**HONOURS COMPLEX VARIABLES**
**SOLUTIONS TO EXERCISES**

# 1. Holomorphic functions

## 1.1. Complex numbers.

**Lemma 1.1.3.** Let $u, w, z \in \mathbb{C}$, where $z = x + iy$ for $x, y \in \mathbb{R}$. Then

(i) $z + w = w + z$ and $zw = wz$ (commutativity of addition and multiplication);
(ii) $u + (z + w) = (u + z) + w$ and $u(zw) = (uz)w$ (associativity of addition and multiplication);
(iii) $u(z + w) = uz + uw$ (distributivity);
(iv) $z + 0 = z$, and there exists a unique complex number $-z := -x + i(-y)$ such that $z + (-z) = 0$; and
(v) $1z = z$, and if $z \neq 0$ then there exists a unique complex number $1/z = z^{-1} := x(x^2 + y^2)^{-1} - iy(x^2 + y^2)^{-1}$ such that $z(1/z) = 1$.

*Proof.* Suppose in addition that $w = a + ib$ and $u = c + id$.

(i) By definition of complex addition, and commutativity of real addition, we have that

$$\begin{aligned} z + w = (x + iy) + (a + ib) = (x + a) + i(y + b) &= (a + x) + i(b + y) \\ &= (a + ib) + (x + iy) \\ &= w + z. \end{aligned}$$

Similarly, by definition of complex multiplication, and commutativity of real addition and multiplication, we have that

$$\begin{aligned} zw = (x + iy)(a + ib) = (xa - yb) + i(xb + ya) &= (ax - by) + i(ay + bx) \\ &= (a + ib)(x + iy) \\ &= wz. \end{aligned}$$

(ii) By definition of complex addition, and associativity of real addition, we have that

$$\begin{aligned} u + (z + w) = (c + id) + ((x + iy) + (a + ib)) &= (c + id) + ((x + a) + i(y + b)) \\ &= c + (x + a) + i(d + (y + b)) \\ &= (c + x) + a + i((d + y) + b) \\ &= ((c + x) + i(d + y)) + (a + ib) \\ &= ((c + id) + (x + iy)) + (a + ib) \\ &= (u + z) + w. \end{aligned}$$

Similarly, by definition of complex multiplication, commutativity of real addition and multiplication, and distributivity of real addition and multiplication, we have

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that

$$\begin{aligned} u(zw) &= (c+id)((x+iy)(a+ib)) \\ &= (c+id)\left((xa-yb)+i(xb+ya)\right) \\ &= c(xa-by)-d(xb+ya)+i\left(c(xb+ya)+d(xa-by)\right) \\ &= cxa-cby-dxb-dya+i\left(cxb+cya+dxa-dby\right) \\ &= (cx-dy)a-(cy+dx)b+i\left((cx-dy)b+(cy+dx)a\right) \\ &= \left((cx-dy)+i(cy+dx)\right)(a+ib) \\ &= \left((c+id)(x+iy)\right)(a+ib) \\ &= (uz)w. \end{aligned}$$

(iii) The definitions of complex multiplication and addition, commutativity of real addition and multiplication, and distributivity of real addition and multiplication imply that

$$\begin{aligned} u(z+w) &= (c+id)((x+iy)+(a+ib)) \\ &= (c+id)\left((x+a)+i(y+b)\right) \\ &= c(x+a)-d(y+b)+i\left(c(y+b)+d(x+a)\right) \\ &= (cx-dy)+(ca-db)+i\left(cy+dx)+(cb+da)\right) \\ &= \left((cx-dy)+i(cy+dx)\right)+\left((ca-db)+i(cb+da)\right) \\ &= (c+id)(x+iy)+(c+id)(a+ib) \\ &= uz+uw. \end{aligned}$$

(iv) By definition of complex addition and the real additive identity,

$$(x+iy)+0=(x+iy)+(0+0i)=(x+0)+i(y+0)=x+iy,$$

and

$$z+(-z)=(x+iy)+(-x+i(-y))=(x-x)+i(y-y)=0+i0=0,$$

and uniqueness of $-z$ follows since by definition, two complex numbers are equal if and only if their real and imaginary parts are equal, so if $(x+iy)+(x'+iy')=0$, then $x+x'=0$ and $y+y'=0$, which implies that $x'=-x$ and $y'=-y$.

(v) By definition of complex multiplication and the real multiplicative identity,

$$1z=(1+0i)(x+iy)=1x-0y+i(1y+0x)=x+iy,$$

and

$$\begin{aligned} (x+iy)\left(\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}\right) &= \frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+i\left(\frac{-xy}{x^2+y^2}+\frac{yx}{x^2+y^2}\right) \\ &= \frac{x^2+y^2}{x^2+y^2}+i\left(\frac{-xy+yx}{x^2+y^2}\right) \\ &= 1. \end{aligned}$$

Uniqueness follows since, if $z'\in\mathbb{C}$ satisfies $zz'=1$, then the definition of the multiplicative identity, and commutativity and associativity of complex multiplication imply that

$$z'=1z'=\left(\frac{1}{z}z\right)z'=\frac{1}{z}(zz')=\frac{1}{z}1=\frac{1}{z}. \qquad \square$$

**Lemma 1.1.6.** Let $\theta,\phi\in\mathbb{R}$, and $n\in\mathbb{Z}$. Then

(i) $e^{i(\theta+\phi)}=e^{i\theta}e^{i\phi}$;

(ii) $e^{-i\theta}=\frac{1}{e^{i\theta}}$; and

(iii) $e^{in\theta} = \left(e^{i\theta}\right)^n$.

The last statement is de Moivre's formula

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n .$$

*Proof.* See workshop 2, question 1. □

**Exercise 1.1.9.** Prove the following identities:

(i) $(1+i)^4 = -4$;
(ii) $(1+i)^{13} = -2^6(1+i)$;
(iii) $(1+i\sqrt{3})^6 = 2^6$;
(iv) $\frac{(1+i\sqrt{3})^3}{(1-i)^2} = -4i$; and
(v) $\frac{(1+i)^5}{(\sqrt{3}+i)^2} = -\sqrt{2}e^{-i\pi/12}$.

*Solution.* (i) We have that $(1+i)^4 = \left((1+i)^2\right)^2 = ((1-1)+2i)^2 = (2i)^2 = -4$.

(ii) By part (a), $(1+i)^{13} = \left((1+i)^4\right)^3(1+i) = (-4)^3(1+i) = -2^6(1+i)$.

(iii) Dividing each side of the conclusion by $2^6$, it suffices to prove that $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^6 = 1$. This follows since $\frac{1}{2}+i\frac{\sqrt{3}}{2} = e^{i\pi/3}$, so $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^6 = \left(e^{i\pi/3}\right)^6 = e^{6i\pi/3} = e^{2\pi i} = 1$.

(iv) Since, as observed above, $1+i\sqrt{3} = 2e^{i\pi/3}$, we have that

$$\frac{(1+i\sqrt{3})^3}{(1-i)^2} = \frac{\left(2e^{i\pi/3}\right)^3}{1-1-2i} = \frac{2^3e^{3i\pi/3}}{-2i} = i2^2e^{i\pi} = -4i.$$

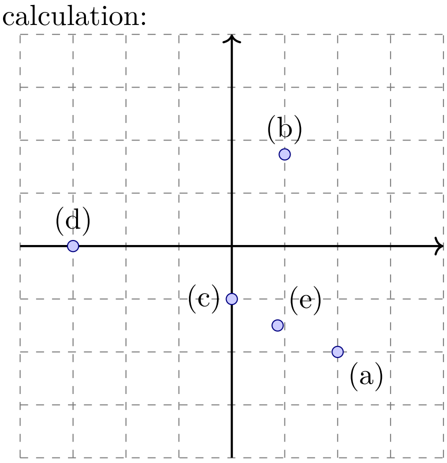
(v) From part (a), we see that

$$\frac{(1+i)^5}{(\sqrt{3}+i)^2} = \frac{(1+i)^4(1+i)}{\left(\sqrt{3}+i\right)^2} = \frac{-4(1+i)}{\left(2e^{i\pi/6}\right)^2} = \frac{-4\sqrt{2}e^{i\pi/4}}{2^2e^{2i\pi/6}} = -\sqrt{2}e^{i(\pi/4-\pi/3)} = -\sqrt{2}e^{-i\pi/12}.$$

**Exercise 1.1.10.** Show the position of the points $2-2i$, $1+\sqrt{3}i$, $-i$, $-3$, and $\frac{\sqrt{3}}{2}-\frac{3}{2}i$ on the complex plane, and express them in polar form.

*Solution.* None of these should require much calculation:

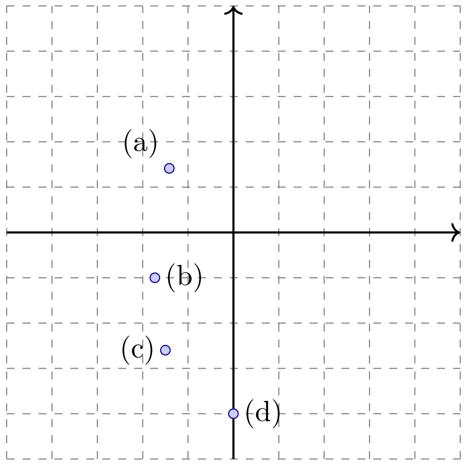
(a) $2-2i = 2\sqrt{2}e^{-i\pi/4}$;
(b) $1+\sqrt{3}i = 2e^{i\pi/3}$;
(c) $-i = e^{-i\pi/2}$;
(d) $-3 = 3e^{i\pi}$; and
(e) $\frac{\sqrt{3}}{2}-\frac{3}{2}i = \sqrt{3}e^{-i\pi/3}$.




**Exercise 1.1.11.** Show the position of the points $2e^{3\pi i/4}$, $2e^{7\pi i/6}$, $3e^{4\pi i/3}$, and $4e^{3\pi i/2}$ on the complex plane, and represent them in cartesian form.

*Solution.* Again, none of these should need serious calculation:

(a) $2e^{3\pi i/4} = -\sqrt{2} + \sqrt{2}i$;
(b) $2e^{7\pi i/6} = -\sqrt{3} - i$;
(c) $3e^{4\pi i/3} = -i\frac{3\sqrt{3}}{2} - \frac{3}{2}$; and
(d) $4e^{3\pi i/2} = -4i$.




**Exercise 1.1.12.** Find the square roots of the complex numbers $5 - 12i$ and $8 + 4\sqrt{5}i$.

*Solution.* We can write $5 - 12i = 13(\cos\theta + i\sin\theta)$, for some $\theta \in \mathbb{R}$, where equating real and imaginary parts shows that $\cos\theta = 5/13$ and $\sin\theta = -12/13$. Suppose that $z^2 = 5-12i$. Then $z = \pm\sqrt{13}(\cos(\theta/2)+i\sin(\theta/2))$. We cannot easily calculate the value of $\theta$, hence neither the value of $\theta/2$, but using the trigonometric half-angle formulae, we can calculate directly the cartesian form. From

$$2\sin^2(\theta/2) = 1 - \cos\theta, \quad \text{and} \quad 2\cos^2(\theta/2) = 1 + \cos\theta,$$

we have that

$$\sin(\theta/2) = \pm\frac{1}{\sqrt{2}}\sqrt{1-\cos\theta} = \pm\frac{1}{\sqrt{2}}\sqrt{1-(5/13)} = \pm\frac{\sqrt{8}}{\sqrt{2}\sqrt{13}} = \pm\frac{2}{\sqrt{13}}, \text{ and}$$
$$\cos(\theta/2) = \pm\frac{1}{\sqrt{2}}\sqrt{1+\cos\theta} = \pm\frac{1}{\sqrt{2}}\sqrt{1+(5/13)} = \pm\frac{\sqrt{18}}{\sqrt{2}\sqrt{13}} = \pm\frac{3}{\sqrt{13}}.$$

Since $5 - 12i$ lies in the bottom right quadrant of the complex plane, so too (dividing the argument by two) does one of its square roots, which allows us to determine the signs of the real and imaginary parts that we need to choose from the above possible four combinations. In conclusion, we see that our solutions are

$$z = \pm\sqrt{13}\left(\frac{3}{\sqrt{13}} - i\frac{2}{\sqrt{13}}\right) = \pm(3-2i).$$

Now write $8 + 4\sqrt{5}i = 12(\cos\theta + i\sin\theta)$ for some $\theta \in \mathbb{R}$, where equating real and imaginary parts shows that $\cos\theta = 8/12 = 2/3$ and $\sin\theta = 4\sqrt{5}/12 = \sqrt{5}/3$. Suppose that $z^2 = 8 + 4\sqrt{5}i$. Then $z = \pm 2\sqrt{3}(\cos(\theta/2) + i\sin(\theta/2))$, where

$$\sin(\theta/2) = \pm\frac{1}{\sqrt{2}}\sqrt{1-\cos\theta} = \pm\frac{1}{\sqrt{2}}\sqrt{1-(2/3)} = \pm\frac{1}{\sqrt{2}\sqrt{3}}, \text{ and}$$
$$\cos(\theta/2) = \pm\frac{1}{\sqrt{2}}\sqrt{1+\cos\theta} = \pm\frac{1}{\sqrt{2}}\sqrt{1+(2/3)} = \pm\frac{\sqrt{5}}{\sqrt{2}\sqrt{3}}.$$

Since $8 + 4\sqrt{5}i$ lies in the top right quadrant of the complex plane, so too (dividing the argument by two) does one of its square roots, which allows us to determine the signs of the real and imaginary parts that we need to choose from the above possible four combinations. In conclusion, we see that our solutions are

$$z = \pm 2\sqrt{3}\left(\frac{\sqrt{5}}{\sqrt{2}\sqrt{3}} + i\frac{1}{\sqrt{2}\sqrt{3}}\right) = \pm\left(\sqrt{2}\sqrt{5} + \sqrt{2}i\right).$$

**Exercise 1.1.13.** Find all the fourth roots of 1, and all the seventh roots of $-1$, and sketch them in the complex plane.

*Solution.* Suppose $z = re^{i\theta}$ satisfies $z^4 = 1$, for some $r > 0$ and $\theta \in \mathbb{R}$. Then since

$$r^4 e^{4i\theta} = \left(re^{i\theta}\right)^4 = z^4 = 1 = e^{2\pi ik},$$

for $k \in \mathbb{Z}$, we have that $r = 1$ and $\theta = k\pi/2$ for $k \in \mathbb{Z}$. This gives four distinct values $z = e^{ik\pi/2}$, for $k = 0, 1, 2, 3$, which are $z = 1$, $i$, $-1$, and $-i$ respectively. These form the vertices of a square centred at the origin with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$ respectively.

Similarly, now suppose $z = re^{i\theta}$ satisfies $z^7 = -1$, for some $r > 0$ and $\theta \in \mathbb{R}$. Then since

$$r^7 e^{7i\theta} = \left(re^{i\theta}\right)^7 = z^7 = -1 = e^{i\pi+2\pi ik},$$

for $k \in \mathbb{Z}$, we have that $r = 1$ and $\theta = \pi/7 + 2\pi k/7$ for $k \in \mathbb{Z}$. This gives seven distinct values $z = e^{i\pi/7+2\pi ik/7}$, for each value of $k = 0, 1, 2, 3, 4, 5, 6$. These form the vertices of a regular heptagon, centred at the origin, with one vertex at $(-1, 0)$.

**Lemma 1.1.14.** Let $z, w \in \mathbb{C}$. Then

(i) $|z| = 0$ if and only if $z = 0$;
(ii) $|\overline{z}| = |z|$;
(iii) $|zw| = |z||w|$;
(iv) $\overline{\overline{z}} = z$;
(v) $|z|^2 = z\overline{z}$;
(vi) $\overline{z + w} = \overline{z} + \overline{w}$;
(vii) $\overline{(zw)} = (\overline{z})(\overline{w})$;
(viii) $|\mathrm{Re}(z)| \leq |z|$ and $|\mathrm{Im}(z)| \leq |z|$; and
(ix) $\mathrm{Re}(z) = \frac{z+\overline{z}}{2}$ and $\mathrm{Im}(z) = \frac{z-\overline{z}}{2i}$.

*Proof.* Except for part (i), we assume that the complex numbers $z$ and $w$ we are considering are non-zero. The results are trivial otherwise. Let $z = x + iy = r_1 e^{i\theta_1}$ and $w = a + ib = r_2 e^{i\theta_2}$, for $x, y, a, b, \theta_1, \theta_2 \in \mathbb{R}$ and $r_1, r_2 > 0$.

(i) $z = x + iy = 0$ if and only if $x = y = 0$ if and only if $|z|^2 = x^2 + y^2 = 0$, if and only if $|z| = 0$.
(ii) $|\overline{z}|^2 = x^2 + (-y)^2 = x^2 + y^2 = |z|^2$, so since $|z|, |\overline{z}|$ are non-negative real numbers, this implies that $|\overline{z}| = |z|$.
(iii) $|zw| = \left|(r_1 e^{i\theta_1})(r_2 e^{i\theta_2})\right| = \left|(r_1 r_2) e^{i(\theta_1+\theta_2)}\right| = r_1 r_2 = \left|r_1 e^{i\theta_1}\right| \left|r_2 e^{i\theta_2}\right| = |z|\,|w|$.
(iv) $\overline{\overline{z}} = x - i(-y) = x + iy = z$.
(v) $z\overline{z} = \left(r_1 e^{i\theta_1}\right) \overline{\left(r_1 e^{i\theta_1}\right)} = r_1 e^{i\theta_1} r_1 e^{-i\theta_1} = r_1^2 e^{i(\theta_1-\theta_1)} = r_1^2 = \left|r_1 e^{i\theta_1}\right|^2 = |z|^2$.
(vi) $\overline{z + w} = \overline{(x + iy) + (a + ib)} = \overline{(x + a) + i(y + b)} = (x + a) - i(y + b) = (x - iy) + (a - ib) = \overline{z} + \overline{w}$.
(vii) $\overline{(zw)} = \overline{(r_1 e^{i\theta_1})(r_2 e^{i\theta_2})} = \overline{r_1 r_2 e^{i(\theta_1+\theta_2)}} = r_1 r_2 e^{-i(\theta_1+\theta_2)} = \left(r_1 e^{-i\theta_1}\right)\left(r_2 e^{-i\theta_2}\right) = \overline{(r_1 e^{i\theta_1})} \cdot \overline{(r_2 e^{i\theta_2})} = (\overline{z})(\overline{w})$.
(viii) $|\mathrm{Re}(z)|^2 = x^2 \leq x^2 + y^2 = |z|^2$, so since $|\mathrm{Re}(z)|$ and $|z|$ are non-negative real numbers, this implies that $|\mathrm{Re}(z)| \leq |z|$. Similarly for $\mathrm{Im}(z)$.
(ix) $z + \overline{z} = (x + iy) + (x - iy) = (x + x) + i(y - y) = 2x$, hence $\mathrm{Re}(z) = x = (z + \overline{z})/2$. Similarly $z - \overline{z} = (x + iy) - (x - iy) = (x - x) + i(y + y) = 2iy$, hence $\mathrm{Im}(z) = y = (z - \overline{z})/2i$.

□

**Proposition 1.1.19.** Let $z, w \in \mathbb{C}$ be non-zero. Then

(i) $\arg(zw) = \arg(z) + \arg(w)$ and $\arg(\overline{z}) = -\arg(z)$, where for any subsets $A, B \subseteq \mathbb{R}$ and $\lambda \in \mathbb{R}$, we write

$$A + B = \{\, a + b : a \in A,\ b \in B \,\} \quad \text{and} \quad \lambda A = \{\, \lambda a : a \in A \,\};$$

and

(ii) $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$ and $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$, where both equations hold modulo $2\pi$; that is, there exist $k, l \in \mathbb{Z}$ such that

$$\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w) + 2k\pi, \quad \text{and} \quad \operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z) + 2l\pi.$$

*Proof.* (i) Suppose that $z = |z|\, e^{i\theta}$ and $w = |w|\, e^{i\phi}$ in exponential form, for some $\theta, \phi \in \mathbb{R}$. By definition $\theta \in \arg(z)$ and $\phi \in \arg(w)$. Then $zw = |z|\, e^{i\theta}\, |w|\, e^{i\phi} = |z|\, |w|\, e^{i(\theta+\phi)}$, so $\theta + \phi \in \arg(zw)$ by definition. Thus

$$\begin{aligned}\arg(z) &= \{\, \theta + 2k\pi : k \in \mathbb{Z}\,\},\\ \arg(w) &= \{\, \phi + 2l\pi : l \in \mathbb{Z}\,\}, \text{ and}\\ \arg(zw) &= \{\, (\theta + \phi) + 2m\pi : m \in \mathbb{Z}\,\}.\end{aligned}$$

So $\psi \in \arg(z) + \arg(w)$ if and only if there exist $k, l \in \mathbb{Z}$ such that $\psi = (\theta + 2k\pi) + (\phi + 2l\pi) = (\theta+\phi) + 2(k+l)\pi$, if and only if $\psi = (\theta+\phi) + 2m\pi$ for some $m \in \mathbb{Z}$, i.e. if and only if $\psi \in \arg(zw)$. Furthermore, recall that cos is an even function and sin is an odd function, i.e. $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$ for all $x \in \mathbb{R}$. Then by definition of the complex conjugate we have that

$$\overline{re^{i\theta}} = \overline{r\cos\theta + ir\sin\theta} = r\cos\theta - ir\sin\theta = r\cos(-\theta) + ir\sin(-\theta) = re^{-i\theta}.$$

Thus $\theta \in \arg(z)$ if and only if $-\theta \in \arg(\overline{z})$, i.e. $\arg(\overline{z}) = -\arg(z)$.

(ii) This just follows from the previous part: since $\operatorname{Arg}(z) \in \arg(z)$, $\operatorname{Arg}(w) \in \arg(w)$, and $\operatorname{Arg}(zw) \in \arg(zw)$, we have that $\operatorname{Arg}(z) + \operatorname{Arg}(w) \in \arg(z) + \arg(w) = \arg(zw)$, so $\operatorname{Arg}(z) + \operatorname{Arg}(w)$ and $\operatorname{Arg}(zw)$ are both elements of $\arg(zw)$, all of the elements of which differ by an integer multiple of $2\pi$. Therefore there exists $k \in \mathbb{Z}$ such that $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w) + 2k\pi$.

Similarly $\operatorname{Arg}(\overline{z}) \in \arg(\overline{z}) = -\arg(z)$, and $-\operatorname{Arg}(z) \in -\arg(z)$, so there exists $l \in \mathbb{Z}$ such that $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z) + 2l\pi$.

□

## 1.2. Topology of the complex plane.

**Lemma 1.2.3.** Let $z_0 \in \mathbb{C}$, and $\varepsilon > 0$. Then $D_\varepsilon(z_0)$ and $D'_\varepsilon(z_0)$ are open, and $\overline{D}_\varepsilon(z_0)$ is closed.

*Proof.* Let $z \in D_\varepsilon(z_0)$. Then by definition $|z - z_0| < \varepsilon$, so we can choose $r > 0$ such that $r \le \varepsilon - |z - z_0|$. We claim that $D_r(z) \subseteq D_\varepsilon(z_0)$. Let $y \in D_r(z)$. Then by definition $|y - z| < r$, so

$$|y - z_0| \le |y - z| + |z - z_0| < r + |z - z_0| \le \varepsilon - |z - z_0| + |z - z_0| = \varepsilon,$$

as required.

Now let $z \in D'_\varepsilon(z_0)$. Choose $r > 0$ such that $r \le \min\{\varepsilon - |z - z_0|, |z - z_0|\}$, which is a positive number since $z \ne z_0$. We claim that $D_r(z) \subseteq D'_\varepsilon(z_0)$. Let $y \in D_r(z)$. Then as above, $y \in D_\varepsilon(z_0)$. Furthermore, $|y - z| < r \le |z - z_0|$, so $y \ne z_0$. So $y \in D_\varepsilon(z_0) \setminus \{z_0\} = D'_\varepsilon(z_0)$, as required.

Finally, to show $\overline{D}_\varepsilon(z_0)$ is closed, we show that its complement $\mathbb{C} \setminus \overline{D}_\varepsilon(z_0)$ is open. So let $z \in \mathbb{C} \setminus \overline{D}_\varepsilon(z_0)$. Then by definition $|z - z_0| \not\le \varepsilon$, so $|z - z_0| > \varepsilon$. Choose $r > 0$ such that $r \le |z - z_0| - \varepsilon$. We claim that $D_r(z) \subseteq \mathbb{C} \setminus \overline{D}_\varepsilon(z_0)$, i.e. that $D_r(z) \cap \overline{D}_\varepsilon(z_0) = \emptyset$. Let $y \in D_r(z)$. By definition $|z - y| < r$, so by the reverse triangle inequality

$$\begin{aligned}|y - z_0| = |(y - z) - (z_0 - z)| \ge |(|y - z| - |z_0 - z|)| &\ge |z_0 - z| - |y - z|\\ &> |z_0 - z| - r\\ &\ge |z_0 - z| - (|z_0 - z| - \varepsilon)\\ &= \varepsilon,\end{aligned}$$

as required. □

**Lemma 1.2.6.** Let $S \subseteq \mathbb{C}$. Then $S$ is closed if and only if $S = \overline{S}$.

*Proof.* Suppose first that $S = \overline{S}$. Let $z \in \mathbb{C} \setminus S$. Then by assumption $z \in \overline{S}$, so $z$ is not a limit point of $S$. Therefore by definition, there exists $\varepsilon > 0$ such that $D'_\varepsilon(z) \cap S = \emptyset$. That is, $D'_\varepsilon(z) \subseteq \mathbb{C} \setminus S$. Since $z \in \mathbb{C} \setminus S$, in fact we have that $D_\varepsilon(z) \subseteq \mathbb{C} \setminus S$. Hence $\mathbb{C} \setminus S$ is open, and thus by definition $S$ is closed.

Now suppose that $S \neq \overline{S}$. Since $S \subseteq \overline{S}$ by definition, this implies that there exists $z \in \overline{S} \setminus S$. So $z \in \mathbb{C} \setminus S$ is such that $D'_\varepsilon(z) \cap S \neq \emptyset$ for all $\varepsilon > 0$. In particular there does not exist $\varepsilon > 0$ such that $D_\varepsilon(z) \subseteq \mathbb{C} \setminus S$. That is to say that $\mathbb{C} \setminus S$ is not open, thus $S$ is not closed. Taking the contrapositive, we see that if $S$ is closed, then $S = \overline{S}$, as required. □

**Lemma 1.2.9.** Let $z_n \in \mathbb{C}$ be a complex sequence, where $z_n = a_n + ib_n$ for $a_n, b_n \in \mathbb{R}$, for each $n$. Then $z = \lim_{n\to\infty} z_n$ if and only if $\mathrm{Re}(z) = \lim_{n\to\infty} a_n$ and $\mathrm{Im}(z) = \lim_{n\to\infty} b_n$. In particular a complex sequence converges if and only if the (real) sequences of its real and imaginary parts converge.

*Proof.* Suppose that $z = \lim_{n\to\infty} z_n$. Let $\varepsilon > 0$. Then by definition there exists $N \in \mathbb{N}$ such that $|z_n - z| < \varepsilon$ whenever $n \geq N$. Then for $n \geq N$, we have that

$$|\mathrm{Re}(z) - \mathrm{Re}(z_n)| = |\mathrm{Re}(z - z_n)| \leq |z - z_n| < \varepsilon,$$

and

$$|\mathrm{Im}(z) - \mathrm{Im}(z_n)| = |\mathrm{Im}(z - z_n)| \leq |z - z_n| < \varepsilon,$$

i.e. $\lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z)$ and $\lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z)$, as required.

On the other hand, suppose that

$$\lim_{n\to\infty} \mathrm{Re}(z_n) = \mathrm{Re}(z) \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Im}(z_n) = \mathrm{Im}(z),$$

and let $\varepsilon > 0$. Then by definition there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned}
|\mathrm{Re}(z_n) - \mathrm{Re}(z)| &< \frac{\varepsilon}{2} \text{ whenever } n \geq N_1, \text{ and} \\
|\mathrm{Im}(z_n) - \mathrm{Im}(z)| &< \frac{\varepsilon}{2} \text{ whenever } n \geq N_2.
\end{aligned}$$

Let $N := \max\{N_1, N_2\}$, and consider $n \geq N$. Then since $n \geq N_1$ and $n \geq N_2$, we have for $z = x + iy$ that

$$\begin{aligned}
|z_n - z| &= |(\mathrm{Re}(z_n) + i\,\mathrm{Im}(z_n)) - (\mathrm{Re}(z) + i\,\mathrm{Im}(z))| \\
&= |(\mathrm{Re}(z_n) - \mathrm{Re}(z)) + i(\mathrm{Im}(z_n) - \mathrm{Im}(z))| \\
&\leq |\mathrm{Re}(z_n) - \mathrm{Re}(z)| + |\mathrm{Im}(z_n) - \mathrm{Im}(z)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence $\lim_{n\to\infty} z_n = z$, as required. □

**Lemma 1.2.10.** Let $S \subseteq \mathbb{C}$ and $z \in \mathbb{C}$. Then $z \in \overline{S}$ if and only if there exists a sequence $z_n \in S$ such that $z = \lim_{n\to\infty} z_n$.

*Proof.* Suppose first that $z \in \overline{S}$. Then if $z \in S$, we define $z_n = z$ for all $n \in \mathbb{N}$, and evidently we have a sequence $z_n \in S$ such that $z = \lim_{n\to\infty} z_n$. If $z \in \overline{S} \setminus S$, then $z$ must be a limit point of $S$, by definition of $\overline{S}$. Then by definition, for each $n \geq 1$, there exists $z_n \in S \cap D'_{1/n}(z)$. Then $z_n \in S$, and $|z_n - z| < 1/n \to 0$, so $z = \lim_{n\to\infty} z_n$, as required.

On the other hand, suppose now that such a sequence exists. If for some $n \in \mathbb{N}$, $z_n = z$, then by assumption $z = z_n \in S \subseteq \overline{S}$, as required. If not, then for $z_n \neq z$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. By definition of convergence, there exists $N \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon \text{ whenever } n \geq N.$$

In particular $z_N \in D_\varepsilon(z) \cap S$, and since $z_N \neq z$, in fact $z_N \in D'_\varepsilon(z) \cap S$. In particular this set is non-empty, as required. □

**Lemma 1.2.12.** Let $z_n \in \mathbb{C}$ be a sequence. Then $z_n$ has a limit if and only if $z_n$ is Cauchy.

*Proof.* Suppose that $z_n \to z \in \mathbb{C}$ as $n \to \infty$. We show that $z_n$ is Cauchy. Let $\varepsilon > 0$. Then by definition of convergence, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $|z_n - z| < \varepsilon/2$. Then for $n, m \geq N$, we have, by the triangle inequality, that

$$|z_n - z_m| = |(z_n - z) - (z_m - z)| \leq |z_n - z| + |z_m - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as required.

Now suppose that $z_n$ is Cauchy. For each $n \in \mathbb{N}$ define two real sequences by $a_n = \mathrm{Re}(z_n)$ and $b_n = \mathrm{Im}(z_n)$. We easily see that both these sequences are Cauchy: for any $\varepsilon > 0$, choosing $N \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ whenever $n, m \geq N$, we have that

$$|a_n - a_m| \leq |z_n - z_m| < \varepsilon \quad \text{and} \quad |b_n - b_m| \leq |z_n - z_m| < \varepsilon$$

whenever $n, m \geq N$. Therefore by the completeness of the real numbers, there exist $a, b \in \mathbb{R}$ such that $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Let $\varepsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$\begin{aligned} |a_n - a| &< \frac{\varepsilon}{2} \text{ whenever } n \geq N_1, \text{ and} \\ |b_n - b| &< \frac{\varepsilon}{2} \text{ whenever } n \geq N_2. \end{aligned}$$

Then for $n \geq N := \max\{N_1, N_2\}$, we have by the triangle inequality that

$$\begin{aligned} |z_n - (a + ib)| = |(a_n + ib_n) - (a + ib)| = |(a_n - a) + i(b_n - b)| &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

i.e. $z_n \to a + ib$. So $z_n$ does indeed have a limit. □

**Lemma 1.2.15** (Bolzano–Weierstrass)**.** Let $z_n \in \mathbb{C}$ be a bounded sequence. Then $z_n$ has a convergent subsequence, i.e. there exist indices $n_k \in \mathbb{N}$ for $k \in \mathbb{N}$ and $z \in \mathbb{C}$ such that $z_{n_k} \to z$ as $k \to \infty$.

*Proof.* Note that this is not just a case of applying the (real) Bolzano–Weierstrass theorem directly to the real and imaginary parts of the sequence $z_n$. This would indeed give a convergent subsequence of the real parts, and a convergent subsequence of the imaginary parts, but in general these will be different subsequences: perhaps every other element of the real parts was chosen, but perhaps every third element of the imaginary parts. So we cannot infer that a subsequence of the complex terms converges, without considering a "subsubsequence", as follows.

Suppose $M > 0$ is such that $|z_n| \leq M$ for all $n \in \mathbb{N}$. Define real sequences $a_n = \mathrm{Re}(z_n)$ and $b_n = \mathrm{Im}(z_n)$. Then $|a_n| \leq |z_n| \leq M$ for all $n \in \mathbb{N}$, so $a_n$ is a bounded real sequence. By the (real) Bolzano–Weierstrass theorem, there therefore exists a convergent subsequence $a_{n_k}$ of $a_n$, say, with limit $a$ as $k \to \infty$. Now, $|b_{n_k}| \leq |z_{n_k}| \leq M$ for all $k \in \mathbb{N}$, so $b_{n_k}$ is a bounded real sequence. Again, the (real) Bolzano–Weierstrass theorem implies that there exists a convergent subsequence $b_{n_{k_l}}$ of $b_{n_k}$, say, with limit

$b$ as $l \to \infty$. Since a subsequence of a convergent sequence converges to the same limit as the original sequence, we know that $a_{n_{k_l}}$ converges to $a$ as $l \to \infty$. Therefore the subsequence $z_{n_{k_l}} = a_{n_{k_l}} + ib_{n_{k_l}}$ converges to $a+ib$. So $z_n$ does indeed have a convergent subsequence. $\square$

## 1.3. Complex-valued functions.

**Lemma 1.3.3.** Let $S \subseteq \mathbb{C}$, $z_0 = x_0 + iy_0 \in \overline{S}$, $f\colon S \to \mathbb{C}$ be of the form $f = u + iv$, and $a_0 \in \mathbb{C}$. Then $a_0 = \lim_{z\to z_0} f(z)$ if and only if $\operatorname{Re}(a_0) = \lim_{(x,y)\to(x_0,y_0)} u(x,y)$ and $\operatorname{Im}(a_0) = \lim_{(x,y)\to(x_0,y_0)} v(x,y)$.

*Proof.* Suppose that $a_0 = \lim_{z\to z_0} f(z)$. Let $\varepsilon > 0$. Then by definition there exists $\delta > 0$ such that $|f(z) - a_0| < \varepsilon$ whenever $z \in S$ satisfies $0 < |z - z_0| < \delta$. Then for $(x, y) \in \mathbb{R}^2$ such that $z = x + iy \in S$ and $0 < |(x, y) - (x_0, y_0)| < \delta$, we have that $0 < |z - z_0| < \delta$, so

$$|\operatorname{Re}(a_0) - u(x,y)| = |\operatorname{Re}(a_0) - \operatorname{Re}(f(x+iy))| = |\operatorname{Re}(a_0 - f(x+iy))| \leq |a_0 - f(z)| < \varepsilon,$$

and

$$|\operatorname{Im}(a_0) - v(x,y)| = |\operatorname{Im}(a_0) - \operatorname{Im}(f(x+iy))| = |\operatorname{Im}(a_0 - f(x+iy))| \leq |a_0 - f(z)| < \varepsilon,$$

i.e. $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \operatorname{Re}(a_0)$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = \operatorname{Im}(a_0)$, as required.

On the other hand, suppose that

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = \operatorname{Re}(a_0) \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = \operatorname{Im}(a_0),$$

and let $\varepsilon > 0$. Then by definition there exist $\delta_1, \delta_2 > 0$ such that

$$|u(x,y) - \operatorname{Re}(a_0)| < \frac{\varepsilon}{2} \text{ whenever } x + iy \in S \text{ and } 0 < |(x,y) - (x_0,y_0)| < \delta_1, \text{ and}$$
$$|v(x,y) - \operatorname{Im}(a_0)| < \frac{\varepsilon}{2} \text{ whenever } x + iy \in S \text{ and } 0 < |(x,y) - (x_0,y_0)| < \delta_2.$$

Let $\delta := \min\{\delta_1, \delta_2\}$, and consider $z = x + iy \in S$ such that $0 < |z - z_0| < \delta$. Then since $0 < |(x, y) - (x_0, y_0)| < \delta \leq \delta_1$ and $0 < |(x, y) - (x_0, y_0)| < \delta \leq \delta_2$, we have that

$$\begin{aligned}
|f(z) - a_0| &= |(u(x,y) + iv(x,y)) - (\operatorname{Re}(a_0) + i\operatorname{Im}(a_0))| \\
&= |(u(x,y) - \operatorname{Re}(a_0)) + i(v(x,y) - \operatorname{Im}(a_0))| \\
&\leq |u(x,y) - \operatorname{Re}(a_0)| + |v(x,y) - \operatorname{Im}(a_0)| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence $\lim_{z\to z_0} f(z) = a_0$, as required. $\square$

**Lemma 1.3.4.** Let $S \subseteq \mathbb{C}$, $z_0 \in \overline{S}$, $f\colon S \to \mathbb{C}$ satisfy $\lim_{z\to z_0} f(z) = a_0$ for some $a_0 \in \mathbb{C}$, and $w_n \in S$ be a sequence such that $w_n \to z_0$. Then $\lim_{n\to\infty} f(w_n) = a_0$.

