Fourier Series

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GLOSSARY

Bounded variation A function f has *bounded variation* on a closed interval $[a, b]$ if there exists a positive constant B such that, for all finite sets of points $a = x_0 < x_1 < \cdots < x_N = b$, the inequality $\sum_{i=1}^{N} |\mathrm{f}(x_i) - \mathrm{f}(x_{i-1})| \leq B$ is satisfied. Jordan proved that a function has bounded variation if and only if it can be expressed as the difference of two nondecreasing functions.

Countably infinite set A set is *countably infinite* if it can be put into one-to-one correspondence with the set of natural numbers $(1, 2, \ldots, n, \ldots)$. Examples: The integers and the rational numbers are countably infinite sets.

Continuous function If $\lim_{x \to c} \mathrm{f}(x) = \mathrm{f}(c)$, then the function f is *continuous* at the point c. Such a point is called a *continuity point* for f. A function which is continuous at all points is simply referred to as continuous.

Lebesgue measure zero A set S of real numbers is said to have *Lebesgue measure zero* if, for each $\epsilon > 0$, there exists a collection $\{(a_i, b_i)\}_{i=1}^{\infty}$ of open intervals such that $S \subset \cup_{i=1}^{\infty} (a_i, b_i)$ and $\sum_{i=1}^{\infty} (b_i - a_i) \leq \epsilon$. Examples: All finite sets, and all countably infinite sets, have Lebesgue measure zero.

Odd and even functions A function f is *odd* if $\mathrm{f}(-x) = -\mathrm{f}(x)$ for all x in its domain. A function f is *even* if $\mathrm{f}(-x) = \mathrm{f}(x)$ for all x in its domain.

One-sided limits $\mathrm{f}(x-)$ and $\mathrm{f}(x+)$ denote limits of $\mathrm{f}(t)$ as t tends to x from the left and right, respectively.

Periodic function A function f is *periodic,* with *period* $P > 0$, if the identity $\mathrm{f}(x + P) = \mathrm{f}(x)$ holds for all x. Example: $\mathrm{f}(x) = |\sin x|$ is periodic with period π.

FOURIER SERIES has long provided one of the principal methods of analysis for mathematical physics, engineering, and signal processing. It has spurred generalizations and applications that continue to develop right up to the present. While the original theory of Fourier series applies to periodic functions occurring in wave motion, such as with light and sound, its generalizations often relate to wider settings, such as the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis.

I. HISTORICAL BACKGROUND

There are antecedents to the notion of Fourier series in the work of Euler and D. Bernoulli on vibrating strings, but the theory of Fourier series truly began with the profound work of Fourier on heat conduction at the beginning of the 19th century. Fourier deals with the problem of describing the evolution of the temperature $T(x, t)$ of a thin wire of length π, stretched between $x = 0$ and $x = \pi$, with a constant zero temperature at the ends: $T(0, t) = 0$ and $T(\pi, t) = 0$. He proposed that the initial temperature $T(x, 0) = \mathrm{f}(x)$ could be expanded in a series of sine functions:

$$\mathrm{f}(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{1}$$

with

$$b_n = \frac{2}{\pi} \int_0^{\pi} \mathrm{f}(x) \sin nx \, dx. \tag{2}$$

A. Fourier Series

Although Fourier did not give a convincing proof of convergence of the infinite series in Eq. (1), he did offer the conjecture that convergence holds for an "arbitrary" function f. Subsequent work by Dirichlet, Riemann, Lebesgue, and others, throughout the next two hundred years, was needed to delineate precisely which functions were expandable in such trigonometric series. Part of this work entailed giving a precise definition of function (Dirichlet), and showing that the integrals in Eq. (2) are properly defined (Riemann and Lebesgue). Throughout this article we shall state results that are always true when Riemann integrals are used (except for Section IV where we need to use results from the theory of Lebesgue integrals).

In addition to positing Eqs. (1) and (2), Fourier argued that the temperature $T(x, t)$ is a solution to the following *heat equation with boundary conditions:*

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{\partial^2 T}{\partial x^2}, && 0 < x < \pi,\ t > 0 \\ T(0, t) = T(\pi, t) &= 0, && t \geq 0 \\ T(x, 0) &= f(x), && 0 \leq x \leq \pi. \end{aligned}$$

Making use of Eq. (1), Fourier showed that the solution $T(x, t)$ satisfies

$$T(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2 t} \sin nx. \tag{3}$$

This was the first example of the use of Fourier series to solve *boundary value problems* in partial differential equations. To obtain Eq. (3), Fourier made use of D. Bernoulli's method of *separation of variables,* which is now a standard technique for solving boundary value problems.

A good, short introduction to the history of Fourier series can be found in *The Mathematical Experience*. Besides his many mathematical contributions, Fourier has left us with one of the truly great philosophical principles: "The deep study of nature is the most fruitful source of knowledge."

II. DEFINITION OF FOURIER SERIES

The Fourier sine series, defined in Eqs. (1) and (2), is a special case of a more general concept: the Fourier series for a *periodic function.* Periodic functions arise in the study of wave motion, when a basic waveform repeats itself periodically. Such periodic waveforms occur in musical tones, in the plane waves of electromagnetic vibrations, and in the vibration of strings. These are just a few examples. Periodic effects also arise in the motion of the planets, in AC electricity, and (to a degree) in animal heartbeats.

A function f is said to have period P if $\mathrm{f}(x + P) = \mathrm{f}(x)$ for all x. For notational simplicity, we shall restrict our discussion to functions of period 2π. There is no loss of generality in doing so, since we can always use a simple change of scale $x = (P/2\pi)t$ to convert a function of period P into one of period 2π.

If the function f has period 2π, then its *Fourier series* is

$$c_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\} \tag{4}$$

with *Fourier coefficients* c_0, a_n, and b_n defined by the integrals

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{f}(x) \, dx \tag{5}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{f}(x) \cos nx \, dx, \tag{6}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \mathrm{f}(x) \sin nx \, dx. \tag{7}$$

[*Note:* The sine series defined by Eqs. (1) and (2) is a special instance of Fourier series. If f is initially defined over the interval $[0, \pi]$, then it can be extended to $[-\pi, \pi]$ (as an odd function) by letting $\mathrm{f}(-x) = -\mathrm{f}(x)$, and then extended periodically with period $P = 2\pi$. The Fourier series for this odd, periodic function reduces to the sine series in Eqs. (1) and (2), because $c_0 = 0$, each $a_n = 0$, and each b_n in Eq. (7) is equal to the b_n in Eq. (2).]

