + (-4)^k(1 - i)^l \\ &= (-4)^k \left((1 + i)^l + (1 - i)^l\right) \\ &= \begin{cases} 2(-4)^k & l = 0, 1; \\ 0 & l = 2; \\ -4(-4)^k & l = 3. \end{cases}\end{aligned}$$

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Question 2. Let $z = x + iy$. Find the real and imaginary parts of the following complex numbers, in terms of x and y:
(a) z^4;
(b) $\frac{z-1}{z+1}$; and
(c) $1/z^2$.

Again, there is no trickery here. However, it will be more efficient, if nothing else, to do as much algebra as you can in terms of z, and associated quantities like $\overline{z}$ and $|z|$, before switching to x and y.

Solution 2. (a) $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, so

$$z^4 = (x^2-y^2+2ixy)^2 = (x^2-y^2)^2 - 4x^2y^2 + 4(x^2-y^2)xyi = x^4 - 6x^2y^2 + y^4 + 4(x^2-y^2)xyi,$$

thus $\mathrm{Re}(z^4) = x^4 - 6x^2y^2 + y^4$ and $\mathrm{Im}(z^4) = 4(x^2-y^2)xy$.

(b) Recall the formulae $x^2 + y^2 = |z|^2 = z\overline{z}$ and $y = \mathrm{Im}(z) = (z - \overline{z})/2i$. Then multiplying the top and bottom of the fraction by the complex conjugate of the denominator gives us

$$\frac{z-1}{z+1} = \frac{z-1}{z+1} \cdot \frac{\overline{z}+1}{\overline{z}+1} = \frac{z\overline{z} + z - \overline{z} - 1}{|z+1|^2} = \frac{|z|^2 + 2i\,\mathrm{Im}(z) - 1}{(x+1)^2 + y^2} = \frac{x^2+y^2-1+2iy}{(x+1)^2+y^2},$$

so $\mathrm{Re}\left(\frac{z-1}{z+1}\right) = \frac{x^2+y^2-1}{(x+1)^2+y^2}$ and $\mathrm{Im}\left(\frac{z-1}{z+1}\right) = \frac{2y}{(x+1)^2+y^2}$.

(c) Recalling that $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$, we see that

$$\frac{1}{z^2} = \frac{\overline{z^2}}{|z^2|^2} = \frac{\overline{z}^2}{|z|^4} = \frac{(x-iy)^2}{(x^2+y^2)^2} = \frac{x^2-y^2-2ixy}{(x^2+y^2)^2},$$

so $\mathrm{Re}\left(\frac{1}{z^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2}$ and $\mathrm{Im}\left(\frac{1}{z^2}\right) = -\frac{2xy}{(x^2+y^2)^2}$.

Question 3. By solving the equation $z^2 = w$ for a complex number $z = x + iy$, find the square roots of the following complex numbers w:
(a) i;
(b) $-i$; and
(c) $\frac{1-i\sqrt{3}}{2}$.

This is not necessarily the easiest way to find square roots of complex numbers, but it is not too painful. You should think about whether part (b) really needs much work, having done part (a).

Solution 3. Since $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, we find the square roots of each w by equating the real and imaginary parts of the numbers $x^2 - y^2 + 2ixy$ and $w = \mathrm{Re}(w) + i\,\mathrm{Im}(w)$.

(a) The two equations are $x^2 - y^2 = 0$ and $2xy = 1$. The first equation implies that $y = \pm x$, but $y = -x$ would imply, by the second equation, that $0 \geq -x^2 = xy = 1/2 > 0$, which is a contradiction. So $y = x$ and hence $2x^2 = 1$, so that $x = y = \pm\frac{1}{\sqrt{2}}$. Thus the square roots are $\pm(1+i)/\sqrt{2}$.

(b) For any $z, w \in \mathbb{C}$, we have that if $z^2 = w$, then $(iz)^2 = i^2z^2 = -z^2 = -w$. So if z is a square root of w, then iz is a square root of $-w$. So by part (a), the square roots of $-i$ are $\pm i(1+i)/\sqrt{2} = \pm(1-i)/\sqrt{2}$.

(c) The two equations are $x^2 - y^2 = 1/2$ and $2xy = -\sqrt{3}/2$. The second equation implies that $x \neq 0$ and $y = -\sqrt{3}/4x$. Substituting this into the first equation gives the equation $x^2 - 3/(16x^2) = 1/2$, which rearranges to the quartic equation $16x^4 - 8x^2 - 3 = 0$. This factorizes as $(4x^2+1)(4x^2-3) = 0$, which implies that $x^2 = -1/4$ or $x^2 = 3/4$, but since x is real, evidently the only possibility is that

$x^2 = 3/4$, so $x = \pm\sqrt{3}/2$. This implies that $y = \mp 1/2$. Hence the square roots are $\pm(\sqrt{3} - i)/2$.

Question 4. Find all values of the fourth roots of -1, i.e. $z \in \mathbb{C}$ such that $z^4 = -1$.

With the question appropriately framed, you should be able to answer this without much work, using the previous question.

Solution 4. Suppose $z^4 = -1$. Then $z^2 = w$ where $w^2 = -1$. But then $w = \pm i$, so we just need to find the square roots of $\pm i$. From the previous question then the four values are

$$z = \frac{1+i}{\sqrt{2}},\ \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}},\ \frac{-1-i}{\sqrt{2}}.$$

Question 5. Find the modulus of each of the following complex numbers:

(a) $-2i(3+i)(2+4i)(1+i)$; and

(b) $\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$.

Only multiply these values out if you really think you have to...

Solution 5. (a) Since $|-2i| = 2$, $|3+i| = \sqrt{10} = \sqrt{2}\sqrt{5}$, $|2+4i| = \sqrt{20} = 2\sqrt{5}$, and $|1+i| = \sqrt{2}$, repeatedly using the fact that $|zw| = |z|\,|w|$, we have that

$$|-2i(3+i)(2+4i)(1+i)| = 2(\sqrt{2}\sqrt{5})(2\sqrt{5})\sqrt{2} = 2^3 \cdot 5 = 40.$$

(b) Similarly, since $|3+4i| = 5$, $|-1+2i| = \sqrt{5}$, $|-1-i| = \sqrt{2}$, and $|3-i| = \sqrt{10} = \sqrt{2}\sqrt{5}$, we have that

$$\left|\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right| = \frac{5\sqrt{5}}{\sqrt{2}(\sqrt{2}\sqrt{5})} = \frac{5}{2}.$$

Question 6. Show that there does not exist $z \in \mathbb{C}$ such that

$$|z| = 1 = |z + 2(1+i)|\,.$$

As is so often the case, a picture might be a good starting place. Persuade yourself geometrically that no such point exists, and use this to motivate a proof. **[5 marks]**

Solution 6. Suppose for a contradiction that such a $z \in \mathbb{C}$ existed. Then by the triangle inequality, since $|-z| = |z|$, we have that

$$2\sqrt{2} = |2(1+i)| = |-z + (z + 2(1+i))| \le |-z| + |z + 2(1+i)| = 1 + 1 = 2,$$

which is a contradiction, since $\sqrt{2} > 1$.

Question 7. (a) Prove that for all $z \in \mathbb{C}$, we have

$$|z| \le |\mathrm{Re}(z)| + |\mathrm{Im}(z)| \le \sqrt{2}\,|z|\,.$$

(b) (i) Find a value of $z \in \mathbb{C}$ such that $|z| = |\mathrm{Re}(z)| + |\mathrm{Im}(z)| = \sqrt{2}\,|z|$.

(ii) Find a value of $z \in \mathbb{C}$ such that $|z| < |\mathrm{Re}(z)| + |\mathrm{Im}(z)| < \sqrt{2}\,|z|$.

It is usually easier to work with the square of the modulus rather than the modulus itself. The second part of this question asks you to show that the inequalities are "sharp": that neither of them may, in general, be improved to a stronger statement, in the form of either a strict inequality, or an equality. **[10=(5+(2+3)) marks]**

Solution 7. (a) Let $z = x + iy \in \mathbb{C}$. Then by the triangle inequality and the fact that $|i| = 1$, we have that

$$|z| = |x + iy| \le |x| + |iy| = |x| + |i|\,|y| = |x| + |y| = |\mathrm{Re}(z)| + |\mathrm{Im}(z)|\,.$$

Furthermore, since $0 \le (|x| - |y|)^2 = |x|^2 - 2\,|x|\,|y| + |y|^2$, and so $2\,|x|\,|y| \le |x|^2 + |y|^2$, we have that

$$(|\mathrm{Re}(z)| + |\mathrm{Im}(z)|)^2 = |x|^2 + 2\,|x|\,|y| + |y|^2 \le 2\,|x|^2 + 2\,|y|^2 = 2\,|z|^2\,.$$

Since both $|\mathrm{Re}(z)| + |\mathrm{Im}(z)|$ and $2\,|z|$ are non-negative real numbers, taking square roots gives us the required inequality.

(b) (i) $z = 0$ satisfies $|z| = |0| = \sqrt{2}\,|0| = \sqrt{2}\,|z|$; that $|\mathrm{Re}(z)| + |\mathrm{Im}(z)|$ equals this common value too is of course trivial, but it also follows by the first part of the question.

(ii) $z = 1 + 2i$ satisfies

$$|z| = \sqrt{5} < 3 = 1 + 2 = |\mathrm{Re}(z)| + |\mathrm{Im}(z)| = 3 < \sqrt{10} = \sqrt{2}\sqrt{5} = \sqrt{2}\,|z|\,.$$

Question 8. The usual order relation $<$ on $\mathbb{R}$ satisfies the two conditions

(a) if $x \neq 0$, then precisely one of $0 < x$ or $0 < -x$ holds; and
(b) if $0 < x, y$, then $0 < x + y$ and $0 < xy$.

Show that there does not exist a relation $<$ on $\mathbb{C}$ which satisfies both these conditions.

Thus there is no natural ordering on the complex numbers. Inequalities hold between real numbers, as usual, but no meaningful inequalities hold between complex numbers.

[5 marks]

Solution 8. Suppose for a contradiction that we have such an order $<$ on $\mathbb{C}$. Since $i \neq 0$, the first condition implies that $0 < i$ or $0 < -i$. In either case, we infer from the second condition that $0 < (-i)^2 = i^2 = -1$. Note that this is *not* a contradiction yet, since there is no reason to assume that this hypothetical order on $\mathbb{C}$ is the same as the usual order on $\mathbb{R}$. But the same argument shows that $0 < (-1)^2 = 1^2 = 1$. This is a contradiction with the first condition, since now both $0 < 1$ and $0 < -1$.

HONOURS COMPLEX VARIABLES WORKSHOP 2 SOLUTIONS 21$^{\text{st}}$ JANUARY 2019

Solutions to questions 5 and 6 are due 11.00, Monday 4$^{\text{th}}$ February 2019.

The aim of this workshop is to become familiar with manipulating and calculating the arguments of complex numbers. We shall investigate what general properties do and do not hold of the multi-valued argument function arg and the single-valued principal value Arg. We also see how to use exponential forms both to compute explicit complex numbers and prove general statements.

Question 1. Let $\theta, \phi \in \mathbb{R}$, and $n \in \mathbb{Z}$. Prove that

(a) $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$;
(b) $e^{-i\theta} = \frac{1}{e^{i\theta}}$; and
(c) $e^{in\theta} = \left(e^{i\theta}\right)^n$.

The last statement abbreviates de Moivre's formula:

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n .$$

This establishes that the exponential form notation, introduced formally just as an abbreviation for the polar form, can indeed be manipulated using the laws of exponentials with which you are familiar.

Solution 1. (a) We use the trigonometric addition formulae to see that

$$\begin{aligned} e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi) &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \sin\phi\cos\theta) \\ &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= e^{i\theta}e^{i\phi}. \end{aligned}$$

(b) From part (a) applied to θ and $-\theta$ we have that

$$e^{i\theta}e^{-i\theta} = e^{i(\theta-\theta)} = e^{i0} = \cos 0 + i\sin 0 = 1,$$

so $e^{-i\theta} = 1/e^{i\theta}$.

(c) We use induction on n to show the result for all positive n. The base case $n = 1$ is evidently true. For the inductive step, suppose for some $n \geq 2$ that the result is true for $n-1$. Then, using part (a) applied to $(n-1)\theta$ and θ, and then the inductive hypothesis, we have that

$$e^{in\theta} = e^{i((n-1)\theta+\theta)} = e^{i(n-1)\theta}e^{i\theta} = \left(e^{i\theta}\right)^{n-1}e^{i\theta} = \left(e^{i\theta}\right)^n.$$

Now suppose $n < 0$. Then $e^{i\theta n} = e^{i(-\theta)(-n)}$, where $-n > 0$, so the result we just proved for positive powers implies that $e^{i\theta n} = \left(e^{-i\theta}\right)^{-n}$. We then apply part (b) to see that $\left(e^{-i\theta}\right)^{-n} = \left(1/e^{i\theta}\right)^{-n}$. But then since $(1/z)^k = 1/z^k$ for any non-zero $z \in \mathbb{C}$ and integer $k > 0$, we conclude that

$$e^{i\theta n} = \left(e^{-i\theta}\right)^{-n} = \left(\frac{1}{e^{i\theta}}\right)^{-n} = \frac{1}{\left(e^{i\theta}\right)^{-n}} = \left(e^{i\theta}\right)^n.$$

The result for 0 follows by direct application of the definition:

$$e^{i0\theta} = e^{i0} = \cos 0 + i\sin 0 = 1 = \left(e^{i\theta}\right)^0;$$

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or one could combine the previous results and observe that

$$\left(e^{i\theta}\right)^0 = 1 = e^{i\theta}\frac{1}{e^{i\theta}} = e^{i\theta}e^{-i\theta} = e^{i\theta-i\theta} = e^{i0}.$$

Question 2. Let $z, w \in \mathbb{C}$ be non-zero. For each of the following statements, either prove that it is true or give a counterexample to show that it is false:

(a) $\arg(-1) = -\arg(-1)$;
(b) $\arg(z^2) = 2\arg(z)$; and
(c) $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$.

Some pictures might give some inspiration. Consider carefully, for part (c), what range of values Arg *may take.*

Solution 2. (a) This is true.

Proof. We have that, by definition,

$$\arg(-1) = \left\{\theta \in \mathbb{R} : e^{i\theta} = -1\right\} = \{\, \pi + 2k\pi : k \in \mathbb{Z}\,\}, \text{ and}$$
$$-\arg(-1) = \{\, -\theta : \theta \in \arg(-1)\,\} = \{-(\pi + 2k\pi) : k \in \mathbb{Z}\,\}.$$

So $\theta \in \arg(-1)$ if and only if there exists $k \in \mathbb{Z}$ such that $\theta = \pi + 2k\pi$. But $\theta = \pi + 2k\pi$ for some integer k if and only if $\theta = -\pi + 2(k+1)\pi = -(\pi + 2(-k-1)\pi)$, where $-k-1$ is also an integer, i.e. if and only if $\theta \in -\arg(-1)$. $\square$

(b) This is false. Let $z = -1$. Then $e^{i0} = 1 = z^2$, so $0 \in \arg(z^2)$, but $e^{i0/2} = 1 \neq -1 = z$, so $0/2 \notin \arg(z)$, thus $0 \notin 2\arg(z)$.

(c) This is false. Let $z = -1 + i = w$. Then $\operatorname{Arg}(z) = \operatorname{Arg}(w) = 3\pi/4$, but by definition $\operatorname{Arg}(zw) \in (-\pi, \pi]$, so, since $zw = (-1+i)^2 = -2i$,

$$\operatorname{Arg}(zw) = \operatorname{Arg}(-2i) = -\frac{\pi}{2} \neq \frac{3\pi}{2} = \frac{3\pi}{4} + \frac{3\pi}{4} = \operatorname{Arg}(z) + \operatorname{Arg}(w).$$

Question 3. Let $z, w \in \mathbb{C}$ be non-zero. Prove that

(a) $\arg(zw) = \arg(z) + \arg(w)$; and
(b) $\arg(1/z) = \arg(\overline{z}) = -\arg(z)$.

This demonstrates that arguments, when considered as a set of all possible values, do indeed behave like exponents.

Solution 3. (a) Suppose that $z = |z|\, e^{i\theta}$ and $w = |w|\, e^{i\phi}$ in exponential form, for some $\theta, \phi \in \mathbb{R}$. By definition $\theta \in \arg(z)$ and $\phi \in \arg(w)$. Then by question 1, $zw = |z|\, e^{i\theta}\, |w|\, e^{i\phi} = |z|\,|w|\, e^{i(\theta+\phi)}$, so $\theta + \phi \in \arg(zw)$ by definition. Thus

$$\arg(z) = \{\, \theta + 2k\pi : k \in \mathbb{Z}\,\},$$
$$\arg(w) = \{\, \phi + 2l\pi : l \in \mathbb{Z}\,\}, \text{ and}$$
$$\arg(zw) = \{\, (\theta + \phi) + 2m\pi : m \in \mathbb{Z}\,\}.$$

So $\psi \in \arg(z) + \arg(w)$ if and only if there exist $k, l \in \mathbb{Z}$ such that $\psi = (\theta + 2k\pi) + (\phi + 2l\pi) = (\theta + \phi) + 2(k+l)\pi$, if and only if $\psi = (\theta + \phi) + 2m\pi$ for some $m \in \mathbb{Z}$, i.e. if and only if $\psi \in \arg(zw)$.

(b) Recall that cos is an even function and sin is an odd function, i.e. $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$ for all $x \in \mathbb{R}$. Then by definition of the complex conjugate we have that

$$\overline{re^{i\theta}} = \overline{r\cos\theta + ir\sin\theta} = r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta}.$$

Thus $\theta \in \arg(z)$ if and only if $-\theta \in \arg(\overline{z})$, i.e. $\arg(\overline{z}) = -\arg(z)$. Furthermore, by question 1,

$$\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta},$$

so $\theta \in \arg(z)$ if and only if $-\theta \in \arg(1/z)$, thus $\arg(1/z) = -\arg(z)$, as required.

Question 4. Prove that $(1+i)^{95} = 2^{47}(1-i)$.

In what form is it best to write a complex number so that you can most easily calculate large powers of it?

Solution 4. Writing $1+i = \sqrt{2}e^{i\pi/4}$, we see, using question 1, that

$$\begin{aligned}(1+i)^{95} = \left(\sqrt{2}e^{i\pi/4}\right)^{95} = \sqrt{2}^{95} e^{95i\pi/4} = \sqrt{2}^{2\cdot 47+1} e^{(11\cdot 8+7)i\pi/4} &= 2^{47}\sqrt{2}e^{11(2i\pi)+7i\pi/4} \\ &= 2^{47}\sqrt{2}\left(e^{2i\pi}\right)^{11} e^{7i\pi/4} \\ &= 2^{47}\sqrt{2}e^{7i\pi/4} \\ &= 2^{47}(1-i).\end{aligned}$$

Question 5. Find all the sixth roots of $i/(1+i)$, i.e. all solutions $z \in \mathbb{C}$ to the equation $z^6 = i/(1+i)$.

[5 marks]

You can—as you possibly did in workshop 1—find square roots of complex numbers by considering cartesian forms, and comparing real and imaginary parts of an appropriate equation. This method would, in principle, work here too, but the algebra would be ghastly. You are recommended not to use the cartesian form here. How many solutions do you expect?

Solution 5. The six roots are $2^{-1/12}e^{i(\pi/24+l\pi/3)}$ for $l = 0,1,2,3,4,5$.

Proof. Suppose $z = re^{i\theta} \in \mathbb{C}$ satisfies $z^6 = i/(1+i)$, for some $r > 0$ and $\theta \in \mathbb{R}$. Then writing the numerator and denominator of $i/(1+i)$ in exponential form,

$$r^6 e^{6i\theta} = \left(re^{i\theta}\right)^6 = z^6 = \frac{i}{1+i} = \frac{e^{i\pi/2}}{\sqrt{2}e^{i\pi/4}} = \frac{1}{\sqrt{2}}e^{i(\pi/2-\pi/4)} = \frac{1}{\sqrt{2}}e^{i\pi/4}.$$

So $r^6 = 1/\sqrt{2}$, and $6\theta \in \{\,\pi/4 + 2k\pi : k \in \mathbb{Z}\,\}$. Hence $r = \left(2^{-1/2}\right)^{1/6} = 2^{-1/12}$, and

$$\begin{aligned}\theta \in \{\,(\pi/4+2k\pi)/6 : k \in \mathbb{Z}\,\} &= \{\,\pi/24 + k\pi/3 : k \in \mathbb{Z}\,\} \\ &= \{\,\pi/24 + l\pi/3 + 2\pi k : l = 0,1,2,3,4,5,\ k \in \mathbb{Z}\,\}.\end{aligned}$$

So $z \in \left\{\, 2^{-1/12}e^{i(\pi/24+l\pi/3)} : l = 0,1,2,3,4,5 \,\right\}$. □

Question 6. Let m be a positive integer, and $\theta \in (0, 2\pi)$. Prove that

$$2\sum_{k=0}^{m} \cos(k\theta) = 1 + \frac{\sin((m+\frac{1}{2})\theta)}{\sin(\theta/2)}.$$

[5 marks]

You may—and are advised to—assume that $\sum_{k=0}^m z^k = (1-z^{m+1})/(1-z)$, for any $z \neq 1$. A connection between this piece of information, about powers of complex numbers, and trigonometric functions of an integer multiple of a variable, as in the left-hand side of the equation to be proved, is provided by de Moivre's formula, which you established in question 1 and may also assume.