*Proof.* Let $\varepsilon > 0$. By definition there exists $\delta > 0$ such that

$$|f(z) - a_0| < \varepsilon \text{ whenever } z \in S \text{ satisfies } 0 < |z - z_0| < \delta.$$

Since $w_n \to z_0$, by definition there exists $N \in \mathbb{N}$ such that

$$|w_n - z_0| < \delta \text{ whenever } n \geq N.$$

Since $|w_n - z_0| > 0$ for all $n$, for $n \geq N$, we have that

$$|f(w_n) - a_0| < \varepsilon,$$

as required. $\square$

**Lemma 1.3.5** (Algebra of limits)**.** Let $S \subseteq \mathbb{C}$, $z_0 \in \overline{S}$, and $f, g\colon S \to \mathbb{C}$ be such that $\lim_{z\to z_0} f(z) = a_0$ and $\lim_{z\to z_0} g(z) = b_0$. Then

(i) $\lim_{z\to z_0}(f(z)+g(z)) = a_0 + b_0$;
(ii) $\lim_{z\to z_0}(f(z)g(z)) = a_0 b_0$; and
(iii) if $b_0 \neq 0$, then $\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{a_0}{b_0}$.

*Proof.* Let $\varepsilon > 0$.

(i) By definition there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}|f(z)-a_0| &< \frac{\varepsilon}{2} \text{ whenever } z \in S \text{ and } 0 < |z-z_0| < \delta_1, \text{ and}\\ |g(z)-b_0| &< \frac{\varepsilon}{2} \text{ whenever } z \in S \text{ and } 0 < |z-z_0| < \delta_2.\end{aligned}$$

Let $\delta := \min\{\delta_1,\delta_2\}$, and consider $z \in S$ such that $0 < |z-z_0| < \delta$. Then since $|z-z_0| < \delta \le \delta_1$ and $|z-z_0| < \delta \le \delta_2$ we have that

$$\begin{aligned}|(f(z)+g(z))-(a_0+b_0)| = |(f(z)-a_0)+(g(z)-b_0)| &\le |f(z)-a_0| + |g(z)-b_0|\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon,\end{aligned}$$

i.e. $\lim_{z\to z_0}(f(z)+g(z)) = a_0 + b_0$.

(ii) Similarly, by definition there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}&|f(z)-a_0| < \min\left\{\frac{\varepsilon}{2|b_0|+1}, 1\right\} \text{ whenever } z \in S \text{ and } 0 < |z-z_0| < \delta_1, \text{ and}\\ &|g(z)-b_0| < \frac{\varepsilon}{2(|a_0|+1)} \text{ whenever } z \in S \text{ and } 0 < |z-z_0| < \delta_2.\end{aligned}$$

Let $\delta := \min\{\delta_1,\delta_2\}$, and consider $z \in S$ such that $0 < |z-z_0| < \delta$. Then since $|z-z_0| < \delta \le \delta_1$ and $|z-z_0| < \delta \le \delta_2$ we have that $|f(z)| = |f(z)-a_0+a_0| \le |f(z)-a_0| + |a_0| < 1 + |a_0|$, and therefore

$$\begin{aligned}|(f(z)g(z))-(a_0b_0)| &= |(f(z)-a_0)b_0 - f(z)(b_0-g(z))|\\ &\le |(f(z)-a_0)b_0| + |f(z)(b_0-g(z))|\\ &= |f(z)-a_0|\,|b_0| + |f(z)|\,|b_0-g(z)|\\ &\le |f(z)-a_0|\,|b_0| + (1+|a_0|)\,|b_0-g(z)|\\ &< \frac{\varepsilon}{2|b_0|+1}|b_0| + (1+|a_0|)\frac{\varepsilon}{2(|a_0|+1)}\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon,\end{aligned}$$

i.e. $\lim_{z\to z_0}(f(z)g(z)) = a_0b_0$.

(iii) We show that $\lim_{z\to z_0}(1/g(z)) = 1/b_0$. The full result then follows by this result and part (ii). By definition there exists $\delta > 0$ such that

$$|g(z)-b_0| < \min\left\{\frac{|b_0|}{2}, \frac{|b_0|^2\varepsilon}{2}\right\} \text{ whenever } z \in S \text{ and } 0 < |z-z_0| < \delta.$$

Consider $z \in S$ such that $0 < |z-z_0| < \delta$. Then by the reverse triangle inequality

$$|g(z)| = |g(z)-b_0+b_0| \ge ||g(z)-b_0| - |b_0|| \ge |b_0| - |g(z)-b_0| > |b_0| - \frac{|b_0|}{2} = \frac{|b_0|}{2}.$$

Furthermore,

$$\left|\frac{1}{g(z)} - \frac{1}{b_0}\right| = \left|\frac{g(z)-b_0}{g(z)b_0}\right| \le \frac{|g(z)-b_0|}{|b_0|^2/2} < \frac{|b_0|^2\varepsilon/2}{|b_0|^2/2} = \varepsilon,$$

i.e. $\lim_{z\to z_0}(1/g(z)) = 1/b_0$. □

**Lemma 1.3.7.** Let $f\colon \mathbb{C} \to \mathbb{C}$ be such that $f = u + iv$, and $z_0 = x_0 + iy_0 \in \mathbb{C}$. Then $f$ is continuous at $z_0$ if and only if $u$ and $v$ are continuous at $(x_0, y_0)$.

*Proof.* This follows by the definition and lemma 1.3.3 applied with $a_0 = f(z_0)$: $f$ is continuous at $z_0$ by definition if and only if $\lim_{z\to z_0} f(z) = f(z_0)$, which by lemma 1.3.3 is true if and only if $u(x_0, y_0) = \operatorname{Re}(f(z_0)) = \lim_{(x,y)\to(x_0,y_0)} u(x, y)$ and $v(x_0, y_0) = \operatorname{Im}(f(z_0)) = \lim_{(x,y)\to(x_0,y_0)} v(x, y)$, i.e. both $u$ and $v$ are continuous at $(x_0, y_0)$. $\square$

**Lemma 1.3.8.** Let $f\colon \mathbb{C} \to \mathbb{C}$. Then $f$ is continuous if and only if the preimage $f^{-1}(U) = \{ z \in \mathbb{C} : f(z) \in U \}$ is open for all open $U \subseteq \mathbb{C}$.

*Proof.* Suppose that $f$ is continuous and $U \subseteq \mathbb{C}$ is open. We are required to prove that $f^{-1}(U)$ is open, so let $z_0 \in f^{-1}(U)$. Then by definition $f(z_0) \in U$, which by assumption is an open set, so there exists $\varepsilon > 0$ such that $D_\varepsilon(f(z_0)) \subseteq U$. But since $f$ is continuous at $z_0$, by definition there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. We claim $D_\delta(z_0) \subseteq f^{-1}(U)$. Consider $z \in D_\delta(z_0)$. Then by definition, $|z - z_0| < \delta$, so precisely by the choice of $\delta$, we have that $|f(z) - f(z_0)| < \varepsilon$. But then by definition $f(z) \in D_\varepsilon(f(z_0))$, which is a subset of $U$, by choice of $\varepsilon$. So $f(z) \in U$, but that is exactly to say that if $z \in f^{-1}(U)$, as required.

Suppose now that the preimage of every open set is open. We are required to prove that $f$ is continuous, so let $z_0 \in \mathbb{C}$, and $\varepsilon > 0$. Then $D_\varepsilon(f(z_0))$ is an open ball, and therefore is an open set, by lemma 1.2.3. Therefore by assumption its preimage $f^{-1}(D_\varepsilon(f(z_0)))$ is an open set. Since clearly $f(z_0) \in D_\varepsilon(f(z_0))$, we have that $z_0 \in f^{-1}(D_\varepsilon(f(z_0)))$ by definition, therefore by definition of being open, there exists $\delta > 0$ such that $D_\delta(z_0) \subseteq f^{-1}(D_\varepsilon(f(z_0)))$. We claim this $\delta$ is as required in the definition of continuity for the given $\varepsilon$. Suppose $z \in \mathbb{C}$ satisfies $|z - z_0| < \delta$. Then by definition $z \in D_\delta(z_0)$, which is a subset of $f^{-1}(D_\varepsilon(f(z_0)))$ by choice of $\delta$. So $z \in f^{-1}(D_\varepsilon(f(z_0)))$, but that is exactly to say that $f(z) \in D_\varepsilon(f(z_0))$, i.e. by definition that $|f(z) - f(z_0)| < \varepsilon$. So $f$ is indeed continuous at $z_0$. $\square$

**Lemma 1.3.9.** Let $f, g\colon \mathbb{C} \to \mathbb{C}$ be continuous at $z_0 \in \mathbb{C}$. Then

(i) the function $f + g$ is continuous at $z_0$;
(ii) the function $fg$ is continuous at $z_0$; and
(iii) if $g(z_0) \neq 0$, then the function $\frac{f}{g}$ is continuous at $z_0$.

*Proof.* These follow by the definition of continuity and the algebra of limits, lemma 1.3.5. $\square$

**Lemma 1.3.10.** Let $S \subseteq \mathbb{C}$ be a closed and bounded set, and $f\colon S \to \mathbb{C}$ be continuous. Then $f(S)$ is closed and bounded.

*Proof.* We first show that $f(S)$ is bounded. Suppose for a contradiction that $f(S)$ is not bounded. In particular, for all $n \in \mathbb{N}$, it is not the case that $|f(z)| \leq n$ for all $z \in S$. That is, for each $n$, there exists $z_n \in S$ such that $|f(z_n)| > n$. This defines a sequence $z_n \in S$. Since by assmption $S$ is a bounded set, the sequence $z_n$ is a bounded sequence, hence lemma 1.2.15 implies that there is a convergent subsequence, $z_{n_k}$, say, such that $z_{n_k} \to z$ as $k \to \infty$, for some $z \in \mathbb{C}$. Then $z$ is a limit point of $S$, so $z \in \overline{S}$. But by assumption $S$ is closed, so $\overline{S} = S$, so $z \in S$. Therefore $f$ is defined at $z$, and is continuous at $z$. Therefore by definition there exists $\delta > 0$ such that

$$|f(w) - f(z)| < 1 \text{ whenever } s \in S \text{ and } |w - z| < \delta.$$

In turn there exists $N \in \mathbb{N}$ such that

$$|z_{n_k} - z| < \delta \text{ whenever } k \geq N.$$

Then for $k \geq N$, we have that $|f(z_{n_k}) - f(z)| < 1$, in particular, $|f(z_{n_k})| < |f(z)| + 1$. But by choice of the points $z_{n_k}$, we have that $|f(z_{n_k})| > n_k$ for all $k$. This is a contradiction, so $f(S)$ is indeed bounded.

We now show that $f(S)$ is closed. By lemma 1.2.6 it suffices to show that $\overline{f(S)} = f(S)$. By definition is always true that $f(S) \subseteq \overline{f(S)}$, so it suffices to show that $\overline{f(S)} \subseteq f(S)$. So let $z \in \overline{f(S)}$. Then by **??** there exists a sequence $z_n \in f(S)\setminus\{z\}$ such that $z_n \to z$. Then by definition, there exists a sequence $w_n \in S$ such that $f(w_n) = z_n$. Since $S$ is bounded, the sequence $w_n$ is bounded, therefore has a convergent subsequence, by lemma 1.2.15, $w_{n_l} \to w \in \mathbb{C}$, say. But then $w \in \overline{S}$ by **??**, thus $w \in S$, since $S$ is closed. By continuity of $f$, we have that $f(w) = f(\lim_{l\to\infty} w_{n_l}) = \lim_{l\to\infty} f(w_{n_l}) = \lim_{l\to\infty} z_{n_l} = \lim_{n\to\infty} z_n = z$, since every subsequence of a convergent sequence converges to the same limit as the sequence itself. But that is exactly to say that $z \in f(S)$, as required. ☐

## 1.4. Complex differentiability and holomorphicity.

**Lemma 1.4.2.** Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of $z_0$, and $f\colon U \to \mathbb{C}$ be differentiable at $z_0$. Then $f$ is continuous at $z_0$.

*Proof.* Define $\phi\colon U \to \mathbb{C}$ by

$$\phi(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & z \neq z_0, \\ f'(z_0) & z = z_0. \end{cases}$$

Then precisely by the definition of differentiability,

$$\lim_{z\to z_0} \phi(z) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = \phi(z_0),$$

i.e. $\phi$ is continuous at $z_0$. But rearranging the definition of $\phi$ shows that for all $z \in U$ we have $f(z) = \phi(z)(z - z_0) + f(z_0)$, which by the Algebra of Limits is continuous at $z_0$. ☐

**Lemma 1.4.3.** Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ be a neighbourhood of $z_0$, and $f, g\colon U \to \mathbb{C}$ be differentiable at $z_0$. Then

(i) the function $f + g$ is differentiable at $z_0$, with

$$(f + g)'(z_0) = f'(z_0) + g'(z_0);$$

(ii) the function $fg$ is differentiable at $z_0$, with

$$(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0);$$

and

(iii) if $g(z_0) \neq 0$, then the function $\frac{f}{g}$ is differentiable at $z_0$, with

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

*Proof.* These follow fairly readily just by the Algebra of Limits.

(i) By definition

$$\begin{aligned}(f + g)'(z_0) = \lim_{z\to z_0} \frac{(f + g)(z) - (f + g)(z_0)}{z - z_0} &= \lim_{z\to z_0} \frac{(f(z) + g(z)) - (f(z_0) + g(z_0))}{z - z_0} \\ &= \lim_{z\to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} + \frac{g(z) - g(z_0)}{z - z_0}\right) \\ &= \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} + \lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} \\ &= f'(z_0) + g'(z_0).\end{aligned}$$

(ii) Similarly, using also that $g$ is continuous at $z_0$, since it is differentiable at $z_0$, by lemma 1.4.2, we have that

$$\begin{aligned}(fg)'(z_0) &= \lim_{z\to z_0} \frac{(fg)(z)-(fg)(z_0)}{z-z_0}\\ &= \lim_{z\to z_0} \frac{f(z)g(z)-f(z_0)g(z_0)}{z-z_0}\\ &= \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}g(z)+f(z_0)\frac{g(z)-g(z_0)}{z-z_0}\right)\\ &= \lim_{z\to z_0}\left(\frac{f(z)-f(z_0)}{z-z_0}\right)\lim_{z\to z_0} g(z)+f(z_0)\lim_{z\to z_0}\left(\frac{g(z)-g(z_0)}{z-z_0}\right)\\ &= f'(z_0)g(z_0)+f(z_0)g'(z_0).\end{aligned}$$

(iii) Finally, it suffices to prove that $(1/g)'(z_0) = -g'(z_0)/g(z_0)^2$, and the full result follows by this result and part (ii). Using again that $g$ is continuous at $z_0$, we have by definition that

$$\begin{aligned}\left(\frac{1}{g}\right)'(z_0) = \lim_{z\to z_0}\frac{1}{z-z_0}\left(\frac{1}{g(z)}-\frac{1}{g(z_0)}\right) &= \lim_{z\to z_0}\left(\frac{g(z_0)-g(z)}{z-z_0}\cdot\frac{1}{g(z)g(z_0)}\right)\\ &= \lim_{z\to z_0}\left(\frac{g(z_0)-g(z)}{z-z_0}\right)\lim_{z\to z_0}\left(\frac{1}{g(z)g(z_0)}\right)\\ &= -g'(z_0)\frac{1}{g(z_0)^2}.\end{aligned}$$

□

**Lemma 1.4.4** (Chain rule)**.** Let $z_0 \in \mathbb{C}$, $U$ be a neighbourhood of $z_0$, $g\colon U \to \mathbb{C}$ be such that $g(U)$ is a neighbourhood of $g(z_0)$, and $f\colon g(U) \to \mathbb{C}$. Suppose furthermore that $g$ is differentiable at $z_0$ and $f$ is differentiable at $g(z_0)$. Then the composition $f \circ g\colon U \to \mathbb{C}$ is differentiable at $z_0$ and

$$(f\circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

*Proof.* Inspired by the proof of lemma 1.4.2, we define $\phi\colon U \to \mathbb{C}$ and $\psi\colon g(U) \to \mathbb{C}$ by

$$\phi(z) = \begin{cases}\frac{g(z)-g(z_0)}{z-z_0} & z \neq z_0,\\ g'(z_0) & z = z_0;\end{cases} \quad\text{and } \psi(w) = \begin{cases}\frac{f(w)-f(g(z_0))}{w-g(z_0)} & w \neq g(z_0),\\ f'(g(z_0)) & w = g(z_0).\end{cases}$$

Then $\phi$ and $\psi$ are continuous at $z_0$ and $g(z_0)$ respectively, $\phi(z_0) = g'(z_0)$, $\psi(g(z_0)) = f'(g(z_0))$, and for all $z \in U$, and $w \in g(U)$, we may write

$$g(z) = \phi(z)(z-z_0)+g(z_0), \text{ and } f(w) = \psi(w)(w-g(z_0))+f(g(z_0)).$$

Let $z \in U$. Then

$$f(g(z)) = \psi(g(z))(g(z)-g(z_0))+f(g(z_0)) = \psi(g(z))\phi(z)(z-z_0)+f(g(z_0)).$$

If $z \neq z_0$, this rearranges to

$$\frac{f(g(z))-f(g(z_0))}{z-z_0} = \psi(g(z))\phi(z).$$

Since $g$ is differentiable at $z_0$, it is continuous at $z_0$, by lemma 1.4.2, so $g(z) \to g(z_0)$ as $z \to z_0$. Continuity of $\psi$ at $g(z_0)$ implies therefore that $\psi(g(z)) \to \psi(g(z_0)) = f'(g(z_0))$ as $z \to z_0$. Continuity of $\phi$ at $z_0$ implies that $\phi(z) \to \phi(z_0) = g'(z_0)$ as $z \to z_0$. Then the algebra of limits implies that

$$\lim_{z\to z_0}\frac{f(g(z))-f(g(z_0))}{z-z_0} = \lim_{z\to z_0}\psi(g(z))\phi(z) = f'(g(z_0))g'(z_0),$$

which shows that $(f\circ g)'(z_0) = f'(g(z_0))g'(z_0)$. □

**Exercise 1.4.15.** Let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $u(x, y) = x^3 - 3xy^2 + y$.

(i) Prove that $u$ is harmonic.
(ii) Prove that $u$ is the real part of a holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ by constructing a harmonic conjugate $v\colon \mathbb{R}^2 \to \mathbb{R}$.
(iii) Write this holomorphic function $f$ explicitly as a function of $z$, where $z = x + iy$.

*Solution.* (i) We just calculate that

$$\frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 6x - 6x = 0$$

for all $(x, y) \in \mathbb{R}^2$, so $u$ is harmonic by definition.

(ii) For a function $v\colon \mathbb{R}^2 \to \mathbb{R}$ such that $f = u + iv$ is holomorphic, we need the Cauchy–Riemann equations to be satisfied everywhere, i.e. that

$$\begin{aligned} \frac{\partial v}{\partial y}(x, y) &= \frac{\partial u}{\partial x}(x, y) = 3x^2 - 3y^2, \text{ and} \\ \frac{\partial v}{\partial x}(x, y) &= -\frac{\partial u}{\partial y}(x, y) = 6xy - 1 \end{aligned}$$

for all $(x, y) \in \mathbb{R}^2$. Integrating the first equation with respect to $y$ gives that

$$v(x, y) = 3x^2 y - y^3 + \phi(x)$$

for some function $\phi\colon \mathbb{R} \to \mathbb{R}$. Integrating the second equation with respect to $x$ gives that

$$v(x, y) = 3x^2 y - x + \psi(y)$$

for some function $\psi\colon \mathbb{R} \to \mathbb{R}$. Combining these expressions we see that

$$v(x, y) = 3x^2 y - x - y^3 + \alpha$$

for some constant $\alpha \in \mathbb{R}$.

(iii) So the required holomorphic function $f$ is of the form

$$\begin{aligned} f(z) = f(x + iy) = u(x, y) + iv(x, y) &= x^3 - 3xy^2 + y + i(3x^2 y - x - y^3 + \alpha) \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 - ix - (i^2)y + i\alpha \\ &= (x + iy)^3 - i(x + iy - \alpha) \\ &= z^3 - i(z - \alpha). \end{aligned}$$

## 1.5. Polynomials and rational functions.

**Exercise 1.5.9.** Determine the sets on which the following functions are holomorphic:

(i) $f(z) = 1/(z^2 - 1)$;
(ii) $f(z) = z^2 + i\,|z|^2$;
(iii) $f(z) = z/(z^2 + 1)$;
(iv) $f(z) = \operatorname{Re}(z^2)$; and
(v) $f(z) = (z + 2i)/(z^2 + 4)$.

*Solution.* (i) $f$ is a rational function in which the denominator $z^2 - 1$ equals 0 if and only if $z = \pm 1$. Therefore by lemma 1.5.6 $f$ is holomorphic on $\mathbb{C} \setminus \{\pm 1\}$.

(ii) $f$ is nowhere holomorphic. Suppose $f$ is differentiable at $z$. Then $|z|^2 = (f(z) - z^2)/i$ is differentiable at $z$ by lemma 1.4.3. Therefore by example 1.4.11, $z = 0$. Therefore $f$ can only be differentiable at a single point, so is nowhere holomorphic.

(iii) $f$ is a rational function in which the denominator $z^2 + 1$ equals 0 if and only if $z = \pm i$. Therefore by lemma 1.5.6 $f$ is holomorphic on $\mathbb{C} \setminus \{\pm i\}$.

(iv) We can write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $u(x,y) = x^2 - y^2$ and $v(x,y) = 0$. If $f$ is differentiable at $z = x+iy$, then the Cauchy–Riemann equations imply that

$$2x = \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 0, \quad \text{and} \quad -2y = \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) = 0,$$

hence $(x,y) = 0$. Therefore $f$ can only be differentiable at a single point, so is nowhere holomorphic.

(v) We can simplify $f$ in the following way:

$$f(z) = \frac{z+2i}{z^2+4} = \frac{z+2i}{(z+2i)(z-2i)} = \frac{1}{z-2i},$$

in which form it is a rational function in which the denominator $z - 2i$ equals 0 if and only if $z = 2i$. Therefore by lemma 1.5.6 $f$ is holomorphic on $\mathbb{C} \setminus \{2i\}$.

**Exercise 1.5.10.** Let $P\colon \mathbb{C} \to \mathbb{C}$ be the polynomial $P(z) = \sum_{n=0}^{N} a_n z^n$, where all the coefficients $a_0, \dots, a_N$ are real, and let $z_0 \in \mathbb{C}$ satisfy $P(z_0) = 0$. Prove that $P(\overline{z_0}) = 0$.

*Solution.* We simply use various properties of complex conjugation: that the complex conjugate of a real number is the same real number, and that complex conjugation commutes with addition and multiplication (lemma 1.1.14 (vi),(vii)):

$$0 = \overline{0} = \overline{P(z_0)} = \overline{\left(\sum_{n=0}^{N} a_n z_0^n\right)} = \sum_{n=0}^{N} \overline{a_n z_0^n} = \sum_{n=0}^{N} \overline{a_n}\,\overline{z_0^n} = \sum_{n=0}^{N} a_n \overline{z_0}^n = P(\overline{z_0}),$$

where it was crucial that all the coefficients $a_n$ are real, so that $\overline{a_n} = a_n$.

## 1.6. The complex exponential and related functions.

**Lemma 1.6.5.** Let $z = x + iy \in \mathbb{C}$. Then

(i) The complex sine and cosine functions are holomorphic at $z$ with derivatives

$$\cos'(z) = -\sin(z) \quad \text{and} \quad \sin'(z) = \cos(z);$$

(ii) $\cos^2(z) + \sin^2(z) = 1$; and
(iii) $\cos(z + 2\pi) = \cos(z)$ and $\sin(z + 2\pi) = \sin(z)$.

*Proof.* (i) By lemmas 1.4.3 and 1.4.4 (algebra of derivatives and chain rule) and proposition 1.6.2(i) we just calculate that

$$\cos'(z) = \frac{i\exp'(iz) - i\exp'(-iz)}{2} = \frac{i(\exp(iz) - \exp(-iz))}{2} = -\frac{\exp(iz) - \exp(-iz)}{2i} = -\sin(z),$$

and

$$\sin'(z) = \frac{i\exp'(iz) - (-i)\exp'(-iz)}{2i} = \frac{i(\exp(iz) + \exp(-iz))}{2i} = \frac{\exp(iz) + \exp(-iz)}{2} = \cos(z).$$

(ii) This is just algebra and the use of proposition 1.6.2(ii):

$$\begin{aligned}
\cos^2(z) + \sin^2(z) &= \left(\frac{\exp(iz) + \exp(-iz)}{2}\right)^2 + \left(\frac{\exp(iz) - \exp(-iz)}{2i}\right)^2 \\
&= \frac{\exp^2(iz) + 2\exp(iz)\exp(-iz) + \exp^2(-iz)}{4} \\
&\qquad + \frac{\exp^2(iz) - 2\exp(iz)\exp(-iz) + \exp^2(-iz)}{-4} \\
&= \frac{1}{4}\left(\exp^2(iz) - \exp^2(iz) + 4\exp(iz)\exp(-iz) + \exp^2(-iz) - \exp^2(-iz)\right) \\
&= \frac{1}{4} 4\exp(iz - iz) \\
&= 1.
\end{aligned}$$

(iii) This uses proposition 1.6.2(iii):

$$\begin{aligned}\cos(z+2\pi) = \frac{\exp(i(z+2\pi)) + \exp(-i(z+2\pi))}{2} &= \frac{\exp(iz+2\pi i) + \exp(-iz-2\pi i)}{2} \\ &= \frac{\exp(iz) + \exp(-iz)}{2} \\ &= \cos(z),\end{aligned}$$

and

$$\begin{aligned}\sin(z+2\pi) = \frac{\exp(i(z+2\pi)) - \exp(-i(z+2\pi))}{2i} &= \frac{\exp(iz+2\pi i) - \exp(-iz-2\pi i)}{2i} \\ &= \frac{\exp(iz) - \exp(-iz)}{2i} \\ &= \sin(z).\end{aligned}$$

□

**Lemma 1.6.6.** Let $z, w \in \mathbb{C}$. Then

(i) $\sin(z + \pi/2) = \cos(z)$;
(ii) $\sin(z + w) = \sin(z)\cos(w) + \cos(z)\sin(w)$; and
(iii) $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$.

*Proof.* (i)

$$\begin{aligned}\sin(z+\pi/2) &= \frac{\exp(i(z+\pi/2)) - \exp(-i(z+\pi/2))}{2i} \\ &= \frac{\exp(iz+i\pi/2) - \exp(-iz-i\pi/2)}{2i} \\ &= \frac{\exp(iz)\exp(i\pi/2) - \exp(-iz)\exp(-i\pi/2)}{2i} \\ &= \frac{\exp(iz)i - \exp(-iz)(-i)}{2i} \\ &= \frac{\exp(iz) + \exp(-iz)}{2} \\ &= \cos(z).\end{aligned}$$

(ii)

$$\begin{aligned}&\sin(z)\cos(w) + \cos(z)\sin(w) \\ &= \frac{\exp(iz) - \exp(-iz)}{2i} \cdot \frac{\exp(iw) + \exp(-iw)}{2} + \frac{\exp(iz) + \exp(-iz)}{2} \cdot \frac{\exp(iw) - \exp(-iw)}{2i} \\ &= \frac{\exp(iz)\exp(iw) + \exp(iz)\exp(-iw) - \exp(-iz)\exp(iw) - \exp(-iz)\exp(-iw)}{4i} \\ &\quad + \frac{\exp(iz)\exp(iw) - \exp(iz)\exp(-iw) + \exp(-iz)\exp(iw) - \exp(-iz)\exp(-iw)}{4i} \\ &= \frac{\exp(iz+iw) + \exp(iz-iw) - \exp(-iz+iw) - \exp(-iz-iw)}{4i} \\ &\quad + \frac{\exp(iz+iw) - \exp(iz-iw) + \exp(-iz+iw) - \exp(-iz-iw)}{4i} \\ &= \frac{2\exp(i(z+w)) - 2\exp(-i(z+w))}{4i} \\ &= \sin(z+w).\end{aligned}$$

(iii)

$$
\begin{aligned}
&\cos(z)\cos(w) - \sin(z)\sin(w)\\
&= \frac{\exp(iz)+\exp(-iz)}{2}\cdot\frac{\exp(iw)+\exp(-iw)}{2} - \frac{\exp(iz)-\exp(-iz)}{2i}\cdot\frac{\exp(iw)-\exp(-iw)}{2i}\\
&= \frac{\exp(iz)\exp(iw)+\exp(iz)\exp(-iw)+\exp(-iz)\exp(iw)+\exp(-iz)\exp(-iw)}{4}\\
&\quad - \frac{\exp(iz)\exp(iw)-\exp(iz)\exp(-iw)-\exp(-iz)\exp(iw)+\exp(-iz)\exp(-iw)}{-4}\\
&= \frac{\exp(iz+iw)+\exp(iz-iw)+\exp(-iz+iw)+\exp(-iz-iw)}{4}\\
&\quad + \frac{\exp(iz+iw)-\exp(iz-iw)-\exp(-iz+iw)+\exp(-iz-iw)}{4}\\
&= \frac{2\exp(i(z+w))+2\exp(-i(z+w))}{4}\\
&= \cos(z+w).
\end{aligned}
$$

□

**Lemma 1.6.7.** Let $z = x + iy \in \mathbb{C}$. Then

$$\sin(x+iy) = \sin x\cosh y + i\cos x\sinh y \quad\text{and}\quad \cos(x+iy) = \cos x\cosh y - i\sin x\sinh y.$$

*Proof.* We just apply the definitions, using that cos is an even function and that sin is an odd function:

$$
\begin{aligned}
\sin(x+iy) &= \frac{\exp(i(x+iy))-\exp(-i(x+iy))}{2i}\\
&= \frac{\exp(ix-y)-\exp(-ix+y)}{2i}\\
&= \frac{e^{-y}(\cos x+i\sin x)-e^{y}(\cos(-x)+i\sin(-x))}{2i}\\
&= \frac{e^{-y}\cos x+ie^{-y}\sin x-e^{y}\cos x+ie^{y}\sin x}{2i}\\
&= \frac{-ie^{-y}\cos x+e^{-y}\sin x+ie^{y}\cos x+e^{y}\sin x}{2}\\
&= \sin x\left(\frac{e^{-y}+e^{y}}{2}\right)+i\cos x\left(\frac{e^{y}-e^{-y}}{2}\right)\\
&= \sin x\cosh y+i\cos x\sinh y,
\end{aligned}
$$

and

$$
\begin{aligned}
\cos(x+iy) &= \frac{\exp(i(x+iy))+\exp(-i(x+iy))}{2}\\
&= \frac{\exp(ix-y)+\exp(-ix+y)}{2}\\
&= \frac{e^{-y}(\cos x+i\sin x)+e^{y}(\cos(-x)+i\sin(-x))}{2}\\
&= \frac{e^{-y}\cos x+ie^{-y}\sin x+e^{y}\cos x-ie^{y}\sin x}{2}\\
&= \cos x\left(\frac{e^{-y}+e^{y}}{2}\right)+i\sin x\left(\frac{e^{-y}-e^{y}}{2}\right)\\
&= \cos x\cosh y-i\sin x\sinh y.
\end{aligned}
$$

□

**Lemma 1.6.10.** Let $z \in \mathbb{C}$. Then

$$\sinh(iz) = i\sin(z) \quad\text{and}\quad \cosh(iz) = \cos(z).$$

*Proof.* This follows by definition:

$$\sinh(iz) = \frac{\exp(iz) - \exp(-iz)}{2} = i\frac{\exp(iz) - \exp(-iz)}{2i} = i\sin(z), \text{ and}$$
$$\cosh(iz) = \frac{\exp(iz) + \exp(-iz)}{2} = \cos(z).$$
□

**Exercise 1.6.12.** Determine the sets on which the following functions $f$ are holomorphic:

(i) $f(z) = \exp(z^3)$; and
(ii) $f(z) = \exp(-|z|^2)$.

*Solution.* (i) $f$ is the composition $f_2 \circ f_1$ of two functions $f_1(z) = z^3$ and $f_2(z) = \exp(z)$, both of which are holomorphic on $\mathbb{C}$, therefore by the chain rule $f$ is holomorphic on $\mathbb{C}$.

(ii) We can write $f(z) = f(x+iy) = u(x,y)+iv(x,y)$ for functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ where $u(x,y) = e^{-(x^2+y^2)}$ and $v(x,y) = 0$. Suppose $f$ is differentiable at $z = x+iy$. Then the Cauchy–Riemann equations imply that

$$-2xe^{-(x^2+y^2)} = \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) = 0 \text{ and}$$
$$-2ye^{-(x^2+y^2)} = \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) = 0,$$

hence that $(x,y) = (0,0)$. So $f$ can only be differentiable at one point, thus is nowhere holomorphic.

**Exercise 1.6.13.** Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = f(x+iy) = e^{x^2-y^2}\left(\cos(2xy) + i\sin(2xy)\right).$$

(i) Use the Cauchy–Riemann equations to prove that $f$ is holomorphic on $\mathbb{C}$.
(ii) Give an alternative proof that $f$ is holomorphic on $\mathbb{C}$ by writing $f(z)$ explicitly in terms of $z$.

*Solution.* (i) We can write $f(z) = f(x+iy) = u(x,y)+iv(x,y)$ for functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ where $u(x,y) = e^{x^2-y^2}\cos(2xy)$ and $v(x,y) = e^{x^2-y^2}\sin(2xy)$. For any $(x,y) \in \mathbb{R}^2$, we see that

$$\frac{\partial u}{\partial x}(x,y) = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) = \frac{\partial v}{\partial y}(x,y), \text{ and}$$
$$\frac{\partial u}{\partial y}(x,y) = -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy) = -\frac{\partial v}{\partial x}(x,y).$$

Therefore the Cauchy–Riemann equations hold and the partial derivatives are continuous on $\mathbb{C}$. This implies by theorem 1.4.6 that $f$ is differentiable on $\mathbb{C}$, and therefore is holomorphic on $\mathbb{C}$.

(ii) We simply observe that we can write

$$f(z) = f(x+iy) = \exp(x^2 - y^2 + 2ixy) = \exp((x+iy)^2) = \exp(z^2),$$

thus $f$ is a composition $f_2 \circ f_1$ of the two functions $f_1(z) = z^2$ and $f_2(z) = \exp(z)$, both of which are holomorphic on $\mathbb{C}$, and therefore $f$ is holomorphic on $\mathbb{C}$.

## 1.7. The complex logarithm.

**Exercise 1.7.14.** Determine the sets on which the following functions are holomorphic:

(i)
  (i) $\mathrm{Log}(z-i)$;
  (ii) $\mathrm{Log}(z-(1+i))$;
  (iii) $\mathrm{Log}_0(z-1)$;

(iv) $\operatorname{Log}_0(z - i)$;
(v) $\operatorname{Log}_0(z - (1 + i))$;
(vi) $\operatorname{Log}_{-\pi/4}(z - 1)$;
(vii) $\operatorname{Log}_{-\pi/4}(z - i)$;

(ii) (i) $2 \operatorname{Log}(1/z)$;
(ii) $\operatorname{Log}(1/z^2)$;

(iii) (i) $\operatorname{Log}\left(1 - \frac{1}{z}\right)$; and
(ii) $\operatorname{Log}\left(1 - \frac{1}{z^2}\right)$.

*Solution.* (i) Each of these is solved via exactly the same method. We are asked to find the domain of holomorphicity of a function $\operatorname{Log}_\phi(g(z))$. The function $\operatorname{Log}_\phi$ is holomorphic on the cut plane $D_{0,\phi} = \mathbb{C} \setminus L_{0,\phi}$. Therefore the function $\operatorname{Log}_\phi(g(z))$ is holomorphic for all $z$ such that $g(z) \in D_{0,\phi}$, or, equivalently, for all $z$ such that $g(z) \notin L_{0,\phi}$. Since in all cases the function $g(z)$ is a straightforward translation, this is easy to solve. I won't give details for all of them. You should start to feel that you can just write down the answer by the time you've done a few of them, when the function inside the logarithm is something as simple as in these cases.