It is more common nowadays to express Fourier series in an algebraically simpler form involving complex exponentials. Following Euler, we use the fact that the complex exponential $e^{i\theta}$ satisfies $e^{i\theta} = \cos\theta + i\sin\theta$. Hence

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{i\theta}),$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

From these equations, it follows by elementary algebra that Formulas (5)–(7) can be rewritten (by rewriting each term separately) as

$$c_0 + \sum_{n=1}^{\infty} \{c_n e^{inx} + c_{-n} e^{-inx}\} \tag{8}$$

with c_n defined for all integers n by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{f}(x) e^{-inx}\, dx. \tag{9}$$

The series in Eq. (8) is usually written in the form

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{10}$$

We now consider a couple of examples. First, let f_1 be defined over $[-\pi, \pi]$ by

$$\mathrm{f}_1(x) = \begin{cases} 1 & \text{if } |x| < \pi/2 \\ 0 & \text{if } \pi/2 \le |x| \le \pi \end{cases}$$

and have period 2π. The graph of f_1 is shown in Fig. 1; it is called a *square wave* in electric circuit theory. The constant c_0 is

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{f}_1(x)\, dx$$
$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1\, dx = \frac{1}{2}.$$

While, for $n \neq 0$,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{f}_1(x) e^{-inx}\, dx$$
$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-inx}\, dx$$
$$= \frac{1}{2\pi} \frac{e^{-in\pi/2} - e^{in\pi/2}}{-in}$$
$$= \frac{\sin(n\pi/2)}{n\pi}.$$

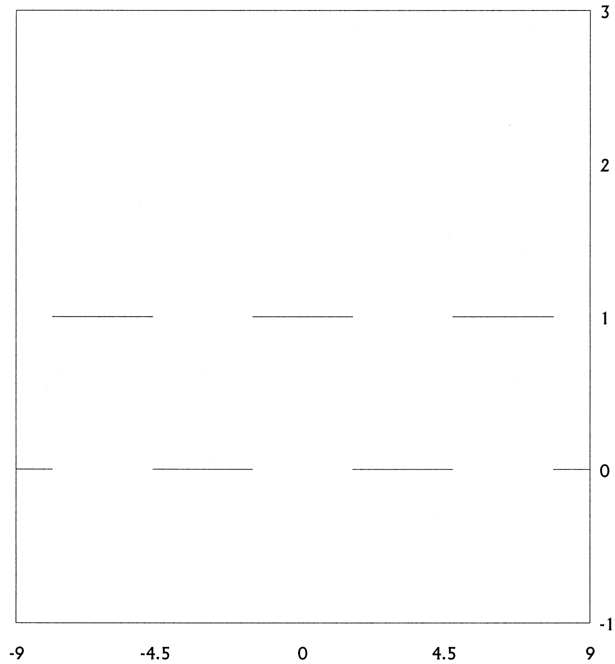

FIGURE 1 Square wave.

Thus, the Fourier series for this square wave is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n\pi}(e^{inx} + e^{-inx})$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2\sin(n\pi/2)}{n\pi} \cos nx. \tag{11}$$

Second, let $\mathrm{f}_2(x) = x^2$ over $[-\pi, \pi]$ and have period 2π, see Fig. 2. We shall refer to this wave as a *parabolic wave.* This parabolic wave has $c_0 = \pi^2/3$ and c_n, for $n \neq 0$, is

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 e^{-inx}\, dx$$
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \cos nx\, dx - \frac{i}{2\pi} \int_{-\pi}^{\pi} x^2 \sin nx\, dx$$
$$= \frac{2(-1)^n}{n^2}$$

after an integration by parts. The Fourier series for this function is then

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2}(e^{inx} + e^{-inx})$$
$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx. \tag{12}$$

We will discuss the convergence of these Fourier series, to f_1 and f_2, respectively, in Section III.

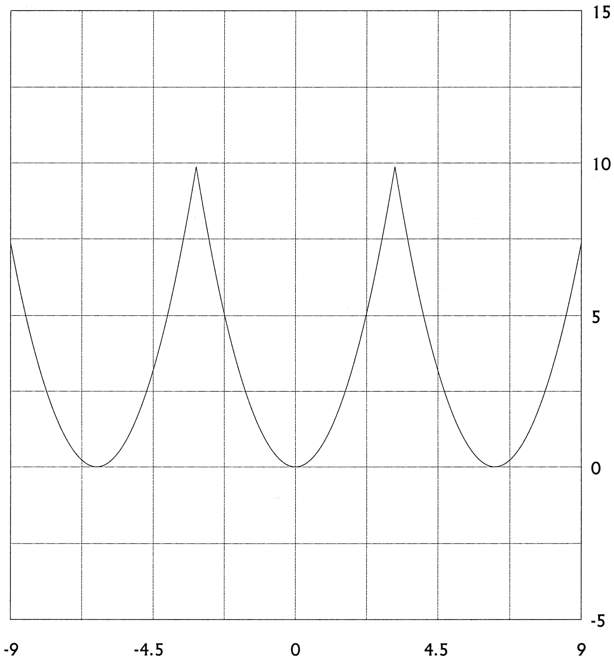

FIGURE 2 Parabolic wave.

Returning to the general Fourier series in Eq. (10), we shall now discuss some ways of interpreting this series. A complex exponential $e^{inx} = \cos nx + i \sin nx$ has a smallest period of $2\pi/n$. Consequently it is said to have a *frequency* of $n/2\pi$, because the form of its graph over the interval $[0, 2\pi/n]$ is repeated $n/2\pi$ times within each unit-length. Therefore, the integral in Eq. (9) that defines the Fourier coefficient c_n can be interpreted as a *correlation* between f and a complex exponential with a precisely located frequency of $n/2\pi$. Thus the whole collection of these integrals, for all integers n, specifies the *frequency content* of f over the set of frequencies $\{n/2\pi\}_{n=-\infty}^{\infty}$. If the series in Eq. (10) converges to f, i.e., if we can write

$$\mathrm{f}(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{13}$$

then f is being expressed as a superposition of elementary functions $c_n e^{inx}$ having frequency $n/2\pi$ and amplitude c_n. (The validity of Eq. (13) will be discussed in the next section.) Furthermore, the correlations in Eq. (9) are *independent* of each other in the sense that correlations between distinct exponentials are zero:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases} \tag{14}$$

This equation is called the *orthogonality property* of complex exponentials.

The orthogonality property of complex exponentials can be used to give a derivation of Eq. (9). Multiplying Eq. (13) by e^{-imx} and integrating term-by-term from $-\pi$ to π, we obtain

$$\int_{-\pi}^{\pi} \mathrm{f}(x)e^{-imx}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{inx}e^{-imx}\,dx.$$

By the orthogonality property, this leads to

$$\int_{-\pi}^{\pi} \mathrm{f}(x)e^{-imx}\,dx = 2\pi c_m,$$

which justifies (in a formal, nonrigorous way) the definition of c_n in Eq. (9).

We close this section by discussing two important properties of Fourier coefficients, *Bessel's inequality* and the *Riemann-Lebesgue lemma.*

Theorem 1 (Bessel's Inequality): If $\int_{-\pi}^{\pi} |\mathrm{f}(x)|^2\,dx$ is finite, then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathrm{f}(x)|^2\,dx. \tag{15}$$

Bessel's inequality can be proved easily. In fact, we have

$$0 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \mathrm{f}(x) - \sum_{n=-N}^{N} c_n e^{inx} \right|^2 dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\mathrm{f}(x) - \sum_{m=-N}^{N} c_m e^{imx} \right)\left(\overline{\mathrm{f}(x)} - \sum_{n=-N}^{N} \overline{c_n} e^{-inx} \right) dx.$$

Multiplying out the last integrand above, and making use of Eqs. (9) and (14), we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| \mathrm{f}(x) - \sum_{n=-N}^{N} c_n e^{inx} \right|^2 dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathrm{f}(x)|^2\,dx - \sum_{n=-N}^{N} |c_n|^2. \tag{16}$$

Thus, for all N,

$$\sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathrm{f}(x)|^2\,dx \tag{17}$$

and Bessel's inequality (15) follows by letting $N \to \infty$.

Bessel's inequality has a physical interpretation. If f has *finite energy,* in the sense that the right side of Eq. (15) is finite, then the sum of the moduli-squared of the Fourier coefficients is also finite. In Section IV, we shall see that the inequality in Eq. (15) is actually an equality, which says that the sum of the moduli-squared of the Fourier coefficients is precisely the same as the energy of f.