Solution 6. Since $\theta \in (0, 2\pi)$, we know that $e^{i\theta} \neq 1$, so we may apply the hint to see, using de Moivre's formula, that

$$\sum_{k=0}^{m} e^{ik\theta} = \sum_{k=0}^{m} \left(e^{i\theta}\right)^k = \frac{1-e^{i(m+1)\theta}}{1-e^{i\theta}} = \frac{-e^{i\theta/2}\left(e^{i(m+\frac{1}{2})\theta} - e^{-i\theta/2}\right)}{-e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)} = \frac{e^{i(m+\frac{1}{2})\theta} - e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}},$$

where we have taken out a factor of $-e^{i\theta/2}$ precisely so that we have a term of the form $e^{i(m+\frac{1}{2})\theta}$ on the right-hand side. Now, $e^{i\theta/2} - e^{-i\theta/2} = e^{i\theta/2} - \overline{e^{i\theta/2}} = 2i\sin(\theta/2)$, so we can multiply by 2 and further simplify the expression above to see that

$$\begin{aligned}
2\sum_{k=0}^{m} e^{ik\theta} &= \frac{e^{i(m+\frac{1}{2})\theta} - e^{-i\theta/2}}{i\sin(\theta/2)} \\
&= \frac{ie^{-i\theta/2} - ie^{i(m+\frac{1}{2})\theta}}{\sin(\theta/2)} \\
&= \frac{i\left(\cos(-\theta/2) + i\sin(-\theta/2)\right) - i\left(\cos((m+\frac{1}{2})\theta) + i\sin((m+\frac{1}{2})\theta)\right)}{\sin(\theta/2)} \\
&= \frac{i\cos(-\theta/2) - \sin(-\theta/2) - i\cos((m+\frac{1}{2})\theta) + \sin((m+\frac{1}{2})\theta)}{\sin(\theta/2)} \\
&= \frac{-\sin(\theta/2) + \sin((m+\frac{1}{2})\theta)}{\sin(\theta/2)} + i\frac{\cos(-\theta/2) - \cos((m+\frac{1}{2})\theta)}{\sin(\theta/2)}.
\end{aligned}$$

This is an equation between two complex numbers, so the real and imaginary parts must be equal. Considering the real parts, and using that sin is an odd function, we see that

$$\begin{aligned}
2\sum_{k=0}^{m}\cos(k\theta) = 2\sum_{k=0}^{m}\mathrm{Re}\left(e^{ik\theta}\right) = \mathrm{Re}\left(2\sum_{k=0}^{m} e^{ik\theta}\right) &= \frac{-\sin(-\theta/2)}{\sin(\theta/2)} + \frac{\sin((m+\frac{1}{2})\theta)}{\sin(\theta/2)} \\
&= 1 + \frac{\sin((m+\frac{1}{2})\theta)}{\sin(\theta/2)},
\end{aligned}$$

as required.

HONOURS COMPLEX VARIABLES WORKSHOP 3 SOLUTIONS
28^{th} JANUARY 2019

Solutions to questions 7 and 8 are due 11.00, Monday 4^{th} February 2019.

This workshop aims to clarify the notions of and relationships between differentiability, the Cauchy-Riemann equations, and holomorphicity. You should get a feel for how to verify whether a given function satisfies each of these properties, by various means.

Question 1. Let $f, g\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = |z|$ and $g(z) = z/(1 + |z|)$.

(a) Show that f is continuous everywhere on $\mathbb{C}$, but nowhere holomorphic.
(b) Hence, or otherwise, show that g is continuous everywhere on $\mathbb{C}$, but nowhere holomorphic.

Lemma 1.4.2 says that differentiability implies continuity. This question shows that the converse is false. Continuity in part (a) can be easily shown from the definition, and the non-holomorphicity can be swiftly shown via Example 1.4.11. I strongly recommend using part (a) to prove part (b): continuity should be straightforward; but think carefully how you could apply part (a) to show non-holomorphicity.

Solution 1. (a) Let $z_0 \in \mathbb{C}$, and $\varepsilon > 0$. Let $\delta = \varepsilon$, and suppose $z \in \mathbb{C}$ satisfies $|z - z_0| < \delta$. Then by the reverse triangle inequality, lemma 1.1.16, we have that

$$|f(z) - f(z_0)| = ||z| - |z_0|| \leq |z - z_0| < \delta = \varepsilon.$$

Hence f is continuous at z_0.

Suppose f was holomorphic at z_0. Then since the function $z \mapsto z^2$ is everywhere holomorphic, the function $z \mapsto |z|^2$ would be holomorphic at z_0, which contradicts example 1.4.11. Thus f is nowhere holomorphic.

(b) We have just shown that $|z|$ is a continuous function, so since $1 + |z| \neq 0$ for all $z \in \mathbb{C}$, and z and 1 are also continuous functions, the algebra of continuous functions implies that g is continuous everywhere on $\mathbb{C}$.

Suppose g was holomorphic at z_0. Note that, rearranging the definition of g, we have that

$$f(z) = |z| = \frac{z - g(z)}{g(z)},$$

so if $g(z_0) \neq 0$, the algebra of holomorphic functions, lemma 1.5.2, implies that f too is holomorphic at z_0. This contradicts part (a), so we must have that $g(z_0) = 0$. But then evidently $z_0 = 0$. So the only point at which g can be holomorphic is 0, and since by definition a function cannot be holomorphic at a single point only, g is nowhere holomorphic.

Question 2. Determine whether the following functions $f\colon \mathbb{C} \to \mathbb{C}$ are differentiable at 0:

(a) $f(z) = \mathrm{Re}(z) + \mathrm{Im}(z)$; and
(b) $f(z) = (\mathrm{Re}(z))\,(\mathrm{Im}(z))$.

Showing that the Cauchy–Riemann equations fail to hold at a point implies that a function is not differentiable at that point. Showing that they do hold at a point does not *imply, without further justification, that the function is differentiable at that point. In some cases it is easiest just to use the definition of the derivative via difference quotients.*

Richard Gratwick, January 2019, file: Workshop-3-solutions.

Solution 2. (a) We can write $f = u + iv$, where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ are defined by $u(x, y) = x + y$ and $v(x, y) = 0$. So at any point $(x, y) \in \mathbb{R}^2$

$$\frac{\partial u}{\partial x}(x, y) = 1 \neq 0 = \frac{\partial v}{\partial y}(x, y), \quad \text{and} \quad \frac{\partial u}{\partial y}(x, y) = 1 \neq 0 = -\frac{\partial v}{\partial x}(x, y).$$

In particular this is true at $(x, y) = (0, 0)$. So the Cauchy–Riemann equations fail to hold at $(0, 0)$, and hence f is not differentiable at $(0, 0)$.

(b) We see that for any non-zero $z \in \mathbb{C}$, we have, by lemma 1.1.14(viii), that

$$0 \leq \left|\frac{f(z) - f(0)}{z - 0}\right| = \left|\frac{(\mathrm{Re}(z))(\mathrm{Im}(z))}{z}\right| = \frac{|\mathrm{Re}(z)|\,|\mathrm{Im}(z)|}{|z|} \leq \frac{|z|\,|z|}{|z|} = |z| \to 0 \quad \text{as } z \to 0.$$

So f is differentiable at 0: $f'(0)$ exists and equals 0.

Question 3. Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^3$. Show that there does not exist a point z on the straight line segment between 1 and i such that

$$\frac{f(i) - f(1)}{i - 1} = f'(z).$$

Thus this attempted complex analogue of the real mean value theorem is not true.

Solution 3. We see that

$$\frac{f(i) - f(1)}{i - 1} = \frac{i^3 - 1}{i - 1} = \frac{-i - 1}{i - 1} = \frac{(-i - 1)(-i - 1)}{2} = \frac{2i}{2} = i.$$

Suppose $f'(z) = 3z^2 = i$. Then $z^2 = i/3$, so $z = e^{i\pi/4}/\sqrt{3}$ or $z = e^{5i\pi/4}/\sqrt{3}$. Evidently no point on the line segment between 1 and i has argument $5\pi/4$. The only point on the line segment that has argument $\pi/4$ is the point $(1 + i)/2$, which has modulus $\sqrt{2}/2 = 1/\sqrt{2} \neq 1/\sqrt{3}$. So no point z on the line segment exists such that $f'(z) = i$.

Question 4. Determine where the following functions f are holomorphic:

(a) $f\colon \mathbb{C} \to \mathbb{C}$ defined by $f(z) = \overline{z}^2$;
(b) $f\colon \mathbb{C} \to \mathbb{C}$ defined by $f(z) = (\mathrm{Re}(z))^2 + i(\mathrm{Im}(z))^2$;
(c) $f\colon \{\, z \in \mathbb{C} : \mathrm{Re}(z) \neq 0 \,\} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} 1 & \mathrm{Re}(z) > 0, \\ -1 & \mathrm{Re}(z) < 0; \end{cases}$$

and
(d) $f\colon \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} 1 & \mathrm{Re}(z) \geq 0, \\ -1 & \mathrm{Re}(z) < 0. \end{cases}$$

Once again, the Cauchy–Riemann equations are your friend, since they allow you to spot immediately where a function is not differentiable, which in turn will tell you where a function is (not) holomorphic.

Solution 4. (a) We can write $f(x + iy) = u(x, y) + iv(x, y)$ where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $u(x, y) = x^2 - y^2$ and $v(x, y) = -2xy$. So

$$\frac{\partial u}{\partial x}(x, y) = 2x, \quad \frac{\partial v}{\partial y}(x, y) = -2x, \quad \frac{\partial u}{\partial y}(x, y) = -2y, \quad \text{and} \quad \frac{\partial v}{\partial x}(x, y) = -2y.$$

Thus $\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y)$ if and only if $x = 0$, and $\frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y)$ if and only if $y = 0$. Thus the Cauchy–Riemann equations hold only at the point $(0, 0)$, hence f can only be differentiable at 0, and hence f is nowhere holomorphic.

(b) We can write $f(x+iy) = u(x,y) + iv(x,y)$, where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $u(x,y) = x^2$ and $v(x,y) = y^2$. Then

$$\frac{\partial u}{\partial x}(x,y) = 2x, \quad \frac{\partial v}{\partial y}(x,y) = 2y, \quad \frac{\partial u}{\partial y}(x,y) = 0, \quad \text{and} \quad \frac{\partial v}{\partial x}(x,y) = 0.$$

Thus $\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$ if and only if $x = y$, and $\frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y)$ for all (x,y). So the Cauchy–Riemann equations hold only on the line $\{ z \in \mathbb{C} : z = x+iy, x = y \}$, which contains no non-empty open sets, hence f is nowhere holomorphic.

(c) f is constant and hence holomorphic on the open set $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$, and constant and hence holomorphic on the open set $\{ z \in \mathbb{C} : \operatorname{Re}(z) < 0 \}$. So f is holomorphic everywhere it is defined.

(d) As in part (c), f is holomorphic everywhere on $\{ z \in \mathbb{C} : \operatorname{Re}(z) \neq 0 \}$. However, f is not continuous at points z such that $\operatorname{Re}(z) = 0$, so f is not differentiable, hence not holomorphic, at these points. So f is holomorphic on the set $\{ z \in \mathbb{C} : \operatorname{Re}(z) \neq 0 \}$.

Question 5. Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \sqrt{|(\operatorname{Re}(z))(\operatorname{Im}(z))|}.$$

Show that f satisfies the Cauchy–Riemann equations at $z = 0$, but is not differentiable at 0.

Theorem 1.4.6 states conditions under which the Cauchy–Riemann equations holding does imply differentiability. It is a slightly fussy statement, with more conditions on the partial derivatives of the real and imaginary parts of the function. This question demonstrates why, by giving an example of a function which does satisfy the Cauchy–Riemann equations at a point, but is not differentiable at that point.

Solution 5. We can write $f(x+iy) = u(x,y) + iv(x,y)$ for some $u, v\colon \mathbb{R}^2 \to \mathbb{R}$, where $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$. Then

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt{|0x|} - 0}{x} = 0 = \frac{\partial v}{\partial y}(0,0),$$

and

$$\frac{\partial u}{\partial y}(0,0) = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y - 0} = \lim_{y \to 0} \frac{\sqrt{|0y|} - 0}{y} = 0 = -\frac{\partial v}{\partial x}(0,0),$$

hence the Cauchy–Riemann equations hold at $(0,0)$.

If the derivative of f at 0 exists, then the limit of the difference quotient $\lim_{z \to 0} \frac{f(z)-f(0)}{z-0}$ must exist and be a fixed complex number, independent of the way in which z tends to 0. We have just calculated the partial derivatives to be 0 at 0, so if the derivative $f'(0)$ exists, then $f'(0) = 0$. However, for $h > 0$, we have that

$$\frac{f(h(1+i)) - f(0)}{h(1+i)} = \frac{\sqrt{|h^2|} - 0}{h(1+i)} = \frac{1}{1+i} = \frac{1-i}{2} \neq 0.$$

So the limit of the difference quotients at 0 does not exist, and hence f is not differentiable at 0.

Question 6. Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is real-valued and holomorphic. Prove that f is constant.

A perhaps surprising result, but not particularly difficult to prove. You may assume that a function $u\colon \mathbb{R}^2 \to \mathbb{R}$ which has partial derivatives identically zero is constant—although it is instructive to try to prove this as well (it is not hard).

Solution 6. We can write $f(x+iy) = u(x,y) + iv(x,y)$, for some $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ where $v(x,y) = 0$. Since f is holomorphic, for every $(x,y) \in \mathbb{R}^2$, the Cauchy–Riemann equations hold, so we have that

$$\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 0, \quad \text{and} \quad \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) = 0.$$

Therefore u and v are both constant, hence so is f, as required. But let us prove that having partial derivatives zero implies being constant. Let $z = x + iy$. Assume that $x, y \geq 0$—the changes to the following argument in the other cases are only notational. Then

$$f(x+iy) - f(x+0i) = \int_0^y \frac{\partial u}{\partial y}(x,t)\,dt = \int_0^y 0\,dt = 0,$$

and

$$f(x+0i) - f(0+0i) = \int_0^x \frac{\partial u}{\partial x}(t,0)\,dt = \int_0^x 0\,dt = 0,$$

so $f(x+iy) = f(x+0i) = f(0)$. That is to say, f is constant.

Question 7. Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic.

(a) Prove that the function $g_1\colon \mathbb{C} \to \mathbb{C}$ defined by $g_1(z) = \overline{f(\overline{z})}$ is holomorphic.
(b) Prove that if the function $g_2\colon \mathbb{C} \to \mathbb{C}$ defined by $g_2(z) = \overline{f(z)}$ is holomorphic, then f is constant.
(c) Prove that if the function $g_3\colon \mathbb{C} \to \mathbb{C}$ defined by $g_3(z) = f(\overline{z})$ is holomorphic, then f is constant.

Here we see that we must treat complex conjugation carefully—it does not interact well with holomorphicity. The Cauchy–Riemann equations will likely be useful for parts (b) and (c), and you may again assume that a function $u\colon \mathbb{R}^2 \to \mathbb{R}$ which has partial derivatives identically zero is constant. They can, however, tell you nothing about part (a). For that, you might like to consider the definition of the derivative.

[5=(3+1+1) marks]

Solution 7. (a) Notice that for non-zero $z \in \mathbb{C}$, we have that

$$\overline{z} \cdot \overline{\left(\frac{1}{z}\right)} = \overline{\left(z \cdot \frac{1}{z}\right)} = \overline{1} = 1,$$

so $\overline{1/z} = 1/\overline{z}$. We use this in the following argument.

Let $z_0 \in \mathbb{C}$. Then for $z \in \mathbb{C}$ not equal to z_0, we see that

$$\begin{aligned}\frac{g_1(z) - g_1(z_0)}{z - z_0} = \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} = \frac{\overline{f(\overline{z}) - f(\overline{z_0})}}{z - z_0} &= \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z - z_0}}\right)} \\ &= \overline{\left(\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}}\right)} \\ &\to \overline{f'(\overline{z_0})} \quad \text{as } z \to z_0.\end{aligned}$$

So g_1 is everywhere differentiable, with $g_1'(z_0) = \overline{f'(\overline{z_0})}$, and hence everywhere holomorphic.

(b) Suppose $f = u + iv$ for $u, v\colon \mathbb{R}^2 \to \mathbb{R}$. Then $g_2 = u_2 + iv_2$, where $u_2 = u$, and $v_2 = -v$. Suppose g_2 is holomorphic. Then, since f too is holomorphic, we apply the Cauchy–Riemann equations to both to see that

$$\frac{\partial u}{\partial x} = \frac{\partial u_2}{\partial x} = \frac{\partial v_2}{\partial y} = -\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x},$$

hence $\frac{\partial u}{\partial x} = 0$. Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u_2}{\partial y} = -\frac{\partial v_2}{\partial x} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

hence $\frac{\partial u}{\partial y} = 0$. Hence too $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$, and it follows that f must be constant.

(c) Suppose $f = u + iv$ for $u, v\colon \mathbb{R}^2 \to \mathbb{R}$. Then $g_3 = u_3 + iv_3$, where $u_3(x, y) = u(x, -y)$, and $v_3(x, y) = v(x, -y)$. Suppose g_3 is holomorphic. Then, since f too is holomorphic, we apply the Cauchy–Riemann equations to both to see that

$$\frac{\partial u}{\partial x} = \frac{\partial u_3}{\partial x} = \frac{\partial v_3}{\partial y} = -\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x},$$

hence $\frac{\partial u}{\partial x} = 0$. Similarly,

$$\frac{\partial u}{\partial y} = -\frac{\partial u_3}{\partial y} = \frac{\partial v_3}{\partial x} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

hence $\frac{\partial u}{\partial y} = 0$. Hence too $\frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$, and it follows that f must be constant.

Question 8. What is the most general form of a polynomial $u(x, y) = ay^3 + by^2x + cyx^2 + dx^3$, where $a, b, c, d \in \mathbb{R}$ and the variables $x, y \in \mathbb{R}$, which is the real part of a holomorphic function? Construct the general form of the corresponding holomorphic function.

Recall lemma 1.4.13 for a property such a polynomial must satisfy. To construct the holomorphic function, you must find a harmonic conjugate for u.

[5 marks]

Solution 8. Let $u(x, y) = ay^3 + by^2x + cyx^2 + dx^3$ be the real part of some holomorphic function. Then by lemma 1.4.13, u must be harmonic. We see that

$$\frac{\partial u}{\partial x}(x, y) = by^2 + 2cyx + 3dx^2, \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}(x, y) = 2cy + 6dx;$$

and

$$\frac{\partial u}{\partial y}(x, y) = 3ay^2 + 2byx + cx^2, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2}(x, y) = 6ay + 2bx.$$

Since u is harmonic we therefore have that

$$0 = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 2cy + 6dx + 6ay + 2bx = 2((c + 3a)y + (3d + b)x)$$

for all $(x, y) \in \mathbb{R}^2$, which implies that $c = -3a$ and $b = -3d$. So u is of the form

$$u(x, y) = ay^3 - 3dy^2x - 3ayx^2 + dx^3.$$

Suppose $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and $f = u + iv$ for u as above. Then by the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial y}(x, y) = \frac{\partial u}{\partial x}(x, y) = -3dy^2 - 6ayx + 3dx^2$$

which, integrating with respect to y, implies that

$$v(x, y) = -dy^3 - 3ay^2x + 3dx^2y + \phi(x),$$

for some function $\phi\colon \mathbb{R} \to \mathbb{R}$. Similarly

$$\frac{\partial v}{\partial x}(x, y) = -\frac{\partial u}{\partial y}(x, y) = -3ay^2 + 6dyx + 3ax^2,$$

which, integrating with respect to x, implies that

$$v(x, y) = -3ay^2x + 3dyx^2 + ax^3 + \psi(y),$$

for some function $\psi\colon \mathbb{R} \to \mathbb{R}$. Combining these two expressions for v, we see that

$$v(x, y) = -dy^3 - 3ay^2x + 3dyx^2 + ax^3 + \alpha,$$

for some $\alpha \in \mathbb{R}$. So

$$f(x + iy) = ay^3 - 3dy^2x - 3ayx^2 + dx^3 + i(-dy^3 - 3ay^2x + 3dyx^2 + ax^3 + \alpha).$$

Since u and v are clearly continuously differentiable on $\mathbb{R}^2$, and, by construction, satisfy the Cauchy–Riemann equations on $\mathbb{R}^2$, theorem 1.4.6 implies that such a function f is indeed holomorphic.

HONOURS COMPLEX VARIABLES WORKSHOP 4 SOLUTIONS 4^{th} FEBRUARY 2019

Solutions to question 6 are due 11.00, Monday 25^{th} February 2019.

The purpose of this workshop is to get some practice with multivalued functions such as the complex logarithm and the complex powers. The only way to get used to these functions is to do some calculations with specific examples—this is your opportunity.

Recall that the functions log and arg are multi-valued functions, of which we can talk about particular holomorphic branches, given a branch cut in the complex plane. The half-line from $z_0 \in \mathbb{C}$ at angle $\phi \in \mathbb{R}$ is denoted $L_{z_0,\phi}$, i.e.

$$L_{z_0,\phi} = \{\, z \in \mathbb{C} : z = z_0 + re^{i\phi} \text{ for } r \geq 0 \,\},$$

and the cut plane $D_{z_0,\phi}$ is defined by $D_{z_0,\phi} = \mathbb{C} \setminus L_{z_0,\phi}$. The branch of logarithm defined by

$$\operatorname{Log}_\phi(z) = \ln|z| + i \operatorname{Arg}_\phi(z),$$

is the branch defined by choosing for the value of the argument $\operatorname{Arg}_\phi(z) \in (\phi, \phi + 2\pi]$. The function Log_ϕ is holomorphic on $D_{0,\phi}$, which we notate D_ϕ. The principal branch, corresponding to $\phi = -\pi$, is written Log and the cut plane $D_{-\pi}$ is written D.