(i) Here $\operatorname{Log} = \operatorname{Log}_{-\pi}$, so $\phi = -\pi$ in the above argument, and $g(z) = z - i$. So $\operatorname{Log}(z - i)$ is holomorphic as long as $z - i \notin L_{0,-\pi}$. But $z - i \in L_{0,\pi}$ if and only if $z \in i + L_{0,-\pi} = L_{i,-\pi}$. So $\operatorname{Log}(z - i)$ is holomorphic on the cut plane with a branch cut from $i$ going in the negative real direction, i.e. on $D_{i,-\pi}$.
(ii) Now $g(z) = z - (1 + i)$, so $\operatorname{Log}(z - (1 + i))$ is holomorphic on $D_{(1+i),-\pi}$.
(iii) Now $\phi = 0$. So $\operatorname{Log}_0(z - 1)$ is holomophic on the cut plane with a branch cut from 1 going in the positive real direction, i.e. on $D_{1,0}$.
(iv) $\operatorname{Log}_0(z - i)$ is holomorphic on $D_{i,0}$.
(v) $\operatorname{Log}_0(z - (1 + i))$ is holomorphic on $D_{(1+i),0}$.
(vi) $\operatorname{Log}_{-\pi/4}(z - 1)$ is holomorphic on $D_{1,-\pi/4}$.
(vii) $\operatorname{Log}_{-\pi/4}(z - i)$ is holomorphic on $D_{i,-\pi/4}$.

(ii) The principle is exactly the same as for part (a), but we have to tread a little more carefully since the function $g$ inside the logarithm is not quite as easy as a translation.

(i) This is not a trick question. The function $2 \operatorname{Log}(1/z)$ is holomorphic if and only if $\operatorname{Log}(1/z)$ is holomorphic, so by example 1.8.11(iii) this is holomorphic on $D_{0,-\pi}$, the non-positive real axis.
(ii) Now we have that $g(z) = 1/z^2$. So $\operatorname{Log}(1/z^2)$ is holomorphic as long as $1/z^2 \notin L_{0,-\pi}$. A complex number $z$ satisfies $1/z^2 = x$ for some negative real number $x$ if and only if $z^2 = 1/x$, if and only if $z = \pm i/\sqrt{|x|}$. Evidently $\operatorname{Log}(1/z^2)$ is not holomorphic at $z = 0$ either. So the set of bad points is $\{ z \in \mathbb{C} : z = \pm i/\sqrt{|x|} \text{ for } x < 0 \text{ or } z = 0 \}$. But this set is just the whole imaginary axis. So $\operatorname{Log}(1/z^2)$ is holomorphic on $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re} z = 0 \}$. What we have here is two branch cuts, each with a branch point at 0, going in the positive and negative imaginary directions. It should be no surprise that we get two branch cuts in this case, since our function $g$ inside the logarithm involved a square, and we therefore had to take square roots to find where the problem points were, at which stage we picked up two solutions.

You might be alarmed to find that $2 \operatorname{Log}(1/z)$ is holomorphic on a different set to $\operatorname{Log}(1/z^2) = \operatorname{Log}((1/z)^2)$. You should not be, because these two functions are not equal—the "usual" laws of logarithms do not hold for particular branches in general, see workshop 4, question 2.

(iii) (i) Now our function is $g(z) = 1 - \frac{1}{z}$. So $\operatorname{Log}\left(1 - \frac{1}{z}\right)$ is holomorphic as long as $1 - \frac{1}{z} \notin L_{0,-\pi}$. A complex number $z$ satisfies $1 - \frac{1}{z} = x$ for a non-positive real number $x$ if and only if $z = \frac{1}{1-x}$ for a non-positive real number $x$. As $x$ runs from 0 through the negative real numbers, the value $\frac{1}{1-x}$ runs through the interval $(0, 1]$ on the real line. So the problematic points $z$ are those points such that $z \in (0, 1]$. Evidently the function is not well-defined at 0, so that is a point of non-holomorphicity too. So $\operatorname{Log}\left(1 - \frac{1}{z}\right)$ is holomorphic everywhere except at real numbers $z \in [0, 1]$, i.e. is holomorphic on $\mathbb{C} \setminus [0, 1]$.

(ii) Now our function is $g(z) = 1 - \frac{1}{z^2}$. So $\operatorname{Log}\left(1 - \frac{1}{z^2}\right)$ is holomorphic as long as $1 - \frac{1}{z^2} \notin L_{0,-\pi}$. Similarly to above, we now end up with problematic points $z$ such that $z^2 \in (0, 1]$, which are exactly those points $z$ that are real and lie in the real interval $[-1, 0) \cup (0, 1]$. Evidently the function is not well-defined at 0, so that is a point of non-holomorphicity too. So $\operatorname{Log}\left(1 - \frac{1}{z^2}\right)$ is holomorphic everywhere except at real numbers $z \in [-1, 1]$, i.e. is holomorphic on $\mathbb{C} \setminus [-1, 1]$.

**Exercise 1.7.15.**

(i)
- (i) Find a branch of $\log(z - 1)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) < 1 \}$.
- (ii) Find a branch of $\log(z + 1)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > -1 \}$.
- (iii) Find a branch of $\log(z^2 - 1)$ that is holomorphic on $\{ z \in \mathbb{C} : -1 < \operatorname{Re}(z) < 1 \}$.

(ii)
- (i) Find a branch of $\log(z - i)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$.
- (ii) Find a branch of $\log(z + i)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$.
- (iii) Find a branch of $\log(z^2 + 1)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$.

*Solution.* (i) (i) A branch $\operatorname{Log}_\phi$ of logarithm is holomorphic on the cut plane $D_{0,\phi}$. Therefore the composition $\operatorname{Log}_\phi(z - 1)$ is holomorphic at points $z$ such that $g(z) = z - 1 \in D_{0,\phi} = \mathbb{C} \setminus L_{0,\phi}$. We have that $z - 1 \in L_{0,\phi}$ if and only if $z \in L_{1,\phi}$. So $\operatorname{Log}_\phi(z - 1)$ is holomorphic on $\mathbb{C} \setminus L_{1,\phi}$. If $\phi \in [-\pi/2, \pi/2]$, then $L_{1,\phi} \cap \{ z \in \mathbb{C} : \operatorname{Re}(z) < 1 \} = \emptyset$, so $\{ z \in \mathbb{C} : \operatorname{Re}(z) < 1 \} \subseteq \mathbb{C} \setminus L_{1,\phi}$. Therefore any such $\phi$ will do, e.g. $\phi = 0$.

(ii) Similarly $\operatorname{Log}_\phi(z + 1)$ is holomorphic on $\mathbb{C} \setminus L_{-1,\phi}$. If $\phi \in [\pi/2, 3\pi/2]$, then $L_{-1,\phi} \cap \{ z \in \mathbb{C} : \operatorname{Re}(z) > -1 \} = \emptyset$, so for such $\phi$, $\operatorname{Log}_\phi(z+1)$ is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > -1 \}$. E.g. $\phi = \pi$ will do.

(iii) The function $f(z) = \operatorname{Log}_0(z - 1) + \operatorname{Log}_\pi(z + 1)$ is holomorphic on $\{ z \in \mathbb{C} : -1 < \operatorname{Re}(z) < 1, \}$ since it is a sum of two functions that are holomorphic there. Moreoever

$$\operatorname{Log}_0(z - 1) + \operatorname{Log}_\pi(z + 1) \in \log(z - 1) + \log(z + 1) = \log((z - 1)(z + 1)) = \log(z^2 - 1),$$

so $f(z) \in \log(z^2 - 1)$, and thus is a branch of $\log(z^2 - 1)$ that is holomorphic on $\{ z \in \mathbb{C} : -1 < \operatorname{Re}(z) < 1, \}$ .

(ii) (i) The function $\operatorname{Log}_\phi(z - i)$ is holomorphic on the cut plane $D_{i,\phi} = \mathbb{C} \setminus L_{i,\phi}$. Choosing $\phi \in [\pi/2, 3\pi/2]$ guarantees that $L_{i,\phi} \cap \{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \} = \emptyset$, so $\operatorname{Log}_\phi(z - i)$ is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$. E.g. $\phi = \pi$ suffices.

(ii) Similarly the function $\operatorname{Log}_\phi(z + i)$ is holomorphic on the cut plane $D_{-i,\phi} = \mathbb{C} \setminus L_{-i,\phi}$. Choosing $\phi \in [\pi/2, 3\pi/2]$ again guarantees that $L_{-i,\phi} \cap \{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \} = \emptyset$, so $\operatorname{Log}_\phi(z + i)$ is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$. So again $\phi = \pi$ suffices.

(iii) As before the function $f(z) = \operatorname{Log}_\pi(z - i) + \operatorname{Log}_\pi(z + i)$ is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$, and

$$\operatorname{Log}_\pi(z - i) + \operatorname{Log}_\pi(z + i) \in \log(z - i) + \log(z + i) = \log((z - i)(z + i)) = \log(z^2 + 1),$$

so $f(z)$ is a branch of $\log(z^2+1)$ that is holomorphic on $\{ z \in \mathbb{C} : \operatorname{Re}(z) > 0 \}$.

**Exercise 1.7.16.** Determine a branch of the function $f(z) = \log(z^2 + 2z + 3)$ that is holomorphic at $z = -1$, and find its derivative at that point.

*Solution.* We can write $f = \log \circ g$ where $g(z) = z^2 + 2z + 3$. We see that $g(-1) = (-1)^2 - 2 + 3 = 2$, so we need to choose a branch of logarithm that is holomorphic at $z = 2$. The principal branch is holomorphic everywhere on the cut plane $\mathbb{C} \setminus (-\infty, 0]$, i.e. the complex plane without the non-positive real axis, so the principal branch is such a branch. Differentiating using the chain rule, we see that for any $z$ such that $g(z)$ is not a non-positive real number, we have that

$$f'(z) = (\operatorname{Log} \circ g)'(z) = \operatorname{Log}'(g(z))g'(z) = \frac{1}{g(z)}(2z + 2),$$

in particular then $f'(-1) = \frac{1}{2}(-2 + 2) = 0$.

**Exercise 1.7.17.** Find all solutions $z \in \mathbb{C}$ of the equation $\cos(z) = \sin(z)$.

*Solution.* Using the definition of the trigonometric functions, we see that $\cos(z) = \sin(z)$ if and only if

$$\frac{\exp(iz) + \exp(-iz)}{2} = \frac{\exp(iz) - \exp(-iz)}{2i},$$

which holds if and only if

$$i \exp(iz) + i \exp(-iz) = \exp(iz) - \exp(-iz),$$

which holds if and only if

$$\exp(iz)(1 - i) = \exp(-iz)(1 + i),$$

which holds if and only if

$$\exp(2iz) = \exp(iz + iz) = \exp(iz) \exp(iz) = \frac{1 + i}{1 - i}.$$

It therefore suffices to find all the solutions to this equation, i.e. by definition, all $z$ such that $2iz \in \log\left(\frac{1+i}{1-i}\right)$. We calculate that

$$\frac{1 + i}{1 - i} = \frac{(1 + i)^2}{2} = \frac{1^2 - 1 + 2i}{2} = i,$$

so

$$\log\left(\frac{1 + i}{1 - i}\right) = \log(i) = \{\, \ln|i| + i \operatorname{Arg}(i) + 2k\pi i : k \in \mathbb{Z} \,\} = \left\{\, i\frac{\pi}{2} + 2k\pi i : k \in \mathbb{Z} \,\right\}.$$

Therefore our solutions are $z \in \mathbb{C}$ such that

$$z \in \frac{1}{2i}\left\{\, i\frac{\pi}{2} + 2k\pi i : k \in \mathbb{Z} \,\right\} = \left\{\, \frac{\pi}{4} + k\pi : k \in \mathbb{Z} \,\right\}.$$

### 1.8. Complex powers.

**Lemma 1.8.8.** Let $\alpha, \beta, z \in \mathbb{C}$, with $z \neq 0$. Then $z^\alpha z^\beta = z^{\alpha+\beta}$, where the principal branch of logarithm is chosen for each power function.

*Proof.* We use the definition of the complex power and proposition 1.6.2(ii) to see that

$$\begin{aligned} z^{\alpha+\beta} = \exp((\alpha + \beta) \operatorname{Log}(z)) &= \exp(\alpha \operatorname{Log}(z) + \beta \operatorname{Log}(z)) \\ &= \exp(\alpha \operatorname{Log}(z)) \exp(\beta \operatorname{Log}(z)) \\ &= z^\alpha z^\beta. \end{aligned}$$

□

**Exercise 1.8.10.** Find the derivative of the principal branch of $z^{1+i}$ at the point $z = i$.

*Solution.* By lemma 1.8.9 we have that $\frac{d}{dz}z^{1+i} = (1+i)z^{(1+i)-1} = (1+i)z^i$ wherever it exists, so the derivative of the principal branch of $z^{1+i}$ at the point $z = i$ is given by the value of the principal branch of $(1+i)z^i$ at the point $i$. We calculate that the principal branch of $z^i$ at $z = i$ is given by

$$i^i = \exp(i\operatorname{Log}(i)) = \exp(i(\ln|i| + i\operatorname{Arg}(i))) = \exp(i(i\pi/2)) = \exp(-\pi/2),$$

so the derivative of the principal branch of $z^{1+i}$ at $z = i$ is $(1+i)\exp(-\pi/2)$.

**Exercise 1.8.11.** Show that it is not true in general that $(zw)^\alpha = z^\alpha w^\alpha$, for the principal branch of the power function in each case.

*Solution.* Let $z = -1+i = w$ and $\alpha = 1/2$. Then $zw = (-1+i)^2 = 1-1-2i = -2i$, so

$$\begin{aligned}\exp\left(\tfrac{1}{2}\operatorname{Log}(zw)\right) = \exp\left(\tfrac{1}{2}\operatorname{Log}(-2i)\right) = \exp\left(\tfrac{1}{2}(\ln|-2i| + i\operatorname{Arg}(-2i))\right) &= \exp\left(\tfrac{1}{2}(\ln 2 - i\tfrac{\pi}{2})\right)\\ &= \exp\left(\tfrac{\ln 2}{2} - i\tfrac{\pi}{4}\right)\\ &= e^{\ln\sqrt{2}}e^{-i\pi/4}\\ &= \sqrt{2}e^{-i\pi/4}\\ &= 1-i.\end{aligned}$$

On the other hand,

$$\begin{aligned}\exp\left(\tfrac{1}{2}\operatorname{Log}(z)\right) = \exp\left(\tfrac{1}{2}\operatorname{Log}(-1+i)\right) &= \exp\left(\tfrac{1}{2}(\ln|-1+i| + i\operatorname{Arg}(-1+i))\right)\\ &= \exp\left(\tfrac{1}{2}(\ln\sqrt{2} + i\tfrac{3\pi}{4})\right)\\ &= \exp\left(\tfrac{\ln\sqrt{2}}{2} + i\tfrac{3\pi}{8}\right),\end{aligned}$$

so by proposition 1.6.2(ii)

$$\begin{aligned}\exp\left(\tfrac{1}{2}\operatorname{Log}(z)\right)\exp\left(\tfrac{1}{2}\operatorname{Log}(w)\right) &= \exp\left(\tfrac{\ln\sqrt{2}}{2} + i\tfrac{3\pi}{8}\right)\exp\left(\tfrac{\ln\sqrt{2}}{2} + i\tfrac{3\pi}{8}\right)\\ &= \exp\left(\tfrac{\ln\sqrt{2}}{2} + i\tfrac{3\pi}{8} + \tfrac{\ln\sqrt{2}}{2} + i\tfrac{3\pi}{8}\right)\\ &= \exp(\ln\sqrt{2} + i\tfrac{3\pi}{4})\\ &= e^{\ln\sqrt{2}}e^{3i\pi/4}\\ &= \sqrt{2}e^{3i\pi/4}\\ &= -1+i.\end{aligned}$$

Thus the value of the principal branch of $(zw)^{1/2}$ at $z = w = -1+i$ is $1-i$, whereas the value of the product of the principal branches $z^{1/2}w^{1/2}$ at $z = w = -1+i$ is $-1+i$.

**Exercise 1.8.14.** Find branches of the following multifunctions that are holomorphic on the given set:

(i) $(z-i)^{1/2}$ on $\{\, z \in \mathbb{C} : \operatorname{Im}(z) < 1 \,\}$; and

(ii) $\left(\frac{z-1}{z+1}\right)^{3/4}$ on $\mathbb{C} \setminus [-1,1]$;

*Solution.* (i) By definition a holomorphic branch $f(z)$ of $(z-i)^{1/2}$ is of the form $f(z) = \exp\left(\frac{1}{2}\operatorname{Log}_\phi(z-i)\right)$ for some $\phi$. Since exp is holomorphic everywere, this is holomorphic as long as $\operatorname{Log}_\phi(z-i)$ is holomorphic, i.e. as long as $z - i \notin L_{0,\phi}$, equivalently as long as $z \notin L_{i,\phi}$. So if we choose $\phi \in [0,\pi]$, then $L_{i,\phi} \cap \{\, z \in \mathbb{C} : \operatorname{Im}(z) < 1 \,\} = \emptyset$, and so $f(z)$ is holomorphic on $\{\, z \in \mathbb{C} : \operatorname{Im}(z) < 1 \,\}$. E.g. $\phi = \pi/2$ suffices.

(ii) The fact we are asked for a branch that is holomorphic except for a set of points on a line segment of finite length indicates that we should be thinking about logarithms involving inversion, since otherwise branch cuts of infinite length will necessarily be appearing. So observe that by definition

$$\left(\frac{z-1}{z+1}\right)^{3/4} = \left(\frac{1-\frac{1}{z}}{1+\frac{1}{z}}\right)^{3/4} = \left\{ \exp\left(\tfrac{3}{4}w\right) : w \in \log\left(\frac{1-\frac{1}{z}}{1+\frac{1}{z}}\right) \right\}.$$

Since

$$\log\left(\frac{1-\frac{1}{z}}{1+\frac{1}{z}}\right) = \log\left(1-\tfrac{1}{z}\right) - \log\left(1+\tfrac{1}{z}\right),$$

it suffices to choose branches of $\log\left(1-\frac{1}{z}\right)$ and $\log\left(1+\frac{1}{z}\right)$ that are holomorphic on $\mathbb{C}\setminus[-1,1]$. In exercise 1.7.14(iii)(a) we saw that $\operatorname{Log}\left(1-\frac{1}{z}\right)$ is holomorphic on $\mathbb{C}\setminus[0,1]$. Similar thinking will show that $\operatorname{Log}\left(1+\frac{1}{z}\right)$ is holomorphic on $\mathbb{C}\setminus[-1,0]$: the points of nonholomorphicity are the point $z=0$ and those points $z$ such that $1+\frac{1}{z}=x$ for non-positive real numbers $x$. This equation is equivalent to $z=\frac{1}{x-1}$, which as $x$ runs through negative real values, gives a set of solutions which runs through the real interval $[-1,0)$. So in particular both $\operatorname{Log}\left(1-\frac{1}{z}\right)$ and $\operatorname{Log}\left(1+\frac{1}{z}\right)$ are holomorphic functions on $\mathbb{C}\setminus[-1,1]$. Therefore the function

$$f(z) = \exp\left(\tfrac{3}{4}\left(\operatorname{Log}\left(1-\tfrac{1}{z}\right) - \operatorname{Log}\left(1+\tfrac{1}{z}\right)\right)\right)$$

is a branch of $\left(\frac{z-1}{z+1}\right)^{3/4}$ that is holomorphic on $\mathbb{C}\setminus[-1,1]$.

**Exercise 1.8.15.** For each of the values $z=-1$, $1+i$, $0$, and $i$, calculate the following values, if defined:

(i) the modulus $|z|$;
(ii) the argument $\arg(z)$;
(iii) the real and imaginary parts;
(iv) the exponential $\exp(z)$;
(v) the complex logarithm $\log(z)$; and
(vi) the complex power $z^z$.

*Solution.* We consider the values in turn:

(i)
 (i) $|-1|=1$;
 (ii) $\arg(-1)=\{\pi+2k\pi : k\in\mathbb{Z}\}$;
 (iii) $\operatorname{Re}(-1)=-1$ and $\operatorname{Im}(-1)=0$;
 (iv) $\exp(-1)=e^{-1}$;
 (v) $\log(-1)=\ln|-1|+i\arg(-1)=\{i\pi+2ik\pi : k\in\mathbb{Z}\}$; and
 (vi) $(-1)^{-1}=1/(-1)=-1$.

(ii)
 (i) $|1+i|=\sqrt{1^2+1^2}=\sqrt{2}$;
 (ii) $\arg(1+i)=\{\pi/4+2k\pi : k\in\mathbb{Z}\}$;
 (iii) $\operatorname{Re}(1+i)=1$ and $\operatorname{Im}(1+i)=1$;
 (iv) $\exp(1+i)=e^1\cos 1+ie^1\sin 1=e(\cos 1+i\sin 1)$;
 (v) $\log(1+i)=\ln|1+i|+i\arg(1+i)=\left\{\ln\sqrt{2}+i\pi/4+2ik\pi : k\in\mathbb{Z}\right\}$; and

(vi)

$$\begin{aligned}
(1+i)^{1+i} &= \{\, \exp((1+i)w) : w \in \log(1+i) \,\} \\
&= \left\{ \exp\left((1+i)\left(\ln\sqrt{2} + i\pi/4 + 2ik\pi\right)\right) : k \in \mathbb{Z} \right\} \\
&= \left\{ \exp\left(\ln\sqrt{2} + i\pi/4 + 2ik\pi - \pi/4 - 2k\pi + i\ln\sqrt{2}\right) : k \in \mathbb{Z} \right\} \\
&= \left\{ \exp\left(\ln\sqrt{2} + i\pi/4\right) \exp(2ik\pi) \exp(-\pi/4 - 2k\pi) \exp\left(i(\ln\sqrt{2})\right) : k \in \mathbb{Z} \right\} \\
&= \left\{ (1+i)e^{-\pi/4-2k\pi} \left(\cos\left(\ln\sqrt{2}\right) + i\sin\left(\ln\sqrt{2}\right)\right) : k \in \mathbb{Z} \right\}.
\end{aligned}$$

(iii)
  (i) $|0| = 0$;
  (ii) $\arg(0)$ is undefined;
  (iii) $\mathrm{Re}(0) = 0$ and $\mathrm{Im}(0) = 0$;
  (iv) $\exp(0) = 1$;
  (v) $\log(0)$ is undefined; and
  (vi) $0^0$ is undefined, since $\log(0)$ is undefined.

(iv)
  (i) $|i| = 1$;
  (ii) $\arg(i) = \{\, \pi/2 + 2k\pi : k \in \mathbb{Z} \,\}$;
  (iii) $\mathrm{Re}(i) = 0$ and $\mathrm{Im}(i) = 1$;
  (iv) $\exp(i) = \cos(1) + i\sin(1)$;
  (v) $\log(i) = \ln|i| + i\arg(i) = \{\, i\pi/2 + 2ik\pi : k \in \mathbb{Z} \,\}$; and
  (vi) $i^i = \left\{ e^{-\pi/2} e^{-2k\pi} : k \in \mathbb{Z} \right\}$ as seen in example 1.8.5(ii).

## 2. Möbius transformations

### 2.1. Conformal maps.

**Exercise 2.1.4.** Sketch the images under the holomorphic function $f(z) = \exp(z)$ of the following subsets of $\mathbb{C}$:

(i) the strip $\{\, z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi \,\}$;
(ii) the half-strip $\{\, z \in \mathbb{C} : \mathrm{Re}(z) < 0 \text{ and } 0 < \mathrm{Im}(z) < \pi \,\}$;
(iii) the half-strip $\{\, z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } 0 < \mathrm{Im}(z) < \pi \,\}$;
(iv) the rectangle $\{\, z \in \mathbb{C} : 1 < \mathrm{Re}(z) < 2 \text{ and } 0 < \mathrm{Im}(z) < \pi \,\}$; and
(v) the half-planes $\{\, z \in \mathbb{C} : \mathrm{Re}(z) > 0 \,\}$ and $\{\, z \in \mathbb{C} : \mathrm{Re}(z) < 0 \,\}$.

*Solution.* (i) $z \in \{\, z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi \,\}$ if and only if $z = x + iy$ where $x \in \mathbb{R}$ is arbitrary and $y \in (0, \pi)$, for which $\exp(z) = e^x e^{iy}$ is therefore a complex number of arbitrary modulus $e^x$ and of argument $y$ between 0 and $\pi$. Thus the image is the upper half plane $\{\, z \in \mathbb{C} : \mathrm{Im}(z) > 0 \,\}$.

(ii) $z \in \{\, z \in \mathbb{C} : \mathrm{Re}(z) < 0 \text{ and } 0 < \mathrm{Im}(z) < \pi \,\}$ if and only if $z = x + iy$ where $x < 0$ and $y \in (0, \pi)$, for which $\exp(z) = e^x e^{iy}$ is a complex number of modulus $e^x$ strictly less than 1 and of argument $y$ between 0 and $\pi$. Thus the image is the set $\{\, z \in \mathbb{C} : |z| < 1 \text{ and } \mathrm{Im}(z) > 0 \,\}$.

(iii) $z \in \{\, z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } 0 < \mathrm{Im}\, z < \pi \,\}$ if and only if $z = x + iy$ where $x > 0$ and $y \in (0, \pi)$, for which $\exp(z) = e^x e^{iy}$ is a complex number of modulus $e^x$ strictly greater than 1 and of argument $y$ between 0 and $\pi$. Thus the image is the set $\{\, z \in \mathbb{C} : |z| > 1 \text{ and } \mathrm{Im}(z) > 0 \,\}$.

(iv) $z \in \{\, z \in \mathbb{C} : 1 < \mathrm{Re}(z) < 2 \text{ and } 0 < \mathrm{Im}(z) < \pi \,\}$ if and only if $z = x + iy$ where $x \in (1, 2)$ and $y \in (0, \pi)$, for which $\exp(z) = e^x e^{iy}$ is a complex number of modulus $e^x \in (e, e^2)$ and of argument $y$ between 0 and $\pi$. Thus the image is the set $\{\, z \in \mathbb{C} : e < |z| < e^2 \text{ and } \mathrm{Im}(z) > 0 \,\}$.

(v) $z \in \{\, z \in \mathbb{C} : \mathrm{Re}(z) > 0 \,\}$ if and only if $z = x + iy$ where $x > 0$ and $y$ is arbitrary, for which $\exp(z) = e^x e^{iy}$ is a complex number of modulus $e^x$ strictly greater than

1 and of arbitrary argument $y$. The image of this half-plane is therefore the set $\{ z \in \mathbb{C} : |z| > 1 \}$. Similarly $z \in \{ z \in \mathbb{C} : \operatorname{Re}(z) < 0 \}$ if and only if $z = x + iy$ where $x < 0$ and $y$ is arbitrary, for which $\exp(z) = e^x e^{iy}$ is a complex number of modulus $e^x$ strictly less than 1 and of arbitrary argument $y$. The image of this half-plane is therefore the set $\{ z \in \mathbb{C} : |z| < 1 \}$.

**Exercise 2.1.5.** Sketch the images under the holomorphic function $f(z) = \cos(z)$ of the following subsets of $\mathbb{C}$:

(i) the half-strip $\{ z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \pi \text{ and } \operatorname{Im}(z) < 0 \}$;
(ii) the half-strip $\{ z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \pi \text{ and } \operatorname{Im}(z) > 0 \}$; and
(iii) the strip $\{ z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \pi \}$.

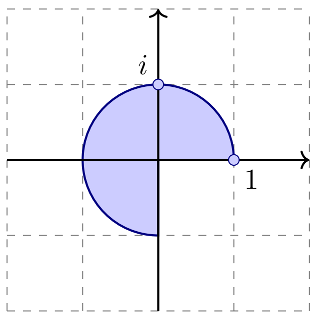
*Solution.* (i) $z \in \{ z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \pi \text{ and } \operatorname{Im}(z) < 0 \}$ if and only if $z = x + iy$ where $x \in (0, \pi)$ and $y < 0$, for which $\cos(z) = \cos x \cosh y - i \sin x \sinh y$ is a complex number of arbitrary real part $\cos x \cosh y$ and of imaginary part $-\sin x \sinh y$ strictly greater than 0. Thus the image is the set $\{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$.

(ii) $z \in \{ z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \pi \text{ and } \operatorname{Im}(z) > 0 \}$ if and only if $z = x + iy$ where $x \in (0, \pi)$ and $y > 0$, for which $\cos(z) = \cos x \cosh y - i \sin x \sinh y$ is a complex number of arbitrary real part $\cos x \cosh y$ and of imaginary part $-\sin x \sinh y$ strictly less than 0. Thus the image is the set $\{ z \in \mathbb{C} : \operatorname{Im}(z) < 0 \}$.

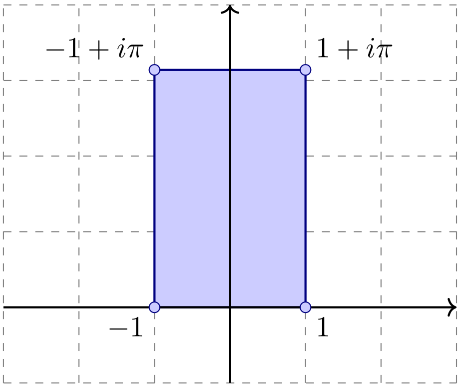
(iii) Notice that since this set contains both the sets in parts (a) and (b), the image must contain the upper and lower half planes. $z \in \{ z \in \mathbb{C} : 0 < \operatorname{Re}(z) < \pi \}$ if and only if $z = x + iy$ where $x \in (0, \pi)$ and $y$ is arbitrary, for which $\cos(z) = \cos x \cosh y - i \sin x \sinh y$ is a complex number, the only condition on which is that if $y = 0$ then $\cos(z) = \cos x \in [-1, 1]$. Thus the image is the set $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im}(z) = 0 \text{ and } |\operatorname{Re}(z)| > 1 \}$.

**Exercise 2.1.6.** Sketch the image under the holomorphic function $f(z) = z^3$ of the set $\{ z \in \mathbb{C} : |z| \leq 1, \text{and } 0 \leq \operatorname{Arg}(z) \leq \pi/2 \}$.

*Solution.* We see that $|f(z)| = \left|z^3\right| = |z|^3 \leq 1$, and furthermore that $\operatorname{Arg}_0(f(z)) = \operatorname{Arg}_0(z^3) = 3 \operatorname{Arg}_0(z) \in [0, 3\pi/2]$ for $z$ in the set under consideration, noting that $\operatorname{Arg}_0(z) = \operatorname{Arg}(z)$ for such $z$. Thus the image is the set $\{ z \in \mathbb{C} : |z| \leq 1 \text{ and } 0 \leq \operatorname{Arg}_0(z) \leq 3\pi/2 \}$, as shown below.



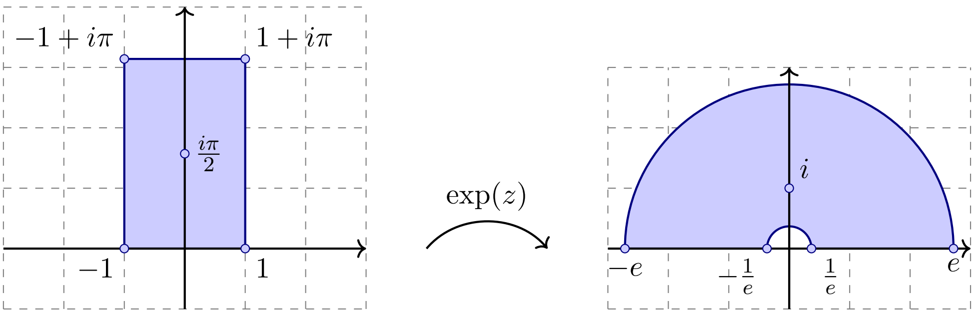

**Exercise 2.1.7.** Sketch the image under the holomorphic function $f(z) = \exp(z)$ of the set shown in the figure:




*Solution.* We begin by establishing where the vertices are mapped to: we have that

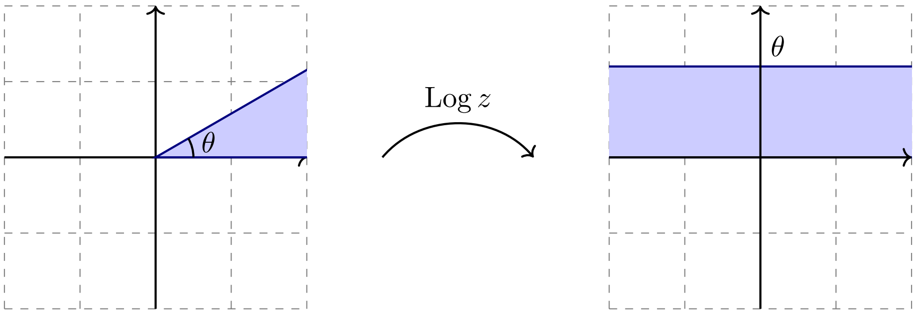
$$\begin{aligned}\exp(-1) &= e^{-1},\\ \exp(1) &= e,\\ \exp(1+i\pi) &= e^1e^{i\pi} = -e, \text{ and}\\ \exp(-1+i\pi) &= e^{-1}e^{i\pi} = -e^{-1}.\end{aligned}$$

Now consider where the edges are mapped to. The edge connecting $-1$ and $1$ is a straight line segment consisting of complex numbers of the form $z = x+iy = x \in [-1, 1]$, for which $\exp(z) = e^x \in [e^{-1}, e^1]$. Similarly the edge connecting $-1+i\pi$ to $1+i\pi$ is a straight line segment consisting of complex numbers of the form $z = x+iy = x+i\pi$ where $x \in [-1, 1]$, for which $\exp(z) = e^xe^{i\pi} = -e^x \in [-e, -e^{-1}]$. The edge connecting $1$ to $1 + i\pi$ is a straight line segment consisting of complex numbers of the form $z = x + iy = 1 + iy$ where $y \in [0, \pi]$, for which $\exp(z) = e^1e^{iy}$ is a complex number of modulus $e$ and of argument $y \in [0, \pi]$. Similarly the line segment connecting $-1$ to $-1 + i\pi$ consists of complex numbers of the form $z = x + iy = -1 + iy$ where $y \in [0, \pi]$, for which $\exp(z) = e^{-1}e^{iy}$ is a complex number of modulus $e^{-1}$ and of argument $y \in [0, \pi]$. This determines the outline of our image; we just need to establish where the interior of our rectangle maps to. We see that $\exp(i\pi/2) = i$ which lies in the interior of the outline of the image. Thus we can draw the following picture:




**Exercise 2.1.8.** For $\theta \in \mathbb{R}$, sketch the image under the holomorphic function $f(z) = \operatorname{Log}(z)$ of the set $\{z \in \mathbb{C} : 0 \leq \operatorname{Arg}(z) \leq \theta\}$.

*Solution.* We may assume that $\theta \in [0, \pi]$: if $\theta < 0$ then the given set is empty, and if $\theta > \pi$, then the condition is just that $0 < \operatorname{Arg}(z) < \pi$. For $z \in \{z \in \mathbb{C} : 0 \leq \operatorname{Arg}(z) \leq \theta\}$, we have that $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ is a complex number of arbitrary real part $\ln|z|$ and of imaginary part $\operatorname{Arg}(z) \in [0, \theta]$. Therefore the image is the set $\{z \in \mathbb{C} : 0 \leq \operatorname{Im}(z) \leq \theta\}$, as shown below.




**Exercise 2.1.9.** Find a holomorphic function $f$ which maps the set $\{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$ onto the set $\{ z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1 \}$.

*Solution.* The previous exercise indicates the way, once we write $\{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \} = \{ z \in \mathbb{C} : \mathrm{Arg}(z) \in (0, \pi) \}$. Thus the function $\mathrm{Log}(z)$ maps this set to the set $\{ z \in \mathbb{C} : 0 < \mathrm{Im}(z) < \pi \}$. Then $f(z) = \mathrm{Log}(z)/\pi$ maps our original set to the set $\{ z \in \mathbb{C} : 0 < \mathrm{Im}(z) < 1 \}$.

### 2.2. Definition of Möbius transformations.

**Lemma 2.2.3.** To a complex matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant $ad - bc = 1$ we associate the Möbius transformation $f_M(z) = \frac{az+b}{cz+d}$. Under this correspondence we have that

$$f_{M_1 M_2} = f_{M_1} \circ f_{M_2} \quad \text{and} \quad f_{M^{-1}} = f_M^{-1}.$$

*Proof.* Suppose $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ and let $z \in \mathbb{C}$. Then

$$\begin{aligned}
(f_{M_1} \circ f_{M_2})(z) = f_{M_1}(f_{M_2}(z)) = f_{M_1}\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) &= \frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + b_1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} \\
&= \frac{a_1 a_2 z + a_1 b_2 + b_1 c_2 z + b_1 d_2}{c_1 a_2 z + c_1 b_2 + d_1 c_2 z + d_1 d_2} \\
&= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)},
\end{aligned}$$

which is associated with the matrix

$$\begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = M_1 M_2.$$

Since the identity funcition $\mathrm{id}(z) = z$ is associated with $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we can apply what we have just proved to see that $f_M \circ f_{M^{-1}} = f_{MM^{-1}} = f_{I_2} = \mathrm{id}$, which is to say that $f_{M^{-1}} = (f_M)^{-1}$ indeed. □

**Exercise 2.2.4.** Let $f(z) = \frac{2z+1}{3z-2}$. Calculate $f(1/z)$ and $f(f(z))$.

*Solution.* We just calculate that

$$f(1/z) = \frac{2(1/z) + 1}{3(1/z) - 2} = \frac{2 + z}{3 - 2z}$$

and

$$f(f(z)) = \frac{2\left(\frac{2z+1}{3z-2}\right) + 1}{3\left(\frac{2z+1}{3z-2}\right) - 2} = \frac{4z + 2 + 3z - 2}{6z + 3 - 6z + 4} = \frac{7z}{7} = z.$$

### 2.4. Deconstructing Möbius transformations.

**Exercise 2.4.4.** Let $f(z) = \frac{1+z}{1-z}$. Sketch the images under $f$ of the real and imaginary axes.

*Solution.* We check the images of three points on the real axis:

$$f(0) = \frac{1+0}{1-0} = 1, \quad f(1) = \frac{1+1}{1-1} = \infty, \quad \text{and} \quad f(-1) = \frac{1-1}{1+1} = 0.$$

Since the real axis is a straight line and $f$ is a Möbius transformation, its image is a circle or a straight line. The three points checked above tell us that the image is a straight line through $f(-1) = 0$ and $f(0) = 1$, i.e. is itself the real axis. We can also see this by observing that $f(z) = (1+z)/(1-z)$ is a real number if $z$ is a real number.