Because of Bessel's inequality, it follows that

$$\lim_{|n|\to\infty} c_n = 0 \tag{18}$$

holds whenever $\int_{-\pi}^{\pi} |\mathrm{f}(x)|^2\, dx$ is finite. The Riemann-Lebesgue lemma says that Eq. (18) holds in the following more general case:

Theorem 2 (Riemann-Lebesgue Lemma): If $\int_{-\pi}^{\pi} |\,\mathrm{f}(x)|\, dx$ is finite, then Eq. (18) holds.

One of the most important uses of the Riemann-Lebesgue lemma is in proofs of some basic pointwise convergence theorems, such as the ones described in the next section.

See Krantz and Walker (1998) for further discussions of the definition of Fourier series, Bessel's inequality, and the Riemann-Lebesgue lemma.

III. CONVERGENCE OF FOURIER SERIES

There are many ways to interpret the meaning of Eq. (13). Investigations into the types of functions allowed on the left side of Eq. (13), and the kinds of convergence considered for its right side, have fueled mathematical investigations by such luminaries as Dirichlet, Riemann, Weierstrass, Lipschitz, Lebesgue, Fejér, Gelfand, and Schwartz. In short, convergence questions for Fourier series have helped lay the foundations and much of the superstructure of mathematical analysis.

The three types of convergence that we shall describe here are *pointwise*, *uniform*, and *norm* convergence. We shall discuss the first two types in this section and take up the third type in the next section.

All convergence theorems are concerned with how the *partial sums*

$$S_N(x) := \sum_{n=-N}^{N} c_n\, e^{inx}$$

converge to $\mathrm{f}(x)$. That is, *does* $\lim_{N\to\infty} S_N = \mathrm{f}$ *hold in some sense*?

The question of pointwise convergence, for example, concerns whether $\lim_{N\to\infty} S_N(x_0) = \mathrm{f}(x_0)$ for each fixed x-value x_0. If $\lim_{N\to\infty} S_N(x_0)$ does equal $\mathrm{f}(x_0)$, then we say that the *Fourier series for* f *converges to* $\mathrm{f}(x_0)$ *at* x_0.

We shall now state the simplest pointwise convergence theorem for which an elementary proof can be given. This theorem assumes that a function is Lipschitz at each point where convergence occurs. A function is said to be *Lipschitz at a point* x_0 if, for some positive constant A,

$$|\mathrm{f}(x) - \mathrm{f}(x_0)| \le A\,|x - x_0| \tag{19}$$

holds for all x near x_0 (i.e., $|x - x_0| < \delta$ for some $\delta > 0$). It is easy to see, for instance, that the square wave function f_1 is Lipschitz at all of its continuity points.

The inequality in Eq. (19) has a simple geometric interpretation. Since both sides are 0 when $x = x_0$, this inequality is equivalent to

$$\left|\frac{\mathrm{f}(x) - \mathrm{f}(x_0)}{x - x_0}\right| \le A \tag{20}$$

for all x near x_0 (and $x \neq x_0$). Inequality (20) simply says that the difference quotients of f (i.e., the slopes of its secants) near x_0 are bounded. With this interpretation, it is easy to see that the parabolic wave f_2 is Lipschitz at all points. More generally, if f has a derivative at x_0 (or even just left- and right-hand derivatives), then f is Lipschitz at x_0.

We can now state and prove a simple convergence theorem.

Theorem 3: Suppose f has period 2π, that $\int_{-\pi}^{\pi} |\mathrm{f}(x)|\, dx$ is finite, and that f is Lipschitz at x_0. Then the Fourier series for f converges to $\mathrm{f}(x_0)$ at x_0.

To prove this theorem, we assume that $\mathrm{f}(x_0) = 0$. There is no loss of generality in doing so, since we can always subtract the constant $\mathrm{f}(x_0)$ from $\mathrm{f}(x)$. Define the function g by $g(x) = \mathrm{f}(x)/(e^{ix} - e^{ix_0})$. This function g has period 2π. Furthermore, $\int_{-\pi}^{\pi} |g(x)|\, dx$ is finite, because the quotient $\mathrm{f}(x)/(e^{ix} - e^{ix_0})$ is bounded in magnitude for x near x_0. In fact, for such x,

$$\begin{aligned}\left|\frac{\mathrm{f}(x)}{e^{ix} - e^{x_0}}\right| &= \left|\frac{\mathrm{f}(x) - f(x_0)}{e^{ix} - e^{x_0}}\right| \\ &\le A\left|\frac{x - x_0}{e^{ix} - e^{x_0}}\right|\end{aligned}$$

and $(x - x_0)/(e^{ix} - e^{x_0})$ is bounded in magnitude, because it tends to the reciprocal of the derivative of e^{ix} at x_0.

If we let d_n denote the nth Fourier coefficient for $g(x)$, then we have $c_n = d_{n-1} - d_n e^{ix_0}$ because $\mathrm{f}(x) = g(x)(e^{ix} - e^{ix_0})$. The partial sum $S_N(x_0)$ then telescopes:

$$\begin{aligned}S_N(x_0) &= \sum_{n=-N}^{N} c_n e^{inx_0} \\ &= d_{-N-1}e^{-iNx_0} - d_N\, e^{i(N+1)x_0}.\end{aligned}$$

Since $d_n \to 0$ as $|n| \to \infty$, by the Riemann-Lebesgue lemma, we conclude that $S_N(x_0) \to 0$. This completes the proof.

It should be noted that for the square wave f_1 and the parabolic wave f_2, it is not necessary to use the general Riemann-Lebesgue lemma stated above. That is because for those functions it is easy to see that $\int_{-\pi}^{\pi} |g(x)|^2\, dx$ is finite for the function g defined in the proof of Theorem 3. Consequently, $d_n \to 0$ as $|n| \to \infty$ follows from Bessel's inequality for g.

In any case, Theorem 3 implies that the Fourier series for the square wave f_1 converges to f_1 at all of its points of continuity. It also implies that the Fourier series for the parabolic wave f_2 converges to f_2 at all points. While this may settle matters (more or less) in a pure mathematical sense for these two waves, it is still important to examine specific partial sums in order to learn more about the nature of their convergence to these waves.

For example, in Fig. 3 we show a graph of the partial sum S_{100} superimposed on the square wave. Although Theorem 3 guarantees that $S_N \to f_1$ as $N \to \infty$ at each continuity point, Fig. 3 indicates that this convergence is at a rather slow rate. The partial sum S_{100} differs significantly from f_1. Near the square wave's jump discontinuities, for example, there is a severe spiking behavior called *Gibbs' phenomenon* (see Fig. 4). This spiking behavior does *not* go away as $N \to \infty$, although the width of the spike does tend to zero. In fact, the peaks of the spikes overshoot the square wave's value of 1, tending to a limit of about 1.09. The partial sum also oscillates quite noticeably about the constant value of the square wave at points away from the discontinuities. This is known as *ringing.*

These defects do have practical implications. For instance, oscilloscopes—which generate wave forms as combinations of sinusoidal waves over a limited range of frequencies—cannot use S_{100}, or any partial sum S_N, to produce a square wave. We shall see, however, in Section V that a clever modification of a partial sum does produce an acceptable version of a square wave.