Question 1. Let $z \in \mathbb{C}$ be non-zero.

(a) Show that $\log(z)$ and $\log(-z)$ have no common values.

(b) What, then, is wrong with the following argument? We have the set equalities

$$\log(z) + \log(z) = \log(zz) = \log(z^2) = \log((-z)(-z)) = \log(-z) + \log(-z),$$

hence $2\log(z) = 2\log(-z)$, and therefore $\log(z) = \log(-z)$.

You have encountered similar dubious thinking in workshop 2 on arguments.

Solution 1. (a) Recall (see lemma 1.7.3(i)) that $\log(z) = \ln|z| + i\arg(z)$. Let $\theta \in \arg(z)$, so $\ln|z| + i\theta \in \log(z)$. Then

$$|-z|\, e^{i\theta} - (-z) = |z|\, e^{i\theta} + z = z + z = 2z \neq 0.$$

This seems a strange thing to write, but what does it say? It says that $|-z|\, e^{i\theta} \neq -z$, i.e. that $\theta \notin \arg(-z)$. Therefore $\ln|z| + i\theta = \ln|-z| + i\theta \notin \log(-z)$. Swapping the roles of z and $-z$ shows that no element of $\log(-z)$ lies in $\log(z)$.

(b) The first line of mathematics is correct, and uses two applications of lemma 1.7.3(ii). The implication "hence" is incorrect, since the two sets $\log(z) + \log(z)$ and $2\log(z)$ are not equal (and neither are the two sets $\log(-z) + \log(-z)$ and $2\log(-z)$ equal). Rather, we have that

$$\log(z) + \log(z) = \{\, u + w : u \in \log(z) \text{ and } w \in \log(z) \,\},$$

whereas $2\log(z) = \{\, 2u : u \in \log(z) \,\}$. Certainly, since $2u = u + u$, we have that $2\log(z) \subseteq \log(z) + \log(z)$, but $\log(z) + \log(z) \subsetneq 2\log(z)$ in general. For example, since $\exp(i\pi) = -1 = \exp(-i\pi)$, we have that $\pm i\pi \in \log(-1)$. So $0 = i\pi + (-i\pi) \in \log(-1) + \log(-1)$. But $\exp(0/2) = 1 \neq -1$, so $0/2 \notin \log(-1)$, thus $0 \notin 2\log(-1)$.

Question 2. Is it true that $\operatorname{Log}(zw) = \operatorname{Log}(z) + \operatorname{Log}(w)$ for all non-zero $z, w \in \mathbb{C}$? Give a proof or a counterexample.

Consider workshop 2, question 2(c), which asked you a similar question about Arg.

Richard Gratwick, January 2019, file: Workshop-4-solutions.

Solution 2. No. The same counterexample which shows that it is not true in general that $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$ will suffice, and for the same reason, since by definition $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$. Let $z = -1 + i = w$. Then

$$\begin{aligned}\operatorname{Im}(\operatorname{Log}(zw)) = \operatorname{Arg}(zw) = \operatorname{Arg}(-2i) = -\frac{\pi}{2} \neq \frac{3\pi}{2} = \frac{3\pi}{4} + \frac{3\pi}{4} &= \operatorname{Arg}(z) + \operatorname{Arg}(w) \\ &= \operatorname{Im}(\operatorname{Log}(z)) + \operatorname{Im}(\operatorname{Log}(w)) \\ &= \operatorname{Im}(\operatorname{Log}(z) + \operatorname{Log}(w)),\end{aligned}$$

so $\operatorname{Log}(zw) \neq \operatorname{Log}(z) + \operatorname{Log}(w)$.

Question 3. Determine the set on which the function $f(z) = \operatorname{Log}(3z - i)$ is holomorphic, and calculate $f'(z)$ wherever it exists.

This is nothing more than chain rule, see lemma 1.7.10 for the first part.

Solution 3. The function Log is holomorphic on $D = D_{0,-\pi}$. The function f can be written as $f(z) = \operatorname{Log}(g(z))$ where $g(z) = 3z - i$ is holomorphic on $\mathbb{C}$. So lemma 1.7.10 implies that f is holomorphic on

$$\begin{aligned}g^{-1}(D) = \{\, z \in \mathbb{C} : g(z) \in D \,\} = \{\, z \in \mathbb{C} : 3z - i \in D \,\} &= D_{i/3,-\pi} \\ &= \mathbb{C} \setminus \{\, z \in \mathbb{C} : z = x + i/3, x \leq 0 \,\}.\end{aligned}$$

By chain rule, lemma 1.4.4, the derivative is given by

$$\frac{d}{dz}(\operatorname{Log}(3z - i)) = \frac{d}{dz}(\operatorname{Log}(g(z))) = \operatorname{Log}'(g(z))g'(z) = \frac{3}{3z - i}.$$

Question 4. Find the values of the following specified branches of the relevant multifunction at the given point, where in each case the branch is holomorphic on $D_{-\pi/2} = \mathbb{C} \setminus \{\, z \in \mathbb{C} : z = iy, y \leq 0 \,\}$:

(a) $\log(1 + i)$, using the holomorphic branch of log that satisfies $\log(1) = 0$;
(b) $(-1 - i)^{1/2}$, using the holomorphic branch of $z^{1/2}$ that satisfies $1^{1/2} = 1$; and
(c) $(-2)^{1/3}$, using the holomorphic branch of $z^{1/3}$ that satisfies $1^{1/3} = \exp(2\pi i/3)$.

This offers you some practice calculating with branches of multifunctions, in particular in establishing which branch you are working with. The position of the branch cut in the definition of $D_{-\pi/2}$ and the value of the branch at the given point uniquely define the interval of length 2π from which the arguments for that branch are drawn.

Solution 4. The branches of logarithm that are holomorphic on $D_{-\pi/2}$ are the branches

$$\operatorname{Log}_{-\pi/2+2k\pi}(z) = \ln|z| + i\operatorname{Arg}_{-\pi/2+2k\pi}(z) = \ln|z| + i\operatorname{Arg}_{-\pi/2}(z) + 2k\pi i$$

for $k \in \mathbb{Z}$. Branches of the powers are defined via branches of logarithm, so to answer these questions we simply need to find the value of k in each case. Since each condition used to specify the branch refers to the value of the multifunction at the point 1, we record here that when we substitute the value 1 into our multifunctions, we substitute the data $|1| = 1$ and $\operatorname{Arg}_{-\pi/2}(1) = 0$.

(a) We apply the condition on the value of $\log(1)$ to see that

$$0 = \ln|1| + i0 + 2k\pi i = 2k\pi i,$$

so $k = 0$. Hence, since $|1 + i| = \sqrt{2}$ and $\operatorname{Arg}_{\pi/2}(1 + i) = \pi/4$, we have that

$$\log(1 + i) = \ln\sqrt{2} + i\frac{\pi}{4} = \frac{1}{2}\ln 2 + i\frac{\pi}{4}.$$

(b) The branch of $z^{1/2}$ is of the form

$$\begin{aligned} z^{1/2} = \exp\left(\tfrac{1}{2}\operatorname{Log}_{-\pi/2+2k\pi}(z)\right) &= \exp\left(\tfrac{1}{2}(\ln|z| + i\operatorname{Arg}_{-\pi/2}(z) + 2k\pi i)\right) \\ &= \exp\left(\tfrac{1}{2}\ln|z|\right)\exp\left(\tfrac{1}{2}(i\operatorname{Arg}_{-\pi/2}(z) + 2k\pi i)\right) \\ &= \sqrt{|z|}\exp\left(\tfrac{1}{2}i\operatorname{Arg}_{-\pi/2}(z) + k\pi i\right) \\ &= \sqrt{|z|}\exp\left(\tfrac{1}{2}i\operatorname{Arg}_{-\pi/2}(z)\right)\exp(k\pi i) \\ &= (-1)^k\sqrt{|z|}\exp\left(\tfrac{1}{2}i\operatorname{Arg}_{-\pi/2}(z)\right), \end{aligned}$$

where, without loss of generality, $k = 0$ or 1. We apply the condition on the value of $1^{1/2}$ to see that

$$1 = 1^{1/2} = (-1)^k\sqrt{|1|}\exp(\tfrac{1}{2}i0) = (-1)^k,$$

so $k = 0$, and $z^{1/2} = \sqrt{|z|}\exp(\frac{1}{2}i\operatorname{Arg}_{-\pi/2}(z))$. So since $|-1-i| = \sqrt{2}$ and $\operatorname{Arg}_{-\pi/2}(-1-i) = 5\pi/4$, we have that

$$(-1-i)^{1/2} = \sqrt[4]{2}\exp(\tfrac{5}{8}\pi i).$$

(c) Similarly, the branch of $z^{1/3}$ is of the form

$$\begin{aligned} z^{1/3} = \exp\left(\tfrac{1}{3}(\ln|z| + i\operatorname{Arg}_{-\pi/2}(z) + 2k\pi i)\right) &= \exp\left(\tfrac{1}{3}\ln|z|\right)\exp\left(\tfrac{1}{3}(i\operatorname{Arg}_{-\pi/2}(z) + 2k\pi i)\right) \\ &= \sqrt[3]{|z|}\exp\left(\tfrac{1}{3}i\operatorname{Arg}_{-\pi/2}(z) + \tfrac{2}{3}k\pi i\right) \end{aligned}$$

where, without loss of generality, $k = 0$, 1, or 2. We apply the condition on the value of $1^{1/3}$ to see that

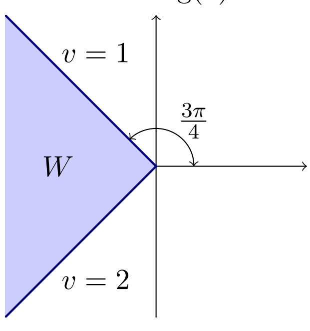
$$\exp(2\pi i/3) = 1^{1/3} = \sqrt[3]{|1|}\exp\left(\tfrac{1}{3}i0 + \tfrac{2}{3}k\pi i\right) = \exp\left(\tfrac{2}{3}k\pi i\right),$$

so $k = 1$, and $z^{1/3} = \sqrt[3]{|z|}\exp\left(\frac{1}{3}i\operatorname{Arg}_{-\pi/2}(z) + \frac{2}{3}\pi i\right)$. So since $|-2| = 2$ and $\operatorname{Arg}_{-\pi/2}(-2) = \pi$, we have that

$$(-2)^{1/3} = \sqrt[3]{2}\exp\left(\tfrac{1}{3}i\pi + \tfrac{2}{3}i\pi\right) = \sqrt[3]{2}\exp(i\pi) = -\sqrt[3]{2}.$$

Question 5. (a) Let $U = \{\, z \in \mathbb{C} : 1 < |z| < 2 \,\}$. Find a real-valued function $u = u(x, y)$ which is harmonic on U and which takes value 1 when $|z| = 1$, and value 2 when $|z| = 2$.

(b) Let $W \subseteq \mathbb{C}$ be the set pictured below. Find a real-valued function $v = v(x, y)$ which is harmonic on the interior of W and which takes value 1 when $\operatorname{Arg}(z) = 3\pi/4$ (on the upper edge of W), and value 2 when $\operatorname{Arg}(z) = -3\pi/4$ (on the lower edge of W).



Three facts might be useful: that the real part of a complex logarithm is constant if $|z|$ is constant; that the imaginary part of a complex logarithm is constant if the corresponding branch of $\arg(z)$ is constant; and that a linear combination of harmonic functions (which include constant functions) is harmonic.

Solution 5. (a) The hint suggests we think about the function $\ln|z|$, which is the real part of the principal branch of the logarithm, $\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$. This has a branch cut along the non-positive real axis, $L_{0,-\pi}$. Everywhere else, i.e. on $D_{0,-\pi}$, the Cauchy–Riemann equations imply that the real part $\ln|z|$ is harmonic (see lemma 1.4.13). However, the set U intersects $L_{0,-\pi}$. So we appear to have a problem. However, the *same* function $\ln|z|$ appears as the real part of the logarithm, even if we make a *different* branch cut. For example, $\mathrm{Log}_0(z) = \ln|z| + i\,\mathrm{Arg}_0(z)$ has a branch cut along $L_{0,0}$, and is holomorphic everywhere else. So by the same argument, $\ln|z|$ is harmonic everywhere on $D_{0,0}$. Combining these two arguments, we see that $\ln|z|$ is harmonic on $D_{0,0} \cup D_{0,-\pi} = \mathbb{C} \setminus \{0\}$, i.e. everywhere except the origin. It is certainly not harmonic at the origin, because it is not defined at the origin. This does not matter because U does not contain the origin.

Now we just have to fix the boundary conditions. This reduces to finding a linear combination $a\ln|z| + b$ for constants $a, b \in \mathbb{R}$ such that $a\ln 1 + b = 1$ and $a\ln 2 + b = 2$. This immediately gives that $b = 1$, and hence $a = 1/\ln 2$. Then the function

$$u(x,y) = \frac{\ln|z|}{\ln 2} + 1,$$

where $z = x + iy$, is harmonic on U, and satisfies the given boundary conditions.

(b) We consider now the imaginary part of a suitable holomorphic function. Since the imaginary part of a branch of logarithm is precisely a branch of the argument, it will certainly be constant on points in the complex plane with the same argument. We just have to be careful to choose a branch which is holomorphic on the given set W. The branch of logarithm Log_0 has a branch cut along $L_{0,0}$, which does not intersect W. So $\mathrm{Log}_0(z) = \ln|z| + i\,\mathrm{Arg}_0(z)$ is a function which is holomorphic on the interior of W, and therefore the imaginary part Arg_0 is harmonic on the interior of W (see again lemma 1.4.13).

Again, it is now just a matter of fixing the boundary conditions. We want constants $a, b \in \mathbb{R}$ such that $a\,\mathrm{Arg}(z) + b$ equals 1 when $\mathrm{Arg}(z) = 3\pi/4$, and $a\,\mathrm{Arg}(z) + b$ equals 2 when $\mathrm{Arg}(z) = -3\pi/4$. These are the conditions as stated in the question, referring to the principal value of the argument function—we need to translate these into conditions referring to the branch Arg_0 which we are using. So our conditions are that $a\,\mathrm{Arg}_0(z) + b$ equals 1 when $\mathrm{Arg}_0(z) = 3\pi/4$, and that $a\,\mathrm{Arg}_0(z) + b$ equals 2 when $\mathrm{Arg}_0(z) = 5\pi/4$. These are satisfied by choosing $a = 2/\pi$ and $b = -1/2$. Then the function

$$v(x,y) = \frac{2}{\pi}\,\mathrm{Arg}_0(z) - \frac{1}{2},$$

where $z = x + iy$, is harmonic on the interior of W, and satisfies the given boundary conditions.

Question 6. Consider the multivalued function $f(z) = \left(z^2 + 1\right)^{1/2}$.

(a) Define a branch of f that is holomorphic on the set $D_1(0) = \{\, z \in \mathbb{C} : |z| < 1 \,\}$.
(b) Define a branch of f that is holomorphic on the set $\mathbb{C} \setminus \overline{D}_1(0) = \{\, z \in \mathbb{C} : |z| > 1 \,\}$.

For part (a), observe that $z^2 + 1 = (z + i)(z - i)$. *There is nothing to stop you using two different branches of a multivalued function in one definition—just make it very clear what branch you are using for what purpose. For part (b), observe that* $z^2 + 1 = z^2\left(1 + z^{-2}\right)$ *for non-zero* z. *The trick of writing the same function in different ways when considering different domains will be used often in the course. Example 1.8.13 might offer some inspiration.*

[10 = (5+5) marks]

Solution 6. (a) By definition a branch $f(z)$ of $\left(z^2+1\right)^{1/2}$ is given by $f(z) = \exp\left(\frac{1}{2}g(z)\right)$ for some branch of logarithm $g(z) \in \log(z^2+1)$. Since lemma 1.7.3(ii) implies that

$$\log\left(z^2+1\right) = \log((z+i)(z-i)) = \log(z+i) + \log(z-i),$$

it suffices to find branches of $\log(z+i)$ and $\log(z-i)$ that are holomorphic on $D_1(0)$.

The function $\operatorname{Log}_{\phi_1}(z+i)$ is a branch of $\log(z+i)$ that is holomorphic on $\{\, z \in \mathbb{C} : z+i \in D_{0,\phi_1} \,\} = D_{-i,\phi_1}$. Choosing ϕ_1 such that the half-line goes in the non-positive imaginary half of the plane implies that $D_1(0) \subseteq D_{-i,\phi_1}$, e.g. $\phi_1 = -\pi/2$ suffices.

Similarly, $\operatorname{Log}_{\phi_2}(z-i)$ is a branch of $\log(z-i)$ that is holomorphic on $\{\, z \in \mathbb{C} : z-i \in D_{0,\phi_2} \,\} = D_{i,\phi_2}$. Choosing ϕ_2 such that the half-line goes in the non-negative imaginary half of the plane implies that $D_1(0) \subseteq D_{i,\phi_2}$, e.g. $\phi_2 = \pi/2$ suffices.

So the function $g(z) = \operatorname{Log}_{-\pi/2}(z+i) + \operatorname{Log}_{\pi/2}(z-i)$ is holomorphic on $D_1(0)$ and satisfies

$$g(z) \in \log(z+i) + \log(z-i) = \log\left(z^2+1\right).$$

Therefore the function

$$f(z) = \exp\left(\frac{1}{2}\left(\operatorname{Log}_{-\pi/2}(z+i) + \operatorname{Log}_{\pi/2}(z-i)\right)\right)$$

is a branch of $\left(z^2+1\right)^{1/2}$ that is holomorphic on $D_1(0)$.

(b) Writing $z^2+1 = z^2\left(1+z^{-2}\right)$, we begin by observing that we only have to worry about defining a branch of $\left(1+z^{-2}\right)^{1/2}$ that is holomorphic on $\{\, z \in \mathbb{C} : |z| > 1 \,\}$: suppose we have such a branch of the form $f_3(z) = \exp\left(\frac{1}{2}g(z)\right)$ for some $g(z) \in \log\left(1+z^{-2}\right)$. Then, since by definition $\exp(\operatorname{Log}(z)) = z$, we have that

$$\begin{aligned}\exp\left(\tfrac{1}{2}\left(2\operatorname{Log}(z) + g(z)\right)\right) = \exp\left(\operatorname{Log}(z) + \tfrac{1}{2}g(z)\right) &= \exp(\operatorname{Log}(z))\exp\left(\tfrac{1}{2}g(z)\right)\\ &= z\exp\left(\tfrac{1}{2}g(z)\right).\end{aligned}$$

The right-hand side is a product of two functions that are holomorphic on $\{\, z \in \mathbb{C} : |z| > 1 \,\}$, and is therefore itself holomorphic on $\{\, z \in \mathbb{C} : |z| > 1 \,\}$. The left-hand side is of the form $\exp\left(\frac{1}{2}h(z)\right)$ where

$$\begin{aligned}h(z) = 2\operatorname{Log}(z) + g(z) \in 2\log(z) + \log\left(1+z^{-2}\right) &\subseteq \log(z) + \log(z) + \log\left(1+z^{-2}\right)\\ &= \log\left(z^2(1+z^{-2})\right).\end{aligned}$$

That is to say that the left-hand side shows that the function is a branch of $\left(z^2\left(1+z^{-2}\right)\right)^{1/2} = \left(z^2+1\right)^{1/2}$, and the right-hand side shows that the function is holomorphic on $\{\, z \in \mathbb{C} : |z| > 1 \,\}$, so we are done.

A branch of $\left(1+z^{-2}\right)^{1/2}$ is of the form $\exp\left(\frac{1}{2}g(z)\right)$ where $g(z) \in \log\left(1+z^{-2}\right)$. The function $\operatorname{Log}_{\phi_3}\left(1+z^{-2}\right)$ is holomorphic except for $z=0$ and values z such that $1+z^{-2} \in L_{0,\phi_3}$. We need to choose ϕ_3 such that $1+z^{-2} \in L_{0,\phi_3}$ implies $|z| \leq 1$, or equivalently, that $\left|z^{-2}\right| \geq 1$. If $1+z^{-2} \in L_{0,\phi_3}$ then $1+z^{-2} = re^{i\phi_3}$ for some $r \geq 0$. Rearranging, we have that $z^{-2} = re^{i\phi_3} - 1$. Therefore we just need to choose ϕ_3 such that $\left|re^{i\phi_3} - 1\right| \geq 1$. Geometrically this follows if ϕ_3 is chosen so that the half-line goes in the non-positive real part of the plane. So e.g. $\phi_3 = -\pi$ will suffice, corresponding to the principal branch of the logarithm. So the fuction

$$f_3(z) = \exp\left(\tfrac{1}{2}\operatorname{Log}\left(1+z^{-2}\right)\right)$$

is a branch of $\left(1+z^{-2}\right)^{1/2}$ that is holomorphic on $\{\, z \in \mathbb{C} : |z| > 1 \,\}$.

By the preceding argument, we define $f(z) = z\exp\left(\frac{1}{2}\operatorname{Log}\left(1+z^{-2}\right)\right)$ and we are done.