To find the image of the imaginary axis, we note that

$$f(i) = \frac{1+i}{1-i} = \frac{(1+i)^2}{2} = \frac{1-1+2i}{2} = i.$$

So the image of the imaginary axis is a circle or a straight line which passes through $f(i) = i$ and through $f(0) = 1$. But $f$ is conformal, so must preserve the angle between the axes at their point of intersection, i.e. the angle between the images of the axes at $f(0) = 1$ must equal the angle between the axes at 0, i.e. $\pi/2$. This then determines the image of the imaginary axis: a circle or line which passes through $i$ and 1 and which has a tangent perpendicular to the real axis at the point 1. This implies that the image is the unit circle centred at 0, i.e. $\{ z \in \mathbb{C} : |z| = 1 \}$.

### 2.5. The cross-ratio.

**Exercise 2.5.4.** Find Möbius transformations $f$ which satisfy:

(i) $f(0) = i$, $f(1) = 1$, and $f(-1) = -1$;
(ii) $f(0) = -i$, $f(1) = 1$, and $f(-1) = -1$;
(iii) $f(0) = 0$, $f(1) = -1$, and $f(-1) = 1$;
(iv) $f(0) = \infty$, $f(1) = -1$, and $f(\infty) = 1$; and
(v) $f(0) = 1$, $f(1) = \infty$, and $f(\infty) = i$.

*Solution.* (i) Suppose $f(z) = (az+b)/(cz+d)$ is as required. Then the three conditions imply that

$$i = f(0) = \frac{b}{d}, \quad 1 = f(1) = \frac{a+b}{c+d}, \quad \text{and} \quad -1 = f(-1) = \frac{-a+b}{-c+d}.$$

Since we can multiply all the terms $a, b, c, d$ by any non-zero constant and get the same transformation, we observe that certainly $d \neq 0$, so we can multiply by $d^{-1}$ and assume that $d = 1$, which implies that $b = i$. Then the remaining conditions can be rearranged to state that

$$c + 1 = c + d = a + b = a + i \quad \text{and} \quad c - 1 = -(-c + d) = -a + b = -a + i,$$

which imply that $c = i$ and $a = 1$. So $f(z) = (z + i)/(iz + 1)$.

(ii) Similarly, suppose $f(z) = (az + b)/(cz + d)$ is as required. Then

$$-i = f(0) = \frac{b}{d}, \quad 1 = f(1) = \frac{a+b}{c+d}, \quad \text{and} \quad -1 = f(-1) = \frac{-a+b}{-c+d}.$$

Again we may assume that $d = 1$, so $b = -i$. Then the remaining conditions become

$$c + 1 = c + d = a + b = a - i \quad \text{and} \quad c - 1 = -(-c + d) = -a + b = -a - i,$$

which imply that $c = -i$ and $a = 1$. So $f(z) = (z - i)/(-iz + 1)$.

(iii) We can just write this one down: $f(z) = -z$ does the trick.

(iv) Suppose $f(z) = (az+b)/(cz+d)$ is as required. Then our three conditions are that

$$\infty = f(0) = \frac{b}{d}, \quad -1 = f(1) = \frac{a+b}{c+d} \quad \text{and} \quad 1 = f(\infty) = \frac{a}{c}.$$

We therefore immediately see that $d = 0$ and that $a = c$. Since $d$ and $c$ cannot both be zero, we can assume that $c = 1$, thus our remaining condition becomes that $-1 = -a = -(c+d) = a+b = 1+b$, hence that $b = -2$. So $f(z) = (z-2)/z$.

(v) Suppose $f(z) = (az+b)/(cz+d)$ is as required. Then our three conditions are that

$$1 = f(0) = \frac{b}{d}, \quad \infty = f(1) = \frac{a+b}{c+d} \quad \text{and} \quad i = f(\infty) = \frac{a}{c}.$$

Since $d \neq 0$ we may assume that $d = 1$, and hence $b = 1$. We see that $c+d = 0$, so $c = -1$. The final condition then implies that $a = -i$. Thus $f(z) = (-iz+1)/(-z+1)$.

## 3. Complex integration

### 3.1. Complex integrals.

**Lemma 3.1.2.** Let $[a,b] \subseteq \mathbb{R}$ be an interval, and let $f, g\colon [a,b] \to \mathbb{C}$ be integrable, and $\alpha, \beta \in \mathbb{C}$. Then

(i) $\alpha f + \beta g$ is integrable, and

$$\int_a^b (\alpha f(t) + \beta g(t))\, dt = \alpha \int_a^b f(t)\, dt + \beta \int_a^b g(t)\, dt;$$

(ii) if $f$ is continuous and $f = \frac{dF}{dt}$ for a differentiable function $F\colon [a,b] \to \mathbb{C}$, then

$$\int_a^b f(t)\, dt = F(b) - F(a);$$

and

(iii)

$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt.$$

*Proof.* Suppose $f = u_1 + iv_1$ and $g = u_2 + iv_2$ for some $u_1, u_2, v_1, v_2\colon [a,b] \to \mathbb{R}$, and that $\alpha = a_1 + ib_1$ and $\beta = a_2 + ib_2$.

(i) By definition, for any $t \in [a,b]$, we have that

$$\begin{aligned}
(\alpha f + \beta g)(t) &= \alpha f(t) + \beta g(t) \\
&= (a_1 + ib_1)(u_1(t) + iv_1(t)) + (a_2 + ib_2)(u_2(t) + iv_2(t)) \\
&= (a_1 u_1(t) - b_1 v_1(t)) + i(b_1 u_1(t) + a_1 v_1(t)) \\
&\quad + (a_2 u_2(t) - b_2 v_2(t)) + i(b_2 u_2(t) + a_2 v_2(t)) \\
&= a_1 u_1(t) - b_1 v_1(t) + a_2 u_2(t) - b_2 v_2(t) \\
&\quad + i(b_1 u_1(t) + a_1 v_1(t) + b_2 u_2(t) + a_2 v_2(t)).
\end{aligned}$$

Since real integrable functions form a vector space, the real and imaginary parts of this function are integrable, therefore by definition the complex function itself is integrable. Furthermore by definition and linearity of the real interal, the integral is given by

$$\begin{aligned}
\int_a^b (\alpha f + \beta g)(t)\,dt &= \int_a^b a_1u_1(t) - b_1v_1(t) + a_2u_2(t) - b_2v_2(t)\,dt \\
&\quad + i\int_a^b b_1u_1(t) + a_1v_1(t) + b_2u_2(t) + a_2v_2(t)\,dt \\
&= a_1\int_a^b u_1(t)\,dt - b_1\int_a^b v_1(t)\,dt + a_2\int_a^b u_2(t)\,dt - b_2\int_a^b v_2(t)\,dt \\
&\quad + i\left(b_1\int_a^b u_1(t)\,dt + a_1\int_a^b v_1(t)\,dt + b_2\int_a^b u_2(t)\,dt + a_2\int_a^b v_2(t)\,dt\right) \\
&= (a_1 + ib_1)\left(\int_a^b u_1(t) + i\int_a^b v_1(t)\,dt\right) \\
&\quad + (a_2 + ib_2)\left(\int_a^b u_2(t)\,dt + i\int_a^b v_2(t)\,dt\right) \\
&= \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt.
\end{aligned}$$

(ii) Suppose that $F = U + iV$ for differentiable $U, V\colon [a,b] \to \mathbb{R}$. Then $U' = u_1$ and $V' = v_1$, so the real fundamental theorem of calculus implies that

$$\begin{aligned}
\int_a^b f(t)\,dt = \int_a^b u_1(t)\,dt + i\int_a^b v_1(t)\,dt &= \int_a^b U'(t)\,dt + i\int_a^b V'(t)\,dt \\
&= U(b) - U(a) + i(V(b) - V(a)) \\
&= U(b) + iV(b) - (U(a) + iV(a)) \\
&= F(b) - F(a).
\end{aligned}$$

(iii) If $\int_a^b f(t)\,dt = 0$, then the result is trivial. Otherwise, using exponential form, we have for some $R > 0$ and $\theta \in \mathbb{R}$ that $\int_a^b f(t)\,dt = Re^{i\theta}$, thus $|\int_a^b f(t)\,dt| = R$. Then, using the linearity of integration which we have just proved, we have that

$$\left|\int_a^b f(t)\,dt\right| = R = e^{-i\theta}Re^{i\theta} = e^{-i\theta}\int_a^b f(t)\,dt = \int_a^b e^{-i\theta}f(t)\,dt.$$

Let $h\colon [a,b] \to \mathbb{C}$ be defined by $h(t) = e^{-i\theta}f(t)$. Then we have by definition of the complex integral that

$$\left|\int_a^b f(t)\,dt\right| = \int_a^b h(t)\,dt = \int_a^b \mathrm{Re}(h(t))\,dt + i\int_a^b \mathrm{Im}(h(t))\,dt.$$

But the left-hand side is real, so the imaginary part of the right-hand side is zero, i.e. $|\int_a^b f(t)\,dt| = \int_a^b \mathrm{Re}(h(t))\,dt$. But then since $\mathrm{Re}(h(t)) \le |h(t)|$ for every $t \in [a,b]$, we have that

$$\left|\int_a^b f(t)\,dt\right| = \int_a^b \mathrm{Re}(h(t))\,dt \le \int_a^b |h(t)|\,dt = \int_a^b \left|e^{-i\theta}f(t)\right|\,dt = \int_a^b |f(t)|\,dt. \qquad \square$$

**3.2. Contour integrals.**

**Lemma 3.2.8.** Let $\Gamma$ be an arc of a circle of radius $r$ traced through an angle $\theta$. Then $\ell(\Gamma) = r\theta$.

*Proof.* The curve is parametrized by $\gamma\colon [t_0, t_0 + \theta] \to \mathbb{C}$, where $\gamma(t) = z_0 + r\exp(it)$ for some $z_0 \in \mathbb{C}$ and $t_0 \in \mathbb{R}$. We see that

$$\gamma(t) = \mathrm{Re}(z_0) + r\cos t + i(\mathrm{Im}(z_0) + r\sin t),$$

and hence

$$\gamma'(t) = -r\sin t + ir\cos t.$$

Therefore by definition

$$\ell(\Gamma) = \int_{t_0}^{t_0+\theta} \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt = \int_{t_0}^{t_0+\theta} r\, dt = r\theta. \qquad \square$$

**Lemma 3.2.11.** Let $\Gamma$ be a curve in $\mathbb{C}$, $f, g\colon \Gamma \to \mathbb{C}$ be continuous, and $\alpha, \beta \in \mathbb{C}$. Then

(i)
$$\int_\Gamma (\alpha f(z) + \beta g(z))\, dz = \alpha \int_\Gamma f(z)\, dz + \beta \int_\Gamma g(z)\, dz;$$

(ii) if $\gamma_1\colon [t_0, t_1] \to \mathbb{C}$ and $\tilde{\gamma}\colon [\tilde{t}_0, \tilde{t}_1] \to \mathbb{C}$ are two parametrizations of $\Gamma$, such that there exists an injective differentiable function $\lambda\colon [\tilde{t}_0, \tilde{t}_1] \to [t_0, t_1]$ with $\lambda'(t) > 0$ for all $t \in [\tilde{t}_0, \tilde{t}_1]$ such that $\tilde{\gamma}(t) = \gamma(\lambda(t))$ for all $t \in [\tilde{t}_0, \tilde{t}_1]$, then
$$\int_{\tilde{t}_0}^{\tilde{t}_1} f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\, dt = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\, dt;$$
and

(iii) if $\Gamma$ is parametrized by $\gamma\colon [0,1] \to \mathbb{C}$, then the curve we shall notate as $-\Gamma$, which runs in the opposite direction to $\Gamma$ but along the same path, parametrized by $\tilde{\gamma}(t)\colon [0,1] \to \mathbb{C}$ defined by $\tilde{\gamma}(t) = \gamma(1-t)$, satisfies
$$\int_{-\Gamma} f(z)\, dz = -\int_\Gamma f(z)\, dz.$$

*Proof.* (i) Suppose $\Gamma$ is parametrized by $\gamma\colon [t_0, t_1] \to \mathbb{C}$. Then by definition, and lemma 3.1.2(i), we have that

$$\begin{aligned}
\int_\Gamma (\alpha f(z) + \beta g(z))\, dz &= \int_{t_0}^{t_1} (\alpha f(\gamma(t)) + \beta g(\gamma(t)))\, \gamma'(t)\, dt \\
&= \int_{t_0}^{t_1} \alpha f(\gamma(t))\gamma'(t) + \beta g(\gamma(t))\gamma'(t)\, dt \\
&= \alpha \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)\, dt + \beta \int_{t_0}^{t_1} g(\gamma(t))\gamma'(t)\, dt \\
&= \alpha \int_\Gamma f(z)\, dz + \beta \int_\Gamma g(z)\, dz.
\end{aligned}$$

Parts (ii) and (iii) are in the lecture notes. $\square$

### 3.3. Independence of path.

**Exercise 3.3.10.** Calculate the contour integral of the following functions $f$ along the given contours $\Gamma$:

(i) $f(z) = 3z^2 - 5z + i$, where $\Gamma$ is the straight line segment from $i$ to 1;
(ii) $f(z) = \exp(z)$, where $\Gamma$ is the upper half circle of radius 1, from 1 to $-1$;
(iii) $f(z) = 1/z$, where $\Gamma$ is any contour in the right half-plane from $-3i$ to $3i$; and
(iv) $f(z) = 1/(1+z^2)$, where $\Gamma$ is the straight line segment from 1 to $1+i$.

*Solution.* (i) The function $f$ has an antiderivative $F(z) = z^3 - \frac{5}{2}z^2 + iz$ on the domain $\mathbb{C}$, so by the path-independence lemma,

$$\begin{aligned}
\int_\Gamma f(z)\, dz = F(1) - F(i) = \left(1 - \frac{5}{2} + i\right) - \left(i^3 - \frac{5}{2}i^2 + i^2\right) &= \left(-\frac{3}{2} + i\right) - \left(-i + \frac{3}{2}\right) \\
&= -3 + 2i.
\end{aligned}$$

(ii) The function $f$ has an antiderivative $F(z) = \exp(z)$ on the domain $\mathbb{C}$, so by the path-independence lemma,

$$\int_\Gamma f(z)\,dz = F(-1) - F(1) = \exp(-1) - \exp(1) = e^{-1} - e^{1}.$$

(iii) The function $f$ has an antiderivative $F(z) = \operatorname{Log}(z)$ on the domain $\mathbb{C} \setminus (-\infty, 0]$, so by the path-independence lemma,

$$\begin{aligned}\int_\Gamma f(z)\,dz = F(3i) - F(-3i) &= \operatorname{Log}(3i) - \operatorname{Log}(-3i)\\ &= (\ln|3i| + i\operatorname{Arg}(3i)) - (\ln|-3i| + i\operatorname{Arg}(-3i))\\ &= \left(\ln 3 + i\frac{\pi}{2}\right) - \left(\ln 3 - i\frac{\pi}{2}\right)\\ &= i\pi.\end{aligned}$$

(iv) We can use partial fractions to write the integrand as

$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)} = \frac{i/2}{z+i} - \frac{i/2}{z-i},$$

so our integral becomes

$$\int_\Gamma \frac{1}{1+z^2}\,dz = \int_\Gamma \frac{i/2}{z+i} - \frac{i/2}{z-i}\,dz = \frac{i}{2}\left(\int_\Gamma \frac{1}{z+i}\,dz - \int_\Gamma \frac{1}{z-i}\,dz\right).$$

Considering the two integrals separately, we see that $f_1(z) = \frac{1}{z+i}$ has an antiderivative $F_1(z) = \operatorname{Log}(z+i)$ on the cut plane $D_{-i,-\pi}$, which is a domain that contains the contour, so the path-independence lemma implies that

$$\begin{aligned}\int_\Gamma f_1(z)\,dz = F_1(1+i) - F_1(1) &= \operatorname{Log}((1+i)+i) - \operatorname{Log}(1+i)\\ &= \operatorname{Log}(1+2i) - \operatorname{Log}(1+i).\end{aligned}$$

Similarly, $f_2(z) = \frac{1}{z-i}$ has an antiderivative $F_2(z) = \operatorname{Log}(z-i)$ on the cut plane $D_{-i,-\pi}$, which is a domain that contains the contour, so the path-independence lemma implies that

$$\begin{aligned}\int_\Gamma f_2(z)\,dz = F_2(1+i) - F_2(1) &= \operatorname{Log}((1+i)-i) - \operatorname{Log}(1-i)\\ &= \operatorname{Log}(1) - \operatorname{Log}(1-i).\end{aligned}$$

So

$$\begin{aligned}\int_\Gamma \frac{1}{z^2+1}\,dz &= \frac{i}{2}\left(\int_\Gamma f_1(z)\,dz - \int_\Gamma f_2(z)\,dz\right)\\ &= \frac{i}{2}\left((\operatorname{Log}(1+2i) - \operatorname{Log}(1+i)) - (\operatorname{Log}(1) - \operatorname{Log}(1-i))\right).\end{aligned}$$

To compute this value, we write the terms in the logarithms in exponential form:

$$\begin{aligned}1+2i &= \sqrt{5}e^{i\theta} \quad \text{where } \tan\theta = 2\\ 1-i &= \sqrt{2}e^{-i\pi/4}\\ 1+i &= \sqrt{2}e^{i\pi/4}.\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{Log}(1+2i) &= \ln\left|\sqrt{5}\right| + i\arctan(2) = \frac{1}{2}\ln 5 + i\arctan(2)\\
\operatorname{Log}(1-i) &= \ln\left|\sqrt{2}\right| - i\frac{\pi}{4} = \frac{1}{2}\ln 2 - i\frac{\pi}{4}\\
\operatorname{Log}(1+i) &= \ln\left|\sqrt{2}\right| + i\frac{\pi}{4} = \frac{1}{2}\ln 2 + i\frac{\pi}{4},
\end{aligned}$$

thus, since $\operatorname{Log}(1) = 0$, we have that

$$\begin{aligned}
\int_\Gamma \frac{1}{z^2+1}\,dz &= \frac{i}{2}\left(\operatorname{Log}(1+2i) - \operatorname{Log}(1+i) - \operatorname{Log}(1) + \operatorname{Log}(1-i)\right)\\
&= \frac{i}{2}\left(\left(\frac{1}{2}\ln 5 + i\arctan(2)\right) - \left(\frac{1}{2}\ln 2 + i\frac{\pi}{4}\right) + \left(\frac{1}{2}\ln 2 - i\frac{\pi}{4}\right)\right)\\
&= \frac{\pi}{4} - \frac{1}{2}\arctan(2) + \frac{i}{4}\ln 5.
\end{aligned}$$

**Exercise 3.3.11.** Evaluate the contour integral $\int_\Gamma |z|^2\,dz$ where $\Gamma$ is the square with vertices at $0$, $1$, $1+i$, and $i$, traversed anti-clockwise, starting at $0$.

*Solution.* Let $\Gamma_1$ be the line segment from $0$ to $1$, $\Gamma_2$ be the line segment from $1$ to $1+i$, $\Gamma_3$ be the line segment from $1+i$ to $i$, and $\Gamma_4$ be the line segment from $i$ to $0$. Then by definition

$$\int_\Gamma |z|^2\,dz = \sum_{i=1}^{4}\int_{\Gamma_i}|z|^2\,dz.$$

So we consider each line segment $\Gamma_i$ in turn. The line $\Gamma_1$ is parametrized by $\gamma_1\colon [0,1]\to\mathbb{C}$ given by $\gamma(t) = t$, so

$$\int_{\Gamma_1}|z|^2\,dz = \int_0^1 |\gamma_1(t)|^2\,\gamma_1'(t)\,dt = \int_0^1 |t|^2\,dt = \int_0^1 t^2\,dt = \left.\frac{1}{3}t^3\right|_{t=0}^{1} = \frac{1}{3}.$$

The line $\Gamma_2$ is parametrized by $\gamma_2\colon [0,1]\to\mathbb{C}$ given by $\gamma(t) = 1+it$, so

$$\int_{\Gamma_2}|z|^2\,dz = \int_0^1 |\gamma_2(t)|^2\,\gamma_2'(t)\,dt = \int_0^1 |1+it|^2\,i\,dt = i\int_0^1 1+t^2\,dt = i\left.\left(t+\frac{1}{3}t^3\right)\right|_{t=0}^{1} = i\frac{4}{3}.$$

The line $\Gamma_3$ is parametrized by $\gamma_3\colon [0,1]\to\mathbb{C}$ given by $\gamma(t) = 1-t+i$, so

$$\begin{aligned}
\int_{\Gamma_3}|z|^2\,dz = \int_0^1 |\gamma_3(t)|^2\,\gamma_3'(t)\,dt = \int_0^1 |1-t+i|^2\,(-1)\,dt &= -\int_0^1 (2-2t+t^2)\,dt\\
&= -\left.\left(2t-t^2+\frac{1}{3}t^3\right)\right|_{t=0}^{1}\\
&= -\frac{4}{3}.
\end{aligned}$$

The line $\Gamma_4$ is parametrized by $\gamma_4\colon [0,1]\to\mathbb{C}$ given by $\gamma(t) = (1-t)i$, so

$$\begin{aligned}
\int_{\Gamma_4}|z|^2\,dz = \int_0^1 |\gamma_4(t)|^2\,\gamma_4'(t)\,dt = \int_0^1 |(1-t)i|^2\,(-i)\,dt &= -i\int_0^1 1-2t+t^2\,dt\\
&= -i\left.\left(t-t^2+\frac{1}{3}t^3\right)\right|_{t=0}^{1} = -\frac{i}{3}.
\end{aligned}$$

So

$$\int_\Gamma |z|^2\,dz = \frac{1}{3} + i\frac{4}{3} - \frac{4}{3} - i\frac{1}{3} = -1+i.$$

### 3.4. Cauchy's Integral Theorem.

**Exercise 3.4.13.** Evaluate the contour integral of the function $f(z) = z^2 + 3z$ along the contour $\Gamma$ which joins 2 to $2i$, defined to be:

(i) the arc of the circle of radius 2 centred at 0, traversed anticlockwise;
(ii) the arc of the circle of radius 2 centred at 0, traversed clockwise; and
(iii) the straight line between the two points.

*Solution.* The function $f$ has an antiderivative $F(z) = \frac{z^3}{3} + \frac{3z^2}{2}$ on the domain $\mathbb{C}$, and so by the path-independence lemma, the integral along $\Gamma$ is the same whichever path between the two points 2 and $2i$ we take, thus the value is the same in each case:

$$\begin{aligned}\int_\Gamma f(z)\,dz = F(2i) - F(2) &= \left(\frac{(2i)^3}{3} + \frac{3(2i)^2}{2}\right) - \left(\frac{2^3}{3} + \frac{3\cdot 2^2}{2}\right)\\ &= \left(\frac{-8i}{3} + \frac{-12}{2}\right) - \left(\frac{8}{3} + \frac{12}{2}\right)\\ &= \frac{-44}{3} - \frac{8i}{3}.\end{aligned}$$

**Exercise 3.4.14.** Evaluate the contour integral of the function $f(z) = 1/(z-2)$ along the contour $\Gamma = C_r(z_0)$, where

(i) $r = 4$ and $z_0 = 2$;
(ii) $r = 4$ and $z_0 = 10$; and
(iii) $r = 10$ and $z_0 = 0$.

*Solution.* We apply theorem 3.4.11 repeatedly.

(i) Since the point $z = 2$ lies inside the contour $C_4(2)$ we have that $\int_{C_4(2)} f(z)\,dz = 2\pi i$.
(ii) Since the point $z = 2$ does not lie inside the contour $C_4(10)$ we have that $\int_{C_4(10)} f(z)\,dz = 0$.
(iii) Since the point $z = 2$ lies inside the contour $C_{10}(0)$ we have that $\int_{C_{10}(0)} f(z)\,dz = 2\pi i$.

**Exercise 3.4.15.** What is the value of $\int_\Gamma z^{-n}\,dz$, for $n \neq 1$, where $\Gamma$ is a loop not passing through 0?

*Solution.* For every $n \neq 1$, the function $z^{-n}$ has an antiderivative $F_n(z) = \frac{1}{-n+1} z^{-n+1}$ on the domain $\mathbb{C} \setminus \{0\}$, which by assumption contains the contour $\Gamma$, so by the path-independence lemma, we have that $\int_\Gamma z^{-n}\,dz = 0$ since $\Gamma$ is a closed contour.

**Exercise 3.4.16.** For $a > 1$, define $I = \int_0^{2\pi} \frac{1}{a+\cos t}\,dt$. By considering $z = \exp(it)$, whence $\cos t = (z + \overline{z})/2$, convert $I_a$ to a contour integral, and hence evaluate it.

*Solution.* This takes a little cunning. Observe that if $z = \exp(it)$, then in fact $\cos t = (z + \overline{z})/2 = (z + z^{-1})/2$, so

$$\frac{1}{a + \cos t} = \frac{1}{a + (z + z^{-1})/2} = \frac{2z}{2az + z^2 + 1}.$$

Therefore

$$\begin{aligned}I_a = \int_0^{2\pi} \frac{1}{a+\cos t}\,dt &= \int_0^{2\pi} \frac{2\exp(it)}{2a\exp(it) + (\exp(it))^2 + 1}\,dt\\ &= \frac{1}{i}\int_0^{2\pi} \frac{2(\exp(it))'}{2a\exp(it) + \exp^2(it) + 1}\,dt\\ &= -2i\int_{C_1(0)} \frac{1}{2az + z^2 + 1}\,dz,\end{aligned}$$

since $\exp(it)$ for $t \in [0, 2\pi]$ is exactly a parametrization of the contour $C_1(0)$. So we now just have to evaluate this contour integral.

The integrand is holomorphic except at the zeros of the denominator $z^2 + 2az + 1$. By the quadratic formula these occur at the points

$$\frac{-2a \pm \sqrt{4a^2 - 4}}{2} = -a \pm \sqrt{a^2 - 1},$$

which are both real since $a > 1$. Evidently $-a - \sqrt{a^2 - 1} < -a < -1$, so this root does not lie inside the contour. But then since the product of the two roots is 1 (the constant term in the quadratic), the other root $-a + \sqrt{a^2 - 1}$ must indeed lie inside the contour. Then since the integrand is of the form $f(z)/(z - z_0)$ where $f(z) = 1/(z - (-a - \sqrt{a^2 - 1}))$ is holomorphic inside and on the contour $C_1(0)$, and $z_0 = -a + \sqrt{a^2 - 1}$ lies inside $C_1(0)$, the Cauchy Integral Formula implies that

$$\begin{aligned}\int_{C_1(0)} \frac{1}{z^2 + 2az + 1}\, dz &= \int_{C_1(0)} \frac{1/(z - (-a - \sqrt{a^2 - 1}))}{z - (-a + \sqrt{a^2 - 1})}\, dz \\ &= 2\pi i \frac{1}{(-a + \sqrt{a^2 - 1}) - (-a - \sqrt{a^2 - 1}))} \\ &= 2\pi i \frac{1}{2\sqrt{a^2 - 1}}.\end{aligned}$$

So, going back to our original integral, we have that

$$I_a = \int_0^{2\pi} \frac{1}{a + \cos t}\, dt = -2i \int_{C_1(0)} \frac{1}{2az + z^2 + 1}\, dz = -2i \left( 2\pi i \frac{1}{2\sqrt{a^2 - 1}} \right) = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

### 3.5. Cauchy's Integral Formula.

**Exercise 3.5.12.** Let $\Gamma$ be the contour parametrized by $\gamma\colon [0, 2\pi] \to \mathbb{C}$ where $\gamma(t) = 2\cos t + i \sin 2t$. Sketch $\Gamma$, and evaluate the contour integrals $\int_\Gamma f(z)\, dz$ of the following functions:

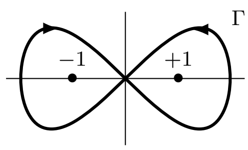
(i) $f(z) = 1/(z - 1)$;
(ii) $f(z) = 1/(z + 1)$;
(iii) $f(z) = 1/(z^2 - 1)$; and
(iv) $f(z) = (2z^2 - z + 1)/(z - 1)^2(z + 1)$.

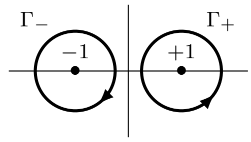
*Solution.* The periodicity of the trigonometric functions guarantees that this is a closed curve starting and ending at $z = 2$. It is clear that as the $x$-coordinate goes from 2 to $-2$ and back to 2 once, the $y$-coordinate goes from 0 to 1 to $-1$ and back to 0 twice. Therefore the resulting curve is a figure-of-eight with a self-intersection at the origin at $t = \pi/2$ and $t = 3\pi/2$, as in the figure below.

The four functions which we are asked to integrate along this contour are holomorphic everywhere except at possibly $z = \pm 1$ (and at only one of these points in two cases). Therefore by Cauchy's Integral Theorem,

$$\int_\Gamma f(z)\, dz = \int_{\Gamma_+} f(z)\, dz + \int_{\Gamma_-} f(z)\, dz$$

for each of the functions $f$, where the contours $\Gamma_+$ and $\Gamma_-$ are the circles indicated: $\Gamma_+$ and $\Gamma_-$ do not intersect, and the points $\pm 1$ lie in the interior of $\Gamma_\pm$. Notice that the contour $\Gamma_-$ is negatively oriented.



(i) So theorem 3.4.10 implies immediately that

$$\int_\Gamma \frac{1}{z-1}\,dz = \int_{\Gamma_+} \frac{1}{z-1}\,dz + \int_{\Gamma_-} \frac{1}{z-1}\,dz = 2\pi i + 0 = 2\pi i.$$

(ii) Similarly

$$\int_\Gamma \frac{1}{z+1}\,dz = \int_{\Gamma_+} \frac{1}{z+1}\,dz + \int_{\Gamma_-} \frac{1}{z+1}\,dz = 0 - 2\pi i = -2\pi i,$$

where the minus sign arises from the negative orientation of $\Gamma_-$.

(iii) Using partial fractions, we can write

$$\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)} = \frac{1/2}{z-1} - \frac{1/2}{z+1},$$

so the previous two calculations imply that

$$\begin{aligned}\int_\Gamma \frac{1}{z^2-1}\,dz = \int_\Gamma \frac{1/2}{z-1} - \frac{1/2}{z+1}\,dz = \frac{1}{2}\left(\int_\Gamma \frac{1}{z-1}\,dz - \int_\Gamma \frac{1}{z+1}\,dz\right) &= \frac{1}{2}(2\pi i + 2\pi i)\\ &= 2\pi i.\end{aligned}$$

(iv) Again we use partial fractions to write

$$\frac{2z^2-z+1}{(z-1)^2(z+1)} = \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{z+1}.$$

Since $1/(z-1)^2$ is of the form $g(z)/(z-z_0)^2$ for $g(z)=1$ and $z_0=1$, the Generalized Cauchy Integral Formula implies that

$$\int_{\Gamma_+} \frac{1}{(z-1)^2}\,dz = 0,$$

and since $1/(z-z_0)^2$ is holomorphic inside and on $\Gamma_-$, the Cauchy Integral Theorem implies that

$$\int_{\Gamma_-} \frac{1}{(z-1)^2}\,dz = 0.$$

Therefore

$$\int_\Gamma \frac{1}{(z-1)^2}\,dz = \int_{\Gamma_+} \frac{1}{(z-1)^2}\,dz + \int_{\Gamma_-} \frac{1}{(z-1)^2}\,dz = 0.$$

Using this and our previous calculations, we have that

$$\int_\Gamma \frac{2z^2-z+1}{(z-1)^2(z+1)}\,dz = \int_\Gamma \frac{1}{z-1}\,dz + \int_\Gamma \frac{1}{(z-1)^2}\,dz + \int_\Gamma \frac{1}{z+1}\,dz = 2\pi i + 0 - 2\pi i = 0.$$

**Exercise 3.5.13.** Evaluate the contour integral

$$\int_{C_3(0)} \frac{\exp(iz)}{(z^2+1)^2}\,dz.$$

*Solution.* We use partial fractions to write

$$\frac{1}{(z^2+1)^2} = \frac{i/4}{z+i} - \frac{1/4}{(z+i)^2} - \frac{i/4}{z-i} - \frac{1/4}{(z-i)^2}.$$

By the (Generalized) Cauchy Integral Formula we have that

$$\begin{aligned}\int_{C_3(0)} \frac{\exp(iz)}{(z^2+1)^2}\,dz &= \frac{i}{4}\int_{C_3(0)} \frac{\exp(iz)}{z+i}\,dz - \frac{1}{4}\int_{C_3(0)} \frac{\exp(iz)}{(z+i)^2}\,dz \\ &\quad - \frac{i}{4}\int_{C_3(0)} \frac{\exp(iz)}{z-i}\,dz - \frac{1}{4}\int_{C_3(0)} \frac{\exp(iz)}{(z-i)^2}\,dz \\ &= \frac{1}{4}\Bigg( i(2\pi i)\exp(-i^2) - 2\pi i \left(\frac{d}{dz}\exp(iz)\right)\Bigg|_{z=-i} \\ &\quad - i(2\pi i)\exp(i^2) - 2\pi i \left(\frac{d}{dz}\exp(iz)\right)\Bigg|_{z=i}\Bigg) \\ &= \frac{1}{4}\left(-2\pi\exp(1) - 2\pi i(i\exp(-i^2)) + 2\pi\exp(-1) - 2\pi i(i\exp(i^2))\right) \\ &= \frac{1}{4}\left(-2\pi e^1 + 2\pi e^1 + 2\pi e^{-1} + 2\pi e^{-1}\right) \\ &= \frac{\pi}{e}.\end{aligned}$$

**Exercise 3.5.14.** Evaluate the contour integral

$$\int_{C_r(z_0)} \frac{z+i}{z^3+2z^2}\,dz,$$

where:

(i) $r = 1$ and $z_0 = 0$;
(ii) $r = 2$ and $z_0 = -2+i$; and
(iii) $r = 1$ and $z_0 = 2i$.

*Solution.* Let

$$f(z) = \frac{z+i}{z^3+2z^2} = \frac{(z+i)}{z^2(z+2)},$$

and notice that $f$ is holomorphic except at the points $z = 0, -2$.

(i) We can write $f(z) = f_1(z)/(z-z_0)^{n+1}$ where $f_1(z) = (z+i)/(z+2)$ is holomorphic inside and on $C_1(0)$, $z_0 = 0$ lies inside $C_1(0)$, and $n = 1$. Therefore the Generalized Cauchy Integral Formula implies that

$$\begin{aligned}\int_{C_1(0)} f(z)\,dz = 2\pi i f_1'(z_0) = 2\pi i \left(\frac{d}{dz}\left(\frac{z+i}{z+2}\right)\right)\Bigg|_{z=0} &= 2\pi i \left(\frac{(z+2)-(z+i)}{(z+2)^2}\right)\Bigg|_{z=0} \\ &= 2\pi i \frac{2-i}{4} \\ &= \frac{\pi}{2} + \pi i.\end{aligned}$$

(ii) We can write $f(z) = f_2(z)/(z-z_0)$ where $f_2(z) = (z+i)/z^2$ is holomorphic inside and on $C_2(-2+i)$, and $z_0 = -2$ lies inside $C_2(-2+i)$. Therefore the Cauchy Integral Formula implies that

$$\int_{C_2(-2+i)} f(z)\,dz = 2\pi i f_2(z_0) = 2\pi i \frac{-2+i}{4} = -\frac{\pi}{2} - i\pi.$$

(iii) Since $f$ is holomorphic inside and on the contour, Cauchy's Integral Theorem implies that

$$\int_{C_1(2i)} f(z)\,dz = 0.$$

**Exercise 3.5.15.** Evaluate the contour integral $\int_{C_2(0)} f(z)\,dz$ where $f$ is defined as:

(i) $f(z) = \sin(3z)/(z - (\pi/2))$;
(ii) $f(z) = z\exp(z)/(2z-3)$;
(iii) $f(z) = \cos(z)/(z^3 + 9z)$;
(iv) $f(z) = (5z^2 + 2z + 1)/(z - i)^3$; and
(v) $f(z) = \exp(-z)/(z+1)^2$.