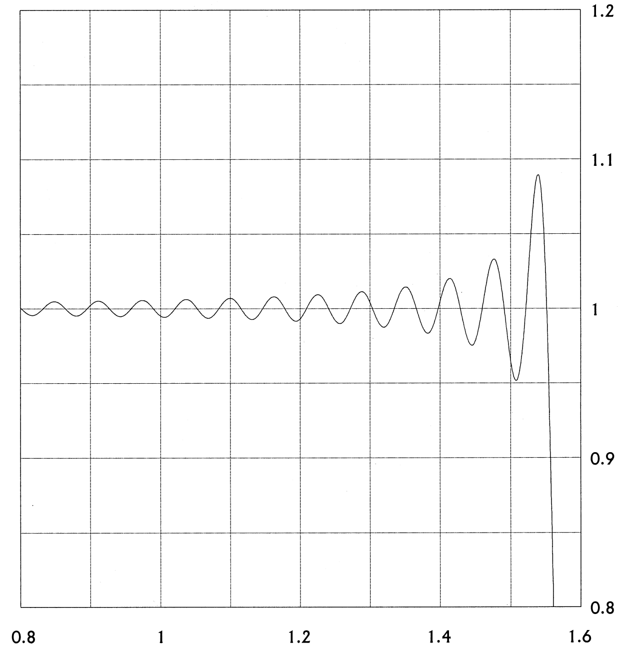

FIGURE 4 Gibbs' phenomenon and ringing for square wave.

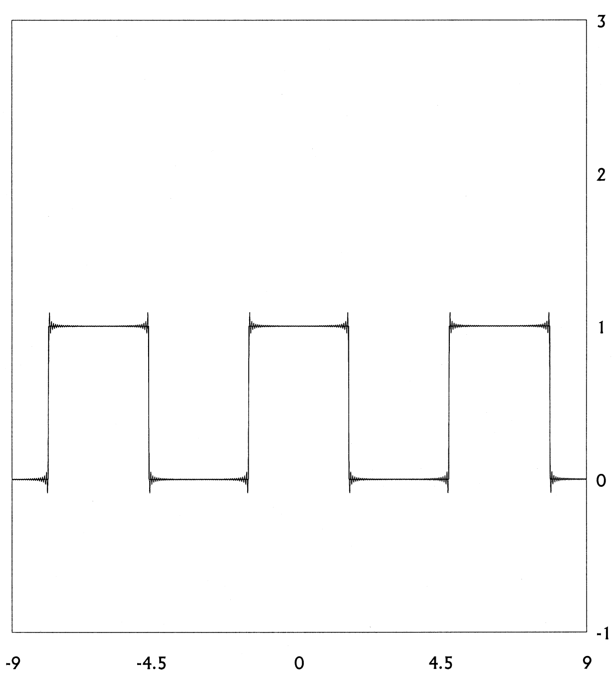

FIGURE 3 Fourier series partial sum S_{100} superimposed on square wave.

The cause of ringing and Gibbs' phenomenon for the square wave is a rather slow convergence to zero of its Fourier coefficients (at a rate comparable to $|n|^{-1}$). In the next section, we shall interpret this in terms of energy and show that a partial sum like S_{100} does not capture a high enough percentage of the energy of the square wave f_1.

In contrast, the Fourier coefficients of the parabolic wave f_2 tend to zero more rapidly (at a rate comparable to n^{-2}). Because of this, the partial sum S_{100} for f_2 is a much better approximation to the parabolic wave (see Fig. 5). In fact, its partial sums S_N exhibit the phenomenon of *uniform convergence.*

We say that the Fourier series for a function f *converges uniformly* to f if

$$\lim_{N\to\infty} \left\{ \max_{x\in[-\pi,\pi]} |f(x) - S_N(x)| \right\} = 0. \tag{21}$$

This equation says that, for large enough N, we can have the maximum distance between the graphs of f and S_N as small as we wish. Figure 5 is a good illustration of this for the parabolic wave.

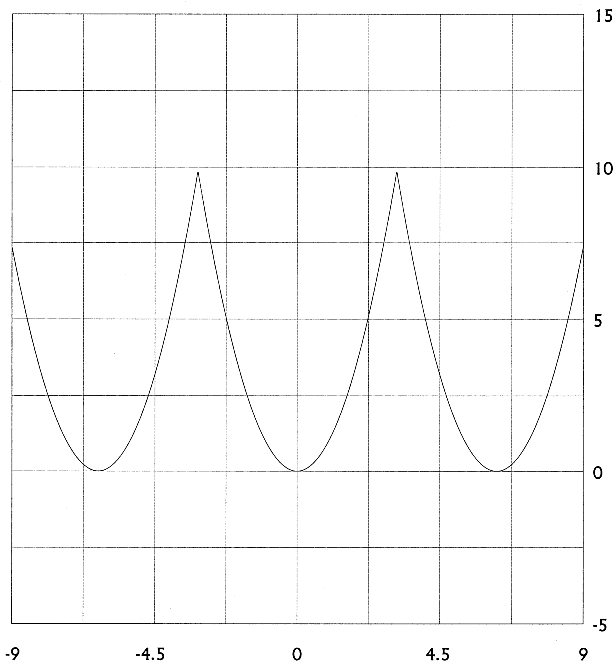

FIGURE 5 Fourier series partial sum S_{100} for parabolic wave.

We can verify Eq. (21) for the parabolic wave as follows. By Eq. (21) we have

$$\begin{aligned}|\mathrm{f}_2(x) - S_N(x)| &= \left|\sum_{n=N+1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx\right| \\ &\le \sum_{n=N+1}^{\infty} \left|\frac{4(-1)^n}{n^2}\cos nx\right| \\ &\le \sum_{n=N+1}^{\infty} \frac{4}{n^2}.\end{aligned}$$

Consequently

$$\begin{aligned}\max_{x\in[-\pi,\pi]} |\mathrm{f}_2(x) - S_N(x)| &\le \sum_{n=N+1}^{\infty} \frac{4}{n^2} \\ &\to 0 \quad \text{as } N \to \infty\end{aligned}$$

and thus Eq. (21) holds for the parabolic wave f_2.

Uniform convergence for the parabolic wave is a special case of a more general theorem. We shall say that f is *uniformly Lipschitz* if Eq. (19) holds for all points using the same constant A. For instance, it is not hard to show that a continuously differentiable, periodic function is uniformly Lipschitz.

Theorem 4: Suppose that f has period 2π and is uniformly Lipschitz at all points, then the Fourier series for f converges uniformly to f.

A remarkably simple proof of this theorem is described in Jackson (1941). More general uniform convergence theorems are discussed in Walter (1994).

Theorem 4 applies to the parabolic wave f_2, but it does not apply to the square wave f_1. In fact, the Fourier series for f_1 cannot converge uniformly to f_1. That is because a famous theorem of Weierstrass says that a uniform limit of continuous functions (like the partial sums S_N) must be a continuous function (which f_1 is certainly not). The Gibbs' phenomenon for the square wave is a conspicuous failure of uniform convergence for its Fourier series.

Gibbs' phenomenon and ringing, as well as many other aspects of Fourier series, can be understood via an integral form for partial sums discovered by Dirichlet. This integral form is

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(x-t)D_N(t)\,dt \tag{22}$$

with *kernel* D_N defined by

$$D_N(t) = \frac{\sin(N+1/2)t}{\sin(t/2)}. \tag{23}$$

This formula is proved in almost all books on Fourier series (see, for instance, Krantz (1999), Walker (1988), or Zygmund (1968)). The kernel D_N is called *Dirichlet's kernel.* In Fig. 6 we have graphed D_{20}.