HONOURS COMPLEX VARIABLES WORKSHOP 5 SOLUTIONS
11^{th} FEBRUARY 2019

Solutions to questions 5 and 6 are due 11.00, Monday 25^{th} February 2019.

This workshop aims to get you thoroughly acquainted with conformal maps and, in particular, Möbius transformations. You should be able to determine the images of certain subsets of the complex plane under various conformal maps, and, vice versa, be able to construct appropriate conformal maps which map a given set to another given set.

Question 1. (a) Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = (z-1)/(z+1)$. Find the image under f of the sets
(i) $U_1 = \{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$; and
(ii) $U_2 = \{ z \in \mathbb{C} : |z| < 1 \}$.
(b) Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \exp(iz)$. Find the image under f of the the sets
(i) $U_1 = \{ z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \pi \}$; and
(ii) $U_2 = \{ z \in \mathbb{C} : -\pi/2 < \operatorname{Re}(z) < \pi/2, \text{ and } \operatorname{Im}(z) > 0 \}$.
(c) Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^{1/2}$, where the principal branch of the square root function is taken. Find the image under f of the the sets
(i) $U_1 = \{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$; and
(ii) $U_2 = \{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ or } \operatorname{Im}(z) \neq 0 \}$.

Think geometrically. The function in part (a) is a Möbius transformation. What particular geometric properties does it therefore have?

Solution 1. (a) Since f is a Möbius transformation, corollary 2.4.3 implies that f maps circlines to circlines.
(i) The easiest approach is to observe that the real part of z is positive if and only if z is in the right half of the complex plane, if and only if the distance from z to 1 is less than the distance from z to -1, if and only if $|z-1| < |z+1|$. Thus $z \in U_1$ if and only if

$$|f(z)| = \left|\frac{z-1}{z+1}\right| = \frac{|z-1|}{|z+1|} < 1,$$

i.e. $f(U_1) = \{ z \in \mathbb{C} : |z| < 1 \}$.
An alternative approach is to check the image under f of the boundary of U_1, the line $\{ z \in \mathbb{C} : \operatorname{Re}(z) = 0 \}$. We know that this image is either a circle or a line. We take three points on the line and check their images:

$$f(i) = \frac{i-1}{i+1} = \frac{(i-1)(1-i)}{2} = -\frac{(1-i)^2}{2} = i,$$
$$f(0) = -1, \text{ and}$$
$$f(-i) = \frac{-i-1}{-i+1} = \frac{(-i-1)(1+i)}{2} = -\frac{(1+i)^2}{2} = -i.$$

Since these three points lie on either a circle or a line, they must lie on the circle of radius 1 centred at the origin. We now just check whether U_1 is mapped to the interior or exterior of the circle by checking the image of any point in U_1. Since $f(1) = 0$, we see indeed that $f(U_1) = \{ z \in \mathbb{C} : |z| < 1 \}$.

Richard Gratwick, January 2019, file: Workshop-5-solutions.

(ii) We check the image of the boundary of U_2, $\{ z \in \mathbb{C} : |z| = 1 \}$. Since this is a circle, the image under f must be a line or a circle. We check the images of three points on the circle, all of which we evaluated in part (i):

$$f(i) = i, \quad f(1) = 0, \quad \text{and} \quad f(-i) = -i.$$

Therefore the circle is mapped to the straight line $\{ z \in \mathbb{C} : \mathrm{Re}(z) = 0 \}$. We need to check which side of the line the disc is mapped to: since $f(0) = -1$, we see that $f(U_2) = \{ z \in \mathbb{C} : \mathrm{Re}(z) < 0 \}$.

(b) We have that $f(x + iy) = \exp(i(x + iy)) = e^{-y}e^{ix}$, i.e. the image under f of $z \in \mathbb{C}$ is a complex number of modulus $e^{-\mathrm{Im}(z)}$ and argument $\mathrm{Re}(z)$.

(i) We therefore see that $z \in f(U_1)$ if and only if the argument of z lies between 0 and π, therefore $f(U_1) = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$.

(ii) Similarly we see that $z \in f(U_2)$ if and only if the modulus of z is a negative power of e, i.e. $|z| < 1$, and the argument of z lies between $-\pi/2$ and $\pi/2$, therefore

$$f(U_2) = \left\{ z \in \mathbb{C} : z = re^{i\theta},\ 0 < r < 1,\ \theta \in (-\pi/2, \pi/2) \right\}.$$

(c) We have that $f(z) = |z|^{1/2}\, e^{i\,\mathrm{Arg}(z)/2}$.

(i) Since $\mathrm{Re}(z) > 0$ if and only if $\mathrm{Arg}(z) \in (-\pi/2, \pi/2)$, we see that

$$f(U_1) = \{ z \in \mathbb{C} : \mathrm{Arg}(z) \in (-\pi/4, \pi/4) \}.$$

(ii) The only points in the complex plane which do not lie in U_2 are those points on the non-positive real axis, i.e. zero and those points with $\mathrm{Arg}(z) = \pi$. Thus $z \in U_2$ if and only if $\mathrm{Arg}(z) \in (-\pi, \pi)$, so

$$f(U_2) = \{ z \in \mathbb{C} : \mathrm{Arg}(z) \in (-\pi/2, \pi/2) \} = \{ z \in \mathbb{C} : \mathrm{Re}(z) > 0 \}.$$

Question 2. For each of the sets U_i defined below, find a conformal map f_i which maps U_i onto the unit disc $D_1(0) = \{ z \in \mathbb{C} : |z| < 1 \}$:

(a) $U_1 = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$;
(b) $U_2 = \{ z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1 \}$;
(c) $U_3 = \{ z \in \mathbb{C} : \mathrm{Im}(z), \mathrm{Re}(z) > 0 \}$; and
(d) $U_4 = \{ z \in \mathbb{C} : -\pi/4 < \mathrm{Arg}(z) < \pi/4 \}$.

I strongly recommend tackling these in roughly the order given, and exploiting the fact that a composition of conformal maps is conformal.

Solution 2. The trick is to use the previous parts of the question as one goes along.

(a) We can argue similarly to how we did in question 1(a)(i), to observe that $\mathrm{Im}(z) > 0$ if and only if $|z - i| < |z + i|$. Thus

$$f_1(z) = \frac{z - i}{z + i}$$

precisely satisfies $|f_1(z)| < 1$ when $z \in U_1$, i.e. $f_1(U) = D_1(0)$.

(b) If we can find a conformal map g_2 so that $g_2(U_2) = U_1$, then we can just compose it with f_1 to get a conformal map $f_2 = f_1 \circ g_2$ such that $f_2(U_2) = D_1(0)$. The function $g_2(z) = \exp(\pi z)$ will work, since for $z = x + iy \in U_2$, $g_2(x + iy) = e^{\pi x}e^{i\pi y}$ is a complex number of arbitrary modulus and argument between 0 and π. So

$$f_2(z) = f_1(g_2(z)) = \frac{\exp(\pi z) - i}{\exp(\pi z) + i}.$$

(c) Again, we simply try to find a conformal map g_3 which maps U_3 to a region we have already dealt with. This is easy, since $z \in U_3$ if and only if $\mathrm{Arg}(z) \in (0, \pi/2)$. So $g_3(z) = z^2$ satisfies

$$g_3(U_3) = \{ z \in \mathbb{C} : \mathrm{Arg}(z) \in (0, \pi) \} = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \},$$

and hence $f_3 = f_1 \circ g_3$ is such that $f_3(U_3) = f_1(g_3(U_3)) = f_1(U_1) = D_1(0)$. So

$$f_3(z) = f_1(g_3(z)) = \frac{z^2 - i}{z^2 + i}.$$

(d) Finally we can easily map U_4 to U_3 by the function $g_4(z) = e^{i\pi/4}z$, so the composition

$$f_4(z) = f_3(g_4(z)) = \frac{(e^{i\pi/4}z)^2 - i}{(e^{i\pi/4}z)^2 + i} = \frac{iz^2 - i}{iz^2 + i} = \frac{z^2 - 1}{z^2 + 1}$$

satisfies $f_4(U_4) = D_1(0)$.

Question 3. For each of the following cases, either construct a Möbius function f with the stated properties, or explain why such a Möbius function does not exist:

(a) f maps the set $\{ z \in \mathbb{C} : |z| = 1 \}$ to the set $\{ z \in \mathbb{C} : \mathrm{Re}(z) = 0 \}$;
(b) f maps the set $\{ z \in \mathbb{C} : |z - 1| = 1 \}$ to the set $\{ z \in \mathbb{C} : \mathrm{Re}(z) = 1 \}$, and maps the set $\{z \in \mathbb{C} : |z + 1| = 1 \}$ to the set $\{ z \in \mathbb{C} : \mathrm{Re}(z) = -1 \}$; and
(c) f maps the set $\{ z \in \mathbb{C} : |z| = 1 \}$ to itself, and maps the set $\{ z \in \mathbb{C} : \mathrm{Re}(z) = 1/\sqrt{2} \}$ to the set $\{ z \in \mathbb{C} : \left|z - \sqrt{2}\right| = 1 \}$.

There are two pertinent facts: Möbius transformations are conformal maps, and they map circlines to circlines. This is part of a question from a 2002 Oxford exam paper.

Solution 3. (a) In question 1(a)(ii) we discovered that $f(z) = (z-1)/(z+1)$ satisfied exactly this property.

(b) Since 0 lies on both the circles, any such transformation f must map 0 to ∞, since the required images are two parallel lines which do not intersect in $\mathbb{C}$. Hence f is of the form $f(z) = (az + b)/cz$ for $c \neq 0$, which we therefore divide through by c, and so assume that f is in fact of form $f(z) = (az + b)/z = a + (b/z)$. We now just try our luck and insist that $f(2) = 1$ and $f(-2) = -1$. This implies that $a + (b/2) = 1$ and $a - (b/2) = -1$, hence $a = 0$ and $b = 2$. So let us consider $f(z) = 2/z$.

We have that $|z - 1| = 1$ if and only if $(z - 1)(\overline{z} - 1) = |z - 1|^2 = 1$, if and only if $|z|^2 - (z + \overline{z}) = 0$. Similarly, $|z + 1| = 1$ if and only if $|z|^2 + (z + \overline{z}) = 0$. We now observe that $\mathrm{Re}(z) = \pm 1$ if and only if $z + \overline{z} = \pm 2$. Using this, let us check where $f(z) = 2/z$ maps our two circles. For z such that $|z - 1| = 1$, we have that

$$f(z) + \overline{f(z)} = \frac{2}{z} + \overline{\left(\frac{2}{z}\right)} = \frac{2}{z} + \frac{2}{(\overline{z})} = \frac{2(\overline{z} + z)}{|z|^2} = 2,$$

and for z such that $|z + 1| = 1$, we have that

$$f(z) + \overline{f(z)} = \frac{2}{z} + \overline{\left(\frac{2}{z}\right)} = \frac{2}{z} + \frac{2}{(\overline{z})} = \frac{2(\overline{z} + z)}{|z|^2} = -2.$$

So f is exactly as required.

(c) No such Möbius transformation exists. The curves $\{ z \in \mathbb{C} : |z| = 1 \}$ and $\{ z \in \mathbb{C} : \mathrm{Re}(z) = 1/\sqrt{2} \}$ intersect at the points $z = (1 \pm i)/\sqrt{2}$, at angles of $\pi/4$. The two circles in the image, however, intersect at these same points, but at an angle of $\pi/2$. Since a Möbius transformation is conformal, i.e. angle-preserving, no such transformation can exist that maps between these pairs of sets as stipulated.

Question 4. Let $f \colon \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ be a Möbius transformation of the form $f(z) = (az + b)/(cz + d)$. Show, just using the definition of a Möbius transformation, that

(a) f is bijective, and the inverse is a Möbius transformation; and
(b) f is conformal, where we extend the definition of conformality to $\tilde{\mathbb{C}}$ by saying that f is conformal at $-d/c$ if $g(z) = 1/f(z)$ is conformal at $-d/c$, and that f is conformal at ∞ if $h(z) = f(1/z)$ is conformal at $z = 0$.

A relatively straightforward question to cement all the relevant definitions in your mind.

Solution 4. (a) We first show that f is injective. By definition the only point that maps to ∞ is $z = -d/c$. Furthermore by definition $f(\infty) = a/c$, and if $z \in \mathbb{C}$ was such that $f(z) = a/c$, then

$$\frac{az+b}{cz+d} = \frac{a}{c},$$

which implies that $caz + cb = acz + ad$, i.e. $ad - bc = 0$, which is a contradiction. Suppose finally that $f(z) = f(w)$, for $z, w \in \mathbb{C}$. Then by definition,

$$\frac{az+b}{cz+d} = \frac{aw+b}{cw+d},$$

which rearranges to imply that

$$aczw + adz + bcw + bd = aczw + cbz + daw + db,$$

and therefore $(ad - bc)z = (ad - bc)w$, which since $ad - bc = 1$, implies that $z = w$, as required.

We now show that f is surjective. By definition $f(-d/c) = \infty$. Let $w \in \mathbb{C}$. Then solving

$$\frac{az+b}{cz+d} = w$$

for $z \in \mathbb{C}$ gives that

$$z = \frac{wd-b}{-wc+a},$$

which is interpreted as ∞ if $w = a/c$. Since $da - (-b)(-c) = ad - bc = 1$, the function $f^{-1}(z) = (dz - b)/(-cz + a)$ is a Möbius transformation which satisfies $f(f^{-1}(z)) = z$ and $f^{-1}(f(z)) = z$.

(b) For $z \in \mathbb{C}$ satisfying $z \neq -d/c$, we simply differentiate f using the usual quotient rule to see that

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} \neq 0,$$

so since f is differentiable with non-zero derivative at every such point, theorem 2.1.2 implies that it is conformal. Suppose $c \neq 0$, and considering the point $z = -d/c \in \mathbb{C}$, we see that $g(z) = 1/f(z) = \frac{cz+d}{az+b}$ has derivative

$$g'(z) = \frac{c(az+b) - a(cz+d)}{(az+b)^2} = \frac{bc-ad}{(az+b)^2} \neq 0$$

as long as $z \neq -b/a$, in particular at $z = -d/c$, since these two values are not equal by assumption, so f is conformal at $-d/c$ by definition. For the point ∞, we consider the function

$$f(1/z) = \frac{a/z+b}{c/z+d} = \frac{bz+a}{dz+c},$$

which has derivative

$$\frac{b(dz+c) - d(bz+a)}{(dz+c)^2} = \frac{cb-da}{(dz+c)^2} \neq 0,$$

for all $z \neq -c/d$, in particular f is conformal at ∞ by definition, if $c \neq 0$. If $c = 0$, then checking conformality at $-d/c$ by definition obliges us to check conformality of $1/f(1/z))$ at 0. We see that

$$\frac{1}{f(1/z)} = \frac{dz+c}{bz+a}$$

has derivative
$$\frac{d(bz+a)-b(dz+c)}{(bz+a)^2} = \frac{ad-bc}{(bz+a)^2} \neq 0,$$
as long as $z \neq -a/b$, in particular at $z = 0$, since $c = 0$ implies that $a \neq 0$.

Question 5. Find a conformal mapping f defined on the cut plane $\mathbb{C} \setminus (-\infty, 0]$ which maps it onto the unit disc $D_1(0) = \{ z \in \mathbb{C} : |z| < 1 \}$.

Have you done question 1? **[5 marks]**

Solution 5. There are various ways of doing this. Defining $g(z) = z^{1/2}$, using the principal branch of the square root function, we can map the cut plane $\mathbb{C} \setminus (-\infty, 0]$, which is the set of points with $\operatorname{Arg}(z) \in (-\pi, \pi)$, to the set
$$\{ z \in \mathbb{C} : \operatorname{Arg}(z) \in (-\pi/2, \pi/2) \} = \{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}.$$
Now we just notice from question 1(a)(i) that the function $f_1(z) = (z-1)/(z+1)$ maps this set to $D_1(0)$. So if we form the composition $f = f_1 \circ g$, we get the function
$$f(z) = \frac{z^{1/2}-1}{z^{1/2}+1}$$
which maps $\mathbb{C} \setminus (-\infty, 0]$ to $D_1(0)$.

Question 6. Let $f\colon \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$ be inversion, i.e. $f(z) = 1/z$.

(a) Briefly explain why f maps a circle which passes through the origin to a straight line.

(b) Let $a, b \in \mathbb{C}$ be non-zero. Show that
$$|f(a) - f(b)| = \frac{|a-b|}{|a|\,|b|}.$$

(c) Hence, or otherwise, deduce that if the vertices A, B, C, D of a quadrilateral lie on a circle (ordered anticlockwise), then the sum of the products of the lengths of the pairs of opposite sides is equal to the product of the lengths of the diagonals, i.e. $|AB|\,|CD| + |BC|\,|DA| = |AC|\,|BD|$ (Ptolemy's Theorem).

The first two parts are not trick questions—they should be as easy as they look. They are included mainly to record those two facts, which might be relevant in the third part. For this third part, I would consider your quadrilateral as being drawn on the complex plane with one of its vertices at the origin. One geometric observation will then let the solution drop out almost effortlessly. **[5=(1+1+3) marks]**

Solution 6. (a) The function f is a Möbius transformation, and therefore maps circlines to circlines. Since $f(0) = \infty$, it maps any circle passing through the origin to a circle or a line, which contains a point at infinity. This cannot be a circle, so it is indeed a straight line.

(b) This really is as easy as it looks:
$$|f(a) - f(b)| = \left| \frac{1}{a} - \frac{1}{b} \right| = \left| \frac{b-a}{ab} \right| = \frac{|b-a|}{|a|\,|b|}.$$

(c) Taking the hint, I put the vertex A at the origin of the complex plane, and let the other vertices B, C, and D be represented by complex numbers b, c, and d respectively. The image under f of the circle, since it passes through the origin, is a straight line, so the points $f(b)$, $f(c)$, and $f(d)$ lie on a straight line in the complex plane. The pertinent observation is then that since c lies between b and d on the circle, $f(c)$ lies between $f(b)$ and $f(d)$ on the line, and therefore the distances between the points obey
$$|f(b) - f(d)| = |f(b) - f(c)| + |f(c) - f(d)|,$$

which, combining with part (b), implies that

$$\frac{|b-d|}{|b|\,|d|} = \frac{|b-c|}{|b|\,|c|} + \frac{|c-d|}{|c|\,|d|}.$$

Multiplying through by $|b|\,|c|\,|d|$, we gain

$$|b-d|\,|c| = |b-c|\,|d| + |c-d|\,|b|\,,$$

which since, in our quadrilateral, $|b-d|$ and $|c|$ are the lengths of the diagonals, and $|b|$, $|d|$, $|b-c|$, and $|c-d|$ are the sidelengths, is exactly the statement required.

HONOURS COMPLEX VARIABLES WORKSHOP 6 SOLUTIONS 25th FEBRUARY 2019

Solutions to questions 5 and 6 are due 11.00, Monday 11th March 2019.

The purpose of this workshop is to get used to the definition of the integral of a complex function along a curve. This is one of the most important definitions in the course, and underpins much of the remaining material. This sheet offers you practice in evaluating such integrals, and in manipulating them and using properties we have proved about them. We also investigate the definition of a domain, which is also critical to a number of results in the course.

Some of these exercises are taken from *Introduction to Complex Analysis* by H. A. Priestley.

Question 1. Evaluate the following integrals $\int_\Gamma f(z)\,dz$, where f and Γ are defined by

(a) $f(z) = z^2$, and $\gamma\colon [-\pi/2, \pi/2] \to \mathbb{C}$ is given by $\gamma(t) = \exp(it)$;
(b) $f(z) = \mathrm{Re}(z)$, and $\gamma\colon [0, 1] \to \mathbb{C}$ is given by $\gamma(t) = t + it$; and
(c) $f(z) = 1/z$, and $\gamma\colon [0, 8\pi] \to \mathbb{C}$ is given by $\gamma(t) = \exp(-it)$.

Nothing clever here: some straightforward examples to make sure you know how to apply the definition of the integral along a curve.

Solution 1. (a) Using the definition and lemma 3.1.2(ii), we see that

$$\begin{aligned}\int_\Gamma f(z)\,dz = \int_{-\pi/2}^{\pi/2} (\gamma(t))^2\gamma'(t)\,dt = \int_{-\pi/2}^{\pi/2} (\exp(it))^2 i\exp(it)\,dt &= i\int_{-\pi/2}^{\pi/2} \exp(3it)\,dt \\ &= \frac{1}{3}\left(\exp(3i\pi/2) - \exp(-3i\pi/2)\right) \\ &= \frac{1}{3}(-i - i) \\ &= -\frac{2i}{3}.\end{aligned}$$

(b) Similarly

$$\begin{aligned}\int_\Gamma f(z)\,dz = \int_0^1 \mathrm{Re}(\gamma(t))\gamma'(t)\,dt = \int_0^1 \mathrm{Re}(t + it)(1 + i)\,dt = \int_0^1 t(1 + i)\,dt &= \frac{1 + i}{2}\left(1^2 - 0^2\right) \\ &= \frac{1 + i}{2}.\end{aligned}$$

(c) The easiest way to see this is to realize that this contour just goes around the unit circle centred at the origin four times in the clockwise direction. Example 3.2.5 tells us that the integral of $1/z$ around the unit circle once in the anti-clockwise direction is $2\pi i$, so this integral should be $(-4)(2\pi i) = -8\pi i$. It is easy to check this from the definition too:

$$\int_\Gamma f(z)\,dz = \int_0^{8\pi} \frac{1}{\gamma(t)}\gamma'(t)\,dt = \int_0^{8\pi} \frac{1}{\exp(-it)}(-i\exp(-it))\,dt = \int_0^{8\pi} (-i)\,dt = -8\pi i.$$

Question 2. Explain why the following contour integrals $\int_\Gamma f(z)\,dz$ are zero:

Richard Gratwick, January 2019, file: Workshop-6-solutions.