*Solution.* (i) The function $f$ is of the form $f(z) = g(z)/(z - z_0)$ where $g(z) = \sin(3z)$ is holomorphic inside and on $C_2(0)$, and $z_0 = \pi/2$ lies inside $C_2(0)$. Therefore the Cauchy Integral Formula implies that

$$\int_{C_2(0)} f(z)\,dz = \int_{C_2(0)} \frac{\sin(3z)}{z - (\pi/2)}\,dz = 2\pi i \sin(3\pi/2) = -2\pi i.$$

(ii) The function $f$ is of the form $f(z) = g(z)/(z - z_0)$ where $g(z) = z\exp(z)/2$ is holomorphic inside and on $C_2(0)$, and $z_0 = 3/2$ lies inside $C_2(0)$. Therefore the Cauchy Integral Formula implies that

$$\int_{C_2(0)} f(z)\,dz = \int_{C_2(0)} \frac{z\exp(z)}{2z-3}\,dz = \int_{C_2(0)} \frac{z\exp(z)/2}{z-(3/2)}\,dz = 2\pi i\left(\frac{3}{2}\frac{\exp(3/2)}{2}\right) = \frac{3\pi i e^{3/2}}{2}.$$

(iii) The function $f$ is of the form $f(z) = g(z)/(z - z_0)$ where $g(z) = \cos(z)/(z^2+9)$ is holomorphic inside and on $C_2(0)$, and $z_0 = 0$ lies inside $C_2(0)$. Therefore the Cauchy Integral Formula implies that

$$\int_{C_2(0)} f(z)\,dz = \int_{C_2(0)} \frac{\cos(z)}{z^3+9z}\,dz = \int_{C_2(0)} \frac{\cos(z)/(z^2+9)}{z}\,dz = 2\pi i\frac{\cos(0)}{9} = \frac{2\pi i}{9}.$$

(iv) The function $f$ is of the form $f(z) = g(z)/(z - z_0)^{n+1}$ where $g(z) = 5z^2 + 2z + 1$ is holomorphic inside and on $C_2(0)$, $z_0 = i$ lies inside $C_2(0)$, and $n = 2$. Therefore the Generalized Cauchy Integral Formula implies that

$$\int_{C_2(0)} f(z)\,dz = \int_{C_2(0)} \frac{5z^2+2z+1}{(z-i)^3}\,dz = 2\pi i\frac{1}{2!}\left(\frac{d^2}{dz^2}(5z^2+2z+1)\right)\bigg|_{z=i} = 10\pi i.$$

(v) The function $f$ is of the form $f(z) = g(z)/(z - z_0)^{n+1}$ where $g(z) = \exp(-z)$ is holomorphic inside and on $C_2(0)$, $z_0 = -1$ lies inside $C_2(0)$, and $n = 1$. Therefore the Generalized Cauchy Integral Formula implies that

$$\int_{C_2(0)} f(z)\,dz = \int_{C_2(0)} \frac{\exp(-z)}{(z+1)^2}\,dz = 2\pi i\left(\frac{d}{dz}(\exp(-z))\right)\bigg|_{z=-1} = 2\pi i(-\exp(1)) = -2\pi i e^1.$$

**Exercise 3.5.16.** For each of the following functions, explain why $\int_{C_2(0)} f(z)\,dz = 0$:

(i) $f(z) = z/(z^2 + 25)$;
(ii) $\exp(-z)(2z+1)$;
(iii) $\mathrm{Log}(z+3)$;
(iv) $\cos(z)/(z^2 - 6z + 10)$; and
(v) $\sec(z/2)$.

*Solution.* (i) The function $f$ is holomorphic everywhere except at the points $z = \pm 5i$, both of which lie outside the contour, so $f$ is holomorphic inside and on the loop $C_2(0)$, and Cauchy's Integral Theorem implies that $\int_{C_2(0)} f(z)\,dz = 0$.

(ii) The function $f$ is a product of two holomorphic functions, therefore is itself holomorphic on $\mathbb{C}$, therefore Cauchy's Integral Theorem implies that $\int_{C_2(0)} f(z)\,dz = 0$.

(iii) The function Log is holomorphic on the cut plane $D_{-\pi}$, so the composition $f(z) = \mathrm{Log}(z+3)$ of Log with the function $z \mapsto z+3$ is holomorphic on the cut plane $D_{-3,-\pi}$. Therefore $f$ is holomorphic inside and on the loop $C_2(0)$, so Cauchy's Integral Theorem implies that $\int_{C_2(0)} f(z)\,dz = 0$.

(iv) The function $f$ is holomorphic everywhere except at the points where the denominator is zero, which by the quadratic formula are the points

$$z = \frac{6 \pm \sqrt{36 - 40}}{2} = 3 \pm i,$$

both of which lie outside the contour. Therefore $f$ is holomorphic inside and on the loop $C_2(0)$, so Cauchy's Integral Theorem implies that $\int_{C_2(0)} f(z)\, dz = 0$.

(v) The function sec is holomorphic except at those points where cos is zero. We can solve this equation directly:

$$0 = \cos(z) = \frac{\exp(iz) + \exp(-iz)}{2}$$

implies that $\exp(-iz) = -\exp(iz)$, which implies that $\exp(2iz) = -1$. This implies that

$$2iz \in \log(-1) = \{\ln|-1| + \pi i + 2k\pi i : k \in \mathbb{Z}\} = \{\, \pi i + 2k\pi i : k \in \mathbb{Z}\,\},$$

i.e. that $z = \pi(2k+1)/2$ for an integer $k$. Thus our integrand $f(z) = \sec(z/2)$ is holomorphic except at points $z$ such that $z/2 = \pi(2k+1)/2$, i.e. at $z = \pi(2k+1)$, all of which lie outside our contour. Therefore $f$ is holomorphic inside and on the loop $C_2(0)$, so Cauchy's Integral Theorem implies that $\int_{C_2(0)} f(z)\, dz = 0$.

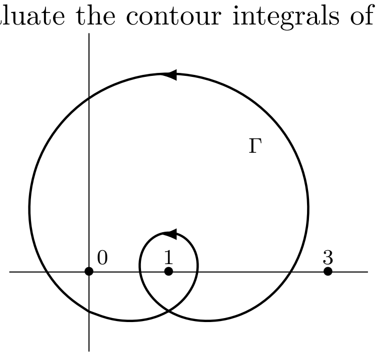
**Exercise 3.5.17.** Evaluate

$$\int_{C_1(0)} \frac{z^2 \exp(z)}{2z + i}\, dz.$$

*Solution.* The function $f$ is of the form $f(z) = g(z)/(z - z_0)$ where $g(z) = z^2 \exp(z)/2$ is holomorphic inside and on the contour $C_1(0)$, and $z_0 = -i/2$ lies inside $C_1(0)$. Therefore the Cauchy Integral Formula implies that

$$\begin{aligned}\int_{C_1(0)} \frac{z^2 \exp(z)}{2z + i}\, dz = \int_{C_1(0)} \frac{z^2 \exp(z)/2}{(z - (-i/2))}\, dz &= 2\pi i \frac{(-i/2)^2 \exp(-i/2)}{2} \\ &= -\frac{\pi i(\cos(-1/2) + i\sin(-1/2))}{4} \\ &= -\frac{\pi(\sin(1/2) + i\cos(1/2))}{4}.\end{aligned}$$

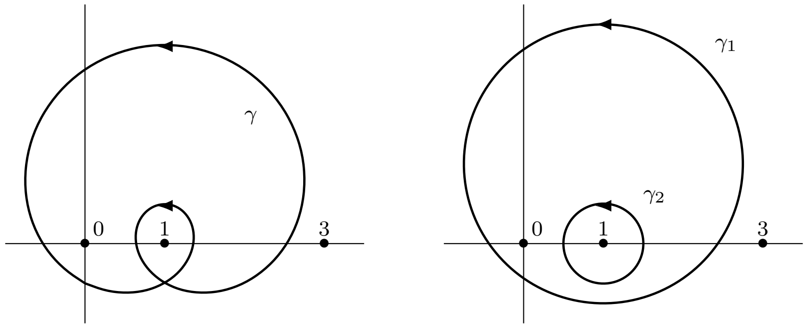
**Exercise 3.5.18.** Evaluate the contour integrals of the following functions, where $\Gamma$ is




the contour depicted:

(i) $f(z) = \cos(z)/z^2(z-1)$; and
(ii) $f(z) = \cos(z)/z^2(z-3)$.

*Solution.* Since the two integrands only fail to be holomorphic at (at most) the points $z = 0, 1, 3$, we can deform the contour given into the two non-intersecting contours indicated below, without changing the value of the integral, by Cauchy's Integral Theorem:




(i) The integrand $f(z) = \cos(z)/z^2(z-1)$ is non-holomorphic at the points $z = 0, 1$, both of which lie inside the contour $\gamma_1$, so we have to split it into partial fractions:

$$\frac{\cos(z)}{z^2(z-1)} = \cos(z)\left(\frac{1}{z-1} - \frac{1}{z^2} - \frac{1}{z}\right).$$

Then we use the (Generalized) Cauchy Integral Formula to see that

$$\begin{aligned}\int_{\gamma_1} \frac{\cos(z)}{z^2(z-1)}\,dz &= \int_{\gamma_1} \cos(z)\left(\frac{1}{z-1} - \frac{1}{z^2} - \frac{1}{z}\right) dz \\ &= \int_{\gamma_1} \frac{\cos(z)}{z-1}\,dz - \int_{\gamma_1} \frac{\cos(z)}{z^2}\,dz - \int_{\gamma_1} \frac{\cos(z)}{z}\,dz \\ &= 2\pi i\left(\cos(1) - \left(\frac{d}{dz}\cos(z)\right)\Big|_{z=0} - \cos(0)\right) \\ &= 2\pi i\left(\cos(1) + \sin(0) - 1\right) \\ &= 2\pi i(\cos(1) - 1).\end{aligned}$$

The other contour is easier, since only one point of non-holomorphicity, $z = 1$, lies inside it: the Cauchy Integral Formula implies that

$$\begin{aligned}\int_{\gamma_2} \frac{\cos(z)}{z^2(z-1)}\,dz &= \int_{\gamma_2} \cos(z)\left(\frac{1}{z-1} - \frac{1}{z^2} - \frac{1}{z}\right) dz \\ &= \int_{\gamma_2} \frac{\cos(z)}{z-1}\,dz - \int_{\gamma_1} \frac{\cos(z)}{z^2}\,dz - \int_{\gamma_1} \frac{\cos(z)}{z}\,dz \\ &= 2\pi i\cos(1) - 0 - 0 \\ &= 2\pi i\cos(1).\end{aligned}$$

Hence

$$\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = 2\pi i(\cos(1) - 1 + \cos(1)) = 2\pi i(2\cos(1) - 1).$$

(ii) The integrand $f(z) = \cos(z)/z^2(z-3)$ is non-holomorphic at the points $z = 0, 3$, of which only $z = 0$ lies inside either contour, and in fact this lies inside only $\gamma_1$. So $f$ is holomorphic inside and on $\gamma_2$, so Cauchy's Integral Theorem implies that

$\int_{\gamma_2} f(z)\,dz = 0$. So the Generalized Cauchy Integral Formula implies that

$$\begin{aligned}\int_\Gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz &= \int_{\gamma_1} \frac{\cos(z)/(z-3)}{z^2}\,dz \\ &= 2\pi i \left(\frac{d}{dz}\left(\frac{\cos(z)}{z-3}\right)\right)\bigg|_{z=0} \\ &= 2\pi i \left(\frac{-(z-3)\sin(z)-\cos(z)}{(z-3)^2}\right)\bigg|_{z=0} \\ &= -\frac{2\pi i}{9}.\end{aligned}$$

### 3.6. Liouville's Theorem and its applications.

**Exercise 3.6.4.** Prove the missing assertion from the proof of the Fundamental Theorem of Algebra: that for a polynomial $P$ of degree $N$, there exists $R > 0$ such that

$$|P(z)| \geq \frac{1}{2}|z|^N \text{ whenever } |z| \geq R.$$

*Proof.* Write $P(z) = z^N + \sum_{n=0}^{N-1} a_n z^n$ for $a_0, \ldots, a_{N-1} \in \mathbb{C}$, and use the reverse triangle inequality and the triangle inequality to see that

$$\begin{aligned}|P(z)| = \left|z^N + \sum_{n=0}^{N-1} a_n z^n\right| = \left|z^N - \left(-\sum_{n=0}^{N-1} a_n z^n\right)\right| &\geq \left|\left|z^N\right| - \left|-\sum_{n=0}^{N-1} a_n z^n\right|\right| \\ &\geq \left|z^N\right| - \left|\sum_{n=0}^{N-1} a_n z^n\right| \\ &\geq |z|^N - \sum_{n=0}^{N-1} |a_n|\,|z|^n \\ &= |z|^N \left(1 - \sum_{n=0}^{N-1} |a_n|\,|z|^{n-N}\right).\end{aligned}$$

Since for each $n = 0, \ldots, N-1$, the exponent $n - N$ is a negative integer, each term $|z|^{n-N} \to 0$ as $|z| \to \infty$. Thus by the algebra of limits,

$$\sum_{n=0}^{N-1} |a_n|\,|z|^{n-N} \to 0 \text{ as } |z| \to \infty.$$

Hence there exists $R > 0$ such that $\sum_{n=0}^{N-1} |a_n|\,|z|^{n-N} < 1/2$ for $|z| \geq R$. Then for $|z| \geq R$, we have that

$$|P(z)| \geq |z|^N \left(1 - \sum_{n=0}^{N-1} |a_n|\,|z|^{n-N}\right) > |z|^N \left(1 - \frac{1}{2}\right) = \frac{1}{2}\,|z|^N. \qquad \square$$

### 3.7. The Maximum Modulus Principle.

**Theorem 3.7.1.** Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$ and $R > 0$ be such that the closed disc $\overline{D}_R(z_0) \subseteq D$, and $f$ be holomorphic on $D$. Then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})\,dt.$$

*Proof.* We just use the Cauchy Integral Formula with the contour $C_R(z_0)$, and compute the integral explicitly using the parametrization $\gamma\colon [0, 2\pi] \to \mathbb{C}$ defined by $\gamma(t) = z_0 + R\exp(it)$:

$$\begin{aligned} f(z_0) = \frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(z)}{z - z_0}\,dz &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - z_0}\gamma'(t)\,dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + R\exp(it))}{z_0 + R\exp(it) - z_0} Ri\exp(it)\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + R\exp(it))\,dt. \end{aligned}$$

□

**Exercise 3.7.4.** Let $D \subseteq \mathbb{C}$ be a domain, and $f$ be holomorphic on $D$, such that $|f(z)|$ is constant on $D$. Show, using the Cauchy–Riemann equations, or otherwise, that $f$ is constant on $D$.

*Solution.* Writing $f = u + iv$ for $u, v\colon D \to \mathbb{R}$, we have by assumption that $u^2 + v^2 = |f(z)|^2$ is constant. Differentiating this, the chain rule then implies that

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0, \quad \text{and} \quad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0,$$

hence

$$u\frac{\partial u}{\partial x} = -v\frac{\partial v}{\partial x} \quad \text{and} \quad u\frac{\partial u}{\partial y} = -v\frac{\partial v}{\partial y}.$$

Since $f$ is holomorphic, the Cauchy–Riemann equations hold everywhere on $D$, thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Then we have that

$$v\frac{\partial u}{\partial y} = -v\frac{\partial v}{\partial x} = u\frac{\partial u}{\partial x}, \quad \text{and} \quad u\frac{\partial u}{\partial y} = -v\frac{\partial v}{\partial y} = -v\frac{\partial u}{\partial x}.$$

Multiplying the first by $u$ and the second by $v$ we have that

$$uv\frac{\partial u}{\partial y} = u^2\frac{\partial u}{\partial x}, \quad \text{and} \quad uv\frac{\partial u}{\partial y} = -v^2\frac{\partial u}{\partial x},$$

from which we infer that

$$0 = \left(u^2\frac{\partial u}{\partial x}\right) - \left(-v^2\frac{\partial u}{\partial x}\right) = \left(u^2 + v^2\right)\frac{\partial u}{\partial x}.$$

This implies either that $|f|^2 = u^2 + v^2 = 0$, in which case $f = 0$, and in particular is constant, or that $\frac{\partial u}{\partial x} = 0$. Then by the Cauchy–Riemann equations and the above relationships, the other three partial derivatives are 0, which, since $D$ is a domain, implies that $u$ and $v$, and hence $f$, are constant.

## 4. Series expansions for holomorphic functions

### 4.1. Infinite series.

**Lemma 4.1.2.** Let $\sum_{j=0}^{\infty} z_j$ be a convergent series. Then $z_n \to 0$ as $n \to \infty$.

*Proof.* Let $\varepsilon > 0$. By definition the sequence of partial sums $S_n = \sum_{j=0}^{n} z_j$ is a convergent sequence. So by lemma 1.2.12 $S_n$ is a Cauchy sequence. So by definition there exists $N \in \mathbb{N}$ such that $|S_n - S_m| < \varepsilon$ whenever $n, m \geq N$. In particular, $n \geq N + 1$ implies that

$$|z_n| = \left|\sum_{j=0}^{n} z_j - \sum_{j=0}^{n-1} z_j\right| = |S_n - S_{n-1}| < \varepsilon.$$

So $z_n \to 0$ as $n \to \infty$. □

**Lemma 4.1.6** (The Comparison Test)**.** Let $z_n \in \mathbb{C}$ be a sequence such that $|z_n| \leq M_n$ for some non-negative real numbers $M_n$, for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$, where $\sum_{j=0}^{\infty} M_j$ is a convergent series. Then $\sum_{j=0}^{\infty} z_j$ is a convergent series.

*Proof.* Let $S_n = \sum_{j=0}^{n} z_j$ be the usual partial sums. We claim that $S_n$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\sum_{j=0}^{\infty} M_j$ converges, by definition the sequence of partial sums of this series converges. So there exists $N_1 \in \mathbb{N}$ such that (recalling that $M_n \geq 0$ for all $n$)

$$\sum_{j=n+1}^{\infty} M_j = \left| \sum_{j=0}^{\infty} M_j - \sum_{j=0}^{n} M_j \right| < \varepsilon \text{ whenever } n \geq N_1.$$

Then for $n, m \geq N := \max\{N_1, n_0\}$, without loss of generality $m > n$, we have, by repeated application of the triangle inequality, that

$$|S_m - S_n| = \left| \sum_{j=0}^{m} z_j - \sum_{j=0}^{n} z_j \right| = \left| \sum_{j=n+1}^{m} z_j \right| \leq \sum_{j=n+1}^{m} |z_j| \leq \sum_{j=n+1}^{m} M_j \leq \sum_{j=n+1}^{\infty} M_j < \varepsilon.$$

Thus the sequence $S_n$ is Cauchy, hence by lemma 1.2.12 $S_n$ converges, i.e. $\sum_{j=0}^{\infty} z_j$ converges. □

**Lemma 4.1.9** (The Ratio Test)**.** Let $z_n \in \mathbb{C}$ be a sequence, and suppose that

$$\lim_{n\to\infty} \left| \frac{z_{n+1}}{z_n} \right| = L.$$

Then

(i) if $L < 1$, the series $\sum_{j=0}^{\infty} z_j$ is convergent; and
(ii) if $L > 1$, the series $\sum_{j=0}^{\infty} z_j$ is divergent.

If $L = 1$, then we can conclude nothing, since the series might be convergent or divergent.

*Proof.* (i) Suppose $L < 1$. By applying the definition of convergence with $\varepsilon = \frac{1-L}{2} > 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \left| \frac{z_{n+1}}{z_n} \right| - L \right| < \frac{1-L}{2} \text{ whenever } n \geq N,$$

in particular

$$\left| \frac{z_{n+1}}{z_n} \right| < L + \frac{1-L}{2} = \frac{L+1}{2}$$

for $n \geq N$. So $|z_{n+1}| < \frac{L+1}{2} |z_n|$ for all $n \geq N$. By an easy induction,

$$|z_{N+k}| < \left( \frac{L+1}{2} \right)^k |z_N|,$$

for all $k \geq 1$. That is, $|z_n| \leq \left(\frac{L+1}{2}\right)^{n-N} |z_N|$ for all $n \geq N$. Since $0 < L < 1$, we have that $\left|\frac{L+1}{2}\right| < 1$, and so the geometric series

$$\sum_{j=0}^{\infty} \left( \left( \frac{L+1}{2} \right)^{j-N} |z_N| \right) = |z_N| \left( \frac{L+1}{2} \right)^{-N} \sum_{j=0}^{\infty} \left( \frac{L+1}{2} \right)^j$$

converges. Therefore the Comparison Test implies that $\sum_{j=0}^{\infty} z_j$ converges.

(ii) Suppose $L > 1$. Note that in particular, the ratio $z_{n+1}/z_n$ is well-defined for sufficiently large $n$, so there exists $N_1$ such that $z_n \neq 0$ for $n \geq N_1$. Applying the definition of convergence with $\varepsilon = \frac{L-1}{2} > 0$, there exists $N_2 \in \mathbb{N}$ such that

$$\left|\left|\frac{z_{n+1}}{z_n}\right| - L\right| < \frac{L-1}{2} \text{ whenever } n \geq N_2,$$

in particular

$$\left|\frac{z_{n+1}}{z_n}\right| > L - \frac{L-1}{2} = \frac{L+1}{2}$$

for $n \geq N_2$. So $|z_{n+1}| > \frac{L+1}{2}|z_n|$ for all $n \geq N := \max\{N_1, N_2\}$. By a similar easy induction,

$$|z_{N+k}| > \left(\frac{L+1}{2}\right)^k |z_N| > 0,$$

for all $k \geq 1$, so $|z_n| > |z_N| > 0$ for all $n \geq N$, and so in particular $z_n$ does not tend to 0 as $n \to \infty$, so $\sum_{j=0}^{\infty} z_j$ does not converge, by (the contrapositive of) lemma 4.1.2.

To see that no conclusion can be made when $L = 1$, consider the two series $\sum_{j=1}^{\infty} \frac{1}{j}$ and $\sum_{j=1}^{\infty} \frac{1}{j^2}$. Then the first series diverges, whereas the second series converges. However, the ratio of successive terms of both series tends to 1:

$$\frac{1/(n+1)}{1/n} = \frac{n}{n+1} = 1 - \frac{1}{n+1} \to 1, \text{ and}$$
$$\frac{1/(n+1)^2}{1/n^2} = \frac{n^2}{(n+1)^2} = \left(\frac{n}{n+1}\right)^2 = \left(1 - \frac{1}{n+1}\right)^2 \to 1$$

as $n \to \infty$. □

**Exercise 4.1.10.** Determine whether the series $\sum_{j=1}^{\infty} z_j$ converges (we index from $j = 1$ to ensure all the terms are well-defined), where the sequence $z_n$ is defined by:

(i) $z_n = 1/(n^4 2^n)$;
(ii) $z_n = (n^2 2^n)/(2n^2 + 2n + 1)$;
(iii) $z_n = (n!)/(n^n)$;
(iv) $z_n = (-1)^n n^2/(3 + 2i)^n$;
(v) $z_n = ((1 + 2i)/(1 - i))^n$;
(vi) $z_n = (ni^n)/(2n + 1)$; and
(vii) $z_n = (n!)/(5n)$.

*Solution.* (i) For each $n$ we have that

$$|z_n| = \left|\frac{1}{n^4 2^n}\right| \leq \left|\frac{1}{2^n}\right| = \left(\frac{1}{2}\right)^n,$$

where the geometric series $\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j$ converges since $|1/2| < 1$, so by the Comparison Test, $\sum_{j=1}^{\infty} z_j$ converges.

(ii) We have that

$$z_n = \frac{n^2 2^n}{2n^2 + 2n + 1} = \frac{n^2}{n^2} \cdot \frac{2^n}{2 + 2n^{-1} + n^{-2}} = \frac{2^n}{2 + 2n^{-1} + n^{-2}} \to \infty \text{ as } n \to \infty,$$

so by the (contrapositive of) lemma 4.1.2, the series $\sum_{j=1}^{\infty} z_j$ does not converge.

(iii) For $n \geq 2$ we have that

$$|z_n| = \frac{n!}{n^n} = \frac{n(n-1)\cdots 2 \cdot 1}{n \cdot n \cdots n \cdot n} = \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{2}{n}\right)\left(\frac{1}{n}\right) \leq 1 \cdot 1 \cdots \frac{2}{n} \cdot \frac{1}{n} = \frac{2}{n^2},$$

where the series $\sum_{j=1}^{\infty} 2j^{-2}$ converges, so by the Comparison Test, the series $\sum_{j=1}^{\infty} z_j$ converges.

(iv) We have that

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{(-1)^{n+1}(n+1)^2/(3+2i)^{n+1}}{(-1)^n n^2/(3+2i)^n}\right| = \left|\frac{(-1)(n+1)^2}{n^2(3+2i)}\right| = \frac{1}{|3+2i|}\left(\frac{n+1}{n}\right)^2$$
$$= \frac{1}{\sqrt{13}}\left(1+\frac{1}{n}\right)^2$$
$$\to \frac{1}{\sqrt{13}} \text{ as } n \to \infty.$$

Since $\frac{1}{\sqrt{13}} < 1$, the Ratio Test implies that the series $\sum_{j=1}^{\infty} z_j$ converges.

(v) We see that

$$\left|\frac{1+2i}{1-i}\right| = \frac{|1+2i|}{|1-i|} = \frac{\sqrt{5}}{\sqrt{2}} > 1,$$

so since $z_n = \left(\frac{1+2i}{1-i}\right)^n$, we have that $z_n$ do not converge to 0 as $n \to \infty$, so by (the contrapositive of) lemma 4.1.2, the series $\sum_{j=1}^{\infty} z_j$ does not converge.

(vi) We have that

$$z_n = \frac{ni^n}{2n+1} = \frac{n}{n} \cdot \frac{i^n}{2+n^{-1}},$$

so

$$z_{4k} = \frac{1}{2+(4k)^{-1}} \to \frac{1}{2} \text{ as } k \to \infty,$$

and therefore in particular $z_n$ does not converge to 0 as $n \to \infty$. So by (the contrapositive of) lemma 4.1.2, the series $\sum_{j=1}^{\infty} z_j$ does not converge.

(vii) We have for $n \geq 3$ that

$$z_n = \frac{n!}{5n} = \frac{n(n-1)\cdots 2\cdot 1}{5n} = \frac{(n-1)\cdots 2\cdot 1}{5} \geq \frac{2}{5},$$

so $z_n$ does not converge to 0 as $n \to \infty$. So by (the contrapositive of) lemma 4.1.2, the series $\sum_{j=1}^{\infty} z_j$ does not converge.

**Exercise 4.1.11.** Evaluate the following convergent series $\sum_{j=0}^{\infty} z_j$, where the sequence $z_n$ is defined by:

(i) $z_n = (i/3)^n$;
(ii) $z_n = 3/(1+i)^n$;
(iii) $z_n = (1/2i)^n$;
(iv) $z_n = (1/3)^{2n}$; and
(v) $z_n = (-1)^n(2/3)^n$.

*Solution.* These are all geometric series. So we repeatedly use the fact that for $|c| < 1$, we have $\sum_{j=0}^{\infty} c^j = \frac{1}{1-c}$. We simply identify the value of $c$ in each case.

(i) $z_n = (i/3)^n = c^n$ where $c = i/3$, so

$$\sum_{j=0}^{\infty} z_j = \sum_{j=0}^{\infty}\left(\frac{i}{3}\right)^j = \frac{1}{1-(i/3)} = \frac{3}{3-i} = \frac{9+3i}{10}.$$

(ii) $z_n = 3/(1+i)^n = 3c^n$ where $c = 1/(1+i)$, so

$$\sum_{j=0}^{\infty} z_j = 3\sum_{j=0}^{\infty}\left(\frac{1}{1+i}\right)^j = \frac{3}{1-(1/(1+i))} = \frac{3(1+i)}{1+i-1} = -i(3+3i) = 3(1-i).$$

(iii) $z_n = (1/2i)^n = c^n$ where $c = (1/2i)$, so
$$\sum_{j=0}^{\infty} z_j = \sum_{j=0}^{\infty} \left(\frac{1}{2i}\right)^j = \frac{1}{1-(1/2i)} = \frac{2i}{2i-1} = \frac{2i(-1-2i)}{5} = \frac{4-2i}{5}.$$

(iv) $z_n = (1/3)^{2n} = c^n$ where $c = (1/3)^2 = 1/9$, so
$$\sum_{j=0}^{\infty} z_j = \sum_{j=0}^{\infty} \left(\frac{1}{9}\right)^j = \frac{1}{1-(1/9)} = \frac{9}{8}.$$

(v) $z_n = (-1)^n(2/3)^n = (-2/3)^n = c^n$ where $c = -2/3$, so
$$\sum_{j=0}^{\infty} z_j = \sum_{j=0}^{\infty} \left(\frac{-2}{3}\right)^j = \frac{1}{1-(-2/3)} = \frac{3}{5}.$$

**Lemma 4.1.17.** Let $S \subseteq \mathbb{C}$, and suppose $f_n\colon S \to \mathbb{C}$ is a sequence of continuous functions, and that $f_n$ converges uniformly to a function $f\colon S \to \mathbb{C}$. Then $f$ is continuous.

*Proof.* Let $z_0 \in S$, and $\varepsilon > 0$. By definition of uniform convergence, there exists $N \in \mathbb{N}$ such that for all $z \in S$,
$$|f_n(z) - f(z)| < \frac{\varepsilon}{3} \text{ whenever } n \geq N.$$
The function $f_N$ is continuous at $z_0$, so by definition there exists $\delta > 0$ such that
$$|f_N(z) - f_N(z_0)| < \frac{\varepsilon}{3} \text{ whenever } z \in S \text{ and } |z - z_0| < \delta.$$
Consider $z \in S$ such that $|z - z_0| < \delta$. Then by the triangle inequality, we have that
$$\begin{aligned}|f(z) - f(z_0)| &= |(f(z) - f_N(z)) + (f_N(z) - f_N(z_0)) + (f_N(z_0) - f(z_0))| \\ &\leq |f(z) - f_N(z)| + |f_N(z) - f_N(z_0)| + |f_N(z_0) - f(z_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$
□

### 4.2. Power series.

**Theorem 4.2.4.** Let $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ be a power series, and suppose that the sequence $\left|\frac{a_j}{a_{j+1}}\right|$ has a limit. Then the radius of convergence $R$ is equal to this limit.

*Proof.* Let $\lim_{j\to\infty} \left|\frac{a_j}{a_{j+1}}\right| = L$. Suppose first that $L$ is a finite non-zero number. Then for all $z \neq z_0$,
$$\left|\frac{a_{j+1}(z-z_0)^{j+1}}{a_j(z-z_0)^j}\right| = \left|\frac{a_{j+1}}{a_j}\right| |z - z_0| \to \frac{1}{L}|z - z_0| \text{ as } j \to \infty.$$
Hence by the Ratio Test, the series converges if $|z - z_0| < L$, and diverges if $|z - z_0| > L$. Then by definition, $L$ is the radius of convergence.

Suppose $L = 0$, and fix $z \neq z_0$. Note in particular that the ratio $a_n/a_{n+1}$ must be well-defined for all sufficiently large $n$, so there exists $N_1 \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq N_1$. There exists $N_2 \in \mathbb{N}$ such that
$$\left|\frac{a_j}{a_{j+1}}\right| < |z - z_0| \text{ whenever } j \geq N_2,$$
and so by an easy induction, letting $N := \max\{N_1, N_2\}$,
$$|a_{N+k}|\,|z - z_0|^k > |a_N| > 0$$

for all $k$, which implies that

$$\left|a_{N+k}(z-z_0)^{N+k}\right| > \left|a_N(z-z_0)^N\right|$$

for all $k$. That is, $|a_n(z-z_0)^n| \geq \left|a_N(z-z_0)^N\right| > 0$ for all $n \geq N$. This implies that the terms $a_j(z-z_0)^j$ of the series do not converge to 0, so the series does not converge. Since $z \neq z_0$ was arbitrary, the power series converges for no point except $z_0$, thus the radius of convergence is $0 = L$.

Finally, suppose that $L = \infty$. Then we see that $\left|\frac{a_{j+1}}{a_j}\right| \to 0$ as $j \to \infty$, so for any $z \neq z_0$, we have that

$$\left|\frac{a_{j+1}(z-z_0)^{j+1}}{a_j(z-z_0)^j}\right| = \left|\frac{a_{j+1}}{a_j}\right| |z-z_0| \to 0 \text{ as } j \to \infty.$$

Hence the series converges at $z$, by the Ratio Test. So the series converges at every $z$, so the radius of convergence is $\infty = L$. ☐

**Exercise 4.2.7.** Find the radius of convergence and the disc of convergence of the power series $\sum_{j=0}^{\infty} a_j(z-z_0)^j$, where the sequence $a_n$ and $z_0 \in \mathbb{C}$ are given by:

(i) $a_n = n$, $z_0 = 0$;
(ii) $a_n = 2^{-n}$, $z_0 = i$;
(iii) $a_n = 1/(n!)$, $z_0 = 0$;
(iv) $a_n = n^3$, $z_0 = 0$;
(v) $a_n = 2^n$, $z_0 = 1$;
(vi) $a_n = (-1)^n n/3^n$, $z_0 = i$; and
(vii) $a_n = n!$, $z_0 = 0$.

*Solution.* (i) We see that

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{n}{n+1}\right| = \frac{1}{1+n^{-1}} \to 1 \text{ as } n \to \infty,$$

so theorem 4.2.4 implies that the radius of convergence is 1, and the power series converges on the disc $D_1(0)$.

(ii) We see that

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{2^{-n}}{2^{-(n+1)}}\right| = 2,$$

so theorem 4.2.4 implies that the radius of convergence is 2, and the power series converges on the disc $D_2(i)$.

(iii) We see that

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{1/(n!)}{1/(n+1)!}\right| = \frac{(n+1)!}{n!} = n+1 \to \infty \text{ as } n \to \infty.$$

Then theorem 4.2.4 implies that the radius of convergence is $\infty$, and the power series converges on $\mathbb{C}$.

(iv) We see that

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{n^3}{(n+1)^3}\right| = \left(\frac{n}{n+1}\right)^3 = \left(\frac{1}{1+n^{-1}}\right)^3 \to 1 \text{ as } n \to \infty,$$

so theorem 4.2.4 implies that the radius of convergence is 1, and the power series converges on the disc $D_1(0)$.

(v) We see that

$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{2^n}{2^{n+1}}\right| = \frac{1}{2},$$

so theorem 4.2.4 implies that the radius of convergence is $1/2$, and the power series converges on the disc $D_{1/2}(1)$.

(vi) We see that
$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{(-1)^n n/3^n}{(-1)^{n+1}(n+1)/3^{n+1}}\right| = \frac{3n}{n+1} = \frac{3}{1+n^{-1}} \to 3 \text{ as } n \to \infty,$$
so theorem 4.2.4 implies that the radius of convergence is 3, and the power series converges on the disc $D_3(i)$.

(vii) We see that
$$\left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{n!}{(n+1)!}\right| = \frac{1}{n+1} \to 0 \text{ as } n \to \infty,$$
so theorem 4.2.4 implies that the radius of convergence is 0, and the power series converges only at the point $z_0 = 0$.

**Exercise 4.2.8.** Determine the radius of convergence and the disc of convergence of the power series
(i) $\sum_{j=0}^{\infty}(z+5i)^{2j}(j+1)^2$; and
(ii) $\sum_{j=0}^{\infty} z^{2j}/(4^j)$.

(i) We have to be careful with this series because only the even coefficients of powers of $(z+5i)$ are non-zero. So we cannot immediately consider the ratio of consecutive coefficients, since this in general will not be well-defined. Consider however defining $w = (z+5i)^2$. Then the series under consideration is the series
$$\sum_{j=0}^{\infty}(j+1)^2 w^j.$$
We then see that
$$\left|\frac{(j+1)^2}{(j+1+1)^2}\right| = \left(\frac{j+1}{j+2}\right)^2 = \left(\frac{1+j^{-1}}{1+2j^{-1}}\right)^2 \to 1 \text{ as } j \to \infty,$$
so theorem 4.2.4 implies that the radius of convergence of the series $\sum_{j=0}^{\infty}(j+1)^2 w^j$ is 1, and it converges for $w = (z+5i)^2$ in the disc $D_1(0)$. Hence the radius of convergence of the original series $\sum_{j=0}^{\infty}(z+5i)^{2j}(j+1)^2$ is 1, and the disc of convergence is $D_1(-5i)$.

(ii) This series can be written as $\sum_{j=0}^{\infty}\frac{(z^2)^j}{4^j} = \sum_{j=0}^{\infty}\left(\frac{z^2}{4}\right)^j$, which is a geometric series in $z^2/4$. Thus the series $\sum_{j=0}^{\infty}\frac{(z^2)^j}{4^j}$ converges if and only if $\left|z^2/4\right| < 1$, i.e. the radius of convergence is 2 and the disc of convergence is $D_2(0)$.

## 4.3. Taylor series.

**Exercise 4.3.8.** Verify the following familiar Taylor series expansions, centred at 0:
$$\exp(z) = \sum_{j=0}^{\infty}\frac{z^j}{j!};$$
$$\cos(z) = \sum_{j=0}^{\infty}(-1)^j\frac{z^{2j}}{(2j)!}; \text{ and}$$
$$\sin(z) = \sum_{j=0}^{\infty}(-1)^j\frac{z^{2j+1}}{(2j+1)!}.$$

*Solution.* The Taylor coefficients for exp centred at 0 are given by
$$\frac{\exp^{(n)}(0)}{n!} = \frac{\exp(0)}{n!} = \frac{1}{n!},$$

and so the Taylor expansion of exp centred at 0 is given by

$$\sum_{j=0}^{\infty} \frac{1}{j!} z^j.$$

Since exp is holomorphic on $\mathbb{C}$, this series converges to $\exp(z)$ for all $z \in \mathbb{C}$.