The most important property of Dirichlet's kernel is that, for all N,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1.$$

From Eq. (23) we can see that the value of 1 follows from cancellation of signed areas, and also that the contribution

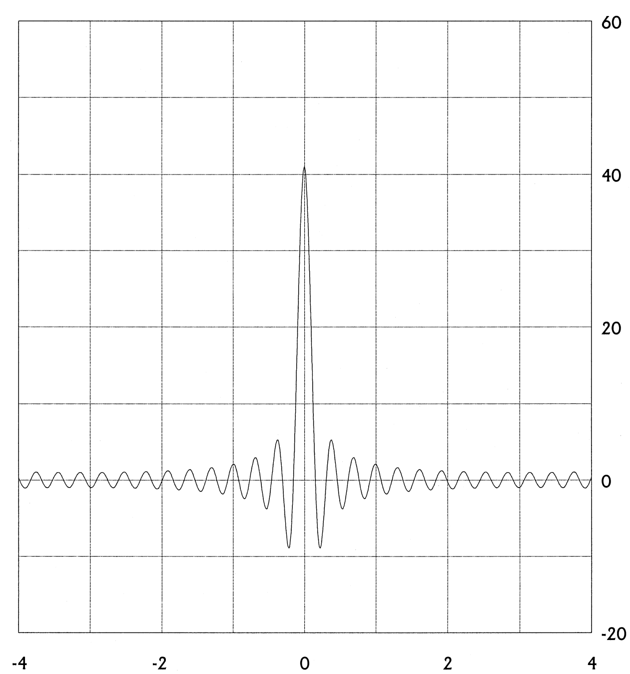

FIGURE 6 Dirichlet's kernel D_{20}.

of the main lobe centered at 0 (see Fig. 6) is significantly greater than 1 (about 1.09 in value).

From the facts just cited, we can explain the origin of ringing and Gibbs' phenomenon for the square wave. For the square wave function f_1, Eq. (22) becomes

$$S_N(x) = \frac{1}{2\pi}\int_{x-\pi/2}^{x+\pi/2} D_N(t)\,dt. \tag{24}$$

As x ranges from $-\pi$ to π, this formula shows that $S_N(x)$ is proportional to the signed area of D_N over an interval of length π centered at x. By examining Fig. 6, which is a typical graph for D_N, it is then easy to see why there is ringing in the partial sums S_N for the square wave. Gibbs' phenomenon is a bit more subtle, but also results from Eq. (24). When x nears a jump discontinuity, the central lobe of D_N is the dominant contributor to the integral in Eq. (24), resulting in a spike which overshoots the value of 1 for f_1 by about 9%.

Our final pointwise convergence theorem was, in essence, the first to be proved. It was established by Dirichlet using the integral form for partial sums in Eq. (22). We shall state this theorem in a stronger form first proved by Jordan.

Theorem 5: If f has period 2π and has bounded variation on $[0, 2\pi]$, then the Fourier series for f converges at all points. In fact, for all x-values,

$$\lim_{N\to\infty} S_N(x) = \tfrac{1}{2}[f(x+) + f(x-)].$$

This theorem is too difficult to prove in the limited space we have here (see Zygmund, 1968). A simple consequence of Theorem 5 is that the Fourier series for the square wave f_1 converges at its discontinuity points to 1/2 (although this can also be shown directly by substitution of $x = \pm\pi/2$ into the series in (Eq. (11)).

We close by mentioning that the conditions for convergence, such as Lipschitz or bounded variation, cited in the theorems above cannot be dispensed with entirely. For instance, Kolmogorov gave an example of a period 2π function (for which $\int_{-\pi}^{\pi} |f(x)|\,dx$ is finite) that has a Fourier series which fails to converge at *every* point.

More discussion of pointwise convergence can be found in Walker (1998), Walter (1994), or Zygmund (1968).

IV. CONVERGENCE IN NORM

Perhaps the most satisfactory notion of convergence for Fourier series is convergence in L^2-norm (also called L^2-convergence), which we shall define in this section. One of the great triumphs of the Lebesgue theory of integration is that it yields necessary and sufficient conditions for L^2-convergence. There is also an interpretation of L^2-norm in terms of a generalized Euclidean distance and this gives a satisfying geometric flavor to L^2-convergence of Fourier series. By interpreting the square of L^2-norm as a type of energy, there is an equally satisfying physical interpretation of L^2-convergence. The theory of L^2-convergence has led to fruitful generalizations such as Hilbert space theory and norm convergence in a wide variety of function spaces.

To introduce the idea of L^2-convergence, we first examine a special case. By Theorem 4, the partial sums of a uniformly Lipschitz function f converge uniformly to f. Since that means that the maximum distance between the graphs of S_N and f tends to 0 as $N \to \infty$, it follows that

$$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - S_N(x)|^2\,dx = 0. \tag{25}$$

This result motivates the definition of L^2-convergence.

If g is a function for which $|g|^2$ has a finite Lebesgue integral over $[-\pi, \pi]$, then we say that g is an L^2-function, and we define its L^2-norm $\|g\|_2$ by

$$\|g\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^2\,dx}\,.$$

We can then rephrase Eq. (25) as saying that $\|f - S_N\|_2 \to 0$ as $N \to \infty$. In other words, *the Fourier series for* f *converges to* f *in L^2-norm.* The following theorem generalizes this result to all L^2-functions (see Rudin (1986) for a proof).

Theorem 6: If f is an L^2-function, then its Fourier series converges to f in L^2-norm.

Theorem 6 says that Eq. (25) holds for every L^2-function f. Combining this with Eq. (16), we obtain *Parseval's equality:*

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx. \tag{26}$$

Parseval's equation has a useful interpretation in terms of energy. It says that the energy of the set of Fourier coefficients, defined to be equal to the left side of Eq. (26), is equal to the energy of the function f, defined by the right side of Eq. (26).

The L^2-norm can be interpreted as a generalized Euclidean distance. To see this take square roots of both sides of Eq. (26): $\sqrt{\sum |c_n|^2} = \|f\|_2$. The left side of this equation is interpreted as a Euclidean distance in an (infinite-dimensional) coordinate space, hence the L^2-norm $\|f\|_2$ is equivalent to such a distance.

As examples of these ideas, let's return to the square wave and parabolic wave. For the square wave f_1, we find that

$$\|f_1 - S_{100}\|_2^2 = \sum_{|n|>100} \frac{\sin^2(n\pi/2)}{(n\pi)^2} = 1.0 \times 10^{-3}.$$

Likewise, for the parabolic wave f_2, we have $\|f_2 - S_{100}\|_2^2 = 2.6 \times 10^{-6}$. These facts show that the energy of the parabolic wave is almost entirely contained in the partial sum S_{100}; their energy difference is almost three orders of magnitude smaller than in the square wave case. In terms of generalized Euclidean distance, we have $\|f_2 - S_{100}\|_2 = 1.6 \times 10^{-3}$ and $\|f_1 - S_{100}\|_2 = 3.2 \times 10^{-2}$, showing that the partial sum is an order of magnitude closer for the parabolic wave.

Theorem 6 has a converse, known as the *Riesz-Fischer theorem.*

Theorem 7 (Riesz-Fischer): If $\sum |c_n|^2$ converges, then there exists an L^2-function f having $\{c_n\}$ as its Fourier coefficients.