(a) f is a continuous function on $\mathbb{C}$, and Γ is the curve parametrized by $\gamma\colon [-1, 2\pi] \to \mathbb{C}$ defined by

$$\gamma(t) = \begin{cases} t+1 & t \in [-1, 0), \\ \exp(it) & t \in [0, \pi/2), \\ -\overline{\exp(it)} & t \in [\pi/2, \pi), \\ (2\pi - t)/\pi & t \in [\pi, 2\pi]; \end{cases}$$

(b) f is a polynomial and Γ is any closed contour; and
(c) $f(z) = \exp(z)$, where Γ is any contour from z_0 to $z_1 = z_0 + 2\pi i$, for any $z_0 \in \mathbb{C}$.

No lengthy explanations are necessary. Just use the basic properties of real and contour integrals, and perhaps also refer to lemma 3.3.9 (path-independence).

Solution 2. (a) The contour consists of:
- the straight line from $(-1+1) = 0$ to $0 + 1 = 1$;
- the arc of the circle of radius one centred at the origin from $\exp(i0) = 1$ to $\exp(i\pi/2) = i$ anticlockwise;
- the arc of the circle of radius one centred at the origin from $-\overline{\exp(i\pi/2)} = i$ to $-\overline{\exp(i\pi)} = 1$ clockwise; and
- the straight line from $(2\pi - \pi)/\pi = 1$ to $(2\pi - 2\pi)/\pi = 0$.

In other words the contour traces the same path from 0 to 1 to i and back again. The integrals in opposite directions along each regular component cancel, so the whole contour integral is zero.

(b) Since f is a polynomial it has an antiderivative on $\mathbb{C}$, so lemma 3.3.9 implies that the contour integral is zero, because the contour is closed.

(c) $f(z) = \exp(z)$ is its own antiderivative, therefore the Fundamental Theorem of Calculus, theorem 3.3.5, and the $2\pi i$-periodicity of exp implies that

$$\int_\Gamma \exp(z)\, dz = \exp(z_1) - \exp(z_0) = \exp(z_0 + 2\pi i) - \exp(z_0) = 0.$$

Question 3. Let f be holomorphic on $\mathbb{C}$, and Γ be a regular curve in $\mathbb{C}$. Is it true that

$$\left| \int_\Gamma f(z)\, dz \right| \le \int_\Gamma |f(z)|\, dz?$$

Consider the definitions of the terms involved. A "yes" must of course come with a proof; a "no" needs just one counterexample. You should compare this carefully with the statement of lemma 3.2.9 (which is of course definitely true).

Solution 3. This is false. Counterexamples may be found almost anywhere, since the left-hand side of the alleged inequality is a real number, whereas the right-hand side is a complex integral, therefore in general is a complex number. Certainly, the *integrand* on the right-hand side is a real number, but the integral is along a curve in the complex plane, and therefore when the integral is computed via a parametrization of this curve, complex terms will, in general, appear.

An easy counterexample is $f(z) = 1$ and Γ the straight line segment from 0 to i, which is parametrized by $\gamma\colon [0, 1] \to \mathbb{C}$ given by $\gamma(t) = it$. Then by definition,

$$\int_\Gamma f(z)\, dz = \int_0^1 \gamma'(t)\, dt = \int_0^1 i\, dt = i,$$

so $\left| \int_\Gamma f(z)\, dz \right| = |i| = 1$, whereas $\int_\Gamma |f(z)|\, dz = \int_\Gamma f(z)\, dz = i$. No inequality holds between these two numbers at all.

Question 4. Prove using lemma 3.2.9 that

(a)
$$\left|\int_{C_2(1)} \frac{1}{z}\,dz\right| \leq 4\pi;$$
and
(b)
$$\left|\int_{C_R^+} \frac{1}{z^4+1}\,dz\right| \leq \frac{R\pi}{R^4-1},$$
where, for $R > 1$, C_R^+ is the semicircular curve in the upper half-plane from R to $-R$.

It is vital to master estimates like these—many contour integrals we evaluate later in the course will require us to find upper bounds of integrals along certain curves in exactly this way. Notice in part (b) that the z^4+1 term in the denominator of the integrand somehow becomes an R^4-1 term in the denominator of the upper bound—in particular there has been a sign change. A use of the reverse triangle inequality is responsible for this.

Solution 4. (a) A point z lies on the curve $C_2(1)$ if and only if $|z-1| = 2$, which implies by the reverse triangle inequality that
$$|z| = |(z-1)+1| = |(z-1)-(-1)| \geq |(|z-1| - |-1|)| \geq |z-1| - 1 = 2-1 = 1.$$
So for all z on $C_2(1)$, we have that $|1/z| \leq 1/1 = 1$, i.e. $\max_{z\in C_2(1)} |1/z| \leq 1$. Since the arclength of a circle of radius R is $2\pi R$, we then have by lemma 3.2.9 that
$$\left|\int_{C_2(1)} \frac{1}{z}\,dz\right| \leq \max_{z\in C_2(1)} \left|\frac{1}{z}\right| \ell(C_2(1)) \leq 1(2\pi(2)) = 4\pi.$$

(b) A point z on the curve C_R^+ satisfies $|z| = R$. Now, to find an upper bound for the modulus of the integrand, we need a lower bound for $\left|z^4+1\right|$. This again uses the reverse triangle inequality: for $z \in C_R^+$, we have that
$$\left|z^4+1\right| = \left|z^4-(-1)\right| \geq \left|\left(\left|z^4\right| - |-1|\right)\right| \geq \left|z^4\right| - 1 = |z|^4 - 1 = R^4-1.$$
Note that the right-hand side is a positive number, since $R > 1$ by assumption. Therefore on C_R^+, we have that
$$\left|\frac{1}{z^4+1}\right| = \frac{1}{|z^4+1|} \leq \frac{1}{R^4-1}.$$
So $\max_{z\in C_R^+} \left|\frac{1}{z^4+1}\right| \leq \frac{1}{R^4-1}$. Since C_R^+ is a semicircle of radius R, its arclength $\ell(C_R^+)$ is half that of the circle of radius R, i.e. is πR. So we have by lemma 3.2.9 that
$$\left|\int_{C_R^+} \frac{1}{z^4+1}\,dz\right| \leq \max_{z\in C_R^+} \left|\frac{1}{z^4+1}\right| \ell(C_R^+) \leq \frac{\pi R}{R^4-1}.$$

Question 5. Let $D, \tilde{D} \subseteq \mathbb{C}$ be domains, and f be holomorphic on D. For each of the following statements, either prove that it is true, or give a counterexample to show that it is false:

(a) $\mathbb{C} \setminus \overline{D}$ is a domain;
(b) $D \cup \tilde{D}$ is a domain;
(c) $D \cap \tilde{D}$ is a domain if it is non-empty;
(d) D is bounded if f is bounded; and
(e) f is bounded if D is bounded.

[5=(1+1+1+1+1) marks]

The word "domain" has a precise definition (see definition 3.3.1), which is crucial to many of the results in the course from this point. It is important, therefore, to understand what it does and does not mean.

Solution 5. (a) This is false. For example $D = \{ z \in \mathbb{C} : -1 < \mathrm{Re}(z) < 1 \}$ is a domain, but $\mathbb{C} \setminus \overline{D} = \{ z \in \mathbb{C} : \mathrm{Re}\, z < -1 \text{ or } \mathrm{Re}(z) > 1 \}$ is not. By the intermediate value theorem, any continuous curve Γ parametrized by $\gamma \colon [t_0, t_1] \to \mathbb{C}$ connecting -3 and 3 must have a point $t \in [t_0, t_1]$ where $\mathrm{Re}(\gamma(t)) = 0$. The curve Γ therefore does not lie in $\mathbb{C} \setminus \overline{D}$.

(b) This is false. For example, $D = \{ z \in \mathbb{C} : \mathrm{Re}(z) > 1 \}$ and $\tilde{D} = \{ z \in \mathbb{C} : \mathrm{Re}(z) < -1 \}$ are both domains, but their union is not, by the argument in the previous part.

(c) This is false. For example, $D = \{ z \in \mathbb{C} : 2 < |z| < 4 \}$ and $\tilde{D} = \{ z \in \mathbb{C} : -1 < \mathrm{Im}(z) < 1 \}$ are both domains, but $D \cap \tilde{D}$ is not. The points -3 and 3 both lie in $D \cap \tilde{D}$, but $z \in D \cap \tilde{D}$ implies that $|\mathrm{Re}(z)| > 1$, so again the argument in part (a) shows that they cannot be connected by a continuous curve.

(d) This is false. For example, $f(z) = 1$ is a bounded holomorphic function on $\mathbb{C}$, which is an unbounded domain.

(e) This is false. For example, $f(z) = 1/z$ is an unbounded holomorphic function on the punctured disc $D_1'(0)$, which is a bounded domain.

Question 6. (a) Show that there does not exist a holomorphic function $f \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $f'(z) = 1/z$ for all non-zero $z \in \mathbb{C}$.

(b) Deduce that there does not exist a holomorphic function $l \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $\exp(l(z)) = z$ for all non-zero $z \in \mathbb{C}$.

[5=(3+2) marks]

This is part of a question from an Oxford 2005 exam paper—in fact one that I sat. We have seen that our definition of the complex logarithm is not holomorphic on $\mathbb{C} \setminus \{0\}$, since it is always non-holomorphic on some half-line from the origin. This question shows that this is not just a problem with the definition we made—it is impossible to define an inverse to the exponential function which is holomorphic on $\mathbb{C} \setminus \{0\}$.

Solution 6. (a) Suppose that such a function f exists. Then by definition the function $1/z$ has an antiderivative on $\mathbb{C} \setminus \{0\}$. Therefore by lemma 3.3.9 (path-independence), we have that $\int_\Gamma \frac{1}{z}\, dz = 0$ for any closed contour in $\mathbb{C} \setminus \{0\}$. But example 3.2.5 shows $\int_{C_R(0)} \frac{1}{z} dz = 2\pi i \neq 0$ for any $R > 0$. This is a contradiction, so no such f exists.

(b) Suppose that such a function l exists. Then since exp is holomorphic, differentiating both sides of the equation $\exp(l(z)) = z$ and using the chain rule implies that

$$z l'(z) = \exp(l(z)) l'(z) = \frac{d}{dz}(\exp(l(z))) = \frac{d}{dz}(z) = 1,$$

therefore $l'(z) = 1/z$, for any non-zero $z \in \mathbb{C}$. This is a contradiction with part (a), so no such function l exists.

HONOURS COMPLEX VARIABLES WORKSHOP 7 SOLUTIONS
4^{th} MARCH 2019

Solutions to questions 4 and 5 are due 11.00, Monday 11^{th} March 2019.

This workshop concentrates on the central results of the course: Cauchy's Integral Theorem, and the (Generalized) Cauchy Integral Formula. They can be used in a variety of ways and have a myriad of consequences, as we will continue to explore in lectures. Here we focus on applying them directly to evaluate integrals, learning when to spot that they apply, and developing tricks so as to be able to apply them even when, at first glance, they do not seem applicable.

Some of these exercises are taken from *Introduction to Complex Analysis* by H. A. Priestley.

Question 1. Explain why the following contour integrals are zero:

(a) $\int_{C_1(1)} \frac{1}{z-3}\, dz$;
(b) $\int_{C_4(i)} \frac{1}{(z-3)^2}\, dz$;
(c) $\int_{C_1(0)} z\,|z|^4\, dz$; and
(d) $\int_{C_1(1)} \frac{1}{1+\exp(z)}\, dz$.

There is more than one reason in some cases. As a start, in each case, consider the integrand, and consider whether you can easily find an antiderivative on a domain containing the contour, or whether it is holomorphic inside and on the contour. Notice that the geometry is subtly different in these two considerations.

Solution 1. (a) The function $f(z) = 1/(z-3)$ is holomorphic everywhere except at $z = 3 \in \operatorname{Ext}(C_1(1))$. Therefore f is holomorphic inside and on the loop $C_1(1)$, therefore Cauchy's Integral Theorem says that the integral is zero.

(b) The function $f(z) = 1/(z-3)^2$ has an antiderivative $F(z) = -1/(z-3)$ on the domain $\mathbb{C} \setminus \{3\}$, so, since the closed contour $C_4(i)$ does not pass through the point $z = 3$, the path-independence lemma implies that the integral is zero.

(c) Since $|z| = 1$ on the contour $C_1(0)$, the integrand is $z\,|z|^4 = z$, which is holomorphic everywhere, in particular inside and on the loop $C_1(0)$, therefore the integral is zero by Cauchy's Integral Theorem.

(d) The function $f(z) = 1/(1+\exp(z))$ is holomorphic everywhere except at points z such that $1+\exp(z) = 0$, i.e. such that $\exp(z) = -1$. So f is holomorphic on the set $\mathbb{C} \setminus \{\, \pi i + 2k\pi i : k \in \mathbb{Z} \,\}$. No points of the form $\pi i + 2k\pi i$ lie inside or on $C_1(1)$, so f is holomorphic inside and on the loop $C_1(1)$, so Cauchy's Integral Theorem implies that the integral is zero.

Question 2. Evaluate the following contour integrals:

(a) $\int_{C_2(0)} \frac{z^3+5}{z-i}\, dz$;
(b) $\int_{C_2(0)} \frac{1}{z^2+z+1}\, dz$; and
(c) $\int_{C_2(0)} \frac{\sin(z)}{z^2+1}\, dz$.

Some practice with the Cauchy Integral Formula. It is important to learn how to manipulate integrands so that they are in a form in which the formula applies—partial fractions might be useful.

Solution 2. (a) Let $f(z) = z^3 + 5$. Then f is holomorphic everywhere, in particular inside and on the loop $C_2(0)$, which contains the point $z = i$ in its interior, so

Richard Gratwick, February 2019, file: Workshop-7-solutions.

Cauchy's Integral Formula asserts that

$$\int_{C_2(0)} \frac{f(z)}{z-i}\,dz = 2\pi i f(i) = 2\pi i(i^3+5) = 2\pi(1+5i).$$

(b) We use partial fractions to write the integrand as

$$\begin{aligned}\frac{1}{z^2+z+1} &= \frac{1}{\left(z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)\left(z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)}\\ &= \frac{1}{\sqrt{3}i}\left(\frac{1}{z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} - \frac{1}{z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}\right).\end{aligned}$$

These two summands define functions that are holomorphic everywhere except at the points $z=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $z=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$ respectively, both of which lie inside the contour. We use of linearity of integration and theorem 3.4.11 to see that

$$\begin{aligned}\int_{C_2(0)} \frac{1}{z^2+z+1}\,dz &= \int_{C_2(0)} \frac{1}{\sqrt{3}i}\left(\frac{1}{z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} - \frac{1}{z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}\right)dz\\ &= \frac{1}{\sqrt{3}i}\left(\int_{C_2(0)} \frac{1}{z-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}\,dz - \int_{C_2(0)} \frac{1}{z-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)}\,dz\right)\\ &= \frac{1}{\sqrt{3}i}(2\pi i - 2\pi i)\\ &= 0.\end{aligned}$$

(c) Similarly we use partial fractions to write

$$\frac{\sin(z)}{z^2+1} = \frac{\sin(z)}{(z+i)(z-i)} = \frac{\sin(z)}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Then our integral can be written as

$$\int_{C_2(0)} \frac{\sin(z)}{z^2+1}\,dz = \frac{1}{2i}\left(\int_{C_2(0)} \frac{\sin(z)}{z-i}\,dz - \int_{C_2(0)} \frac{\sin(z)}{z+i}\,dz\right),$$

where both the integrals on the right-hand side are of the form $\int_{C_2(0)} \frac{f(z)}{z-z_0}\,dz$ for the holomorphic function $f(z)=\sin(z)$ and the points $z_0=\pm i$, both of which lie inside the contour. Therefore Cauchy's Integral Formula implies, using also that sin is an odd function, that

$$\begin{aligned}\int_{C_2(0)} \frac{\sin(z)}{z^2+1}\,dz = \frac{1}{2i}\left(\int_{C_2(0)} \frac{\sin(z)}{z-i}\,dz - \int_{C_2(0)} \frac{\sin(z)}{z+i}\,dz\right) &= \frac{1}{2i}2\pi i(\sin(i)-\sin(-i))\\ &= 2\pi\sin(i)\\ &= 2\pi\frac{\exp(i^2)-\exp(-i^2)}{2i}\\ &= -\pi i(\exp(-1)-\exp(1))\\ &= \pi i\left(e-\tfrac{1}{e}\right).\end{aligned}$$

Question 3. Evaluate

$$\int_{C_1(0)} \frac{\mathrm{Re}(z)}{z-\frac{1}{2}}\,dz.$$

Beware! The function $\mathrm{Re}(z)$ *is non-holomorphic. But look carefully at the set of points described by* $C_1(0)$*, and see if you can find a holomorphic function which is equal to* $\mathrm{Re}(z)$ *on that set. This is a common trick.*

Solution 3. As the remark says, we cannot do very much with the integral while the integrand has an expression of the form $\mathrm{Re}(z)$ in it, since this is non-holomorphic. However, we recall that $\mathrm{Re}(z) = (z+\overline{z})/2$, and, furthermore, that on $C_1(0)$ we have $z\overline{z} = |z|^2 = 1$, thus $\overline{z} = z^{-1}$. Hence $\mathrm{Re}(z) = (z + z^{-1})/2$ on $C_1(0)$. Thus it suffices to evaluate the integral

$$\int_{C_1(0)} \frac{(z+z^{-1})/2}{z-\frac{1}{2}}\,dz = \frac{1}{2}\left(\int_{C_1(0)} \frac{z}{z-\frac{1}{2}}\,dz + \int_{C_1(0)} \frac{1}{z\left(z-\frac{1}{2}\right)}\,dz\right).$$

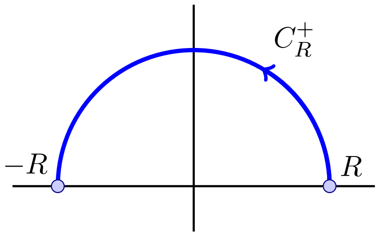
We can use partial fractions to rewrite the second integrand as

$$\frac{1}{z\left(z-\frac{1}{2}\right)} = \frac{2}{z-\frac{1}{2}} - \frac{2}{z}.$$

Since the points $z = 0, 1/2$ lie inside the loop $C_1(0)$, we can then use the Cauchy Integral Formula to see that

$$\begin{aligned}\int_{C_1(0)} \frac{(z+z^{-1})/2}{z-\frac{1}{2}}\,dz &= \frac{1}{2}\left(\int_{C_1(0)} \frac{z}{z-\frac{1}{2}}\,dz + \int_{C_1(0)} \frac{2}{z-\frac{1}{2}}\,dz - \int_{C_1(0)} \frac{2}{z}\,dz\right) \\ &= \frac{1}{2}\left(2\pi i\frac{1}{2} + 2(2\pi i) - 2(2\pi i)\right) \\ &= \frac{\pi i}{2}.\end{aligned}$$

Question 4. Let $f(z) = \dfrac{1}{z^2+4}$. Let $R > 2$ be a real number and let C_R^+ denote the semicircular arc of radius R in the upper half-plane centred at the origin from R to $-R$.



(a) Show that

$$\left|\int_{C_R^+} f(z)\,dz\right| \to 0 \text{ as } R \to \infty.$$

(b) Use this to evaluate the (improper) integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+4}\,dx.$$

[5=(2+3) marks]

A taste of things to come: this kind of problem will turn up repeatedly towards the end of the course, where we will have rather more sophisticated tools at our disposal. This one, however, we can solve now. One of the most powerful and surprising applications of complex analysis is the evaluation of real integrals, which apparently have no connection to complex numbers, via contour integration.

Solution 4. (a) On C_R^+, we have that $|z| = R$, so by the reverse triangle inequality, $\left|z^2+4\right| = \left|z^2-(-4)\right| \geq \left|\left|z^2\right| - |-4|\right| \geq \left|z^2\right| - 4 = R^2 - 4$, which is positive by assumption, so

$$\max_{z\in C_R^+}\left|\frac{1}{z^2+4}\right| \leq \frac{1}{R^2-4}.$$

Since C_R^+ has arclength πR, lemma 3.2.9 implies that

$$\left|\int_{C_R^+} f(z)\,dz\right| \leq \max_{z\in C_R^+}|f(z)|\,\ell(C_R^+) \leq \frac{\pi R}{R^2-4}.$$

Therefore

$$0 \leq \left|\int_{C_R^+} f(z)\,dz\right| \leq \frac{\pi R}{R^2-4} = \frac{1}{R}\cdot\frac{\pi}{1-(2/R)^2} \to 0 \text{ as } R\to\infty.$$

So $\lim_{R\to\infty}\left|\int_{C_R^+} f(z)\,dz\right| = 0$.