Similarly the Taylor coefficients for cos centred at 0 are given by

$$\frac{\cos^{(n)}(0)}{n!} = \begin{cases} \frac{(-1)^{n/2}\cos(0)}{n!} = \frac{(-1)^{n/2}}{n!} & n = 2j, \\ \frac{(-1)^{(n+1)/2}\sin(0)}{n!} = 0 & n = 2j+1, \end{cases}$$

so the Taylor expansion of cos centred at 0 is given by

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} z^{2j}.$$

Since cos is holomorphic on $\mathbb{C}$, this series converges to $\cos(z)$ for all $z \in \mathbb{C}$.

Similarly the Taylor coefficients for sin centred at 0 are given by

$$\frac{\sin^{(n)}(0)}{n!} = \begin{cases} \frac{(-1)^{n/2}\sin(0)}{n!} = 0 & n = 2j, \\ \frac{(-1)^{(n-1)/2}\cos(0)}{n!} = \frac{(-1)^{(n-1)/2}}{n!} & n = 2j+1, \end{cases}$$

so the Taylor expansion of sin centred at 0 is given by

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j+1}.$$

Since sin is holomorphic on $\mathbb{C}$, this series converges to $\sin(z)$ for all $z \in \mathbb{C}$.

**Lemma 4.3.10.** Let $z_0 \in \mathbb{C}$, $R > 0$, $\alpha, \beta \in \mathbb{C}$, and $f, g$ be holomorphic on $D_R(z_0)$. Then

(i) the Taylor series for $\alpha f + \beta g$ centred at $z_0$, valid on $D_R(z_0)$, is the series

$$\sum_{j=0}^{\infty} \frac{\alpha f^{(j)}(z_0) + \beta g^{(j)}(z_0)}{j!}(z - z_0)^j;$$

and

(ii) the Taylor series for $fg$ centred at $z_0$, valid on $D_R(z_0)$, is the series

$$\sum_{j=0}^{\infty} \frac{1}{j!} \left( \sum_{k=0}^{j} \binom{j}{k} f^{(k)}(z_0) g^{(j-k)}(z_0) \right) (z - z_0)^j.$$

*Proof.* (i) We know that the function $\alpha f + \beta g$ is holomorphic on $D_R(z_0)$, by lemma 1.4.3. Therefore the Taylor series for $\alpha f + \beta g$ centred at $z_0$ exists and converges to the function on $D_R(z_0)$. We use the usual addition rule for derivatives to see that the coefficients of this Taylor series are indeed

$$\frac{(\alpha f + \beta g)^{(j)}(z_0)}{j!} = \frac{\alpha f^{(j)}(z_0) + \beta g^{(j)}(z_0)}{j!}.$$

(ii) Similarly we know that the function $fg$ is holomorphic on $D_R(z_0)$, and therefore the Taylor series for $fg$ centred at $z_0$ exists and converges to $fg$ on $D_R(z_0)$. We use the generalized Leibniz formula to see that the coefficients of this Taylor series are indeed

$$\frac{(fg)^{(j)}(z_0)}{j!} = \frac{1}{j!} \sum_{k=0}^{j} \binom{j}{k} f^{(k)}(z_0) g^{(j-k)}(z_0).$$

□

**Exercise 4.3.16.** Find the Taylor series centred at the point $z_0$ of each of the following functions $f$, and determine the set on which the expansion is valid:

(i) $f(z) = \cosh(z)$, $z_0 = 0$;
(ii) $f(z) = \sinh(z)$, $z_0 = 0$;
(iii) $f(z) = 1/(1-z)$, $z_0 = i$;
(iv) $f(z) = \mathrm{Log}(1-z)$, $z_0 = 0$;
(v) $f(z) = 1/(1+z)$, $z_0 = 0$;
(vi) $f(z) = z/(1-z)^2$, $z_0 = 0$;
(vii) $f(z) = z^3 \sin(3z)$, $z_0 = 0$; and
(viii) $f(z) = (1+z)/(1-z)$, $z_0 = i$.

*Solution.* (i) We use the relation $\cosh(z) = \cosh(-i^2 z) = \cosh(i(-iz)) = \cos(-iz)$ from lemma 1.6.10 and the Taylor series of cos centred at 0 established in exercise 4.3.8 to see that

$$\cosh(z) = \cos(-iz) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}(-iz)^{2j} = \sum_{j=0}^{\infty} \frac{(-1)^j(-i)^{2j}}{(2j)!} z^{2j} = \sum_{j=0}^{\infty} \frac{(-1)^j(-1)^j}{(2j)!} z^{2j} = \sum_{j=0}^{\infty} \frac{1}{(2j)!} z^{2j},$$

which is valid wherever the series for cos is valid, i.e. on $\mathbb{C}$.

(ii) Similarly we use the relation $\sinh(z) = \sinh(-i^2 z) = \sinh(i(-iz)) = i\sin(-iz)$ from lemma 1.6.10 and the Taylor series of sin centred at 0 established in exercise 4.3.8 to see that

$$\begin{aligned}\sinh(z) = i\sin(-iz) = i\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}(-iz)^{2j+1} &= \sum_{j=0}^{\infty} \frac{i(-1)^j(-i)^{2j+1}}{(2j+1)!} z^{2j+1} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j(-1)^j}{(2j+1)!} z^{2j+1} \\ &= \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} z^{2j+1},\end{aligned}$$

which is valid wherever the series for sin is valid, i.e. on $\mathbb{C}$.

(iii) We use a geometric series expansion to see that

$$\begin{aligned}\frac{1}{1-z} = \frac{1}{1-i-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\left(\frac{z-i}{1-i}\right)} &= \frac{1}{1-i}\sum_{j=0}^{\infty} \left(\frac{z-i}{1-i}\right)^j \\ &= \sum_{j=0}^{\infty} \frac{1}{(1-i)^{j+1}}(z-i)^j,\end{aligned}$$

which is valid wherever the geometric series is valid, i.e. when $\left|\frac{z-i}{1-i}\right| < 1$, i.e. on $D_{\sqrt{2}}(i)$.

(iv) By lemma 1.7.10, the function $\mathrm{Log}(1-z)$ is holomorphic except for points $z$ such that $1-z$ lies on the non-positive real axis, i.e. for points $z$ which lie on $[1,\infty)$. So $f$ is holomorphic at 0. Computing derivatives, we find that

$$\frac{d^j}{dz^j}\mathrm{Log}(1-z) = \frac{-(j-1)!}{(1-z)^j},$$

so $f^{(j)}(0) = -(j-1)!$ for $j \geq 1$, and $f^{(0)}(0) = f(0) = \mathrm{Log}(1) = 0$. Thus the Taylor series centred at 0 is given by

$$\sum_{j=1}^{\infty} \frac{-(j-1)!}{j!} z^j = \sum_{j=1}^{\infty} \frac{-1}{j} z^j,$$

which is valid on the largest disc centred at 0 on which the function is holomorphic, i.e. on $D_1(0)$.

(v) We use the geometric series expansion to see that

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{j=0}^{\infty} (-z)^j = \sum_{j=0}^{\infty} (-1)^j z^j,$$

which is valid wherever the geometric series is valid, i.e. when $|-z| < 1$, i.e. on $D_1(0)$.

(vi) We note that $\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z}$, so we use term-by-term differentiation to see that

$$\frac{z}{(1-z)^2} = z\frac{d}{dz}\frac{1}{1-z} = z\frac{d}{dz}\sum_{j=0}^{\infty} z^j = z\sum_{j=1}^{\infty} \frac{d}{dz} z^j = z\sum_{j=1}^{\infty} j z^{j-1} = \sum_{j=1}^{\infty} j z^j,$$

which is valid wherever the geometric series which we differentiatied is valid, i.e. when $|z| < 1$, i.e. on $D_1(0)$.

(vii) We use the Taylor series for sin established in example 4.3.8 to see that

$$\begin{aligned} z^3 \sin(3z) = z^3 \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}(3z)^{2j+1} &= z^3 \sum_{j=0}^{\infty} \frac{(-1)^j 3^{2j+1}}{(2j+1)!} z^{2j+1} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j 3^{2j+1}}{(2j+1)!} z^{2j+4}, \end{aligned}$$

which is valid wherever the Taylor series for sin is valid, i.e. on $\mathbb{C}$.

(viii) We use the expansion for part (iii) to see that

$$\begin{aligned} \frac{1+z}{1-z} &= (1+z)\sum_{j=0}^{\infty} \frac{1}{(1-i)^{j+1}}(z-i)^j \\ &= (1+i+(z-i))\sum_{j=0}^{\infty} \frac{1}{(1-i)^{j+1}}(z-i)^j \\ &= \sum_{j=0}^{\infty} \frac{1+i}{(1-i)^{j+1}}(z-i)^j + \sum_{j=0}^{\infty} \frac{1}{(1-i)^{j+1}}(z-i)^{j+1} \\ &= \frac{1+i}{1-i} + \sum_{j=1}^{\infty} \frac{1+i}{(1-i)^{j+1}}(z-i)^j + \sum_{j=1}^{\infty} \frac{1}{(1-i)^{j}}(z-i)^j \\ &= \frac{(1+i)^2}{2} + \sum_{j=1}^{\infty} \left(\frac{1+i}{(1-i)^{j+1}} + \frac{1}{(1-i)^j}\right)(z-i)^j \\ &= i + \sum_{j=1}^{\infty} \frac{1}{(1-i)^j}\left(\frac{1+i}{1-i} + 1\right)(z-i)^j \\ &= i + \sum_{j=1}^{\infty} \frac{i+1}{(1-i)^j}(z-i)^j, \end{aligned}$$

which is valid whenever the series from part (iii) is valid, i.e. on $D_{\sqrt{2}}(i)$.

**Exercise 4.3.17.** Define $f(z) = \sum_{j=0}^{\infty} \frac{j^3}{3^j} z^j$. Compute the contour integrals of the following functions $g$ around the contour $C_1(0)$:

(i) $g(z) = f(z)/z^4$;
(ii) $g(z) = \exp(z) f(z)$; and
(iii) $g(z) = f(z)\sin(z)/z^2$.

*Solution.* Observe that

$$\left|\frac{n^3/3^n}{(n+1)^3 3^{n+1}}\right| = \frac{n^3 3^{n+1}}{3^n (n+1)^3} = 3\left(\frac{n}{n+1}\right)^3 = 3\left(\frac{1}{1+n^{-1}}\right)^3 \to 3 \text{ as } n \to \infty,$$

so the radius of convergence of the power series defining $f$ is 3, and so $f$ is holomorphic on $C_1(0)$. Since then $f$ has a Taylor series centred at 0, by uniqueness of the Taylor series we see that

$$\frac{f^{(j)}(0)}{j!} = \frac{j^3}{3^j}.$$

(i) The integral we are asked to evaluate is of the form $f(z)/(z-z_0)^{n+1}$ where $f$ is holomorphic inside and on $C_1(0)$, $z_0 = 0$ lies in the interior of $C_1(0)$, and $n = 3$. So using the Generalized Cauchy Integral Formula, we see that

$$\int_{C_1(0)} \frac{f(z)}{z^4}\, dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i}{3!} 3! \frac{3^3}{3^3} = 2\pi i.$$

(ii) The function $g(z) = \exp(z) f(z)$ is holomorphic inside and on the contour $C_1(0)$, so $\int_{C_1(0)} g(z)\, dz = 0$.

(iii) The integral we are asked to evaluate is of the form $g(z)/(z-z_0)^{n+1}$ where $g(z) = f(z)\sin(z)$ is holomorphic inside and on $C_1(0)$, $z_0 = 0$ lies in the interior of $C_1(0)$, and $n = 1$. So using the Generalized Cauchy Integral Formula, we see that

$$\begin{aligned}\int_{C_1(0)} \frac{f(z)\sin(z)}{z^2}\, dz = 2\pi i \left.\frac{d}{dz}(f(z)\sin(z))\right|_{z=0} &= 2\pi i \left.(f(z)\cos(z) + f'(z)\sin(z))\right|_{z=0} \\ &= 2\pi i f(0) \\ &= 0.\end{aligned}$$

## 4.4. Laurent series.

**Exercise 4.4.10.** Find the Laurent series for the function $f(z) = 1/(z+z^2)$ which is valid on the following domains:

(i) $A_{0,1}(0)$;
(ii) $A_{1,\infty}(0)$;
(iii) $A_{0,1}(-1)$; and
(iv) $A_{1,\infty}(-1)$.

*Solution.* (i) We use a geometric series expansion to see that

$$\begin{aligned}f(z) = \frac{1}{z+z^2} = \frac{1}{z(1+z)} = \frac{1}{z}\cdot\frac{1}{1+z} = \frac{1}{z}\cdot\frac{1}{1-(-z)} = \frac{1}{z}\sum_{j=0}^{\infty}(-z)^j &= \sum_{j=0}^{\infty}(-1)^j z^{j-1} \\ &= \sum_{j=-1}^{\infty}(-1)^{j+1} z^j,\end{aligned}$$

which is valid for $z \neq 0$ such that the geometric series is valid, i.e. on $A_{0,1}(0)$.

(ii) We use a geometric series expansion to see that

$$\begin{aligned}
f(z) = \frac{1}{z(1+z)} = \frac{1}{z}\cdot\frac{1}{1+z} = \frac{1}{z^2}\cdot\frac{1}{\left(\frac{1}{z}\right)+1} = \frac{1}{z^2}\cdot\frac{1}{1-\left(\frac{-1}{z}\right)} &= \frac{1}{z^2}\sum_{j=0}^{\infty}\left(\frac{-1}{z}\right)^j \\
&= \sum_{j=0}^{\infty}(-1)^j z^{-(j+2)} \\
&= \sum_{j=-\infty}^{-2}(-1)^j z^j,
\end{aligned}$$

which is valid for $z \neq 0$ such that the geometric series is valid, i.e. when $|1/z| < 1$, i.e. on $A_{1,\infty}(0)$.

(iii) We use a geometric series expansion to see that

$$\begin{aligned}
f(z) = \frac{1}{z(1+z)} = \frac{1}{z+1}\cdot\frac{-1}{1-(z+1)} = \frac{-1}{1+z}\sum_{j=0}^{\infty}(z+1)^j &= \sum_{j=0}^{\infty}-(z+1)^{j-1} \\
&= \sum_{j=-1}^{\infty}-(z+1)^j,
\end{aligned}$$

which is valid for $z \neq -1$ such that the geometric series is valid, i.e. on $A_{0,1}(-1)$.

(iv) We use a geometric series expansion to see that

$$\begin{aligned}
f(z) = \frac{1}{z(1+z)} = \frac{1}{z+1}\cdot\frac{1}{(z+1)-1} &= \frac{1}{(z+1)^2}\cdot\frac{1}{1-\left(\frac{1}{z+1}\right)} \\
&= \frac{1}{(1+z)^2}\sum_{j=0}^{\infty}\left(\frac{1}{z+1}\right)^j \\
&= \sum_{j=0}^{\infty}(z+1)^{-(j+2)} \\
&= \sum_{j=-\infty}^{-2}(z+1)^j,
\end{aligned}$$

which is valid for $z \neq -1$ such that the geometric series is valid, i.e. on $A_{1,\infty}(-1)$.

**Exercise 4.4.11.** Find the Laurent series for the function $f(z) = z/(z+1)(z-2)$ which is valid on the following domains:

(i) $A_{0,1}(0)$;
(ii) $A_{1,2}(0)$; and
(iii) $A_{2,\infty}(0)$.

*Solution.* We use partial fractions to write

$$f(z) = \frac{z}{(z+1)(z-2)} = \frac{1/3}{z+1} + \frac{2/3}{z-2}.$$

(i) We use geometric series expansions to see that

$$\frac{1/3}{z+1} = \frac{1}{3}\cdot\frac{1}{1-(-z)} = \frac{1}{3}\sum_{j=0}^{\infty}(-z)^j = \sum_{j=0}^{\infty}\frac{(-1)^j}{3}z^j,$$

which is valid when $|z| < 1$, i.e. on $D_1(0)$. Similarly we see that

$$\frac{2/3}{z-2} = \frac{2}{3} \cdot \frac{-1}{2} \cdot \frac{1}{1-\left(\frac{z}{2}\right)} = \frac{-1}{3} \sum_{j=0}^{\infty} \left(\frac{z}{2}\right)^j = \sum_{j=0}^{\infty} \frac{-1}{3 \cdot 2^j} z^j,$$

which is valid when $|z/2| < 1$, i.e. on $D_2(0)$. So we have that

$$\begin{aligned} f(z) = \frac{1/3}{z+1} + \frac{2/3}{z-2} = \sum_{j=0}^{\infty} \frac{(-1)^j}{3} z^j + \sum_{j=0}^{\infty} \frac{-1}{3 \cdot 2^j} z^j &= \sum_{j=0}^{\infty} \left( \frac{(-1)^j}{3} + \frac{-1}{3 \cdot 2^j} \right) z^j \\ &= \sum_{j=0}^{\infty} \frac{-((-1)^{j+1} 2^j + 1)}{3 \cdot 2^j} z^j, \end{aligned}$$

which is valid wherever both geometric series are valid, i.e. on $D_1(0) \cap D_2(0) = D_1(0)$, which includes the annulus $A_{0,1}(0)$. The expansion is valid at the point $z = 0$, because the function is in fact holomorphic at 0, and therefore the Laurent series is in fact a Taylor series, as we see, since it has no negative powers of $z$.

(ii) The second expansion above is valid on $D_2(0)$, so we can use that again. For the first term we write

$$\frac{1/3}{z+1} = \frac{1}{3} \cdot \frac{1}{z} \cdot \frac{1}{1-\left(\frac{-1}{z}\right)} = \frac{1}{3z} \sum_{j=0}^{\infty} \left(\frac{-1}{z}\right)^j = \sum_{j=0}^{\infty} \frac{(-1)^j}{3} z^{-(j+1)} = \sum_{j=-\infty}^{-1} \frac{(-1)^{j+1}}{3} z^j,$$

which is valid when $|1/z| < 1$, i.e. on $\mathbb{C} \setminus \overline{D}_1(0) = A_{1,\infty}(0)$. So we have that

$$f(z) = \frac{1/3}{z+1} + \frac{2/3}{z-2} = \sum_{j=-\infty}^{-1} \frac{(-1)^{j+1}}{3} z^j + \sum_{j=0}^{\infty} \frac{-1}{3 \cdot 2^j} z^j,$$

which is valid wherever both geometric series are valid, i.e. on $D_2(0) \cap (\mathbb{C} \setminus \overline{D}_1(0)) = A_{1,2}(0)$.

(iii) Now we have to find an alternative expansion for the second term. Write

$$\frac{2/3}{z-2} = \frac{2}{3} \cdot \frac{1}{z} \cdot \frac{1}{1-\left(\frac{2}{z}\right)} = \frac{2}{3z} \sum_{j=0}^{\infty} \left(\frac{2}{z}\right)^j = \frac{2}{3z} \sum_{j=0}^{\infty} 2^j z^{-j} = \sum_{j=0}^{\infty} \frac{2^{j+1}}{3} z^{-(j+1)} = \sum_{j=-\infty}^{-1} \frac{2^{-j}}{3} z^j,$$

which is valid whenever $|2/z| < 1$, i.e. on $\mathbb{C} \setminus \overline{D}_2(0) = A_{2,\infty}(0)$. So we can write

$$\begin{aligned} f(z) = \frac{1/3}{z+1} + \frac{2/3}{z-2} = \sum_{j=-\infty}^{-1} \frac{(-1)^{j+1}}{3} z^j + \sum_{j=-\infty}^{-1} \frac{2^{-j}}{3} z^j &= \sum_{j=-\infty}^{-1} \left( \frac{(-1)^{j+1}}{3} + \frac{2^{-j}}{3} \right) z^j \\ &= \sum_{j=-\infty}^{-1} \frac{-(-2)^j + 1}{3 \cdot 2^j} z^j, \end{aligned}$$

which is valid wherever both geometric series are valid, i.e. on $A_{1,\infty}(0) \cap A_{2,\infty}(0) = A_{2,\infty}(0)$.

**Exercise 4.4.12.** Find the Laurent series for the function $f(z) = (z+1)/z(z-4)^3$ which is valid on the domain $A_{0,4}(4)$.

*Solution.* We need to find the Laurent series for $\frac{z+1}{z} = 1 + \frac{1}{z}$ which is centred at 4. So we use the geometric series expansion to see that

$$\frac{1}{z} = \frac{1}{4-(-(z-4))} = \frac{1}{4} \cdot \frac{1}{1-\left(\frac{-(z-4)}{4}\right)} = \frac{1}{4} \sum_{j=0}^{\infty} \left(\frac{-(z-4)}{4}\right)^j = \sum_{j=0}^{\infty} (-1)^j 4^{-(j+1)} (z-4)^j,$$

which is valid whenever $|-(z-4)/4| < 1$, i.e. on $D_4(4)$. So we may write

$$1 + \frac{1}{z} = 1 + \sum_{j=0}^{\infty} (-1)^j 4^{-(j+1)} (z-4)^j = \left(1 + 4^{-1}\right) + \sum_{j=1}^{\infty} (-1)^j 4^{-(j+1)} (z-4)^j,$$

which is valid on $D_4(4)$. So we can write

$$\begin{aligned} f(z) = \frac{1}{(z-4)^3} \cdot \frac{z+1}{z} &= \frac{1}{(z-4)^3} \left( \left(1 + 4^{-1}\right) + \sum_{j=1}^{\infty} (-1)^j 4^{-(j+1)} (z-4)^j \right) \\ &= \frac{5}{4}(z-4)^{-3} + \sum_{j=1}^{\infty} (-1)^j 4^{-(j+1)} (z-4)^{j-3} \\ &= \frac{5}{4}(z-4)^{-3} + \sum_{j=-2}^{\infty} (-1)^{j+1} 4^{-(j+4)} (z-4)^j, \end{aligned}$$

which is valid on $A_{0,4}(4)$.

**Exercise 4.4.13.** Find the Laurent series of the following functions $f$ on the domain $A_{0,\infty}(0)$:

(i) $f(z) = \sin(2z)/z^3$; and
(ii) $f(z) = z^2 \cos(1/3z)$.

*Solution.* (i) We use the Taylor series for sin centred at 0 to see that

$$f(z) = \frac{\sin(2z)}{z^3} = \frac{1}{z^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} (2z)^{2j+1} = \frac{1}{z^3} \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j+1}}{(2j+1)!} z^{2j+1} = \sum_{j=0}^{\infty} \frac{(-1)^j 2^{2j+1}}{(2j+1)!} z^{2j-2},$$

which is valid on $A_{0,\infty}(0)$.

(ii) We use the Taylor series for cos centred at 0 to see that

$$f(z) = z^2 \cos(1/3z) = z^2 \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left(\frac{1}{3z}\right)^{2j} = z^2 \sum_{j=0}^{\infty} \frac{(-1)^j}{3^{2j}(2j)!} z^{-2j} = \sum_{j=0}^{\infty} \frac{(-1)^j}{3^{2j}(2j)!} z^{-2j+2},$$

which is valid on $A_{0,\infty}(0)$.

# 5. The residue calculus and its applications

## 5.1. The Cauchy Residue Theorem.

**Exercise 5.1.10.** Determine all the isolated singularities of the following functions $f$, and compute the residue at each singularity:

(i) $f(z) = \exp(3z)/(z-2)$;
(ii) $f(z) = (z+1)/(z^2 - 3z + 2)$;
(iii) $f(z) = (\cos(z))/z^2$;
(iv) $f(z) = ((z-1)/(z+1))^3$;
(v) $f(z) = \exp(z)/z(z+1)^3$; and
(vi) $f(z) = (z-1)/\sin(z)$.

*Solution.* (i) We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(3z)$ is holomorphic and non-zero on $\mathbb{C}$, and $h(z) = z - 2$ is holomorphic and has a simple zero at $z = 2$. Therefore the only singularity of $f$ is the isolated singularity at $z = 2$, which by lemma 4.5.11(i) is a simple pole of $f$. Then lemma 5.1.7 implies that

$$\operatorname{Res}(f, 2) = \frac{g(2)}{h'(2)} = \frac{\exp(3 \cdot 2)}{1} = e^6.$$

(ii) The function $f$ is a rational function where the denominator $z^2 - 3z + 2 = (z-1)(z-2)$ has zeros at $z = 1, 2$. Each of these points is a simple zero. The numerator is non-zero at each of these points, so the singularities of $f$ are the two isolated singularities $z = 1, 2$, which by lemma 4.5.11(i) are both simple poles of $f$. Writing first

$$f(z) = \frac{(z+1)/(z-2)}{z-1},$$

we use lemma 5.1.7 to see that

$$\operatorname{Res}(f, 1) = (1+1)/(1-2) = -2;$$

similarly writing

$$f(z) = \frac{(z+1)/(z-1)}{z-2},$$

we use lemma 5.1.7 to see that

$$\operatorname{Res}(f, 2) = (2+1)/(2-1) = 3.$$

(iii) We can write $f(z) = g(z)/h(z)$ where $g(z) = \cos(z)$ is holomorphic on $\mathbb{C}$ and $h(z) = z^2$ is holomorphic on $\mathbb{C}$ and has a zero of order 2 (or a double zero) at $z = 0$, at which $g(z)$ is non-zero. So the only singularity of $f$ is the isolated singularity $z = 0$, which by lemma 4.5.11(i) is a double pole of $f$. Then lemma 5.1.5 implies that

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} \frac{d}{dz} z^2 f(z) = \lim_{z \to 0} \frac{d}{dz} \cos(z) = \lim_{z \to 0} -\sin(z) = 0.$$

(iv) We can write $f(z) = g(z)/h(z)$ where $g(z) = (z-1)^3$ is holomorphic on $\mathbb{C}$ and $h(z) = (z+1)^3$ is holomophic on $\mathbb{C}$ with a zero of order 3 (or a triple zero) at $z = -1$. Therefore the only singularity of $f$ is the isolated singularity at $z = -1$. Since $g(-1) \neq 0$, lemma 4.5.11(i) implies that $z = -1$ is a triple pole of $f$. Then lemma 5.1.5 implies that

$$\operatorname{Res}(f, -1) = \lim_{z \to -1} \frac{1}{2} \frac{d^2}{dz^2} (z+1)^3 f(z) = \lim_{z \to -1} \frac{1}{2} \frac{d^2}{dz^2} (z-1)^3 = \lim_{z \to -1} \frac{1}{2} 6(z-1) = -6.$$

(v) We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(z)$ is holomorphic and non-zero on $\mathbb{C}$, and $h(z) = z(z+1)^3$ is holomorphic on $\mathbb{C}$ and has zeros at $z = -1, 0$. Therefore the singularities of $f$ are at the isolated singularities $z = -1, 0$. The point $z = -1$ is a zero of order 3 (or a triple zero) of $h$, and $z = 0$ is a simple zero of $h$. Since $g$ is non-zero at both points, lemma 4.5.11(i) implies that $z = -1$ is a triple pole of $f$ and $z = 0$ is a simple pole of $f$. We then write

$$f(z) = \frac{\exp(z)/(z+1)^3}{z}$$

and use lemma 5.1.7 to see that

$$\operatorname{Res}(f, 0) = \exp(0)/(0+1)^3 = 1;$$

and use lemma 5.1.5 to see that

$$\begin{aligned}
\operatorname{Res}(f,-1) &= \lim_{z\to -1}\frac{1}{2}\frac{d^2}{dz^2}(z+1)^3 f(z)\\
&= \lim_{z\to -1}\frac{1}{2}\frac{d^2}{dz^2}\frac{\exp(z)}{z}\\
&= \lim_{z\to -1}\frac{1}{2}\frac{d}{dz}\left(\frac{z\exp(z)-\exp(z)}{z^2}\right)\\
&= \lim_{z\to -1}\frac{1}{2}\frac{z^2(z\exp(z)+\exp(z)-\exp(z))-2z(z\exp(z)-\exp(z))}{z^4}\\
&= \frac{1}{2}\frac{(-1)^2\left((-1)\exp(-1)\right)-2(-1)\left((-1)\exp(-1)-\exp(-1)\right)}{(-1)^4}\\
&= \frac{-5}{2e}.
\end{aligned}$$

(vi) We can write $f(z) = g(z)/h(z)$ where $g(z) = z-1$ is holomorphic on $\mathbb{C}$, and $h(z) = \sin(z)$ is holomorphic on $\mathbb{C}$ with zeros at $z = z_k = k\pi$ for $k \in \mathbb{Z}$. Therefore the singularities of $f$ are the isolated singularities $z_k = k\pi$. Since each $z_k$ is a simple zero of $h$, and $g(z_k) \neq 0$ for all $k$, lemma 4.5.11(i) implies that each $z_k$ is a simple pole of $f$. Therefore lemma 5.1.7 implies that

$$\operatorname{Res}(f,z_k) = \frac{g(z_k)}{h'(z_k)} = \frac{z_k-1}{\cos(z_k)} = (-1)^k(k\pi-1).$$

**Exercise 5.1.13.** Evaluate, using the Cauchy Residue Theorem, the following contour integrals of the given function $f$ around the given contour $C_R(0)$:

(i) $f(z) = \sin(z)/(z^2-4)$, $R = 5$;
(ii) $f(z) = \exp(z)/z(z-2)^3$, $R = 3$;
(iii) $f(z) = \tan(z)$, $R = 2\pi$;
(iv) $f(z) = 1/z^2\sin(z)$, $R = 1$;
(v) $f(z) = (3z+2)/(z^4+1)$, $R = 3$; and
(vi) $f(z) = 1/(z^2+z+1)$, $R = 8$;

*Solution.* (i) We can write $f(z) = g(z)/h(z)$ where $g(z) = \sin(z)$ is holomorphic on $\mathbb{C}$ and $h(z) = z^2-4 = (z+2)(z-2)$ is holomorphic on $\mathbb{C}$ with simple zeros at $z = \pm 2$. So $f$ has isolated singularities at $z = \pm 2$, both of which lie inside $C_5(0)$. Since $g(\pm 2) \neq 0$, lemma 4.5.11(i) implies that both points are simple poles of $f$. Writing

$$f(z) = \frac{\sin(z)/(z-2)}{z+2}$$

we use lemma 5.1.7 to see that

$$\operatorname{Res}(f,-2) = \sin(-2)/(-2-2) = \frac{\sin(2)}{4},$$

and similarly writing

$$f(z) = \frac{\sin(z)/(z+2)}{z-2}$$

we use lemma 5.1.7 to see that

$$\operatorname{Res}(f,2) = \sin(2)/(2+2) = \frac{\sin(2)}{4}.$$

So the Cauchy Residue Theorem implies that

$$\int_{C_5(0)} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f,-2)+\operatorname{Res}(f,2)\right) = 2\pi i\left(\frac{\sin(2)}{4}+\frac{\sin(2)}{4}\right) = \sin(2)\pi i.$$

(ii) We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(z)$ is holomorphic on $\mathbb{C}$ and $h(z) = z(z-2)^3$ is holomorphic on $\mathbb{C}$ with a simple zero at $z = 0$ and a zero of order 3 at $z = 2$. Therefore the singularities of $f$ are the isolated singularities $z = 0, 2$, both of which lie inside $C_3(0)$. Since $g$ is non-zero at both singularities, lemma 4.5.11(i) implies that $z = 0$ is a simple pole of $f$, and $z = 2$ is a triple pole of $f$. Then writing

$$f(z) = \frac{\exp(z)/(z-2)^3}{z},$$

lemma 5.1.7 implies that

$$\mathrm{Res}\,(f, 0) = \exp(0)/(0-2)^3 = \frac{-1}{8}.$$

Lemma 5.1.5 implies that, borrowing the calculation from exercise 5.1.10(v),

$$\begin{aligned}
\mathrm{Res}\,(f, 2) &= \lim_{z\to 2} \frac{1}{2}\frac{d^2}{dz^2}(z-2)^3 f(z) \\
&= \lim_{z\to 2} \frac{1}{2}\frac{d^2}{dz^2}\frac{\exp(z)}{z} \\
&= \lim_{z\to 2} \frac{1}{2}\left(\frac{z^2(z\exp(z)+\exp(z)-\exp(z)) - 2z(z\exp(z)-\exp(z))}{z^4}\right) \\
&= \frac{1}{2}\frac{4(2\exp(2)) - 2\cdot 2(2\exp(2)-\exp(2))}{2^4} \\
&= \frac{e^2}{8}.
\end{aligned}$$

So the Cauchy Residue Theorem implies that

$$\int_{C_3(0)} f(z)\,dz = 2\pi i\,(\mathrm{Res}\,(f,0) + \mathrm{Res}\,(f,2)) = 2\pi i\left(\frac{-1}{8} + \frac{e^2}{8}\right) = \frac{\pi i(e^2-1)}{4}.$$

(iii) We can write $f(z) = g(z)/h(z)$ where $g(z) = \sin(z)$ is holomorphic on $\mathbb{C}$, and $h(z) = \cos(z)$ is holomorphic on $\mathbb{C}$, and has zeros at $z_k = \pi(\frac{1}{2} + k)$ for $k \in \mathbb{Z}$. So the singularities of $f$ are the isolated singularities $z_k$ for $k \in \mathbb{Z}$, of which only $z_{-2}, z_{-1}, z_0, z_1$ lie inside $C_{2\pi}(0)$. Each $z_k$ is a simple zero of $h$, so since $g(z_k) \neq 0$, lemma 4.5.11(i) implies that each $z_k$ is a simple pole of $f$. Therefore lemma 5.1.7 implies that

$$\mathrm{Res}\,(f, z_k) = \frac{g(z_k)}{h'(z_k)} = \frac{\sin(z_k)}{\cos'(z_k)} = \frac{\sin(z_k)}{-\sin(z_k)} = -1.$$

Therefore the Cauchy Residue Theorem implies that

$$\int_{C_{2\pi}(0)} f(z)\,dz = 2\pi i \sum_{k=-2}^{1} \mathrm{Res}\,(f, z_k) = 2\pi i(-4) = -8\pi i.$$

(iv) We can write $f(z) = g(z)/h(z)$ where $g(z) = 1$ is holomorphic on $\mathbb{C}$ and $h(z) = z^2\sin(z)$ is holomorphic on $\mathbb{C}$ with zeros at $z_k = k\pi$ for $k \in \mathbb{Z}$. The singularities of $f$ are the isolated singularities $z_k = k\pi$, of which only $z_0 = 0$ lies inside $C_1(0)$. Since $z_0 = 0$ is a zero of order 3 of $h$, and $g(0) \neq 0$, lemma 4.5.11(i) implies that $z_0 = 0$ is a triple pole of $f$. To find the residue it is easiest to compute the Laurent series of $f$ centred at 0:

$$\begin{aligned} f(z) = \frac{1}{z^2\sin(z)} &= \frac{1}{z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)} \\ &= \frac{1}{z^3} \cdot \frac{1}{1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)} \\ &= \frac{1}{z^3}\left(1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)^2 + \cdots\right), \end{aligned}$$

using the geometric series expansion with ratio $\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots$, which is valid for small $|z|$. From this we see that the coefficient of $z^{-1}$ in this series is $\frac{1}{3!} = \frac{1}{6}$. So by the Cauchy Residue Theorem

$$\int_{C_1(0)} f(z)\,dz = 2\pi i \operatorname{Res}(f, 0) = 2\pi i \frac{1}{6} = \frac{\pi i}{3}.$$

(v) We can write $f(z) = g(z)/h(z)$ where $g(z) = 3z + 2$ is holomorphic on $\mathbb{C}$, and $h(z) = z^4 + 1$ is holomorphic on $\mathbb{C}$, and has zeros at $z_k = \omega^{1+2k}$ for $k = 0, 1, 2, 3$, where $\omega = e^{i\pi/4}$ is an 8-th root of unity. So $f$ has isolated singularities at each $z_k$, each of which lies inside $C_3(0)$. Each $z_k$ is a simple zero of $h$, and $g(z_k) \neq 0$ for each $k$, so lemma 4.5.11(i) implies that each $z_k$ is a simple pole of $f$. Lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_k) = \frac{g(z_k)}{h'(z_k)} = \frac{3z_k + 2}{4z_k^3} = \frac{3\omega^{1+2k} + 2}{4\omega^{3+6k}}.$$

So by the Cauchy Residue Theorem, since $\omega^8 = 1$,

$$\begin{aligned} \int_{C_3(0)} f(z)\,dz &= 2\pi i \sum_{k=0}^{3} \operatorname{Res}(f, z_k) \\ &= 2\pi i \left(\frac{3\omega + 2}{4\omega^3} + \frac{3\omega^3 + 2}{4\omega^9} + \frac{3\omega^5 + 2}{4\omega^{15}} + \frac{3\omega^7 + 2}{4\omega^{21}}\right) \\ &= \frac{\pi i}{2}\left(3\left(\omega^{-2} + \omega^2 + \omega^{-2} + \omega^2\right) + 2\left(\omega^{-3} + \omega^{-1} + \omega^1 + \omega^3\right)\right) \\ &= \frac{\pi i}{2}\left(3\left(-\omega^2 + \omega^2 - \omega^2 + \omega^2\right) + 2\left(-\omega^1 - \omega^3 + \omega^1 + \omega^3\right)\right) \\ &= 0. \end{aligned}$$

(vi) We can write $f(z) = g(z)/h(z)$ where $g(z) = 1$ is holomorphic on $\mathbb{C}$ and $h(z) = z^2 + z + 1$ is holomorphic on $\mathbb{C}$ and by the quadratic formula has zeros at $z_\pm = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$. Therefore $f$ has isolated singularities at $z_\pm$, both of which lie inside $C_8(0)$. Each $z_\pm$ is a simple zero of $h$, and $g(z_\pm) \neq 0$, so lemma 4.5.11(i) implies that $z_\pm$ is a simple pole of $f$. Writing

$$f(z) = \frac{1/(z - z_+)}{z - z_-},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_-) = 1/(z_- - z_+);$$

and similarly writing

$$f(z) = \frac{1/(z - z_-)}{z - z_+},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_+) = 1/(z_+ - z_-).$$

We do not need to evaluate these to see that the Cauchy Residue Theorem implies that

$$\int_{C_8(0)} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f, z_-) + \operatorname{Res}(f, z_+)\right) = 2\pi i\left(\frac{1}{z_- - z_+} + \frac{1}{z_+ - z_-}\right) = 0.$$

## 5.2. The Argument Principle and Rouché's Theorem.

**Exercise 5.2.10.** Use Rouché's Theorem to show that the polynomial $z^6 + 4z^2 - 1$ has exactly two zeros in the disc $D_1(0)$.