This theorem is proved in Rudin (1986). Theorem and the Riesz-Fischer theorem combine to give necessary and sufficient conditions for L^2-convergence of Fourier series, conditions which are remarkably easy to apply. This has made L^2-convergence into the most commonly used notion of convergence for Fourier series.

These ideas for L^2-norms partially generalize to the case of L^p-norms. Let p be real number satisfying $p \geq 1$. If g is a function for which $|g|^p$ has a finite Lebesgue integral over $[-\pi, \pi]$, then we say that g is an L^p-function, and we define its L^p-norm $\|g\|_p$ by

$$\|g\|_p = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(x)|^p \, dx\right]^{1/p}.$$

If $\|f - S_N\|_p \to 0$, then we say that the Fourier series for f converges to f in L^p-norm. The following theorem generalizes Theorem 6 (see Krantz (1999) for a proof).

Theorem 8: If f is an L^p-function for $p > 1$, then its Fourier series converges to f in L^p-norm.

Notice that the case of $p = 1$ is not included in Theorem 8. The example of Kolmogorov cited at the end of Section III shows that there exist L^1-functions whose Fourier series do not converge in L^1-norm. For $p \neq 2$, there are no simple analogs of either Parseval's equality or the Riesz-Fischer theorem (which say that we can characterize L^2-functions by the magnitude of their Fourier coefficients). Some partial analogs of these latter results for L^p-functions, when $p \neq 2$, are discussed in Zygmund (1968) (in the context of *Littlewood-Paley* theory).

We close this section by returning full circle to the notion of pointwise convergence. The following theorem was proved by Carleson for L^2-functions and by Hunt for L^p-functions ($p \neq 2$).

Theorem 9: If f is an L^p-function for $p > 1$, then its Fourier series converges to it at almost all points.

By *almost all points,* we mean that the set of points where divergence occurs has Lebesgue measure zero. References for the proof of Theorem 9 can be found in Krantz (1999) and Zygmund (1968). Its proof is undoubtedly the most difficult one in the theory of Fourier series.

V. SUMMABILITY OF FOURIER SERIES

In the previous sections, we noted some problems with convergence of Fourier series partial sums. Some of these problems include Kolmogorov's example of a Fourier series for an L^1-function that diverges everywhere, and Gibbs' phenomenon and ringing in the Fourier series partial sums for discontinuous functions. Another problem is Du Bois Reymond's example of a continuous function whose Fourier series diverges on a countably infinite set of points, see Walker (1968). It turns out that all of these difficulties simply disappear when new summation methods, based on appropriate modifications of the partial sums, are used.

The simplest modification of partial sums, and one of the first historically to be used, is to take *arithmetic means.* Define the Nth arithmetic mean σ_N by $\sigma_N = (S_0 + S_1 + \cdots + S_{N-1})/N$. From which it follows that

$$\sigma_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) c_n \, e^{inx}. \tag{27}$$

The factors $(1 - |n|/N)$ are called *convergence factors.* They modify the Fourier coefficients c_n so that the amplitude of the higher frequency terms (for $|n|$ near N) are damped down toward zero. This produces a great improvement in convergence properties as shown by the following theorem.

Theorem 10: Let f be a periodic function. If f is an L^p-function for $p \geq 1$, then $\sigma_N \to$ f in L^p-norm as $N \to \infty$. If f is a continuous function, then $\sigma_N \to$ f uniformly as $N \to \infty$.

Notice that L^1-convergence is included in Theorem 10. Even for Kolmogorov's function, it is the case that $\|f - \sigma_N\|_1 \to 0$ as $N \to \infty$. It also should be noted that no assumption, other than continuity of the periodic function, is needed in order to ensure uniform convergence of its arithmetic means.

For a proof of Theorem 10, see Krantz (1999). The key to the proof is Fejér's integral form for σ_N:

$$\sigma_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(x-t)F_N(t)\,dt \tag{28}$$

where *Fejér's kernel* F_N is defined by

$$F_N(t) = \frac{1}{N}\left(\frac{\sin Nt/2}{\sin t/2}\right)^2. \tag{29}$$

In Fig. 7 we show the graph of F_{20}. Compare this graph with the one of Dirichlet's kernel D_{20} in Fig. 6. Unlike Dirichlet's kernel, Fejér's kernel is positive $[F_N(t) \geq 0]$, and is close to 0 away from the origin. These two facts are the main reasons that Theorem 10 holds. The fact that Fejér's kernel satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(t)\,dt = 1$$

is also used in the proof.

An attractive feature of arithmetic means is that Gibbs' phenomenon and ringing do not occur. For example, in Fig. 8 we show σ_{100} for the square wave and it is plain that these two defects are absent. For the square wave function f_1, Eq. (28) reduces to

$$\sigma_N(x) = \frac{1}{2\pi}\int_{x-\pi/2}^{x+\pi/2} F_N(t)\,dt.$$

As x ranges from $-\pi$ to π, this formula shows that $\sigma_N(x)$ is proportional to the area of the positive function F_N over an interval of length π centered at x. By examining Fig. 7, which is a typical graph for F_N, it is easy to see why ringing and Gibbs' phenomenon do not occur for the arithmetic means of the square wave.

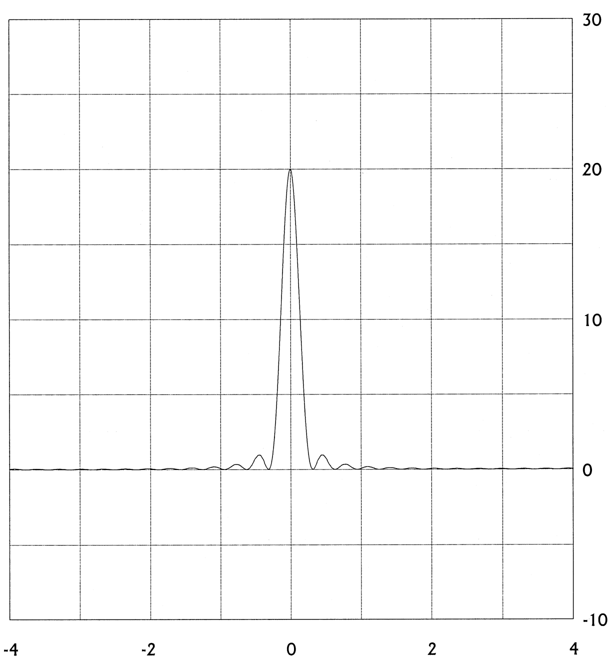

FIGURE 7 Fejér's kernel F_{20}.

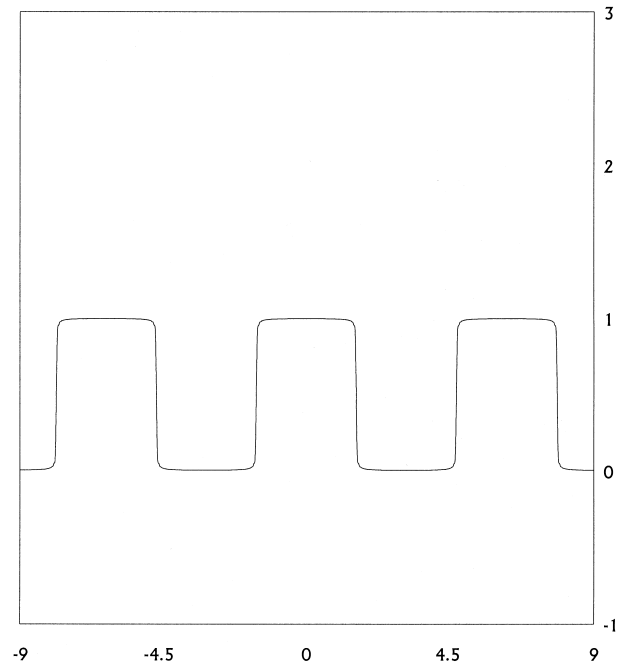

FIGURE 8 Arithmetic mean σ_{100} for square wave.