(b) The point here is that $\int_{-R}^{R}\frac{1}{x^2+4}\,dx$ can be thought of as the contour integral of $\frac{1}{z^2+4}$ along the segment of the real axis from $-R$ to R. Call this contour γ_R. Define a contour Γ_R by following first γ_R and then C_R^+. Then Γ_R is a loop to which we can consider applying Cauchy's Integral Formula to evaluate $\int_{\Gamma_R}\frac{1}{z^2+4}\,dz$. We can write $f(z) = \dfrac{g(z)}{z-2i}$, where $g(z) = \dfrac{1}{z+2i}$ is holomorphic inside and on Γ_R. Therefore Cauchy's Integral Formula implies that

$$\int_{\Gamma_R}\frac{1}{z^2+4}\,dz = \int_{\Gamma_R}\frac{g(z)}{z-2i}\,dz = 2\pi i g(2i) = \frac{2\pi i}{2i+2i} = \frac{\pi}{2}.$$

The crucial fact here is that this is independent of R, so we can now let $R\to\infty$. We know what happens to the integral over the two regular curves of Γ_R: the integral over the semicircular arc C_R^+ goes to 0, by part (a), and the integral over the line segment on the real axis converges by definition to the integral we want. Precisely,

$$\begin{aligned}
I = \lim_{R\to\infty}\int_{\gamma_R}\frac{1}{z^2+4}\,dz &= \lim_{R\to\infty}\left(\int_{\Gamma_R}\frac{1}{z^2+4}\,dz - \int_{C_R^+}\frac{1}{z^2+4}\,dz\right)\\
&= \lim_{R\to\infty}\int_{\Gamma_R}\frac{1}{z^2+4}\,dz - \lim_{R\to\infty}\int_{C_R^+}\frac{1}{z^2+4}\,dz\\
&= \lim_{R\to\infty}\frac{\pi}{2} - 0\\
&= \frac{\pi}{2}.
\end{aligned}$$

Question 5. Suppose f is holomorphic on $D_1(0)$ and is such that $\max_{z\in C_r(0)}|f(z)| \to 0$ as $r\to 1$. Prove that f is identically zero. **[5 marks]**

This might seem rather surprising—there seems little intuitive reason for this to be true. We shall uncover richer results about the "shape" of holomorphic functions in the Maximum Modulus Principle. This is an application of the Cauchy Integral Formula, but on what contour? A good picture should help you choose.

Solution 5. Fix $z_0 \in D_1(0)$, and let $\varepsilon > 0$. We shall show that $|f(z_0)| < \varepsilon$. Since ε is arbitrary, this shows the result. By assumption there exists $r_0 \in (0,1)$ such that

$$\max_{z\in C_r(0)}|f(z)| < \frac{\varepsilon(1-|z_0|)}{2}$$

whenever $r \in [r_0, 1)$. Consider $r = \max\left\{r_0, \frac{|z_0|+1}{2}\right\}$. Then since

$$r \geq \frac{|z_0| + 1}{2} > \frac{|z_0| + |z_0|}{2} = |z_0|,$$

by definition $z_0 \in D_r(0)$. Furthermore, for $z \in D_r(0)$, the reverse triangle inequality implies that

$$|z - z_0| \geq ||z| - |z_0|| \geq |z| - |z_0| = r - |z_0| \geq \frac{|z_0| + 1}{2} - |z_0| = \frac{1 - |z_0|}{2}.$$

So

$$\left|\frac{f(z)}{z - z_0}\right| = |f(z)| \cdot \frac{1}{|z - z_0|} < \frac{\varepsilon(1 - |z_0|)}{2} \cdot \frac{2}{1 - |z_0|} = \varepsilon.$$

So the Cauchy Integral Formula and lemma 3.2.9 imply that

$$\begin{aligned} |f(z_0)| = \left|\frac{1}{2\pi i}\int_{C_r(0)} \frac{f(z)}{z - z_0}\, dz\right| &\leq \frac{1}{2\pi} \max_{z \in C_r(0)} \left|\frac{f(z)}{z - z_0}\right| \ell(C_r(0)) \\ &< \frac{2\pi r \varepsilon}{2\pi} \\ &< \varepsilon, \end{aligned}$$

since $r < 1$.

HONOURS COMPLEX VARIABLES WORKSHOP 8 SOLUTIONS
11th MARCH 2019

Solutions to questions 5 and 6 are due 11.00, Monday 25th March 2019.

We now begin to explore some of the many consquences of the (Generalized) Cauchy Integral Formula. Most of the results on this workshop sheet derive from Liouville's Theorem. We discover various ways of applying this theorem, and prove some rather striking results, which contrast starkly with the possible behaviour of real differentiable functions.

Some of these exercises are taken from *Introduction to Complex Analysis* by H. A. Priestley.

Question 1. Evaluate the following integrals $\int_{C_R(z_0)} f(z)\,dz$, where f and $C_R(z_0)$ are defined by

(a) $f(z) = \dfrac{\exp(z)}{z^3}$, $z_0 = 0$, and $R = 1$;

(b) $f(z) = \dfrac{1}{(z-4)(z+1)^4}$, $z_0 = -1$, and $R = 3$; and

(c) $f(z) = \dfrac{1}{(z+1)^2(z^2+9)}$, $z_0 = 0$, and $R = 2$.

A gentle warm-up exercise in using the Generalized Cauchy Integral Formula. You should find yourself doing more differentiation than integration.

Solution 1. (a) The integrand f is of the form $g(z)/(z-z_0)^{(n+1)}$ where $g(z) = \exp(z)$ is holomorphic everywhere, in particular inside and on the loop $C_1(0)$, and $z_0 = 0 \in \operatorname{Int}(C_1(0))$, and $n = 2$. Therefore the Generalized Cauchy Integral Formula implies that

$$\int_{C_1(0)} \frac{\exp(z)}{z^3}\,dz = \frac{2\pi i}{2!}\,\frac{d^2}{dz^2}\exp(z)\bigg|_{z=0} = \pi i\exp(0) = \pi i.$$

(b) The integrand f is of the form $g(z)/(z-z_0)^{(n+1)}$ where $g(z) = 1/(z-4)$ is holomorphic everywhere except at the point $z = 4$, in particular inside and on the loop $C_3(-1)$, the point $z_0 = -1 \in \operatorname{Int}(C_3(-1))$, and $n = 3$. Therefore the Generalized Cauchy Integral Formula implies that

$$\begin{aligned}\int_{C_3(-1)} \frac{1}{(z-4)(z+1)^4}\,dz = \frac{2\pi i}{3!}\left(\frac{d^3}{dz^3}\left(\frac{1}{z-4}\right)\right)\bigg|_{z=-1} &= \frac{2\pi i}{3!}\left(\frac{-3!}{(z-4)^4}\right)\bigg|_{z=-1}\\ &= \frac{-2\pi i}{(-5)^4}\\ &= \frac{-2\pi i}{625}.\end{aligned}$$

(c) The integrand f is of the form $g(z)/(z-z_0)^{(n+1)}$ where $g(z) = 1/(z^2+9)$ is holomorphic everywhere except at the points $z = \pm 3i$, in particular inside and on the loop $C_2(0)$, the point $z_0 = -1 \in \operatorname{Int}(C_2(0))$, and $n = 1$. Therefore the Generalized Cauchy Integral Formula implies that

$$\begin{aligned}\int_{C_2(0)} \frac{1}{(z+1)^2(z^2+9)}\,dz = 2\pi i\left(\frac{d}{dz}\left(\frac{1}{z^2+9}\right)\right)\bigg|_{z=-1} = 2\pi i\left(\frac{-2z}{(z^2+9)^2}\right)\bigg|_{z=-1} &= \frac{4\pi i}{100}\\ &= \frac{\pi i}{25}.\end{aligned}$$

Richard Gratwick, March 2019, file: Workshop-8-solutions.

Question 2. (a) Let f be holomorphic on $\mathbb{C}$ and satisfy $|f(z)| \to 0$ as $|z| \to \infty$. Prove that $f(z) = 0$ for all $z \in \mathbb{C}$.

(b) Find a function $g\colon \mathbb{R} \to \mathbb{R}$ that is infinitely differentiable and satisfies $|g(x)| \to 0$ as $|x| \to \infty$, but $g(x) \neq 0$ for all $x \in \mathbb{R}$.

A demonstration of one of the many ways in which the shape of complex differentiable functions is much more restricted than that of real differentiable functions. For the proof, you may care to recall that a complex function that is continuous on a closed and bounded set is bounded.

Solution 2. (a) By definition there exists $R > 0$ such that $|f(z)| < 1$ whenever $|z| > R$. The function f is holomorphic and therefore continuous on the closed and bounded set $\overline{D}_R(0)$, and is therefore bounded on this set, i.e. there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \overline{D}_R(0)$. Then $|f(z)| \leq \max\{1, M\}$ for all $z \in \mathbb{C}$, thus f is a bounded holomorphic function on $\mathbb{C}$. Liouville's Theorem then implies that f is constant. Since $|f(z)| \to 0$ as $|z| \to \infty$, we must have that $f(z) = 0$ for all $z \in \mathbb{C}$.

(b) Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = e^{-x^2}$. Then g is infinitely differentiable, $|g(x)| = g(x) \to 0$ as $|x| \to \infty$, but $g(x) \neq 0$ for all $x \in \mathbb{R}$.

Question 3. Suppose that f is holomorphic on $\mathbb{C}$ and is periodic in the real and imaginary directions, i.e. there exist real $a_0, b_0 > 0$ such that $f(z) = f(z + a_0)$ and $f(z) = f(z + ib_0)$ for all $z \in \mathbb{C}$. Prove that f is constant.

The function $\exp(z)$ *is periodic in the imaginary direction, so interesting functions with periodicity in one coordinate direction certainly exist. This question shows that no non-trivial functions exist that are periodic in both coordinate directions. Again, remember that a continuous function on a closed and bounded set is bounded. As a further exercise, you might like to consider whether it is important that the two directions are perpendicular.*

Solution 3. The basic point is that the function f is determined by its values on a rectangle, on which it is bounded. Let

$$S = \{\, z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq a_0, \text{ and } 0 \leq \mathrm{Im}(z) \leq b_0 \,\}.$$

Then S is a closed and bounded set, and f is holomorphic and hence continuous on S. Therefore f is bounded on S, i.e. there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in S$. We claim that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$. Choose $m, n \in \mathbb{Z}$ such that $ma_0 \leq \mathrm{Re}(z) \leq (m+1)a_0$ and $nb_0 \leq \mathrm{Im}(z) \leq (n+1)b_0$. Then $z - (ma_0 + inb_0) \in S$. Repeated application of the periodicity condition on f implies that

$$\begin{aligned} f(z) = f(z - a_0) = f(z - 2a_0) = \cdots = f(z - ma_0) &= f(z - ma_0 - ib_0) \\ &= f(z - ma_0 - 2ib_0) \\ &\ \ \vdots \\ &= f(z - (ma_0 + inb_0)), \end{aligned}$$

(assuming $m, n \geq 0$ just for notational reasons), so

$$|f(z)| = |f(z - (ma_0 + inb_0))| \leq M,$$

since $z - (ma_0 + inb_0) \in S$. So indeed f is bounded on $\mathbb{C}$. Since it is holomorphic on $\mathbb{C}$, Liouville's Theorem implies that f is constant.

Question 4. Let f be holomorphic on $\mathbb{C}$. Prove that f is constant if any one of the following conditions holds:

(a) there exists $M_1 \in \mathbb{R}$ such that $\mathrm{Re}(f(z)) \leq M_1$ for all $z \in \mathbb{C}$;
(b) there exists $M_2 \in \mathbb{R}$ such that $\mathrm{Re}(f(z)) \geq M_2$ for all $z \in \mathbb{C}$;
(c) there exists $M_3 \in \mathbb{R}$ such that $\mathrm{Im}(f(z)) \leq M_3$ for all $z \in \mathbb{C}$; and

(d) there exists $M_4 \in \mathbb{R}$ such that $\operatorname{Im}(f(z)) \geq M_4$ for all $z \in \mathbb{C}$.

You will need a little bit of cunning to get you started. How can you translate a bound on (for example) the real part of a function into a bound on the modulus of a(nother) function? Once you have cracked that, consider carefully how much work you really need to do for the remaining parts.

Solution 4. Let $f = u + iv$ for $u, v \colon \mathbb{R}^2 \to \mathbb{R}$.

(a) By assumption $u(x, y) \leq M_1$ for all $(x, y) \in \mathbb{R}^2$. Define $g(z) = g(x + iy) = \exp(f(z)) = \exp(u(x, y) + iv(x, y)) = e^{u(x,y)}e^{iv(x,y)}$. Then $|g(z)| = e^{u(x,y)} \leq e^{M_1}$ by assumption, for all $z \in \mathbb{C}$, since real exponentiation is a monotonically increasing function. So g is a bounded function on $\mathbb{C}$. Since f and exp are holomorphic on $\mathbb{C}$, by the chain rule g is holomorphic on $\mathbb{C}$. So Liouville's Theorem implies that g is constant. This implies that u is constant, which by the Cauchy–Riemann equations implies that v is constant, hence f is constant.

(b) By assumption $u(x, y) \geq M_2$ for all $(x, y) \in \mathbb{R}^2$, or, equivalently, $-u(x, y) \leq -M_2$ for all $(x, y) \in \mathbb{R}^2$. Define $g(z) = \exp(-f(z))$. Then $|g(z)| = e^{-u(x,y)} \leq e^{-M_2}$ for all $z \in \mathbb{C}$ and the conclusion follows as before.

(c) By assumption $v(x, y) \leq M_3$ for all $(x, y) \in \mathbb{R}^2$. Define $g(z) = \exp(-if(z))$. Then $|g(z)| = e^{v(x,y)} \leq e^{M_3}$ for all $z \in \mathbb{C}$ and the conclusion follows as before.

(d) By assumption $v(x, y) \geq M_4$ for all $(x, y) \in \mathbb{R}^2$, or, equivalently, $-v(x, y) \leq -M_4$ for all $(x, y) \in \mathbb{R}^2$. Define $g(z) = \exp(if(z))$. Then $|g(z)| = e^{-v(x,y)} \leq e^{-M_4}$ for all $z \in \mathbb{C}$ and the conclusion follows as before.

Question 5. Suppose that f is holomorphic on $\mathbb{C}$, and for some integer $N \geq 1$ there exists $C > 0$ such that $|f(z)| \leq C\,|z|^N$ for all $z \in \mathbb{C}$. Prove that the nth derivative $f^{(n)}(z) = 0$ for all $z \in \mathbb{C}$, for all $n \geq N + 1$.

[5 marks]

This an application of the Generalized Cauchy Integral Formula, similar to—although not identical to—that used in the proof of theorem 3.6.1 (notice that this result as stated does not apply here, since the upper bound given on the function is not a constant). Once we have established that holomorphic functions have valid Taylor series expansions, we will be able to conclude that in fact such a function must be a polynomial of degree at most N.

Solution 5. Fix $z_0 \in \mathbb{C}$ and $n \geq N + 1$, and let $R \geq |z_0|$. The Generalized Cauchy Integral Theorem implies that

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi}\int_{C_R(z_0)} \frac{f(z)}{(z - z_0)^{n+1}}\,dz\right| \leq \frac{n!}{2\pi}\max_{z\in C_R(z_0)}\left|\frac{f(z)}{(z - z_0)^{n+1}}\right|\ell(C_R(z_0)).$$

Let $z \in C_R(z_0)$. Then by definition $|z - z_0| = R$. Furthermore, since $R \geq |z_0|$, the triangle inequality implies that

$$|f(z)| \leq C\,|z|^N \leq C\,(|z - z_0| + |z_0|)^N = C\,(R + |z_0|)^N \leq C\,(R + R)^N = 2^N C R^N.$$

Therefore

$$\max_{z\in C_R(z_0)}\left|\frac{f(z)}{(z - z_0)^{n+1}}\right| = \max_{z\in C_R(z_0)}\frac{|f(z)|}{|z - z_0|^{n+1}} \leq \frac{2^N C R^N}{R^{n+1}} = 2^N C R^{N-(n+1)}.$$

Therefore, since $\ell(C_R(z_0)) = 2\pi R$,

$$\begin{aligned}\left|f^{(n)}(z_0)\right| \leq \frac{n!}{2\pi}\max_{z\in C_R(z_0)}\left|\frac{f(z)}{(z - z_0)^{n+1}}\right|\ell(C_R(z_0)) &\leq \frac{n!}{2\pi}2^N C R^{N-(n+1)}2\pi R\\ &= 2^N C R^{N-n} n!\\ &\to 0\end{aligned}$$

as $R \to \infty$, since $n \geq N + 1$. Therefore $f^{(n)}(z_0) = 0$, as required.

Question 6. Suppose that f is holomorphic on $\mathbb{C}$ and satisfies $|f(z)| \to \infty$ as $|z| \to \infty$. Prove that f is surjective.

[5 marks]

Here is a remarkable fact that is certainly not true of real differentiable functions. You are asked to prove an existential assertion, i.e. that some point exists with a certain property. You have no way of guessing what this point might be. So what is your alternative strategy? You may assume the result of question 2(a) if you wish, although you do not have to. This is from a 2002 Oxford exam paper.

Solution 6. Suppose for a contradiction that f is not surjective, so there exists $a \in \mathbb{C}$ such that $f(z) \neq a$ for all $z \in \mathbb{C}$. Then the function $g(z) = 1/(f(z) - a)$ is holomorphic on $\mathbb{C}$. The reverse triangle inequality implies that

$$|f(z) - a| \geq ||f(z)| - |a|| \geq |f(z)| - |a| = |f(z)| \left(1 - (|a| \,/\, |f(z)|)\right),$$

which is a positive number for large enough $|z|$ since $|f(z)| \to \infty$ as $|z| \to \infty$. Therefore, again since $|f(z)| \to \infty$ as $|z| \to \infty$,

$$|g(z)| = \left|\frac{1}{f(z) - a}\right| = \frac{1}{|f(z) - a|} \leq \frac{1}{|f(z)|}\left(\frac{1}{1 - (|a| \,/\, |f(z)|)}\right) \to 0 \text{ as } |z| \to \infty.$$

So question 2(a) implies that $g(z) = 0$ for all $z \in \mathbb{C}$, which is a contradiction. So f is indeed surjective.

HONOURS COMPLEX VARIABLES WORKSHOP 9 SOLUTIONS
18$^{\text{th}}$ MARCH 2019

Solutions to questions 4 and 5 are due 11.00, Monday 25$^{\text{th}}$ March 2019.

This workshop aims to get you familiar with manipulating and calculating with complex series, in particular with determining where they do and do not converge, and calculating and using Taylor series expansions of functions. Many of the tricks and tests for complex series are basically the same as those for real series.

Some of these exercises are taken from *Introduction to Complex Analysis* by H. A. Priestley.

Question 1. For each of the following series, determine the set of points for which the series converges:

(a) $\sum_{j=0}^{\infty} \dfrac{(z-1)^j}{2^{j-1}}$;
(b) $\sum_{j=0}^{\infty} \dfrac{(z-i)^j}{(z+i)^j}$; and
(c) $\sum_{j=0}^{\infty} \exp(jz)$.

An answer of the form "the series converges if $z \in S$" is inadequate, since, for example, such a statement is evidently true but somewhat uninformative if $S = \{1\}$ for part (a). An adequate answer is of the form "the series converges if and only if *$z \in S$". Those three extra words quite change the logical assertion you are making. You might find the basic result about geometric series (lemma 4.1.7) useful.*

Solution 1. (a) We can write the terms of the series as

$$\frac{(z-1)^n}{2^{n-1}} = 2\left(\frac{z-1}{2}\right)^n,$$

so the series is the geometric series $2\sum_{j=0}^{\infty} c^j$ where $c = (z-1)/2$. Therefore this series converges if and only if $|(z-1)/2| < 1$, i.e. if and only if $z \in D_2(1)$.

(b) Again we can write the terms of the sequence as

$$\frac{(z-i)^n}{(z+i)^n} = \left(\frac{z-i}{z+i}\right)^n,$$

so this too is a geometric series of the form $\sum_{j=0}^{\infty} c^j$ where $c = \frac{z-i}{z+i}$. This therefore converges if and only if $\left|\frac{z-i}{z+i}\right| < 1$, i.e. if and only if $\mathrm{Im}(z) > 0$.

(c) By the properties of the exponential function, we can write the terms of the sequence as $\exp(nz) = (\exp(z))^n$, so this too is a geometric series of the form $\sum_{j=0}^{\infty} c^j$, where $c = \exp(z)$. This therefore converges if and only if $|c| < 1$. Writing $z = x + iy$, we recall that $\exp(z) = \exp(x+iy) = e^x e^{iy}$, so $|\exp(z)| = e^x < 1$ if and only if $x < 0$. So the series converges if and only if $\mathrm{Re}(z) < 0$.

Question 2. Determine the radius of convergence R of the power series $\sum_{j=1}^{\infty} a_j z^j$, where a_n is defined by:

(a) $a_n = n^2$;
(b) $a_n = (-1)^n/n^3$;
(c) $a_n = n^{-n}$; and
(d) $a_n = n!$.

Richard Gratwick, March 2019, file: Workshop-9-solutions.

Theorem 4.2.4, derived from the ratio test, will likely be very useful.

Solution 2. (a) We see that

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n^2}{(n+1)^2} = \frac{n^2}{n^2+2n+1} = \frac{n^2}{n^2} \cdot \frac{1}{1+2n^{-1}+n^{-2}} \to 1 \text{ as } n \to \infty,$$

hence theorem 4.2.4 implies that $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1$.

(b) Similarly

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{(-1)^n/n^3}{(-1)^{n+1}/(n+1)^3}\right| = \left|\frac{(-1)^n(n+1)^3}{(-1)^{n+1}n^3}\right| = \frac{n^3+3n^2+3n+1}{n^3}$$
$$= 1+3n^{-1}+3n^{-2}+n^{-3}$$
$$\to 1 \text{ as } n \to \infty,$$

hence theorem 4.2.4 implies that $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1$.