*Solution.* Let $f(z) = z^6 + 4z^2 - 1$ and $g(z) = 4z^2 - 1$. Then $f$ and $g$ are holomorphic inside and on the unit circle $C_1(0)$, and for $z \in C_1(0)$, we see that

$$|f(z) - g(z)| = \left|(z^6 + 4z^2 - 1) - (4z^2 - 1)\right| = \left|z^6\right| = 1,$$

and, using the reverse triangle inequality,

$$|f(z)| = \left|z^6 + 4z^2 - 1\right| \geq \left|z^6 + 4z^2\right| - 1 \geq \left|4z^2\right| - \left|z^6\right| - 1 = 4 - 1 - 1 = 2.$$

So $|f(z) - g(z)| = 1 < 2 \leq |f(z)|$ for $z \in C_1(0)$. The function $g$ has two zeros, at $\pm 1/2$, both of which lie inside $C_1(0)$. Therefore Rouché's theorem implies that $f$ too has exactly two zeros in $\operatorname{Int}(C_1(0)) = D_1(0)$.

**Exercise 5.2.11.** Prove that the equation $z^3 + 9z + 27 = 0$ has no roots in the disc $D_2(0)$.

*Solution.* Let $f(z) = 27$ and $g(z) = z^3 + 9z + 27$. Then $f$ and $g$ are holomorphic inside and on $C_2(0)$, and for $z \in C_2(0)$, we have

$$\begin{aligned}|f(z) - g(z)| = \left|27 - (z^3 + 9z + 27)\right| = \left|z^3 + 9z\right| \leq \left|z^3\right| + 9\,|z| = 8 + 18 = 26 < 27 \\ = |f(z)|\,.\end{aligned}$$

The function $f$ evidently has no zeros inside $C_2(0)$, therefore Rouché's theorem implies that $g$ has no zeros in $\operatorname{Int}(C_2(0)) = D_2(0)$ either.

**Exercise 5.2.12.** Prove that all the roots of the equation $z^6 - 5z^2 + 10 = 0$ lie in the annulus $A_{1,2}(0)$.

*Solution.* We prove that the equation has no solutions in $\overline{D}_1(0)$ and six solutions inside $D_2(0)$, which implies the result.

Let $f(z) = 10$ and $g(z) = z^6 - 5z^2 + 10$. Then $f$ and $g$ are holomorphic inside and on the unit circle $C_1(0)$. For $z \in C_1(0)$, we have that

$$\begin{aligned}|f(z) - g(z)| = \left|10 - (z^6 - 5z^2 + 10)\right| = \left|z^6 - 5z^2\right| = \left|z^2\right|\left|z^4 - 5\right| \leq |z|^4 + 5 = 6 \\ < 10 \\ = |f(z)|\,.\end{aligned}$$

The function $f$ evidently has no zeros inside $C_1(0)$, therefore Rouché's theorem implies that $g$ has no solutions inside $C_1(0)$ either. Furthermore, $g$ has no zeros on $C_1(0)$: suppose for a contradiction that $z \in C_1(0)$ satisfied $z^6 = 5z^2 - 10$. Then $1 = \left|z^6\right| = \left|5z^2 - 10\right|$. But the reverse triangle inequality implies that

$$1 = 2 - 1 = \left|2 - |z|^2\right| \leq \left|2 - z^2\right|,$$

so $\left|5z^2 - 10\right| = 5\left|z^2 - 2\right| \geq 5$, which is a contradiction.

Now let $f(z) = z^6$ and $g(z) = z^6 + 5z^2 - 10$. Then $f$ and $g$ are holomorphic inside and on $C_2(0)$, the circle of radius 2 centred at 0. For $z \in C_2(0)$, we have that

$$|f(z) - g(z)| = \left|z^6 - (z^6 + 5z^2 - 10)\right| = \left|5z^2 - 10\right| \leq 5\left|z^2\right| + 10 = 20 + 10 = 30 < 64 \\ = \left|z^6\right|.$$

Counted with multiplicity, the function $f$ has six zeros inside $C_2(0)$, therefore Rouché's theorem implies $g$ too has six zeros inside $C_2(0)$.

**Exercise 5.2.13.** Find the number of roots of the equation $6z^4 + z^3 - 2z^2 + z - 1 = 0$ in the disc $D_1(0)$.

*Solution.* Define $f(z) = 6z^4$ and $g(z) = 6z^4 + z^3 - 2z^2 + z - 1$. Then $f$ and $g$ are both holomorphic inside and on the unit circle $C_1(0)$. On $C_1(0)$, we have that

$$\begin{aligned} |f(z) - g(z)| = \left|6z^4 - (6z^4 + z^3 - 2z^2 + z - 1)\right| &= \left|z^3 - 2z^2 + z - 1\right| \\ &\leq \left|z^3\right| + 2\left|z^2\right| + |z| + 1 \\ &= 1 + 2 + 1 + 1 \\ &= 5 \\ &< 6 = \left|6z^4\right| = |f(z)|. \end{aligned}$$

Counted with multiplicity, the function $f$ has four zeros inside $C_1(0)$, hence Rouché's theorem implies that $g$ too has four zeros in $\mathrm{Int}(C_1(0)) = D_1(0)$.

**Exercise 5.2.14.** Prove that the equation $z = 2 - \exp(-z)$ has exactly one root in the right half-plane.

*Solution.* Let $R > 4$, and let $\Gamma_1$ be the line segment from $iR$ to $-iR$, and $\Gamma_2$ be the semicircular arc from $-iR$ to $iR$ in the right half plane. Let $\Gamma_R$ be the loop obtained by traversing both contours. Define $f(z) = z - 2$ and $g(z) = \exp(-z) + z - 2$. Then $f$ and $g$ are both holomorphic inside and on $\Gamma_R$. For $z \in \Gamma_R$, we have that

$$|f(z) - g(z)| = |(z-2) - (\exp(-z) + z - 2)| = |\exp(-z)| = e^{-\mathrm{Re}(z)} \leq 1 < 2 \leq |z - 2| \\ = |f(z)|.$$

The function $f$ has exactly one zero inside $\Gamma_R$, so Rouché's theorem implies that $g$ too has exactly one zero inside $\Gamma_R$. This argument holds for all $R > 4$, so letting $R \to \infty$ we see that $g$ has exactly one root in the right half plane.

**Exercise 5.2.19** (Schwarz's Lemma)**.** Let $f$ be holomorphic on $D_1(0)$, and satisfy $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in D_1(0)$. Prove that $|f(z)| \leq |z|$, using the following steps.

(i) Define $F(z) = f(z)/z$ for $z \in D_1'(0)$, and $F(0) = f'(0)$. Show that $F$ is holomorphic on $D_1(0)$.

(ii) Let $z \in D_1(0)$ be non-zero, and suppose $|z| < r < 1$. Use the Maximum Modulus Principle to show that

$$|F(z)| \leq \max_{w \in C_r(0)} \left|\frac{f(w)}{w}\right| = \max_{w \in C_r(0)} \frac{|f(w)|}{r} \leq \frac{1}{r}.$$

(iii) Letting $r \to 1$ in part (ii), deduce that $|f(z)| \leq |z|$ for all $z \in D_1(0)$.

*Solution.* (i) The function $f$ has a Taylor series centred at 0 which is valid on $D_1(0)$, which starts at $j = 1$ since $f(0) = 0$:

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} z^j = \sum_{j=1}^{\infty} \frac{f^{(j)}(0)}{j!} z^j.$$

Therefore for $z \in D_1'(0)$ we have that

$$\frac{f(z)}{z} = \frac{1}{z}\sum_{j=1}^{\infty}\frac{f^{(j)}(0)}{j!}z^j = \sum_{j=1}^{\infty}\frac{f^{(j)}(0)}{j!}z^{j-1} = \sum_{j=0}^{\infty}\frac{f^{(j+1)}(0)}{(j+1)!}z^j.$$

The right-hand side defines a convergent power series $g(z)$ on $D_1(0)$, which is therefore holomorphic on $D_1(0)$, by theorem 4.2.6. By definition $F(z) = f(z)/z = g(z)$ for $z \in D_1'(0)$, and $F(0) = f'(0) = g(0)$. So in fact $F(z) = g(z)$ for all $D_1(0)$, hence $F$ is holomorphic on $D_1(0)$.

(ii) The maximum modulus principle implies that the maximum of $|F(z)|$ on $D_r(0)$ is attained on the boundary, $C_r(0)$, so for all $z \in D_r(0)$, we have that

$$|F(z)| \leq \max_{w\in D_r(0)} |F(w)| \leq \max_{w\in C_r(0)} |F(w)| = \max_{w\in C_r(0)} \frac{|f(w)|}{|w|} = \max_{w\in C_r(0)} \frac{|f(w)|}{r} \leq \frac{1}{r},$$

since $|f(w)| \leq 1$ by assumption.

(iii) Let $z \in D_1'(0)$. Then for all $r$ such that $|z| < r < 1$, the previous part applies, and we see that

$$\frac{|f(z)|}{|z|} = |F(z)| \leq \frac{1}{r}.$$

We can therefore take the limit as $r \to 1$ to see that

$$\frac{|f(z)|}{|z|} \leq \lim_{r\to 1}\frac{1}{r} = 1,$$

hence $|f(z)| \leq |z|$ for all non-zero $z \in D_1(0)$. Since $f(0) = 0$, the result extends to all of $D_1(0)$.

**Exercise 5.2.20.** Let $f$ satisfy the hypothesis of exercise 5.2.19.

(i) Show that if there exists a non-zero point $z_0 \in D_1(0)$ such that $|f(z_0)| = |z_0|$, then $f(z) = e^{i\theta}z$ for some constant $\theta \in \mathbb{R}$.
(ii) Show that if $|f'(0)| = 1$, then $f(z) = e^{i\theta}z$ for some constant $\theta \in \mathbb{R}$.

*Solution.* Define $F$ as in exercise 5.2.19, and recall that $|F(z)| \leq 1$ for all $z \in D_1(0)$.

(i) By the definition of $F$ and the assumption,

$$|F(z_0)| = \left|\frac{f(z_0)}{z_0}\right| = \frac{|f(z_0)|}{|z_0|} = 1,$$

so $F$ attains its maximum modulus at $z_0$. Then the maximum modulus principle implies that $F$ is constant. Since $|F(z)| = 1$, we have in fact that $F(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ for all $z \in D_1(0)$, which in turn implies that $f(z) = e^{i\theta}z$ for all $D_1(0)$.

(ii) Similarly by definition $|F(0)| = |f'(0)| = 1$ implies that $F$ is constant, with the same conclusion.

**Exercise 5.2.21.** Let $h$ be holomorphic on $\mathbb{C}$ and satisfy $|h(z)| < 1$ for all $z$ such that $|z| = 1$. Prove that the equation $h(z) = z$ has exactly one solution, counting multiplicity, in $D_1(0)$. (Such a solution is a *fixed point* of $h$.)

*Solution.* Let $f(z) = z$ and $g(z) = z - h(z)$. Then $f$ and $g$ are holomorphic inside and on the unit circle $C_1(0)$, and by assumption, on $C_1(0)$ we have that

$$|f(z) - g(z)| = |z - (z - h(z))| = |h(z)| < 1 = |z| = |f(z)|.$$

Evidently $f$ has exactly one zero inside $C_1(0)$, so Rouché's theorem implies that $g$ too has exactly one zero inside $C_1(0)$. That is, there is a unique solution to $h(z) = z$ inside $D_1(0)$, as required.

### 5.3. Application: trigonometric integrals.

**Exercise 5.3.3.** Evaluate the following real integrals:

(i) $\int_0^{2\pi} \frac{1}{2+\sin\theta}\,d\theta$;
(ii) $\int_0^{\pi} \frac{8}{5+2\cos\theta}\,d\theta$;
(iii) $\int_{-\pi}^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta$; and
(iv) $\int_0^{\pi} \frac{1}{(3+2\cos\theta)^2}\,d\theta$.

*Solution.* (i) Define

$$f(z) = \frac{1}{iz}\cdot\frac{1}{\left(2+\left(\frac{z-1/z}{2i}\right)\right)} = \frac{-i}{2z+\left(\frac{z^2-1}{2i}\right)} = \frac{2}{4iz+z^2-1}.$$

Then

$$I = \int_0^{2\pi}\frac{1}{2+\sin\theta}\,d\theta = \int_{C_1(0)} f(z)\,dz.$$

The denominator of $f$ has, by the quadratic formula, zeros at

$$z_\pm = \frac{-4i\pm\sqrt{-16+4}}{2} = i(-2\pm\sqrt{3}).$$

The function $f$ therefore has isolated singularities at $z_\pm = i(-2\pm\sqrt{3})$, of which only $z_+ = i(-2+\sqrt{3})$ lies inside $C_1(0)$. The point $z_+$ is a simple zero of the denominator, so lemma 4.5.11(i) implies that it is a simple pole of $f$. Writing

$$f(z) = \frac{2/(z-z_-)}{z-z_+},$$

lemma 5.1.7 implies that

$$\mathrm{Res}\,(f,z_+) = 2/(z_+-z_-) = \frac{2}{i(-2+\sqrt{3})-i(-2-\sqrt{3})} = \frac{2}{2i\sqrt{3}} = \frac{1}{i\sqrt{3}}.$$

So by the Cauchy Residue Theorem

$$I = \int_{C_1(0)} f(z)\,dz = 2\pi i\mathrm{Res}\,(f,z_+) = 2\pi i\frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

(ii) We note that

$$I = \int_0^{\pi}\frac{8}{5+2\cos\theta}\,d\theta = \frac{1}{2}\int_0^{2\pi}\frac{8}{5+2\cos\theta}\,d\theta,$$

and therefore define

$$f(z) = \frac{1}{iz}\cdot\frac{8}{\left(5+2\left(\frac{z+1/z}{2}\right)\right)} = \frac{-8i}{5z+z^2+1},$$

so

$$I = \frac{1}{2}\int_{C_1(0)} f(z)\,dz.$$

Using the quadratic formula, the denominator of $f$ has zeros at

$$z_\pm = \frac{-5\pm\sqrt{25-4}}{2} = \frac{-5\pm\sqrt{21}}{2},$$

so $f$ has isolated singularities at $z_\pm$, of which only $z_+$ lies inside $C_1(0)$. The point $z_+$ is a simple zero of the denominator of $f$, and therefore lemma 4.5.11(i) implies that $z_+$ is a simple pole of $f$. Writing

$$f(z) = \frac{-8i/\left(z-z_-\right)}{z-z_+},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_+) = -8i/(z_+ - z_-) = \frac{-8i}{\left(\left(-5+\sqrt{21}\right)/2\right) - \left(\left(-5-\sqrt{21}\right)/2\right)} = \frac{-8i}{\sqrt{21}}.$$

Then the Cauchy Residue Theorem implies that

$$I = \frac{1}{2}\int_{C_1(0)} f(z)\,dz = \frac{1}{2}2\pi i \operatorname{Res}(f, z_+) = \pi i \frac{-8i}{\sqrt{21}} = \frac{8\pi}{\sqrt{21}}.$$

(iii) Starting down the usual strategy of immediately defining the corresponding complex function, we quickly realize that this might get quite complicated. So we try to simplify the integral a little using usual real integration techniques. First, using $\cos(2\theta) = 1 - 2\sin^2\theta$, we can write

$$I = \int_{-\pi}^{\pi} \frac{1}{1+\sin^2\theta}\,d\theta = \int_{-\pi}^{\pi} \frac{1}{1+\left(\frac{1-\cos(2\theta)}{2}\right)}\,d\theta = \int_{-\pi}^{\pi} \frac{2}{3-\cos(2\theta)}\,d\theta,$$

in which we make the substitution $\phi = 2\theta$ to see that

$$I = \int_{-2\pi}^{2\pi} \frac{1}{3-\cos\phi}\,d\phi = 2\int_{-\pi}^{\pi} \frac{1}{3-\cos\phi}\,d\phi.$$

Now we proceed as usual: define

$$f(z) = \frac{1}{iz}\cdot\frac{1}{3-\left(\frac{z+1/z}{2}\right)} = \frac{2i}{-6z+z^2+1},$$

so

$$I = 2\int_{C_1(0)} f(z)\,dz.$$

By the quadratic formula, the denominator of $f$ is zero at points $z_\pm = \frac{6\pm\sqrt{36-4}}{2} = 3\pm 2\sqrt{2}$. So $f$ has isolated singularities at $z_\pm$, of which only $z_-$ lies inside $C_1(0)$. The point $z_-$ is a simple zero of the denominator, and therefore lemma 4.5.11(i) implies that $z_-$ is a simple pole of $f$. Writing

$$f(z) = \frac{2i/(z-z_+)}{z-z_-},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_-) = 2i/(z_- - z_+) = \frac{2i}{\left(3-2\sqrt{2}\right)-\left(3+2\sqrt{2}\right)} = \frac{2i}{-4\sqrt{2}} = \frac{-i}{2\sqrt{2}}.$$

Then the Cauchy Residue Theorem implies that

$$I = 2\int_{C_1(0)} f(z)\,dz = 2(2\pi i)\operatorname{Res}(f, z_-) = 4\pi i\frac{-i}{2\sqrt{2}} = \sqrt{2}\pi.$$

(iv) Again we have that

$$I = \int_0^{\pi} \frac{1}{(3+2\cos\theta)^2}\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{(3+2\cos\theta)^2}\,d\theta,$$

so defining

$$f(z) = \frac{1}{iz}\cdot\frac{1}{\left(3+2\left(\frac{z+1/z}{2}\right)\right)^2} = \frac{-iz}{\left(3z+z^2+1\right)^2},$$

we have that

$$I = \frac{1}{2}\int_{C_1(0)} f(z)\,dz.$$

By the quadratic formula, the denominator of $f$ has zeros at $z_\pm = \frac{-3\pm\sqrt{9-4}}{2} = \frac{-3\pm\sqrt{5}}{2}$, so $f$ has isolated singularities at $z_\pm$, of which only $z_+$ lies inside $C_1(0)$. The point $z_+$ is a double zero of the denominator of $f$, so lemma 4.5.11(i) implies $z_+$ is a double pole of $f$. Then lemma 5.1.5 implies that

$$\begin{aligned}
\operatorname{Res}(f, z_+) = \lim_{z\to z_+} \frac{d}{dz}(z-z_+)^2 f(z) &= \lim_{z\to z_+} \frac{d}{dz}\frac{-iz}{(z-z_-)^2} \\
&= \lim_{z\to z_+} \frac{(z-z_-)^2(-i) - 2(-iz)(z-z_-)}{(z-z_-)^4} \\
&= \frac{-i(z_+-z_-)^2 + 2iz_+(z_+-z_-)}{(z_+-z_-)^4} \\
&= \frac{-5i + 2\sqrt{5}i\left(\frac{-3+\sqrt{5}}{2}\right)}{25} \\
&= \frac{-3i}{5\sqrt{5}}.
\end{aligned}$$

The Cauchy Residue Theorem implies that

$$I = \frac{1}{2}\int_{C_1(0)} f(z)\,dz = \frac{1}{2}2\pi i \operatorname{Res}(f, z_+) = \pi i \frac{-3i}{5\sqrt{5}} = \frac{3\pi}{5\sqrt{5}}.$$

### 5.4. Improper integrals.

**Exercise 5.4.4.** Evaluate the following integrals:

(i) $\int_0^\infty \frac{x^2+1}{x^4+1}\,dx$;
(ii) $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$;
(iii) $\int_0^\infty \frac{1}{x^3+1}\,dx$ (consider the contour which is the boundary of the sector $\{\, z = re^{i\theta} : 0 \le \theta \le 2\pi/3, \text{ and } 0 \le r \le \rho \,\}$);
(iv) $\text{p. v.}\int_{-\infty}^\infty \frac{1}{x^2+2x+2}\,dx$;
(v) $\text{p. v.}\int_{-\infty}^\infty \frac{x^2}{(x^2+9)^2}\,dx$;
(vi) $\text{p. v.}\int_{-\infty}^\infty \frac{1}{(x^2+1)(x^2+4)}\,dx$; and
(vii) $\text{p. v.}\int_{-\infty}^\infty \frac{x}{(x^2+4x+13)^2}\,dx$.

*Solution.* First we observe that each integrand is of the form $P(x)/Q(x)$ where $P, Q$ are polynomials with $\deg(Q) \ge \deg(P) + 2$. As discussed in lectures, this implies that the integral of the corresponding complex function $P(z)/Q(z)$ around the semicircular arc $C_R^+$ from $R$ to $-R$ in the upper half plane converges to 0 as $R$ tends to $\infty$. We can prove this formally as follows.

Suppose $P(z) = \sum_{n=0}^M a_n z^n$ and $Q(z) = \sum_{n=0}^N b_n z^n$ where $a_n, b_n \in \mathbb{C}$, $a_M, b_N \neq 0$, and $N \ge M+2$. Then by the triangle inequality,

$$|P(z)| = \left|\sum_{n=0}^M a_n z^n\right| \le \sum_{n=0}^M |a_n z^n| \le \left(\max_{n=0,\dots,M} |a_n|\right)(M+1)\,|z|^M,$$

for $|z| \ge 1$. We recall from exercise 3.6.4 that there exists $R_0 > 0$ such that

$$|Q(z)| \ge \frac{|b_N|}{2}\,|z|^N$$

whenever $|z| \geq R_0$. Therefore, for $R \geq \max\{1, R_0\}$, we have, by lemma 3.2.9, that

$$\begin{aligned}
\left| \int_{C_R^+} \frac{P(z)}{Q(z)}\, dz \right| \leq \max_{z \in C_R^+} \left| \frac{P(z)}{Q(z)} \right| \ell(C_R^+) &= \max_{z \in C_R^+} \frac{|P(z)|}{|Q(z)|} \pi R \\
&\leq \frac{\left(\max_{n=0,\ldots,M} |a_n|\right)(M+1)R^M}{\frac{|b_N|}{2} R^N} \pi R \\
&= 2\,|b_N|^{-1} \left( \max_{n=0,\ldots,M} |a_n| \right)(M+1)\pi R^{M-N+1} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

since by assumption $M - N + 1 \leq -1$.

With this observation we can deal with most of the integrals in the same way. For each we let $\gamma_R$ denote the line segment from $-R$ to $R$ on the real axis, $C_R^+$ denote the semicircular arc from $R$ to $-R$ in the upper half plane discussed above, and let $\Gamma_R$ denote the resulting loop gained by composing the two contours. Then for any of the integrands,

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)\, dx = \lim_{R \to \infty} \int_{\gamma_R} f(z)\, dz = \lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz.$$

So in most cases it suffices to calculate $\lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz$.

(i) Let

$$f(z) = \frac{z^2+1}{z^4+1}.$$

Then

$$I = \lim_{R \to \infty} \int_0^R \frac{x^2+1}{x^4+1}\, dx = \frac{1}{2} \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2+1}{x^4+1}\, dx = \frac{1}{2} \lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz.$$

We can write $f(z) = g(z)/h(z)$ where $g(z) = z^2 + 1$ is holomorphic on $\mathbb{C}$ and $h(z) = z^4 + 1$ is holomorphic on $\mathbb{C}$ with zeros at $z_k = \omega^{1+2k}$ where $\omega = e^{i\pi/4}$ for $k = 0, 1, 2, 3$. The function $f$ therefore has isolated singularities at each $z_k$, of which only $z_0$ and $z_1$ lie inside $\Gamma_R$ for $R > 1$. For each $k$, $z_k$ is a simple zero of $h$, and $g(z_k) \neq 0$, so lemma 4.5.11(i) implies that $z_k$ is a simple pole of $f$. Lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)} = \frac{z_0^2+1}{4z_0^3} = \frac{e^{i\pi/2}+1}{4e^{3i\pi/4}} = \frac{1+i}{2\sqrt{2}(-1+i)} = \frac{-(1+i)^2}{4\sqrt{2}} = \frac{-i}{2\sqrt{2}},$$

and

$$\operatorname{Res}(f, z_1) = \frac{g(z_1)}{h'(z_1)} = \frac{z_1^2+1}{4z_1^3} = \frac{e^{3i\pi/2}+1}{4e^{9i\pi/4}} = \frac{1-i}{2\sqrt{2}(1+i)} = \frac{(1-i)^2}{4\sqrt{2}} = \frac{-i}{2\sqrt{2}}.$$

The Cauchy Residue Theorem implies that

$$I = \frac{1}{2} \lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz = \frac{1}{2} 2\pi i \left( \operatorname{Res}(f, z_0) + \operatorname{Res}(f, z_1) \right) = \pi i \left( \frac{-i}{2\sqrt{2}} + \frac{-i}{2\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}}.$$

(ii) Define

$$f(z) = \frac{z^2}{(z^2+1)(z^2+4)},$$

then

$$I = \lim_{R \to \infty} \int_0^R \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{1}{2} \lim_{R \to \infty} \int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}\, dx = \frac{1}{2} \lim_{R \to \infty} \int_{\Gamma_R} f(z)\, dz.$$

We can write $f(z) = g(z)/h(z)$ where $g(z) = z^2$ is holomorphic on $\mathbb{C}$ and $h(z) = (z^2+1)(z^2+4)$ is holomorphic on $\mathbb{C}$ and has zeros at $z = \pm i, \pm 2i$. So $f$ has isolated singularities at $z = \pm i, \pm 2i$, of which only $z = i, 2i$ lie inside $\Gamma_R$ for $R > 2$. Both these points are simple zeros of $h$, and are not zeros of $g$, so lemma 4.5.11(i) implies that they are simple poles of $f$. Writing

$$f(z) = \frac{z^2/(z+i)(z^2+4)}{z-i},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, i) = i^2/(i+i)(i^2+4) = \frac{-1}{2i(-1+4)} = \frac{-1}{6i};$$

similarly writing

$$f(z) = \frac{z^2/(z^2+1)(z+2i)}{z-2i},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, 2i) = (2i)^2/((2i)^2+1)(2i+2i) = \frac{-4}{(-4+1)(4i)} = \frac{-4}{-12i} = \frac{1}{3i}.$$

The Cauchy Residue Theorem implies that

$$I = \frac{1}{2}\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = \frac{1}{2}2\pi i\left(\operatorname{Res}(f,i) + \operatorname{Res}(f,2i)\right) = \pi i\left(\frac{-1}{6i} + \frac{1}{3i}\right) = \frac{\pi}{6}.$$

(iii) Let

$$f(z) = \frac{1}{z^3+1},$$

but as suggested we have to deal with this function a little differently. Let $\Gamma_{1,R}$ be the line segment from 0 to $R$, $\Gamma_{2,R}$ be the circular arc from $R$ to $R\exp(2i\pi/3)$ in the upper half plane, and $\Gamma_{3,R}$ be the line segment from $R\exp(2i\pi/3)$ to 0. Let $\Gamma_R$ be the loop defined by the three contours. We want to evaluate

$$I = \lim_{R\to\infty}\int_0^R \frac{1}{x^3+1}\,dx = \lim_{R\to\infty}\int_{\Gamma_{1,R}} f(z)\,dz.$$

As usual $\int_{\Gamma_{2,R}} f(z)\,dz \to 0$ as $R \to \infty$: the only difference is that the arc is shorter than the usual semicircular arcs we consider, which makes no difference to the argument.

Consider now the contribution of the integral along $\Gamma_{3,R}$. We can parametrize $-\Gamma_{3,R}$ (i.e. the contour traversed in the opposite direction—it is easier for us to write this parametrization down) by the function $\gamma\colon [0,R] \to \mathbb{C}$ given by $\gamma(t) = te^{2\pi i/3}$. Therefore, by definition, recalling we parametrized $\Gamma_{3,R}$ backwards,

$$\begin{aligned}\int_{\Gamma_{3,R}} f(z)\,dz = -\int_0^R f(\gamma(t))\gamma'(t)\,dt = -\int_0^R \frac{e^{2\pi i/3}}{\left(te^{2\pi i/3}\right)^3+1}\,dt &= -e^{2\pi i/3}\int_0^R \frac{1}{t^3+1}\,dt \\ &= -e^{2\pi i/3}\int_{\Gamma_{1,R}} f(z)\,dz.\end{aligned}$$

So

$$\begin{aligned}\int_{\Gamma_R} f(z)\,dz &= \int_{\Gamma_{1,R}} f(z)\,dz + \int_{\Gamma_{2,R}} f(z)\,dz + \int_{\Gamma_{3,R}} f(z)\,dz \\ &= \left(1 - e^{2\pi i/3}\right)\int_{\Gamma_{1,R}} f(z)\,dz + \int_{\Gamma_{2,R}} f(z)\,dz \\ &\to \left(1 - e^{2\pi i/3}\right) I \quad \text{as } R \to \infty.\end{aligned}$$

So let us evaluate the integral around $\Gamma_R$ as usual. The denominator of $f$ has zeros at $z_k = e^{i\pi(1+2k)/3}$ for $k = 0, 1, 2$, so $f$ has isolated singularities at $z_k$, of which only $z_0$ lies inside $\Gamma_R$. Since $z_k$ is a simple zero of the denominator of $f$, lemma 4.5.11(i) implies that $z_k$ is a simple pole of $f$. Then lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_0) = \frac{1}{3z_0^2} = \frac{1}{3e^{2\pi i/3}}.$$

So the Cauchy Residue Theorem implies that

$$\begin{aligned} I = \lim_{R\to\infty} \left(\frac{1}{1-e^{2\pi i/3}}\right) \int_{\Gamma_R} f(z)\, dz &= \left(\frac{1}{1-e^{2\pi i/3}}\right) 2\pi i \operatorname{Res}(f, z_0) \\ &= \frac{2\pi i}{\left(1-e^{2\pi i/3}\right) 3e^{2\pi i/3}} \\ &= \frac{2\pi i}{3\left(e^{2\pi i/3} - e^{-2\pi i/3}\right)} \\ &= \frac{2\pi}{3\sqrt{3}}. \end{aligned}$$

(iv) Let

$$f(z) = \frac{1}{z^2 + 2z + 2},$$

so

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2 + 2x + 2}\, dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz.$$

By the quadratic formula the denominator of $f$ has zeros at $z_\pm = \frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$, so the function $f$ has isolated singularities at $z_\pm$, of which only $z_+$ lies inside $\Gamma_R$, for $R > \sqrt{2}$. Since $z_+$ is a simple zero of the denominator of $f$, lemma 4.5.11(i) implies that $z_+$ is a simple pole of $f$. Writing

$$f(z) = \frac{1/(z - z_-)}{z - z_+},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, z_+) = 1/(z_+ - z_-) = \frac{1}{(-1+i) - (-1-i))} = \frac{1}{2i}.$$

Then the Cauchy Residue Theorem implies that

$$I = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz = 2\pi i \operatorname{Res}(f, z_+) = 2\pi i \frac{1}{2i} = \pi.$$

(v) Let

$$f(z) = \frac{z^2}{(z^2 + 9)^2},$$

so

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{(x^2 + 9)^2}\, dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz.$$

We can write $f(z) = g(z)/h(z)$ where $g(z) = z^2$ is holomorphic on $\mathbb{C}$ and $h(z) = (z^2+9)^2$ is holomorphic on $\mathbb{C}$ with zeros at $z_\pm = \pm 3i$. So $f$ has isolated singularities at $z_\pm$, of which only $z_+$ lies inside $\Gamma_R$ for $R > 3$. Since $g(z_+) \neq 0$, and $z_+$ is a double zero of $h$, lemma 4.5.11(i) implies that $z_+$ is a double pole of $f$. Then lemma 5.1.5 implies that

$$\begin{aligned}\operatorname{Res}(f, z_+) = \lim_{z\to z_+} \frac{d}{dz}(z-z_+)^2 f(z) &= \lim_{z\to z_+} \frac{d}{dz}\frac{z^2}{(z+3i)^2} \\ &= \lim_{z\to z_+} \frac{2z(z+3i)^2 - 2z^2(z+3i)}{(z+3i)^4} \\ &= \frac{2(3i)(6i)^2 - 2(3i)^2(6i)}{(6i)^4} \\ &= \frac{-6^3 i + 3\cdot 6^2 i}{6^4} \\ &= \frac{6^2(-6+3)i}{6^4} \\ &= \frac{-i}{12}.\end{aligned}$$

Therefore the Cauchy Residue Theorem implies that

$$I = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f, z_+) = 2\pi i \frac{-i}{12} = \frac{\pi}{6}.$$

(vi) Let

$$f(z) = \frac{1}{(z^2+1)(z^2+4)},$$

so

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+4)}\,dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz.$$

The denominator of $f$ has zeros at $z = \pm i, \pm 2i$, so $f$ has isolated singularities at $z = \pm i, \pm 2i$, of which only $z = i, 2i$ lie inside $\Gamma_R$ for $R > 2$. Each point $z = i, 2i$ is a simple zero of the denominator of $f$, so lemma 4.5.11(i) implies that $z = i, 2i$ is a simple pole of $f$. Writing

$$f(z) = \frac{1/\left((z+i)(z^2+4)\right)}{z-i},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, i) = 1/(i+i)(i^2+4) = \frac{1}{(2i)(-1+4)} = \frac{1}{6i};$$

and similarly writing

$$f(z) = \frac{1/\left((z^2+1)(z+2i)\right)}{z-2i},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f, 2i) = 1/((2i)^2+1)(2i+2i) = \frac{1}{(-4+1)(4i)} = \frac{-1}{12i}.$$

So the Cauchy Residue Theorem implies that

$$I = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 2\pi i\left(\operatorname{Res}(f, i) + \operatorname{Res}(f, 2i)\right) = 2\pi i \left(\frac{1}{6i} - \frac{1}{12i}\right) = \frac{\pi}{6}.$$

(vii) Let

$$f(z) = \frac{z}{(z^2+4z+13)^2},$$

so

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{x}{(x^2+4x+13)^2}\,dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz.$$

We can write $f(z) = g(z)/h(z)$ where $g(z) = z$ is holomorphic on $\mathbb{C}$ and $h(z) = (z^2+4z+13)^2$ is holomorphic on $\mathbb{C}$ with, by the quadratic formula, zeros at $z_\pm = \frac{-4\pm\sqrt{16-4\cdot 13}}{2} = -2 \pm 3i$. So the function $f$ has isolated singularities at the points $z_\pm$, of which only $z_+$ lies inside $\Gamma_R$ for $R > \sqrt{13}$. Since $z_+$ is a double zero of $h$ and $g(z_+) \neq 0$, lemma 4.5.11(i) implies that $z_+$ is a double pole of $f$. Then lemma 5.1.5 implies that

$$\begin{aligned}\operatorname{Res}(f, z_+) = \lim_{z\to z_+} \frac{d}{dz}(z-z_+)^2 f(z) &= \lim_{z\to z_+} \frac{d}{dz} \frac{z}{(z-z_-)^2} \\ &= \lim_{z\to z_+} \frac{(z-z_-)^2 - 2z(z-z_-)}{(z-z_-)^4} \\ &= \frac{(6i)^2 - 2(-2+3i)6i}{(6i)^4} \\ &= \frac{-36+24i+36}{36^2} \\ &= \frac{i}{54}.\end{aligned}$$

Then the Cauchy Residue Theorem implies that

$$I = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f, z_+) = 2\pi i \frac{i}{54} = -\frac{\pi}{27}.$$

**Exercise 5.4.8.** Evaluate the following integrals:

(i) p. v.$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1}\,dx$;
(ii) $\int_0^\infty \frac{\cos x}{(x^2+1)^2}\,dx$;
(iii) p. v.$\int_{-\infty}^{\infty} \frac{x \sin 3x}{x^2+4}\,dx$;
(iv) p. v.$\int_{-\infty}^{\infty} \frac{\exp(3ix)}{x-2i}\,dx$;
(v) p. v.$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\,dx$;
(vi) p. v.$\int_{-\infty}^{\infty} \frac{\exp(-2ix)}{x^2+4}\,dx$;
(vii) p. v.$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x-3i}\,dx$; and
(viii) $\int_0^\infty \frac{x^3 \sin(2x)}{(x^2+1)^2}\,dx$.