The method of arithmetic means is just one example from a wide range of summation methods for Fourier series. These summation methods are one of the major elements in the area of *finite impulse response filtering* in the fields of electrical engineering and signal processing.

A *summation kernel* K_N is defined by

$$K_N(x) = \sum_{n=-N}^{N} m_n\, e^{inx}. \tag{30}$$

The real numbers $\{m_n\}$ are the *convergence factors* for the kernel. We have already seen two examples: Dirichlet's kernel (where $m_n = 1$) and Fejér's kernel (where $m_n = 1 - |n|/N$).

When K_N is a summation kernel, then we define the modified partial sum of f to be $\sum_{n=-N}^{N} m_n\, c_n\, e^{inx}$. It then follows from Eqs. (14) and (30) that

$$\sum_{n=-N}^{N} m_n\, c_n\, e^{inx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{f}(x-t)K_N(t)\,dt. \tag{31}$$

The function defined by both sides of Eq. (31) is denoted by $K_N * \mathrm{f}$. It is usually more convenient to use the left side of Eq. (31) to compute $K_N * \mathrm{f}$, while for theoretical purposes (such as proving Theorem 11 below), it is more convenient to use the right side of Eq. (31).

We say that a summation kernel K_N is *regular* if it satisfies the following three conditions.

1. For each N,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx = 1.$$

2. There is a positive constant C such that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_N(x)|\,dx \le C.$$

3. For each $0 < \delta < \pi$,

$$\lim_{N\to\infty}\left\{\max_{\delta\le|x|\le\pi} |K_N(x)|\right\} = 0.$$

There are many examples of regular summation kernels. Fejér's kernel, which has $m_n = 1 - |n|/N$, is regular. Another regular summation kernel is Hann's kernel, which has $m_n = 0.5 + 0.5\cos(n\pi/N)$. A third regular summation kernel is de le Vallée Poussin's kernel, for which $m_n = 1$ when $|n| \le N/2$, and $m_n = 2(1 - |m/N|)$ when $N/2 < |m| \le N$. The proofs that these summation kernels are regular are given in Walker (1996). It should be noted that Dirichlet's kernel is *not* regular, because properties 2 and 3 do not hold.

As with Fejér's kernel, all regular summation kernels significantly improve the convergence of Fourier series. In fact, the following theorem generalizes Theorem 10.

Theorem 11: Let f be a periodic function, and let K_N be a regular summation kernel. If f is an L^p-function for $p \ge 1$, then $K_N * \mathrm{f} \to \mathrm{f}$ in L^p-norm as $N \to \infty$. If f is a continuous function, then $K_N * \mathrm{f} \to \mathrm{f}$ uniformly as $N \to \infty$.

For an elegant proof of this theorem, see Krantz (1999).

From Theorem 11 we might be tempted to conclude that the convergence properties of regular summation kernels are all the same. They do differ, however, in the *rates* at which they converge. For example, in Fig. 9 we show $K_{100} * \mathrm{f}_1$ where the kernel is Hann's kernel and f_1 is the square wave. Notice that this graph is a much better approximation of a square wave than the arithmetic mean graph in Fig. 8. An oscilloscope, for example, can easily generate the graph in Fig. 9, thereby producing an acceptable version of a square wave.

Summation of Fourier series is discussed further in Krantz (1999), Walker (1996), Walter (1994), and Zygmund (1968).

VI. GENERALIZED FOURIER SERIES

The classical theory of Fourier series has undergone extensive generalizations during the last two hundred years.

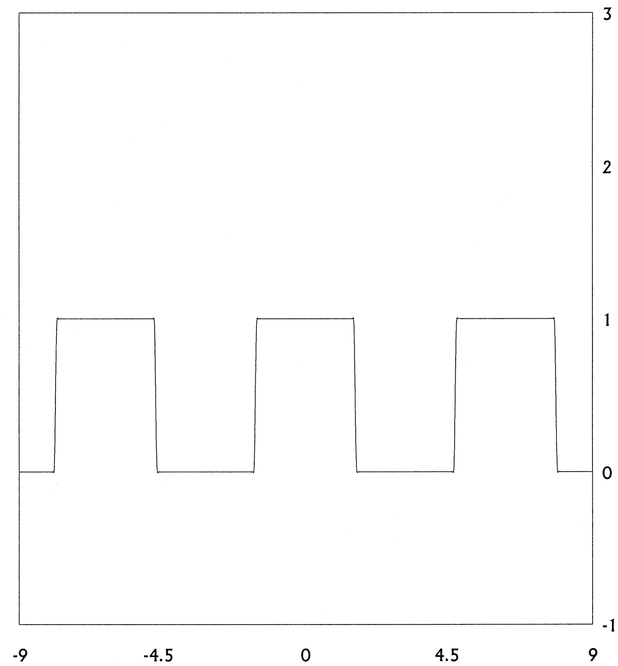

FIGURE 9 Approximate square wave using Hann's kernel.

For example, Fourier series can be viewed as one aspect of a general theory of *orthogonal series expansions.* In this section, we shall discuss a few of the more celebrated orthogonal series, such as Legendre series, Haar series, and wavelet series.

We begin with Legendre series. The first two *Legendre polynomials* are $P_0(x) = 1$, and $P_1(x) = x$. For $n = 2, 3, 4, \ldots$, the nth Legendre polynomial P_n is defined by the recursion relation

$$nP_n(x) = (2n-1)xP_{n-1}(x) + (n-1)P_{n-2}(x).$$

These polynomials satisfy the following *orthogonality relation*

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \begin{cases} 0 & \text{if } m \ne n \\ (2n+1)/2 & \text{if } m = n. \end{cases} \tag{32}$$

This equation is quite similar to Eq. (14). Because of Eq. (32)—recall how we used Eq. (14) to derive Eq. (9) —the *Legendre series* for a function f over the interval $[-1, 1]$ is defined to be

$$\sum_{n=0}^{\infty} c_n P_n(x) \tag{33}$$

with

$$c_n = \frac{2}{2n+1}\int_{-1}^{1} \mathrm{f}(x)P_n(x)\,dx. \tag{34}$$

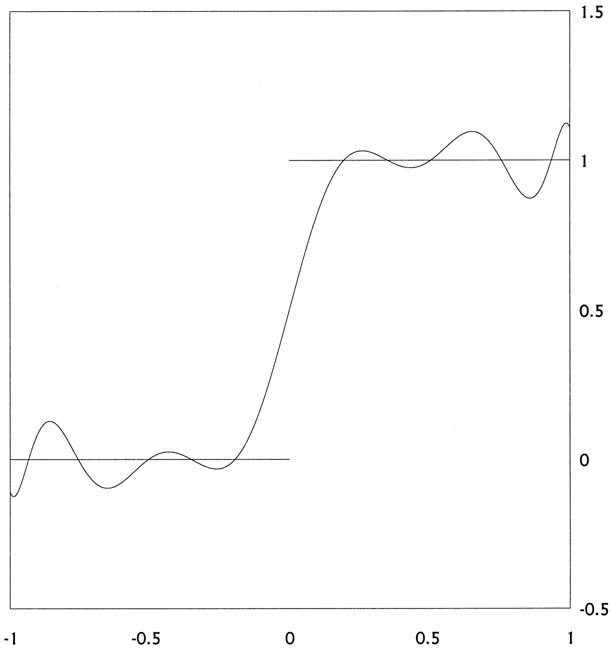

FIGURE 10 Step function and its Legendre series partial sum S_{11}.