(c) We have to be a little cleverer with this one:

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{1/n^n}{1/(n+1)^{n+1}}\right| = \frac{(n+1)^{n+1}}{n^n} = (n+1)\left(\frac{n+1}{n}\right)^n = (n+1)\left(1+\frac{1}{n}\right)^n.$$

At this point we remember that $\left(1+\frac{1}{n}\right)^n \to e^1 > 0$ as $n \to \infty$, therefore theorem 4.2.4 implies that $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \infty$.

(d) This one is easier:

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 \text{ as } n \to \infty,$$

so theorem 4.2.4 implies that $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 0$.

Question 3. Find the Taylor series expansion of the function f centred at the point z_0, and determine the set on which it is valid, where f and z_0 are given by:

(a) $f(z) = \dfrac{1}{3+2z}$, $z_0 = 0$;
(b) $f(z) = \dfrac{1}{(1-z)^2}$, $z_0 = 3$; and
(c) $f(z) = \sin(z)$, $z_0 = 1$.

Use algebraic trickery and known series expansions, as much as you can.

Solution 3. (a) We write

$$\frac{1}{3+2z} = \frac{1}{3}\cdot\frac{1}{1-(-2z/3)} = \frac{1}{3}\sum_{j=0}^{\infty}(-2z/3)^j = \sum_{j=0}^{\infty}\frac{(-2)^j}{3^{j+1}}z^j,$$

using the geometric series expansion of $\frac{1}{1-(-2z/3)}$. This expansion is therefore valid on $D_{3/2}(0)$.

(b) We spot that

$$f(z) = \frac{d}{dz}\left(\frac{1}{1-z}\right),$$

and we therefore first find the series expansion of $1/(1-z)$ centred at $z_0 = 3$:

$$\frac{1}{1-z} = \frac{1}{-2-(z-3)} = \frac{-1}{2}\cdot\frac{1}{1-(-(z-3)/2)} = \frac{-1}{2}\sum_{j=0}^{\infty}(-(z-3)/2)^j$$
$$= \sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{2^{j+1}}(z-3)^j,$$

which is valid on $D_2(3)$. On this set we may differentiate the series term-by-term to get

$$f(z) = \frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{d}{dz}\sum_{j=0}^{\infty}\frac{(-1)^{j+1}}{2^{j+1}}(z-3)^j = \sum_{j=1}^{\infty}\frac{d}{dz}\frac{(-1)^{j+1}}{2^{j+1}}(z-3)^j$$
$$= \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{2^{j+1}}j(z-3)^{j-1}$$
$$= \sum_{j=0}^{\infty}\frac{(-1)^{j}}{2^{j+2}}(j+1)(z-3)^{j}.$$

This expansion is valid wherever the series which we differentiated was valid, i.e. on $D_2(3)$.

(c) We use the usual trigonometric addition formulae to write

$$\begin{aligned} f(z) = \sin(z) &= \sin(1 + (z-1)) \\ &= \sin(1)\cos(z-1) + \cos(1)\sin(z-1) \\ &= \sin(1)\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}(z-1)^{2j} + \cos(1)\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}(z-1)^{2j+1} \\ &= \sum_{j=0}^{\infty} a_j (z-1)^j \end{aligned}$$

where

$$a_k = \begin{cases} \frac{(-1)^{k/2}\sin(1)}{k!} & k = 2j \text{ for some } j \geq 0, \\ \frac{(-1)^{(k-1)/2}\cos(1)}{k!} & k = 2j+1 \text{ for some } j \geq 0. \end{cases}$$

This is valid wherever the trigonometric expansions are valid, i.e. on $\mathbb{C}$.

Question 4. Suppose that f is holomorphic on $\mathbb{C}$ and there exists $C > 0$ such that $|f(z)| \leq C\,|z|^2$ for all $z \in \mathbb{C}$.

Show that $f(z) = cz^2$ for some $c \in \mathbb{C}$ such that $|c| \leq C$.

[5 marks]

Another fact that is certainly not true of real differentiable functions (can you find a counter-example?). Consider the Taylor expansion of f centred at a suitable point. As always, you may assume any result from any previous workshop.

Solution 4. The Taylor series expansion for f centred at 0 is valid on the largest disc centred at 0 on which f is holomorphic. In this case this is the whole complex plane. So the expression

$$f(z) = \sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}z^j$$

is valid for all $z \in \mathbb{C}$. By assumption we are in the situation of workshop 8, question 5, and therefore we can apply that result immediately and see that $f^{(n)}(0) = 0$ for all $n \geq 3$. So in fact

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2$$

for all $z \in \mathbb{C}$. Applying the assumption again we see that $|f(0)| \leq C\,|0|^2 = 0$, so $f(0) = 0$. Since the assumption also implies that

$$\left|\frac{f(z)-f(0)}{z-0}\right| = \left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} \leq \frac{C\,|z|^2}{|z|} = C\,|z| \to 0$$

as $z \to 0$, we have by definition that

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = 0.$$

Thus in fact

$$f(z) = \frac{f''(0)}{2} z^2 = cz^2,$$

where $c = \frac{f''(0)}{2}$, and evidently we must have $|c| \le C$, by assumption.

Question 5. For $\varepsilon > 0$, define $U_\varepsilon = \{ z \in \mathbb{C} : \mathrm{Re}(z) > 1 + \varepsilon \}$, and for integers $n \ge 1$, define $\zeta_n(z) = n^{-z}$, where the principal branch of the logarithm is taken in the definition of the complex power.

(a) Show that $|\zeta_n(z)| < n^{-(1+\varepsilon)}$ for all n and $z \in U_\varepsilon$.
(b) Deduce that $\sum_{n=1}^{\infty} \zeta_n(z)$ converges uniformly on U_ε.
(c) Deduce that the function ζ defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \zeta_n(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is holomorphic on $\{ z \in \mathbb{C} : \mathrm{Re}(z) > 1 \}$.

[5=(1+1+3) marks]

For part (b) you may assume that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$. This function ζ is the celebrated Riemann zeta function. In fact this function can be extended to be defined on $\mathbb{C} \setminus \{1\}$. As an interesting exercise, you might like to prove that if $z \in \mathbb{C} \setminus \{1\}$ satisfies $\zeta(z) = 0$, then either $z = -2k$ for a positive integer k, or $\mathrm{Re}(z) = 1/2$, but please don't spend too long trying.

Solution 5. (a) By definition, for $z = x + iy \in U_\varepsilon$, we have that

$$\begin{aligned}\zeta_n(z) = n^{-z} = \exp(-z\,\mathrm{Log}(n)) = \exp\left(-z\left(\ln|n| + i\,\mathrm{Arg}(n)\right)\right) &= \exp(-z \ln n) \\ &= \exp(-x\ln n - iy\ln n) \\ &= e^{-x\ln n} e^{-iy\ln n} \\ &= n^{-x} e^{-iy\ln n},\end{aligned}$$

so $|\zeta_n(z)| = n^{-x} < n^{-(1+\varepsilon)}$, since $x > 1 + \varepsilon$.

(b) By part (a), $z \in U_\varepsilon$ implies that $|\zeta_n(z)| < n^{-(1+\varepsilon)}$, where the series $\sum_{n=1}^{\infty} n^{-(1+\varepsilon)}$ converges, since $1+\varepsilon > 1$. Therefore the Weierstrass M-test (lemma 4.1.19) exactly states that $\sum_{n=1}^{\infty} \zeta_n(z)$ converges uniformly on U_ε.

(c) Let $z_0 \in \{ z \in \mathbb{C} : \mathrm{Re}(z) > 1 \}$. Then choosing $\varepsilon = (\mathrm{Re}(z_0) - 1)/2 > 0$, we have that

$$1 + \varepsilon = 1 + \frac{(\mathrm{Re}(z_0)) - 1}{2} = \frac{1}{2} + \frac{\mathrm{Re}(z_0)}{2} < \frac{\mathrm{Re}(z_0)}{2} + \frac{\mathrm{Re}(z_0)}{2} = \mathrm{Re}(z_0),$$

so by definition $z_0 \in U_\varepsilon$. By part (b) the function $\zeta(z) = \sum_{n=1}^{\infty} \zeta_n(z)$ converges uniformly on U_ε. By the chain rule $\zeta_n(z) = \exp(-z\ln n)$ is holomorphic on $\mathbb{C}$, in particular on U_ε, for each $n \ge 1$. So $\sum_{n=1}^{N} \zeta_n(z)$ is holomorphic on U_ε for each N, and by definition of ζ converging uniformly on U_ε, this sequence of partial sums $\sum_{n=1}^{N} \zeta_n(z)$ converges uniformly to $\zeta(z)$ on U_ε as $N \to \infty$. Therefore by theorem 4.1.23 applied to the sequence of partial sums, ζ is holomorphic on U_ε. In particular ζ is holomorphic at z_0.

HONOURS COMPLEX VARIABLES WORKSHOP 10 SOLUTIONS
25^{th} MARCH 2019

Questions 1–3 concentrate on computing Laurent expansions, and classifying zeros and singularities of functions. All of this theory will be fundamental to the grand finale of the course: Cauchy's Residue Theorem. Questions 4 and 5 are more conceptual questions on the Identity Theorem and its corollaries (question 4) and applications of Laurent expansions (question 5).

Question 1. Find the Laurent series expansion of the function f that is valid on a punctured disc centred at each of the singularities of f, and determine the radius of the punctured disc in each case, for f defined as follows:

(a) $f(z) = \dfrac{1}{z^2-1}$;

(b) $f(z) = \dfrac{1}{z^2(z-1)}$; and

(c) $f(z) = \dfrac{1-iz}{1+iz}$.

Use geometric series expansions ruthlessly. Don't be alarmed if you end up with only finitely many terms—no-one said that the series expansions must have infinitely many non-zero terms.

Solution 1. (a) The function f has singularities at $z = \pm 1$, and can be written in the form

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{1}{z+1} \cdot \frac{1}{z-1}.$$

First consider the expansion centred at $z = 1$. In the above expression, $\frac{1}{z-1}$ is already of the form of a Laurent expansion centred at 1. So we just calculate that

$$\begin{aligned} \frac{1}{z+1} = \frac{1}{2-(-(z-1))} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{-(z-1)}{2}\right)} &= \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{-(z-1)}{2}\right)^j \\ &= \frac{1}{2}\sum_{j=0}^{\infty}(-1)^j 2^{-j}(z-1)^j \\ &= \sum_{j=0}^{\infty}(-1)^j 2^{-(j+1)}(z-1)^j, \end{aligned}$$

which is valid on $D_2(1)$. So

$$\begin{aligned} f(z) = \frac{1}{z-1}\sum_{j=0}^{\infty}(-1)^j 2^{-(j+1)}(z-1)^j &= \sum_{j=0}^{\infty}(-1)^j 2^{-(j+1)}(z-1)^{j-1} \\ &= \sum_{j=-1}^{\infty}(-1)^{j+1} 2^{-(j+2)}(z-1)^j, \end{aligned}$$

which is valid on $D_2'(1)$.

Richard Gratwick, March 2019, file: Workshop-10-solutions.

Now consider the expansion centred at $z=-1$. Now, $\frac{1}{z+1}$ is already of the form of a Laurent expansion centred at -1. So now we calculate that

$$\begin{aligned}\frac{1}{z-1} = \frac{-1}{2-(z+1)} = \frac{-1}{2}\cdot\frac{1}{1-\left(\frac{z+1}{2}\right)} &= \frac{-1}{2}\sum_{j=0}^{\infty}\left(\frac{z+1}{2}\right)^j \\ &= \frac{-1}{2}\sum_{j=0}^{\infty}2^{-j}(z+1)^j \\ &= \sum_{j=0}^{\infty}-2^{-(j+1)}(z+1)^j,\end{aligned}$$

which is valid on $D_2(-1)$. So

$$\begin{aligned}f(z) = \frac{1}{z+1}\sum_{j=0}^{\infty}-2^{-(j+1)}(z+1)^j &= \sum_{j=0}^{\infty}-2^{-(j+1)}(z+1)^{j-1} \\ &= \sum_{j=-1}^{\infty}-2^{-(j+2)}(z+1)^j,\end{aligned}$$

which is valid on $D_2'(-1)$.

(b) The function f has singularities at $z=0,1$. First consider the expansion centred at $z_0=0$. The we use the geometric series expansion to see that

$$f(z) = \frac{1}{z^2(z-1)} = \frac{1}{z^2}\cdot\frac{1}{z-1} = \frac{1}{z^2}\cdot\frac{-1}{1-z} = \frac{-1}{z^2}\sum_{j=0}^{\infty}z^j = \sum_{j=0}^{\infty}-z^{j-2} = \sum_{j=-2}^{\infty}-z^j,$$

which is valid on $D_1'(0)$. Now consider the expansion centred at $z_0=1$. Note that $\frac{1}{z^2} = \frac{d}{dz}\frac{-1}{z}$, and expand $\frac{1}{z}$ about the point $z_0=1$:

$$\frac{1}{z} = \frac{1}{1-(-(z-1))} = \sum_{j=0}^{\infty}(-(z-1))^j = \sum_{j=0}^{\infty}(-1)^j(z-1)^j,$$

which is valid on $D_1(1)$. So by term-by-term differentiation, we have that

$$\begin{aligned}\frac{1}{z^2} = \frac{d}{dz}\frac{-1}{z} = \frac{d}{dz}\sum_{j=0}^{\infty}(-1)^{j+1}(z-1)^j &= \sum_{j=1}^{\infty}\frac{d}{dz}(-1)^{j+1}(z-1)^j \\ &= \sum_{j=1}^{\infty}(-1)^{j+1}j(z-1)^{j-1} \\ &= \sum_{j=0}^{\infty}(-1)^j(j+1)(z-1)^j,\end{aligned}$$

which is valid wherever the series we differentiated is valid, i.e. on $D_1(1)$. Therefore

$$\begin{aligned}f(z) = \frac{1}{z-1}\cdot\frac{1}{z^2} = \frac{1}{z-1}\sum_{j=0}^{\infty}(-1)^j(j+1)(z-1)^j &= \sum_{j=0}^{\infty}(-1)^j(j+1)(z-1)^{j-1} \\ &= \sum_{j=-1}^{\infty}(-1)^{j+1}(j+2)(z-1)^j,\end{aligned}$$

which is valid on $D_1'(1)$.

(c) The function f is holomorphic everywhere except at $z = i$. We first write the numerator of f as a power series centred at $z_0 = i$, which will of course have only finitely many terms:

$$1 - iz = 1 - i(z - i) + 1 = 2 - i(z - i).$$

Now we see that

$$\frac{1}{1+iz} = \frac{1}{i}\frac{1}{z-i} = \frac{-i}{z-i},$$

which is a (one-term) Laurent expansion centred at $z_0 = i$. Putting them together we have that

$$f(z) = \frac{1-iz}{1+iz} = \frac{-i}{z-i}(2 - i(z-i)) = \frac{-2i}{z-i} - 1,$$

which because it has only finitely many terms is valid on $\mathbb{C} \setminus \{i\}$.

Question 2. Find the order of the zero of the function f at $z_0 = 0$, for f defined as follows:

(a) $f(z) = z^{25}$;
(b) $f(z) = \sin(z^2)$;
(c) $f(z) = z\sin(z^2)$; and
(d) $f(z) = \exp(z) - 1$.

The order of the zero is determined by how many derivatives we need to take to find a non-zero term when evaluated at the zero, see definition 4.5.3. This of course determines which is the first non-zero term in the Taylor series centred at that point.

Solution 2. (a) The function f is written in the form of a Taylor series centred at $z_0 = 0$: every coefficient except that corresponding to $(z-z_0)^{25}$ is zero. In particular $f^{(k)}(0) = 0$ for all $k = 0, \dots, 24$, but $f^{(25)}(0) \neq 0$. So, by definition, 0 is a zero of order 25 of f.

(b) By the chain rule $f'(z) = 2z\cos(z^2)$, so $f'(0) = 0$. But by the product rule and the chain rule,

$$f''(z) = -2z(2z)\sin(z^2) + 2\cos(z^2) = 2(\cos(z^2) - 2z^2\sin(z^2)),$$

so we see that $f''(0) \neq 0$. So, by definition, 0 is a zero of order 2 of f.

(c) Using the product rule and the chain rule repeatedly we see that

$$\begin{aligned}
f'(z) &= 2z^2\cos(z^2) + \sin(z^2); \\
f''(z) &= -2z^2(2z)\sin(z^2) + 4z\cos(z^2) + 2z\cos(z^2) \\
&= -4z^3\sin(z^2) + 6z\cos(z^2); \text{ and} \\
f'''(z) &= -4z^3(2z)\cos(z^2) - 12z^2\sin(z^2) - 6z(2z)\sin(z^2) + 6\cos(z^2);
\end{aligned}$$

from which we see that $f(0) = f'(0) = f''(0) = 0$, but $f'''(0) \neq 0$. So, by definition, 0 is a zero of order 3 of f.

(d) We see that $f'(z) = \exp(z)$, so $f'(0) \neq 0$. So, by definition, 0 is a zero of order 1 of f.

Question 3. Classify the singularities of the following functions f, where f is defined as follows:

(a) $f(z) = \dfrac{1}{\exp(z) - 1}$;
(b) $f(z) = \dfrac{\sin(2\pi z)}{z^3(2z-1)}$;
(c) $f(z) = \sin(1/z)$; and
(d) $f(z) = \dfrac{1}{\exp(1/z) - i}$.

The classification of singularities (points at which the function is not holomorphic) is two-fold. First, we establish whether a singularity is isolated or non-isolated (definition 4.5.1). Then, for any isolated singularity, we establish whether it is a removable singularity, a pole of finite order, or an essential singularity (definition 4.5.7). Lemma 4.5.11 is very useful for the second classification.

Solution 3. (a) The function f is holomorphic except at points z such that $\exp(z) = 1$, i.e. $z_k = 2\pi ik$ for $k \in \mathbb{Z}$. These are evidently all isolated singularities. We can write $f(z) = g(z)/h(z)$ where $g(z) = 1$ and $h(z) = \exp(z) - 1$. Then $g(z_k) \neq 0$ for all k, $h(z_k) = 0$ for all k, but $h'(z_k) = \exp(z_k) \neq 0$ for all k. So each z_k is not a zero of g, and is a zero of order 1 of h. So by lemma 4.5.11(i), z_k is a pole of order 1, or a simple pole, of f, for each k.

(b) The function f is holomorphic except at the points $z = 0, 1/2$. Evidently both of these points are isolated zeros. We can write $f(z) = g(z)/h(z)$ where $g(z) = \sin(2\pi z)$ and $h(z) = z^3(2z - 1) = 2z^4 - z^3$.

Then $g(0) = 0$, but since $g'(z) = 2\pi \cos(2\pi z)$, $g'(0) \neq 0$. So 0 is a zero of order 1 of g. Also $h(0) = h'(0) = h''(0) = 0$, but $h'''(0) \neq 0$. So the point 0 is a zero of order 3 of h. Therefore lemma 4.5.11(ii) implies that $z = 0$ is a pole of order $3 - 1 = 2$ of f, or a double pole.

On the other hand, $g(1/2) = 0$ and $g'(1/2) \neq 0$, so $1/2$ is a zero of g of order 1. Furthermore $h(1/2) = 0$ but $h'(z) = 8z^3 - 3z^2$, so $h'(1/2) \neq 0$. So $z = 1/2$ is a zero of h of order 1. Therefore lemma 4.5.11(ii) implies that $z = 1/2$ is a removable singularity of f.

(c) The function f is holomorphic except at $z = 0$, which is therefore an isolated singularity of f. The Laurent series of f centred at $z = 0$ is given by

$$\sin(1/z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}(1/z)^{2j+1} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{-(2j+1)},$$

which is valid on $\mathbb{C} \setminus \{0\}$. This clearly has infinitely many non-zero coefficients of negative powers of z, thus, by definition, $z = 0$ is an essential singularity of f.

(d) The function f is holomorphic except at the points $z = 0$ and points z such that $\exp(1/z) = i$, i.e. points z_k such that $1/z_k = \frac{i\pi}{2} + 2k\pi i$ for $k \in \mathbb{Z}$. Thus the singularities are at 0 and $z_k = 1/((i\pi/2) + 2k\pi i)$ for integers $k \in \mathbb{Z}$. Since $z_k \to 0$ as $k \to \infty$, we see that $z = 0$ is not an isolated singularity of f. Each z_k is an isolated singularity of f. Writing $f(z) = g(z)/h(z)$, where $g(z) = 1$ and $h(z) = \exp(1/z) - i$, we see that each z_k is not a zero of g, and since $h'(z) = \frac{-1}{z^2}\exp(1/z)$, $h'(z_k) \neq 0$ for all $k \in \mathbb{Z}$. So z_k is a zero of order 1 of h. Thus lemma 4.5.11(i) implies that z_k is a simple pole of f, for each $k \in \mathbb{Z}$.

Question 4. Suppose z_n is a sequence of distinct points in $D_1(0)$ such that $z_n \to 0$ as $n \to \infty$. Determine whether each of the following statements is true or false:

(a) if f is holomorphic on $D_1(0)$ and satisfies $f(z_n) = \sin(z_n)$ for all $n \in \mathbb{N}$, then $f(z) = \sin(z)$ for all $z \in D_1(0)$;
(b) there exists a function f which is holomorphic on $D_1(0)$ such that $f(z_n) = n$ for all $n \in \mathbb{N}$;
(c) there exists a function f which is holomorphic on $D_1(0)$ such that $f(z_n) = 0$ for even $n \in \mathbb{N}$ and $f(z_n) = z_n$ for odd $n \in \mathbb{N}$; and
(d) there exists a function f which is holomorphic on $D_1(0)$ such that $f(z_n) = (-1)^n z_n$ for all $n \in \mathbb{N}$.