*Solution.* Again, we let $\gamma_R$ be the line segment from $-R$ to $R$, $C_R^\pm$ be the semicircular arc in the upper $(+)$ or lower $(-)$ half-plane from $R$ to $-R$, and $\Gamma_R$ be the associated loop.

(i) Define

$$f(z) = \frac{\exp(2iz)}{z^2+1},$$

so

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos(2x)}{x^2+1}\,dx = \operatorname{Re}\left(\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz\right) = \operatorname{Re}\left(\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz\right),$$

if $C_R^+$ is used, since the Jordan Lemma implies that the contribution to the integral around $C_R^+$ goes to 0 as $R \to \infty$. We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(2iz)$ is holomorphic on $\mathbb{C}$ and $h(z) = z^2+1$ is holomorphic on $\mathbb{C}$ and has zeros at $z_\pm = \pm i$. So $f$ has isolated singularities at $\pm i$ of which only $i$ lies inside $\Gamma_R$ for

$R > 1$. Since $i$ is a simple zero of $h$, and $g(i) \neq 0$, lemma 4.5.11(i) implies that $i$ is a simple pole of $f$. Writing

$$f(z) = \frac{\exp(2iz)/(z+i)}{z-i},$$

lemma 5.1.7 implies that

$$\mathrm{Res}\,(f, i) = \exp(2i^2)/(i+i) = \frac{1}{2ie^2}.$$

The Cauchy Residue Theorem implies that

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 2\pi i \mathrm{Res}\,(f, i) = 2\pi i \frac{1}{2ie^2} = \frac{\pi}{e^2},$$

so

$$I = \mathrm{Re}\left(\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz\right) = \frac{\pi}{e^2}.$$

(ii) First we note that

$$I = \lim_{R\to\infty} \int_0^R \frac{\cos x}{(x^2+1)^2}\,dx = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^R \frac{\cos x}{(x^2+1)^2}\,dx.$$

Define

$$f(z) = \frac{\exp(iz)}{(z^2+1)^2},$$

so

$$I = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^R \frac{\cos x}{(x^2+1)^2}\,dx = \frac{1}{2}\mathrm{Re}\left(\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz\right) = \frac{1}{2}\mathrm{Re}\left(\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz\right),$$

if $C_R^+$ is used to close the contour, since the Jordan Lemma implies that the contribution around $C_R^+$ tends to 0 as $R \to \infty$.

We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(iz)$ is holomorphic on $\mathbb{C}$ and $h(z) = (z^2+1)^2$ is holomorphic on $\mathbb{C}$ and has zeros at $z_\pm = \pm i$. So the function $f$ has isolated singularities at $\pm i$, of which only $i$ lies inside $\Gamma_R$ for $R > 1$. Since $i$ is a double zero of $h$ and $g(i) \neq 0$, lemma 4.5.11(i) implies that $i$ is a double pole of $f$. Then lemma 5.1.5 implies that

$$\begin{aligned}
\mathrm{Res}\,(f, i) = \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z) &= \lim_{z\to i} \frac{d}{dz}\frac{\exp(iz)}{(z+i)^2} \\
&= \lim_{z\to i} \frac{i(z+i)^2\exp(iz) - 2\exp(iz)(z+i)}{(z+i)^4} \\
&= \frac{i(2i)^2\exp(-1) - 2\exp(-1)(2i)}{(2i)^4} \\
&= \frac{-4ie^{-1} - 4ie^{-1}}{2^4} \\
&= \frac{1}{2ie}.
\end{aligned}$$

The Cauchy Residue Theorem implies that

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 2\pi i \mathrm{Res}\,(f, i) = 2\pi i \frac{1}{2ie} = \frac{\pi}{e},$$

so

$$I = \frac{1}{2}\mathrm{Re}\left(\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz\right) = \frac{\pi}{2e}.$$

(iii) Define
$$f(z) = \frac{z\exp(3iz)}{z^2+4},$$
so
$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{x\sin(3x)}{x^2+4}\,dx = \mathrm{Im}\left(\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz\right) = \mathrm{Im}\left(\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz\right),$$
if $C_R^+$ is used to close the contour, since the Jordan Lemma implies that the contribution around $C_R^+$ tends to 0 as $R\to\infty$.

We can write $f(z) = g(z)/h(z)$ where $g(z) = z\exp(3iz)$ is holomorphic on $\mathbb{C}$ and $h(z) = z^2+4$ is holomorphic on $\mathbb{C}$ with zeros at $z_\pm = \pm 2i$. So the function $f$ has isolated singularities at $\pm 2i$, of which only $2i$ lies inside $\Gamma_R$ for $R > 2$. Since $2i$ is a simple zero of $h$ and $g(2i) \neq 0$, lemma 4.5.11(i) implies that $2i$ is a simple pole of $f$. Writing
$$f(z) = \frac{z\exp(3iz)/(z+2i)}{z-2i},$$
lemma 5.1.7 implies that
$$\mathrm{Res}\,(f, 2i) = (2i)\exp(3i(2i))/(2i+2i) = \frac{1}{2e^6}.$$
The Cauchy Residue Theorem implies that
$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = 2\pi i \mathrm{Res}\,(f,2i) = 2\pi i\frac{1}{2e^6} = \frac{\pi i}{e^6},$$
so
$$I = \mathrm{Im}\left(\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz\right) = \frac{\pi}{e^6}.$$

(iv) Define
$$f(z) = \frac{\exp(3iz)}{z-2i},$$
so
$$I = \lim_{R\to\infty}\int_{-R}^{R}\frac{\exp(3ix)}{x-2i}\,dx = \lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz$$
if $C_R^+$ is used to close the contour, since the Jordan Lemma implies that the contribution around $C_R^+$ tends to 0 as $R\to\infty$. We don't even need the Residue Theorem to calculate this, since the integral is exactly in the form $g(z)/(z-z_0)$ where $g(z) = \exp(3iz)$ is holomorphic on $\mathbb{C}$ and $z_0 = 2i$ is inside $\Gamma_R$ for $R > 2$. So the Cauchy Integral Formula implies that
$$I = \lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = 2\pi i\exp(3i(2i)) = \frac{2\pi i}{e^6}.$$

(v) Define
$$f(z) = \frac{\exp(iz)}{(z^2+1)(z^2+4)},$$
so
$$I = \lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x}{(x^2+1)(x^2+4)}\,dx = \mathrm{Re}\left(\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz\right) = \mathrm{Re}\left(\int_{\Gamma_R} f(z)\,dz\right),$$
if $C_R^+$ is used to close the contour, since the Jordan Lemma implies that the contribution around $C_R^+$ tends to 0 as $R\to\infty$.

We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(iz)$ is holomorphic on $\mathbb{C}$ and $h(z) = (z^2+1)(z^2+4)$ has zeros at $z = \pm i, \pm 2i$. So the function $f$ has isolated singularities at $\pm i, \pm 2i$, of which only $i, 2i$ lie inside $\Gamma_R$ for $R > 2$. Since $i, 2i$ are simple zeros of $h$ and $g(i), g(2i) \neq 0$, lemma 4.5.11(i) implies that $i, 2i$ are simple poles of $f$. Writing

$$f(z) = \frac{\exp(iz)/(z+i)(z^2+4)}{z-i},$$

lemma 5.1.7 implies that

$$\mathrm{Res}\,(f, i) = \exp(i^2)/(i+i)(i^2+4) = \frac{e^{-1}}{(2i)(-1+4)} = \frac{1}{6ie};$$

similarly writing

$$f(z) = \frac{\exp(iz)/(z+2i)(z^2+1)}{z-2i},$$

lemma 5.1.7 implies that

$$\mathrm{Res}\,(f, 2i) = \exp(i(2i))/(2i+2i)((2i)^2+1) = \frac{e^{-2}}{(4i)(-4+1)} = \frac{-1}{12ie^2}.$$

So the Cauchy Residue Theorem implies that

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 2\pi i\,(\mathrm{Res}\,(f,i) + \mathrm{Res}\,(f,2i)) = 2\pi i \left(\frac{1}{6ie} - \frac{1}{12ie^2}\right) = \pi\left(\frac{1}{3e} - \frac{1}{6e^2}\right).$$

So

$$I = \mathrm{Re}\left(\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz\right) = \pi\left(\frac{1}{3e} - \frac{1}{6e^2}\right).$$

(vi) Let

$$f(z) = \frac{\exp(-2iz)}{z^2+4},$$

so

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{\exp(-2ix)}{x^2+4}\,dx = \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz,$$

if $C_R^-$ is used to close the contour, since the Jordan Lemma implies that the contribution around $C_R^-$ tends to 0 as $R \to \infty$.

We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(-2iz)$ is holomorphic on $\mathbb{C}$ and $h(z) = z^2+4$ is holomorphic on $\mathbb{C}$ with zeros at $\pm 2i$. So the function $f$ has isolated singularities at $\pm 2i$, of which only $-2i$ lies inside $\Gamma_R$ for $R > 2$ (recalling that we have closed the contour with the semicircular arc in the lower half plane). Since $-2i$ is a simple zero of $h$ and $g(-2i) \neq 0$, lemma 4.5.11(i) implies that $-2i$ is a simple pole of $f$. Writing

$$f(z) = \frac{\exp(-2iz)/(z-2i)}{z+2i},$$

lemma 5.1.7 implies that

$$\mathrm{Res}\,(f, -2i) = \exp(-2i(-2i))/(-2i-2i) = \frac{\exp(-4)}{-4i} = \frac{1}{-4ie^4}.$$

Note that the choice of the semicircular arc in the lower half plane means that $\Gamma_R$ is traversed in the negative orientation, hence the change in sign when we apply the Cauchy Residue Theorem to see that

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = -2\pi i \mathrm{Res}\,(f, -2i) = -2\pi i \frac{1}{-4ie^4} = \frac{\pi}{2e^4}.$$

So

$$I = \lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = \frac{\pi}{2e^4}.$$

(vii) We have to be careful with this integral, because since there is a complex term in the denominator, we cannot immediately write it as the real part of the integral with an exponential in place of the cosine in the numerator. Nonetheless, it is true that for $x \in \mathbb{R}$ we have

$$\cos(2x) = \frac{\exp(2ix) + \exp(-2ix)}{2},$$

so we can write

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos(2x)}{x - 3i}\, dx = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{\exp(2ix)}{x - 3i} + \frac{\exp(-2ix)}{x - 3i}\, dx.$$

So we have that

$$\begin{aligned} I &= \frac{1}{2} \lim_{R\to\infty} \left( \int_{\gamma_R} \frac{\exp(2iz)}{z - 3i}\, dz + \int_{\gamma_R} \frac{\exp(-2iz)}{z - 3i}\, dz \right) \\ &= \frac{1}{2} \lim_{R\to\infty} \left( \int_{\Gamma_R^+} \frac{\exp(2iz)}{z - 3i}\, dz + \int_{\Gamma_R^-} \frac{\exp(-2iz)}{z - 3i}\, dz \right), \end{aligned}$$

where $\Gamma_R^\pm$ is the contour we get by closing $\gamma_R$ with the contour $C_R^\pm$, which is chosen for each integral so that the contribution around $C_R^\pm$ goes to 0 as $R \to \infty$. We do not need the Residue Theorem to evaluate these integrals: by the Cauchy Integral Formula we have for $R > 3$ that that

$$\int_{\Gamma_R^+} \frac{\exp(2iz)}{z - 3i}\, dz = 2\pi i \exp(2i(3i)) = 2\pi i e^{-6},$$

and

$$\int_{\Gamma_R^-} \frac{\exp(-2iz)}{z - 3i}\, dz = 0,$$

since the only point at which that integrand is non-holomorphic, $3i$, does not lie in the contour $\Gamma_R^-$ which uses the semicircular contour in the lower half plane. So

$$I = \frac{1}{2} 2\pi i e^{-6} = \frac{\pi i}{e^6}.$$

(viii) Let

$$f(z) = \frac{z^3 \exp(2iz)}{(z^2 + 1)^2},$$

so, since the real integrand $\frac{x^3 \sin(2x)}{(x^2+1)^2}$ is even, we have that

$$\begin{aligned} I = \lim_{R\to\infty} \int_0^R \frac{x^3 \sin(2x)}{(x^2 + 1)^2}\, dx = &= \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 \sin(2x)}{(x^2 + 1)^2}\, dx \\ &= \frac{1}{2} \operatorname{Im} \left( \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 \exp(2ix)}{(x^2 + 1)^2}\, dx \right) \\ &= \frac{1}{2} \operatorname{Im} \left( \lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz \right) \\ &= \frac{1}{2} \operatorname{Im} \left( \lim_{R\to\infty} \int_{\Gamma_R} f(z)\, dz \right), \end{aligned}$$

if $C_R^+$ is used to close the contour, since the Jordan Lemma implies that the contribution around $C_R^+$ tends to 0 as $R \to \infty$.

We can write $f(z) = g(z)/h(z)$ where $g(z) = z^3 \exp(2iz)$ is holomorphic on $\mathbb{C}$ and $h(z) = (z^2 + 1)^2$ is holomorphic on $\mathbb{C}$ with zeros at $z = \pm i$. So $f$ has isolated singularities at $\pm i$, of which only $i$ lies inside $\Gamma_R$ for $R > 1$. Since $i$ is a double zero of $h$ and $g(i) \neq 0$, lemma 4.5.11(i) implies that $i$ is a double pole of $f$. Lemma 5.1.5 implies that

$$\begin{aligned}
\operatorname{Res}(f,i) &= \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z) \\
&= \lim_{z\to i} \frac{d}{dz}\frac{z^3\exp(2iz)}{(z+i)^2} \\
&= \lim_{z\to i} \frac{(z+i)^2\left(2iz^3\exp(2iz)+3z^2\exp(2iz)\right)-2z^3\exp(2iz)(z+i)}{(z+i)^4} \\
&= \frac{(2i)^2\left(2e^{-2}-3e^{-2}\right)-4e^{-2}}{2^4} \\
&= \frac{-4(-e^{-2})-4e^{-2}}{2^4} \\
&= 0.
\end{aligned}$$

The Cauchy Residue Theorem implies that

$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f,i) = 0,$$

so

$$I = \operatorname{Im}\left(\lim_{R\to\infty}\int_{\Gamma_R} f(z)\,dz\right) = 0.$$

**5.5. Application: improper integrals with poles.**

**Exercise 5.5.5.** Evaluate the following integrals:

(i) p. v.$\int_{-\infty}^{\infty} \frac{1}{x^3-1}\,dx$;

(ii) $\int_0^{\infty} \frac{\cos(ax)-\cos(bx)}{x^2}\,dx$ where $0 < a < b$; and

(iii) $\int_0^{\infty} \frac{\sin x}{x(x^2+1)}\,dx$.

*Solution.* (i) Since the integrand is undefined at $x = 1$, we interpret it as

$$I = \text{p. v.}\int_{-\infty}^{\infty} \frac{1}{x^3-1}\,dx = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\left(\int_{-R}^{1-\varepsilon}\frac{1}{x^3-1}\,dx + \int_{1+\varepsilon}^{R}\frac{1}{x^3-1}\,dx\right).$$

Let $f(z) = \frac{1}{z^3-1}$. We can write $f(z) = g(z)/h(z)$ where $g(z) = 1$ is holomorphic, and $h(z) = z^3 - 1$ has zeros at $z_k = e^{2\pi i k/3}$ for $k = 0, 1, 2$. Therefore the function $f$ has singularities at $z_k = e^{2\pi i k/3}$ for $k = 0, 1, 2$.

We would like to integrate this function along the real axis, but $f$ has a singularity at $z = 1$, so we need to consider a contour with an indentation around the point $z = 1$. For $R > 2$ and $\varepsilon \in (0, 1)$ consider the loop $\Gamma_{R,\varepsilon}$ which is constructed as follows: follow the line segment $\gamma^-_{R,\varepsilon}$ from $-R$ to $1-\varepsilon$ along the real axis, the semicircular contour $-C^+_\varepsilon(1)$ from $1-\varepsilon$ to $1+\varepsilon$ in the upper half plane, traversed clockwise (hence the minus sign in the notation), the line segment $\gamma^+_{R,\varepsilon}$ from $1+\varepsilon$ to $R$ along the real axis, and then the semicircular contour $C^+_R$ from $R$ to $-R$ in the upper half plane.

Then

$$\int_{\Gamma_{R,\varepsilon}} f(z)\,dz = \int_{\gamma^-_{R,\varepsilon}} f(z)\,dz + \int_{-C^+_\varepsilon(1)} f(z)\,dz + \int_{\gamma^+_{R,\varepsilon}} f(z)\,dz + \int_{C^+_R} f(z)\,dz.$$

We have defined the contour $\Gamma_{R,\varepsilon}$ so that it has an indentation around $z_0$, and $z_1$ lies inside the contour. We therefore need to calculate the residues of $f = g/h$ at both these points. The points $z_0, z_1$ are simple zeros of the function $h$, and

$g(z_0) = g(z_1) = 1 \neq 0$, so lemma 4.5.11(i) implies that these are simple poles of the function $f$. Lemma 5.1.7 then implies that

$$\begin{aligned}
\operatorname{Res}(f, z_0) &= \frac{g(z_0)}{h'(z_0)} = \frac{1}{3z_0^2} = \frac{1}{3}; \text{ and} \\
\operatorname{Res}(f, z_1) &= \frac{g(z_1)}{h'(z_1)} = \frac{1}{3z_1^2} = \frac{1}{3\left(e^{2\pi i/3}\right)^2} = \frac{e^{2\pi i/3}}{3}.
\end{aligned}$$

The Cauchy Residue Theorem implies that

$$\int_{\Gamma_{R,\varepsilon}} f(z)\, dz = 2\pi i \operatorname{Res}(f, z_1) = \frac{2\pi i e^{2\pi i/3}}{3} = \frac{2\pi i}{3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

Lemma 5.5.3 implies that

$$\lim_{\varepsilon \to 0} \int_{-C_\varepsilon^+(1)} f(z)\, dz = -i\pi \operatorname{Res}(f, 1) = \frac{-i\pi}{3},$$

where the minus sign comes from the fact that $-C_\varepsilon^+(1)$ is traversed the wrong way.

By the same argument as in the solutions to exercise 5.4.4, since $f(z) = P(z)/Q(z)$ where $P(z) = 1$ and $Q(z) = z^3 - 1$, so $\deg(Q) = 3 \geq 2 = 0 + 2 = \deg(P) + 2$, we see that

$$\lim_{R \to \infty} \int_{C_R^+} f(z)\, dz = 0.$$

So, collecting our information, we have that

$$\begin{aligned}
I &= \lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \left( \int_{\gamma_{R,\varepsilon}^-} f(z)\, dz + \int_{\gamma_{R,\varepsilon}^+} f(z)\, dz \right) \\
&= \lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \left( \int_{\Gamma_{R,\varepsilon}} f(z)\, dz - \int_{-C_\varepsilon^+(1)} f(z)\, dz - \int_{C_R^+} f(z)\, dz \right) \\
&= \lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \int_{\Gamma_{R,\varepsilon}} f(z)\, dz - \lim_{\varepsilon \to 0} \int_{-C_\varepsilon^+(1)} f(z)\, dz - \lim_{R \to \infty} \int_{C_R^+} f(z)\, dz \\
&= \frac{2\pi i}{3}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \frac{i\pi}{3} - 0 \\
&= \frac{-\pi}{\sqrt{3}}.
\end{aligned}$$

(ii) Consider the integral

$$\begin{aligned}
&\text{p. v.} \int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\, dx \\
&= \lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \left( \int_{-R}^{-\varepsilon} \frac{\cos(ax) - \cos(bx)}{x^2}\, dx + \int_{\varepsilon}^{R} \frac{\cos(ax) - \cos(bx)}{x^2}\, dx \right).
\end{aligned}$$

The integrand in the question is even, so the integral in the question (from 0 to $\infty$) can be evaluated by calculating the above integral and dividing by 2. The above integral is in turn the real part of the integral

$$I_0 = \lim_{\substack{R \to \infty \\ \varepsilon \to 0}} \left( \int_{-R}^{-\varepsilon} \frac{\exp(iax) - \exp(ibx)}{x^2}\, dx + \int_{\varepsilon}^{R} \frac{\exp(iax) - \exp(ibx)}{x^2}\, dx \right).$$

Let $f(z) = \frac{\exp(iaz)-\exp(ibz)}{z^2}$. We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(iaz) - \exp(ibz)$ is holomorphic, and $h(z) = z^2$ is zero at $z = 0$. Therefore the function $f$ has a singularity at $z = 0$.

So we need to consider a contour with an indentation around the point $z = 0$. For $R > 1$ and $\varepsilon \in (0, 1)$ consider the loop $\Gamma_{R,\varepsilon}$ which is constructed as follows: follow the line segment $\gamma^-_{R,\varepsilon}$ from $-R$ to $-\varepsilon$ along the real axis, the semicircular contour $-C^+_\varepsilon$ from $-\varepsilon$ to $\varepsilon$ in the upper half plane, traversed clockwise (hence the minus sign in the notation), the line segment $\gamma^+_{R,\varepsilon}$ from $\varepsilon$ to $R$ along the real axis, and then the semicircular contour $C^+_R$ from $R$ to $-R$ in the upper half plane.

Then

$$\int_{\Gamma_{R,\varepsilon}} f(z)\,dz = \int_{\gamma^-_{R,\varepsilon}} f(z)\,dz + \int_{-C^+_\varepsilon} f(z)\,dz + \int_{\gamma^+_{R,\varepsilon}} f(z)\,dz + \int_{C^+_R} f(z)\,dz.$$

We have defined the contour $\Gamma_{R,\varepsilon}$ so that it has an indentation around $z = 0$. We therefore need to calculate the residue of $f = g/h$ at this point. The point $z = 0$ is a double zero of the function $h$, and a simple zero of $g$: while

$$g(0) = \exp(ia0) - \exp(ib0) = 1 - 1 = 0,$$

we have that $g'(z) = ia\exp(iaz) - ib\exp(ibz)$, so

$$g'(0) = ia\exp(ia0) - ib\exp(ib0) = i(a - b).$$

So by lemma 4.5.11(ii) the point $z = 0$ is a simple zero of $f$. Note that because this simple pole does *not* arise from a quotient of a non-zero numerator and a denominator with a simple zero, lemma 5.1.7 does not apply. Instead we use lemma 5.1.5 to calculate that

$$\begin{aligned}\operatorname{Res}(f, 0) = \lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{\exp(iaz) - \exp(ibz)}{z} = \lim_{z\to 0} \frac{g(z)}{z} &= \lim_{z\to 0} \frac{g(z) - g(0)}{z - 0} \\ &= g'(0) \\ &= i(a - b).\end{aligned}$$

Since the function $f$ is holomorphic inside and on the contour $\Gamma_{R,\varepsilon}$, Cauchy's Integral Theorem implies that

$$\int_{\Gamma_{R,\varepsilon}} f(z)\,dz = 0.$$

Lemma 5.5.3 implies that

$$\lim_{\varepsilon\to 0} \int_{-C^+_\varepsilon} f(z)\,dz = -i\pi\operatorname{Res}(f, 0) = -i\pi(i(a - b)) = \pi(a - b).$$

where the minus sign comes from the fact that $-C^+_\varepsilon$ is traversed the wrong way.

By the Jordan Lemma, splitting the function $f$ into two summands $\exp(iaz)/z^2$ and $\exp(ibz)/z^2$, we see that

$$\lim_{R\to\infty} \int_{C^+_R} f(z)\,dz = 0.$$

So, collecting our information, we have that

$$\begin{aligned}
I_0 &= \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left( \int_{\gamma^-_{R,\varepsilon}} f(z)\,dz + \int_{\gamma^+_{R,\varepsilon}} f(z)\,dz \right) \\
&= \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left( \int_{\Gamma_{R,\varepsilon}} f(z)\,dz - \int_{-C^+_\varepsilon} f(z)\,dz - \int_{C^+_R} f(z)\,dz \right) \\
&= \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\Gamma_{R,\varepsilon}} f(z)\,dz - \lim_{\varepsilon\to 0} \int_{-C^+_\varepsilon} f(z)\,dz - \lim_{R\to\infty} \int_{C^+_R} f(z)\,dz \\
&= 0 - \pi(a-b) - 0 \\
&= \pi(b-a).
\end{aligned}$$

We now simply recall that the integral $I$ we can in fact wanted was half of the real part of this integral, thus we conclude that

$$I = \int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,dx = \frac{1}{2}\operatorname{Re}(I_0) = \frac{1}{2}\operatorname{Re}(\pi(b-a)) = \frac{\pi}{2}(b-a).$$

(iii) Consider the integral

$$\text{p. v.}\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,dx = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left( \int_{-R}^{-\varepsilon} \frac{\sin x}{x(x^2+1)}\,dx + \int_{\varepsilon}^{R} \frac{\sin x}{x(x^2+1)}\,dx \right).$$

The integrand in the question is even, so the integral in the question (from 0 to $\infty$) can be evaluated by calculating the above integral and dividing by 2. The above integral is in turn the imaginary part of the integral

$$I_0 = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \left( \int_{-R}^{-\varepsilon} \frac{\exp(ix)}{x(x^2+1)}\,dx + \int_{\varepsilon}^{R} \frac{\exp(ix)}{x(x^2+1)}\,dx \right).$$

Let $f(z) = \frac{\exp(iz)}{z(z^2+1)}$. We can write $f(z) = g(z)/h(z)$ where $g(z) = \exp(iz)$ is holomorphic, and $h(z) = z(z^2+1)$ is zero at $z = 0, \pm i$. Therefore the function $f$ has a singularities at $z = 0, \pm i$.

So we need to consider a contour with an indentation around the point $z = 0$. For $R > 1$ and $\varepsilon \in (0,1)$ consider the loop $\Gamma_{R,\varepsilon}$ which is constructed as follows: follow the line segment $\gamma^-_{R,\varepsilon}$ from $-R$ to $-\varepsilon$ along the real axis, the semicircular contour $-C^+_\varepsilon$ from $-\varepsilon$ to $\varepsilon$ in the upper half plane, traversed clockwise (hence the minus sign in the notation), the line segment $\gamma^+_{R,\varepsilon}$ from $\varepsilon$ to $R$ along the real axis, and then the semicircular contour $C^+_R$ from $R$ to $-R$ in the upper half plane.

Then

$$\int_{\Gamma_{R,\varepsilon}} f(z)\,dz = \int_{\gamma^-_{R,\varepsilon}} f(z)\,dz + \int_{-C^+_\varepsilon} f(z)\,dz + \int_{\gamma^+_{R,\varepsilon}} f(z)\,dz + \int_{C^+_R} f(z)\,dz.$$

We have defined the contour $\Gamma_{R,\varepsilon}$ so that it has an indentation around $z = 0$, and the point $z = i$ is in its interior. We therefore need to calculate the residue of $f = g/h$ at these points. Both points are not zeros of the function $g$, and are simple zeros of the function $h$. So lemma 4.5.11(i) implies that they are simple poles of $f$. Lemma 5.1.7 then implies that

$$\operatorname{Res}(f,0) = \frac{g(0)}{h'(0)} = \frac{\exp(i0)}{3(0)^2+1} = 1; \text{ and}$$

$$\operatorname{Res}(f,i) = \frac{g(i)}{h'(i)} = \frac{\exp(i^2)}{3i^2+1} = \frac{-1}{2e}.$$

The Cauchy Residue Theorem implies that

$$\int_{\Gamma_{R,\varepsilon}} f(z)\,dz = 2\pi i \mathrm{Res}\,(f,i) = 2\pi i\frac{-1}{2e} = \frac{-\pi i}{e}.$$

Lemma 5.5.3 implies that

$$\lim_{\varepsilon\to 0}\int_{-C_\varepsilon^+} f(z)\,dz = -i\pi\mathrm{Res}\,(f,0) = -i\pi.$$

where the minus sign comes from the fact that $-C_\varepsilon^+$ is traversed the wrong way.

By the Jordan Lemma, we see that

$$\lim_{R\to\infty}\int_{C_R^+} f(z)\,dz = 0.$$

So, collecting our information, we have that

$$\begin{aligned}
I_0 &= \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\left(\int_{\gamma_{R,\varepsilon}^-} f(z)\,dz + \int_{\gamma_{R,\varepsilon}^+} f(z)\,dz\right)\\
&= \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\left(\int_{\Gamma_{R,\varepsilon}} f(z)\,dz - \int_{-C_\varepsilon^+} f(z)\,dz - \int_{C_R^+} f(z)\,dz\right)\\
&= \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_{R,\varepsilon}} f(z)\,dz - \lim_{\varepsilon\to 0}\int_{-C_\varepsilon^+} f(z)\,dz - \lim_{R\to\infty}\int_{C_R^+} f(z)\,dz\\
&= -\frac{\pi i}{e} + i\pi - 0\\
&= \pi i\left(1-\frac{1}{e}\right).
\end{aligned}$$

We now simply recall that the integral $I$ we can in fact wanted was half of the imaginary part of this integral, thus we conclude that

$$I = \int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx = \frac{1}{2}\,\mathrm{Im}(I_0) = \frac{1}{2}\,\mathrm{Im}\left(\pi i\left(1-\frac{1}{e}\right)\right) = \frac{\pi}{2}\left(1-\frac{1}{e}\right).$$

**5.6. Infinite series.**

**Exercise 5.6.2.** (i) By considering the function $f(z) = \pi\cot(\pi z)/(z^2+1)$, evaluate

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1}.$$

(ii) By considering the function $f(z) = \pi\cot(\pi z)/(z-(1/2))^2$, evaluate

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n-(1/2))^2}.$$

*Solution.* (i) We can write $f(z) = g(z)/h(z)$ where $g(z) = \pi\cos(\pi z)$ is holomorphic on $\mathbb{C}$ and $h(z) = \sin(\pi z)(z^2+1)$ is holomorphic on $\mathbb{C}$ with zeros at $z_n = n$ for $n\in\mathbb{Z}$ and $z=\pm i$. So $f$ has isolated singularities at all integers $n$ and at $\pm i$. The points $n$ and $\pm i$ are simple zeros of $h$, and not zeros of $g$, so lemma 4.5.11(i) implies that $f$ has simple poles at $n$ and $\pm i$. Writing

$$f(z) = \frac{\pi\cos(\pi z)/(z^2+1)}{\sin(\pi z)},$$

lemma 5.1.7 implies that

$$\mathrm{Res}\,(f,n) = \frac{\pi\cos(\pi n)/(n^2+1)}{\pi\cos(\pi n)} = \frac{1}{n^2+1};$$

and similarly writing

$$f(z) = \frac{\pi\cos(\pi z)/(\sin(\pi z)(z+i))}{z-i},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f,i) = \pi\cos(\pi i)/\sin(\pi i)(i+i) = \frac{\pi\cot(\pi i)}{2i};$$

and writing

$$f(z) = \frac{\pi\cos(\pi z)/(\sin(\pi z)(z-i))}{z+i},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f,-i) = \pi\cos(-\pi i)/\sin(-\pi i)(-i-i) = \frac{\pi\cos(\pi i)}{\sin(\pi i)2i} = \frac{\pi\cot(\pi i)}{2i}.$$

Consider the square contour $\Gamma_N$ with vertices at $(N+\frac{1}{2})(1\pm i)$ and $(N+\frac{1}{2})(-1\pm i)$, as used in example 5.6.1. For each $N \geq 1$ the Cauchy Residue Theorem implies that

$$\begin{aligned}\int_{\Gamma_N} f(z)\,dz &= 2\pi i\left(\left(\sum_{n=-N}^{N}\operatorname{Res}(f,n)\right) + \operatorname{Res}(f,i) + \operatorname{Res}(f,-i)\right)\\ &= 2\pi i\left(\left(\sum_{n=-N}^{N}\frac{1}{n^2+1}\right) + 2\frac{\pi\cot(\pi i)}{2i}\right).\end{aligned}$$

As in example 5.6.1, we assume the existence of $K > 0$ such that $|\cot(\pi z)| \leq K$ for all $z \in \Gamma_N$ for all $N \in \mathbb{N}$. So

$$\left|\int_{\Gamma_N} f(z)\,dz\right| \leq \max_{z\in\Gamma_N}\left|\frac{\pi\cot(\pi z)}{z^2+1}\right|\ell(\Gamma_N) \leq \frac{\pi K}{|z^2|-1}4(2N+1) \leq \frac{4(2N+1)\pi K}{N^2-1}$$
$$\to 0 \text{ as } N\to\infty.$$

Taking the limit as $N\to\infty$ of the expression for $\int_{\Gamma_N} f(z)\,dz$, we then have that

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+1} = \frac{-2\pi\cot(\pi i)}{2i} = \frac{-\pi\cot(\pi i)}{i} = \pi\coth(\pi).$$

(ii) We can write $f(z) = g(z)/h(z)$ where $g(z) = \pi\cos(\pi z)$ is holomorphic on $\mathbb{C}$ and $h(z) = \sin(\pi z)(z-(1/2))^2$ is holomorphic on $\mathbb{C}$ with zeros at $z_n = n$ for $n \in \mathbb{Z}$ and $z = 1/2$. So $f$ has isolated singularities at all integers $n$ and at $1/2$. The points $n$ are simple zeros of $h$, and not zeros of $g$, so lemma 4.5.11(i) implies that $f$ has simple poles at $n$. Writing

$$f(z) = \frac{\pi\cos(\pi z)/(z-(1/2))^2}{\sin(\pi z)},$$

lemma 5.1.7 implies that

$$\operatorname{Res}(f,n) = \frac{\pi\cos(\pi n)/(n-(1/2))^2}{\pi\cos(\pi n)} = \frac{1}{(n-(1/2))^2}.$$

The point $1/2$ is a double zero of $h$ but is also a simple zero of $g$. Therefore lemma 4.5.11(ii) implies that it is a simple pole of $f$. We then calculate the residue at 1/2 using lemma 5.1.5 as follows:

$$\begin{aligned}
\operatorname{Res}(f, 1/2) = \lim_{z\to 1/2}(z-1/2)f(z) &= \lim_{z\to 1/2} \frac{\pi\cos(\pi z)}{(z-1/2)\sin(\pi z)} \\
&= \lim_{z\to 1/2} \frac{\pi(\cos(\pi z)-\cos(\pi/2))}{z-1/2}\cdot\frac{1}{\sin(\pi z)} \\
&= \pi \lim_{z\to 1/2} \frac{\cos(\pi z)-\cos(\pi/2)}{z-1/2}\cdot \lim_{z\to 1/2}\frac{1}{\sin(\pi z)} \\
&= \pi \left.\frac{d}{dz}\left(\cos(\pi z)\right)\right|_{z=1/2} \\
&= -\pi^2 \sin(\pi z)|_{z=1/2} \\
&= -\pi^2.
\end{aligned}$$

Consider the square contour $\Gamma_N$ with vertices at $(N+\frac{1}{2})(1\pm i)$ and $(N+\frac{1}{2})(-1\pm i)$, as used in example 5.6.1. For each $N \geq 1$ the Cauchy Residue Theorem implies that

$$\begin{aligned}
\int_{\Gamma_N} f(z)\,dz &= 2\pi i\left(\left(\sum_{n=-N}^{N}\operatorname{Res}(f,n)\right) + \operatorname{Res}(f,1/2)\right) \\
&= 2\pi i\left(\sum_{n=-N}^{N}\frac{1}{(n-(1/2))^2} - \pi^2\right).
\end{aligned}$$

As in example 5.6.1, we assume the existence of $K>0$ such that $|\cot(\pi z)| \leq K$ for all $z\in\Gamma_N$ for all $N\in\mathbb{N}$. So

$$\left|\int_{\Gamma_N} f(z)\,dz\right| \leq \max_{z\in\Gamma_N}\left|\frac{\pi\cot(\pi z)}{(z-(1/2))^2}\right|\ell(\Gamma_N) \leq \frac{\pi K}{(|z|-(1/2))^2}4(2N+1) \leq \frac{4(2N+1)\pi K}{N^2} \to 0 \text{ as } N\to\infty.$$

Taking the limit as $N\to\infty$ of the expression for $\int_{\Gamma_N} f(z), dz$, we then have that

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n-(1/2))^2} = \pi^2.$$

**Lemma 5.6.4.** Let $0\leq k\leq n$ be non-negative integers, and let $\binom{n}{k}$ be the usual binomial coefficient, and let $\Gamma$ be a loop with 0 in its interior. Then

$$\binom{n}{k} = \frac{1}{2\pi i}\int_\Gamma \frac{(1+z)^n}{z^{k+1}}\,dz.$$

*Proof.* We expand using the usual binomial formula and use linearity of integration to see that

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{(1+z)^n}{z^{k+1}}\,dz = \frac{1}{2\pi i}\int_\Gamma z^{-(k+1)}\sum_{j=0}^{n}\binom{n}{j}z^j\,dz &= \frac{1}{2\pi i}\sum_{j=0}^{n}\binom{n}{j}\int_\Gamma z^{j-(k+1)}\,dz \\
&= \frac{1}{2\pi i}\binom{n}{k}2\pi i \\
&= \binom{n}{k}
\end{aligned}$$

since

$$\int_\Gamma z^l\,dz = \begin{cases} 0 & l\neq -1, \\ 2\pi i & l=-1. \end{cases}$$

□