This question demonstrates another way in which holomorphic functions are much more carefully constrained in their behaviour than their real-differentiable counterparts. The Identity Theorem, and in particular corollary 4.6.8, will be useful.

Solution 4. (a) This is true, and is a direct application of the corollary of the identity theorem, corollary 4.6.8, since f and sin are holomorphic on the domain $D_1(0)$, and agree on a sequence of points with a limit inside $D_1(0)$.

(b) This is false. Suppose such a holomorphic function f existed. Then f would in particular be holomorphic at 0, and hence continuous at 0. Then since $z_n \to 0$, we would have that $f(0) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} n$, which does not exist in the complex plane, which is therefore a contradiction.

(c) This is false. Suppose such a holomorphic function f existed. Then $f(z_{2k}) = 0$ for all $k \in \mathbb{N}$, where $z_{2k} \to 0 \in D_1(0)$ as $k \to \infty$, so by corollary 4.6.8, $f(z) = 0$ for all $z \in D_1(0)$. But similarly, $f(z_{2k+1}) = z_{2k+1}$ for all $k \in \mathbb{N}$, where $z_{2k+1} \to 0 \in D_1(0)$, so corollary 4.6.8 implies that $f(z) = z$ for all $z \in D_1(0)$, which is a contradiction.

(d) This is false. Suppose such a holomorphic function f existed. Then $f(z_{2k}) = z_{2k}$ for all $k \in \mathbb{N}$, where $z_{2k} \to 0 \in D_1(0)$ as $k \to \infty$, so by corollary 4.6.8, $f(z) = z$ for all $z \in D_1(0)$. But similarly, $f(z_{2k+1}) = -z_{2k+1}$ for all $k \in \mathbb{N}$, where $z_{2k+1} \to 0 \in D_1(0)$, so corollary 4.6.8 implies that $f(z) = -z$ for all $z \in D_1(0)$, which is a contradiction.

Question 5. Let f be holomorphic on the punctured disc $D_r'(z_0)$, and satisfy $|f(z)| \leq M$ for all $z \in D_r'(z_0)$, for some $M > 0$. Show that f can be (re-)defined at z_0 to define a function that is holomorphic on $D_r(z_0)$.

This is similar in principle to workshop 8, question 5, except now we only know that f has a Laurent expansion (in general) valid on $D_r'(z_0)$. But can you estimate the coefficients in a similar way?

Solution 5. Consider the Laurent expansion of f around z_0:

$$f(z) = \sum_{j=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z_0)^{j+1}} \, dw \right) (z - z_0)^j,$$

which is valid on $D_r'(z_0)$, for any loop $\Gamma \subseteq D_r'(z_0)$ with $z_0 \in \operatorname{Int}(\Gamma)$, in particular for the loop $C_\rho(z_0)$ for $0 < \rho < r$. Using lemma 3.2.9, we can then estimate the coefficients of the Laurent expansion:

$$\left| \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(w)}{(w - z_0)^{j+1}} \, dw \right| \leq \frac{1}{2\pi} \max_{w \in C_\rho(z_0)} \left| \frac{f(w)}{(w - z_0)^{j+1}} \right| \ell(C_\rho(z_0)) \leq \frac{1}{2\pi} \frac{M}{\rho^{j+1}} 2\pi\rho = M\rho^{-j}.$$

Since f is holomorphic arbitrarily close to z_0, we may choose ρ as small as we like, so for $j < 0$, we have that

$$\left| \frac{1}{2\pi i} \int_{C_\rho(z_0)} \frac{f(w)}{(w - z_0)^{j+1}} \, dw \right| \leq \lim_{\rho \to 0} M\rho^{-j} = 0.$$

Thus the jth coefficient in the Laurent expansion around z_0 is zero for all negative j. Then we have that

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j,$$

for the coefficients a_j as above, which is valid on $D_r'(z_0)$. But the right-hand side is a power series, since there are no negative powers of $(z - z_0)$ present, so the right-hand side is holomorphic on $D_r(z_0)$, i.e. including the point z_0, by theorem 4.2.6. But this power series equals f at all points except z_0 itself. So if we (re-)define f to take value a_0 at z_0, then f equals this power series at all points in $D_r(z_0)$, and is therefore holomorphic on $D_r(z_0)$.

HONOURS COMPLEX VARIABLES WORKSHOP 11 SOLUTIONS
1^{st} APRIL 2019

The Cauchy Residue Theorem is the crowning acheivement of this subject. This workshop looks at how to use it and its various applications. The most impressive results are the applications to the evaluation of real integrals.

Question 1. Calculate the residue at each isolated singularity of the function f, where f is defined as follows:

(a) $f(z) = \dfrac{1}{(z+1)^2(z-1)}$;
(b) $f(z) = \dfrac{z}{z^4-1}$;
(c) $f(z) = \dfrac{z}{\sin(z)}$; and
(d) $f(z) = \exp(1/z)$.

The Cauchy Residue Theorem reduces the calculation of integrals to the calculation of residues. It is therefore crucial to be able to calculate these proficiently. The first job is to establish where the singularities are, and classify them. Depending on their nature, you will then likely be able to use lemma 5.1.7 or lemma 5.1.5. This is not guaranteed, however, and sometimes you might have to return to the definition and consider the appropriate Laurent series.

Solution 1. (a) The function f has singularities at $z = \pm 1$, both of which are evidently isolated. Consider the singularity at -1. We may write $f(z) = g_1(z)/h_1(z)$ where $g_1(z) = 1/(z-1)$ and $h_1(z) = (z+1)^2$ are both holomorphic on a neighbourhood of -1. The point -1 is not a zero of g_1, and is a zero of order 2 of h_1. Therefore lemma 4.5.11(i) implies that -1 is a pole of order 2 (or a double pole) of f. We can then calculate the residue by using lemma 5.1.5:

$$\operatorname{Res}(f, -1) = \lim_{z\to -1} \frac{d}{dz}(z+1)^2 f(z) = \lim_{z\to -1} \frac{d}{dz}\frac{1}{z-1} = \lim_{z\to -1} \frac{-1}{(z-1)^2} = \frac{-1}{4}.$$

Now consider the singularity at $z = 1$. We may write $f(z) = g_2(z)/h_2(z)$, where $g_2(z) = 1/(z+1)^2$ and $h_2(z) = z-1$ are both holomorphic on a neighbourhood of 1. The point 1 is not a zero of g_2, and is a simple zero of h_2. Therefore lemma 4.5.11(i) implies that 1 is a simple pole of f. We can calculate the residue more easily using lemma 5.1.7:

$$\operatorname{Res}(f, 1) = \frac{g_2(1)}{h_2'(1)} = \frac{1}{4}.$$

(b) The function f has singularities at $z_k = e^{ki\pi/2}$ for $k = 0, 1, 2, 3$, all of which are evidently isolated. We may write $f(z) = g(z)/h(z)$ where $g(z) = z$ and $h(z) = z^4 - 1$ are both holomorphic everywhere. None of the singularities is a zero of g, and each is a simple zero of h, since $h'(z) = 4z^3$, which is non-zero at each of the singularities. So lemma 4.5.11(i) implies that each singularity is a simple pole of f. Then using lemma 5.1.7,

$$\operatorname{Res}(f, z_k) = \frac{g(z_k)}{h'(z_k)} = \frac{z_k}{4z_k^3} = \frac{1}{4z_k^2} = (-1)^k\frac{1}{4}.$$

Richard Gratwick, March 2019, file: Workshop-11-solutions.

(c) The function f has singularities at $z_k = k\pi$ for $k \in \mathbb{Z}$, all of which are isolated. We may write $f(z) = g(z)/h(z)$ where $g(z) = z$ and $h(z) = \sin(z)$ are holomorphic everywhere. For $k \neq 0$, z_k is not a zero of g, whereas z_0 is a simple zero of g. Each z_k is a simple zero of h. So lemma 4.5.11 implies that z_k is a simple pole of f for $k \neq 0$, and z_0 is a removable singularity f.

By definition, $\operatorname{Res}(f, z_0) = 0$. For $k \neq 0$, we use lemma 5.1.7 to see that

$$\operatorname{Res}(f, z_k) = \frac{g(z_k)}{h'(z_k)} = \frac{z_k}{\cos(z_k)} = (-1)^k k\pi.$$

(d) The function f has a singularity at 0, which is evidently isolated. The residue is most easily calculated by computing the Laurent expansion centred at 0, valid on $\mathbb{C} \setminus \{0\}$:

$$\exp(1/z) = \sum_{j=0}^{\infty} \frac{(1/z)^j}{j!} = \sum_{j=0}^{\infty} \frac{1}{j!} z^{-j},$$

from which we see that the residue of f at 0, i.e. the coefficient of z^{-1} in this expansion, is $1/1! = 1$.

Question 2. Show that $z^4 + 12z + 1$ has exactly three zeros inside the annulus $A_{1,4}(0) = \{ z \in \mathbb{C} : 1 < |z| < 4 \}$.

Rouché's Theorem, theorem 5.2.7, helps you count zeros of functions in particular regions. The hard part is choosing appropriate functions with which to compare the function in question.

Solution 2. We show that the polynomial has exactly four zeros inside $D_4(0)$ and exactly one zero inside $D_1(0)$. This implies that there are exactly three zeros in $D_4(0) \setminus D_1(0)$. On $C_1(0)$ we have, by the reverse triangle inequality, that

$$\left|z^4 + 12z + 1\right| \geq \left||12z| - \left|z^4 + 1\right|\right| \geq |12z| - \left|z^4 + 1\right| \geq 12\,|z| - |z|^4 - 1 = 12 - 1 - 1 = 10,$$

so there are no zeros on $C_1(0)$. Therefore the three zeros in $D_4(0) \setminus D_1(0)$ must in fact lie in $D_4(0) \setminus \overline{D}_1(0) = A_{1,4}(0)$.

First let $f(z) = z^4$ and $g(z) = z^4 + 12z + 1$. Then for $z \in C_4(0)$, we have that

$$\begin{aligned} |f(z) - g(z)| = \left|z^4 - (z^4 + 12z + 1)\right| = |12z + 1| \leq 12\,|z| + 1 = 12 \cdot 4 + 1 = 49 &< 256 \\ &= 4^4 \\ &= |z|^4 \\ &= |f(z)|\,. \end{aligned}$$

The function f has (counting multiplicity) exactly four zeros inside $D_4(0)$, therefore by Rouché's Theorem so too does the function g.

Now let $f(z) = 12z$ and $g(z) = z^4 + 12z + 1$. Then for $z \in C_1(0)$, we have that

$$\begin{aligned} |f(z) - g(z)| = \left|12z - (z^4 + 12z + 1)\right| = \left|z^4 + 1\right| \leq |z|^4 + 1 = 1^4 + 1 = 2 < 12 &= 12\,|z| \\ &= |f(z)|\,. \end{aligned}$$

The function f has exactly one zero inside $D_1(0)$, therefore by Rouché's Theorem so too does the function g.

Question 3. Let $a > 0$. Use the Cauchy Residue Theorem to evaluate the (improper) integral

$$\int_{-\infty}^{\infty} \frac{1}{a^2 + x^2}\,dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{a^2 + x^2}\,dx.$$

You calculated a very similar integral in Workshop 7, using the Cauchy Integral Formula. You should now be able to do it again, rather faster.

Solution 3. Consider the contour Γ_R defined by the traversing first the line segment γ_R from $-R$ to R on the real axis, and the semicircular contour C_R^+ from R to $-R$ in the upper half plane. Then Γ_R is a loop. Let $f(z) = 1/(a^2 + z^2)$. The function f has singularities at $z = \pm ia$, which are evidently isolated, of which only ia lies inside Γ_R for $R > a$. We can write $f(z) = g(z)/h(z)$ where $g(z) = 1$ and $h(z) = a^2 + z^2$ are holomorphic everywhere. The point ai is not a zero of g, and since $h'(z) = 2z$, the point ai is a simple zero of h. Therefore lemma 5.1.7 implies that

$$\operatorname{Res}(f, ai) = \frac{g(ai)}{h'(ai)} = \frac{1}{2ai}.$$

The Cauchy Residue Theorem implies that, for $R > a$,

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f, ai) = 2\pi i \frac{1}{2ai} = \frac{\pi}{a}.$$

As in workshop 7 question 4(a), we observe that for $z \in C_R^+$, we have, by the reverse triangle inequality, that

$$|f(z)| = \left|\frac{1}{z^2 + a^2}\right| = \frac{1}{|z^2 + a^2|} \leq \frac{1}{|z^2| - a^2} = \frac{1}{R^2 - a^2}.$$

Hence

$$\left|\int_{C_R^+} f(z)\,dz\right| \leq \max_{z \in C_R^+} |f(z)|\,\ell(C_R^+) \leq \frac{\pi R}{R^2 - a^2} = \frac{1}{R} \cdot \frac{\pi}{1 - (a/R)^2} \to 0$$

as $R \to \infty$. So

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{a^2 + x^2}\,dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{a^2 + x^2}\,dx &= \lim_{R \to \infty} \left(\int_{\Gamma_R} f(z)\,dz - \int_{C_R^+} f(z)\,dz \right) \\
&= \lim_{R \to \infty} \int_{\Gamma_R} f(z)\,dz - \lim_{R \to \infty} \int_{C_R^+} f(z)\,dz \\
&= \frac{\pi}{a}.
\end{aligned}$$

Question 4. Use the Cauchy Residue Theorem to evaluate the integral

$$\int_0^{2\pi} \frac{1}{5 + 4\cos(2\theta)}\,d\theta.$$

The Cauchy Residue Theorem allows us to calculate trigonometric integrals of this kind, by integrating a suitable function around the unit circle. The hardest part is establishing what a suitable function would be.

Solution 4. Note that for $z = \exp(i\theta)$, we have that

$$\cos(2\theta) = \operatorname{Re}(z^2) = \frac{z^2 + \overline{z^2}}{2} = \frac{z^2 + z^{-2}}{2}.$$

So

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{5 + 4\cos(2\theta)}\,d\theta &= \int_0^{2\pi} \frac{1}{5 + 4\left(\exp(i\theta)^2 + \exp(i\theta)^{-2}\right)/2}\,d\theta \\
&= \int_0^{2\pi} \frac{i\exp(i\theta)}{i\exp(i\theta)} \cdot \frac{1}{5 + 4\left(\exp(i\theta)^2 + \exp(i\theta)^{-2}\right)/2}\,d\theta \\
&= \int_0^{2\pi} \frac{\gamma'(\theta)}{i\gamma(\theta)} \cdot \frac{1}{5 + 4((\gamma(\theta))^2 + (\gamma(\theta))^{-2})/2}\,d\theta \\
&= \int_{C_1(0)} \frac{1}{iz} \cdot \frac{1}{5 + 2\left(z^2 + z^{-2}\right)}\,dz,
\end{aligned}$$

where we have used the definition of the contour integral via the parametrization $\gamma\colon [0,2\pi] \to \mathbb{C}$ given by $\gamma(\theta) = \exp(i\theta)$. We can rearrange the integrand to get

$$\frac{1}{iz} \cdot \frac{1}{5 + 2\left(z^2 + z^{-2}\right)} = \frac{-i}{5z + 2z^3 + 2z^{-1}} = \frac{-iz}{5z^2 + 2z^4 + 2},$$

so let

$$f(z) = \frac{-iz}{5z^2 + 2z^4 + 2}.$$

Then by the quadratic formula, f has singularities at points z such that

$$z^2 = \frac{-5 \pm \sqrt{25 - 16}}{4} = \frac{-5 \pm 3}{4} = -2, -1/2.$$

The square roots of -2 lie outside the unit circle, so will not contribute to the integral. The two square roots of $-1/2$, viz $z_\pm = \pm i/\sqrt{2}$, lie inside the unit circle, so will contribute. Writing $f(z) = g(z)/h(z)$ for $g(z) = -iz$ and $h(z) = 5z^2 + 2z^4 + 2$, we see that $z_\pm$ are not zeros of g, but are simple zeros of h, since $h'(z) = 10z + 8z^3$ which is non-zero at $z_\pm$. So lemma 4.5.11(i) implies that $z_\pm$ are simple poles of f. Then lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_\pm) = \frac{g(z_\pm)}{h'(z_\pm)} = \frac{-iz_\pm}{10z_\pm + 8z_\pm^3} = \frac{-i}{10 + 8z_\pm^2} = \frac{-i}{10 + 8(-1/2)} = \frac{-i}{6}.$$

Therefore the Cauchy Residue Theorem implies that

$$\int_0^{2\pi} \frac{1}{5 + 4\cos(2\theta)}\,d\theta = \int_{C_1(0)} f(z)\,dz = 2\pi i \left(\operatorname{Res}(f, z_+) + \operatorname{Res}(f, z_-)\right) = 2\pi i \frac{-2i}{6} = \frac{2\pi}{3}.$$

Question 5. Use the Cauchy Residue Theorem to evaluate the (improper) integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin x}{(x^2+1)^2}\,dx.$$

The Cauchy Residue Theorem allows us to calculate improper real integrals of this form. Remember that $\sin x = \operatorname{Im}(\exp(ix))$ *for real numbers* x*. As usual, considering an upper-semicircular contour will work, but you might find estimating the contribution from the semicircular arc slightly more intimidating than usual. Parametrize it explicitly and consider carefully what you know about the values of all the terms present.*

Solution 5. Note that, interpreting all integrals as improper integrals, we have

$$\int_{-\infty}^{\infty} \frac{x\exp(ix)}{(x^2+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{x\cos x + ix\sin x}{(x^2+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{x\cos x}{(x^2+1)^2}\,dx + i\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2}\,dx,$$

so

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)^2}\,dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{x\exp(ix)}{(x^2+1)^2}\,dx\right).$$

So define

$$f(z) = \frac{z\exp(iz)}{(z^2+1)^2},$$

and consider as usual the contour Γ_R defined by first traversing the line segment γ_R from $-R$ to R along the real axis, and then the semicircle C_R^+ from R to $-R$ in the upper half plane. The function f has singularities at the points $z = \pm i$, of which only i lies inside Γ_R, for $R > 1$. Writing $f(z) = g(z)/h(z)$ for $g(z) = z\exp(iz)$ and $h(z) = (z^2+1)^2$, we see that i is not a zero of g, and is a zero of order 2 (or a double zero) of h. So by lemma 4.5.11(i), i is a double pole of f. Then lemma 5.1.5 implies that

$$\begin{aligned}\mathrm{Res}\,(f, z_0) = \lim_{z\to i}\frac{d}{dz}(z-i)^2 f(z) &= \lim_{z\to i}\frac{d}{dz}\frac{z\exp(iz)}{(z+i)^2}\\ &= \lim_{z\to i}\frac{(z+i)^2(iz\exp(iz)+\exp(iz)) - 2z\exp(iz)(z+i)}{(z+i)^4}\\ &= \frac{(2i)^2(-\exp(-1)+\exp(-1)) - (2i)^2\exp(-1)}{(2i)^4}\\ &= \frac{1}{4e}.\end{aligned}$$

By the Cauchy Residue Theorem, we then have that

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\mathrm{Res}\,(f,i) = 2\pi i\frac{1}{4e} = \frac{\pi i}{2e}.$$

To estimate the contribution of the semicircular arc to the integral, we parametrize it explicitly, use the reverse triangle inequality and the fact that on $[0,\pi]$, $\sin t \geq 0$, so $\exp(-R\sin t) \leq 1$. We have

$$\begin{aligned}\left|\int_{C_R^+} f(z)\,dz\right| &= \left|\int_0^\pi f(R\exp(it))Ri\exp(it)\,dt\right|\\ &\leq \int_0^\pi \left|\frac{R\exp(it)\exp(Ri\exp(it))}{((R\exp(it))^2+1)^2}Ri\exp(it)\right|\,dt\\ &= \int_0^\pi \left|\frac{R^2\exp(i(R\cos t + iR\sin t))}{(R^2\exp(it)^2+1)^2}\right|\,dt\\ &= \int_0^\pi \frac{\left|R^2\exp(iR\cos t)\exp(-R\sin t)\right|}{\left|R^2\exp(it)^2+1\right|^2}\,dt\\ &\leq \int_0^\pi \frac{R^2\exp(-R\sin t)}{(R^2-1)^2}\,dt\\ &\leq \int_0^\pi \frac{R^2}{(R^2-1)^2}\,dt\\ &= \frac{R^2\pi}{(R^2-1)^2}\\ &\to 0 \text{ as } R\to\infty.\end{aligned}$$

So

$$\begin{aligned}\lim_{R\to\infty}\int_{-R}^R f(z)\,dz &= \lim_{R\to\infty}\left(\int_{\Gamma_R} f(z)\,dz - \int_{C_R^+} f(z)\,dz\right)\\ &= \lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz - \lim_{R\to\infty}\int_{C_R^+} f(z)\,dz\\ &= \frac{\pi i}{2e}.\end{aligned}$$

So

$$\int_{-\infty}^{\infty}\frac{x\sin x}{(x^2+1)^2}\,dx = \lim_{R\to\infty}\int_{-R}^R \frac{x\sin x}{(x^2+1)^2}\,dx = \mathrm{Im}\left(\lim_{R\to\infty}\int_{-R}^R f(z)\,dz\right) = \frac{\pi}{2e}.$$