The partial sum S_N of the series in Eq. (33) is defined to be

$$S_N(x) = \sum_{n=0}^{N} c_n P_n(x).$$

As an example, let $f(x) = 1$ for $0 \le x \le 1$ and $f(x) = 0$ for $-1 \le x < 0$. The Legendre series for this step function is [see Walker (1988)]:

$$\frac{1}{2} + \sum_{k=0}^{\infty} \frac{(-1)^k (4k+3)(2k)!}{4^{k+1}(k+1)!k!} P_{2k+1}(x).$$

In Fig. 10 we show the partial sum S_{11} for this series. The graph of S_{11} is reminiscent of a Fourier series partial sum for a step function. In fact, the following theorem is true.

Theorem 12: If $\int_{-1}^{1} |f(x)|^2\,dx$ is finite, then the partial sums S_N for the Legendre series for f satisfy

$$\lim_{N\to\infty} \int_{-1}^{1} |f(x) - S_N(x)|^2\,dx = 0.$$

Moreover, if f is Lipschitz at a point x_0, then $S_N(x_0) \to f(x_0)$ as $N \to \infty$.

This theorem is proved in Walter (1994) and Jackson (1941). Further details and other examples of *orthogonal polynomial series* can be found in either Davis (1975), Jackson (1941), or Walter (1994). There are many important orthogonal series—such as Hermite, Laguerre, and Tchebysheff—which we cannot examine here because of space limitations.

We now turn to another type of orthogonal series, the Haar series. The defects, such as Gibbs' phenomenon and ringing, that occur with Fourier series expansions can be traced to the unlocalized nature of the functions used for expansions. The complex exponentials used in classical Fourier series, and the polynomials used in Legendre series, are all non-zero (except possibly for a finite number of points) over their domains. In contrast, Haar series make use of localized functions, which are non-zero only over tiny regions within their domains.

In order to define Haar series, we first define the *fundamental Haar wavelet* $H(x)$ by

$$H(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x \le 1. \end{cases}$$

The *Haar wavelets* $\{H_{j,k}(x)\}$ are then defined by

$$H_{j,k}(x) = 2^{j/2} H(2^j x - k)$$

for $j = 0, 1, 2, \ldots;\ k = 0, 1, \ldots, 2^j - 1$. Notice that $H_{j,k}(x)$ is non-zero only on the interval $[k2^{-j}, (k+1)2^{-j}]$, which for large j is a tiny subinterval of $[0, 1]$. As k ranges between 0 and $2^j - 1$, these subintervals partition the interval $[0, 1]$, and the partition becomes finer (shorter subintervals) with increasing j.

The *Haar series* for a function f is defined by

$$b + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} c_{j,k} H_{j,k}(x) \tag{35}$$

with $b = \int_0^1 f(x)\,dx$ and

$$c_{j,k} = \int_0^1 f(x) H_{j,k}(x)\,dx.$$

The definitions of b and $c_{j,k}$ are justified by orthogonality relations between the Haar functions (similar to the orthogonality relations that we used above to justify Fourier series and Legendre series).

A partial sum S_N for the Haar series in Eq. () is defined by

$$S_N(x) = b + \sum_{\{j,k\,|\,2^j+k \le N\}} c_{j,k} H_{j,k}(x).$$

For example, let f be the function on [0, 1] defined as follows

$$f(x) = \begin{cases} x - 1/2 & \text{if } 1/4 < x < 3/4 \\ 0 & \text{if } x \le 1/4 \text{ or } 3/4 \le x. \end{cases}$$

In Fig. 11 we show the Haar series partial sum S_{256} for this function. Notice that there is no Gibbs' phenomenon with this partial sum. This contrasts sharply with the Fourier

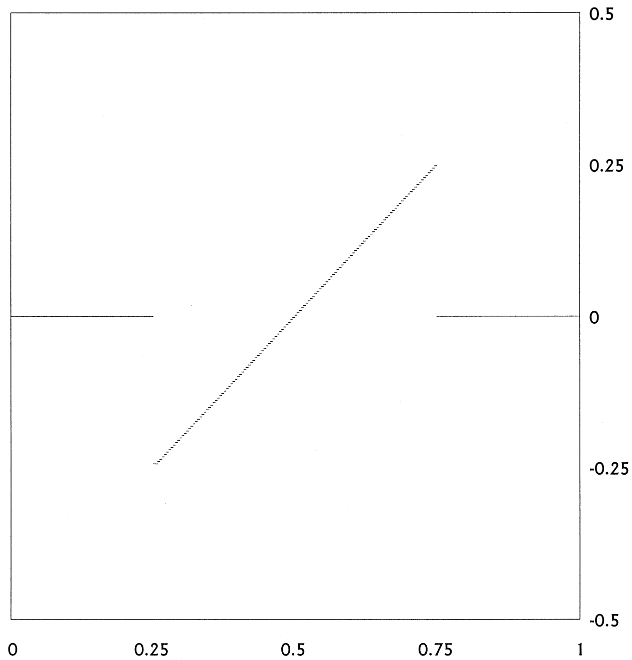

FIGURE 11 Haar series partial sum S_{256}, which has 257 terms.

series partial sum, also using 257 terms, which we show in Fig. 12.

The Haar series partial sums satisfy the following theorem [proved in Daubechies (1992) and in Meyer (1992)].

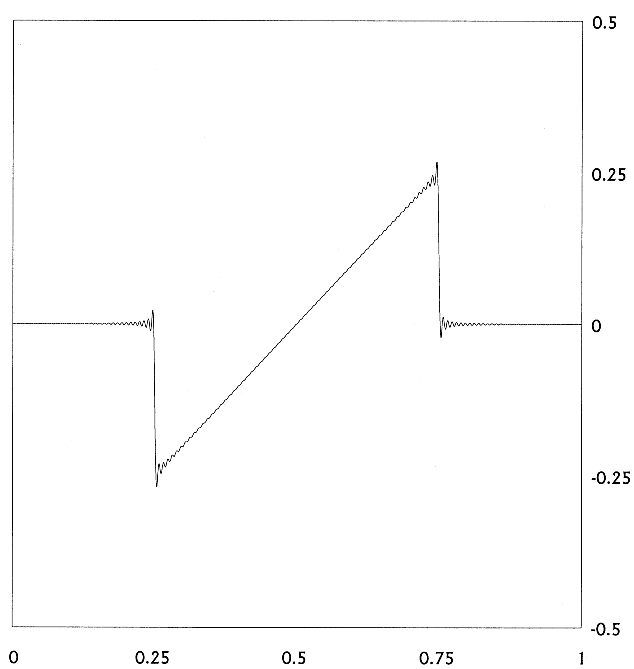

FIGURE 12 Fourier series partial sum S_{128}, which has 257 terms.

Theorem 13: Suppose that $\int_0^1 |f(x)|^p\,dx$ is finite, for $p \geq 1$. Then the Haar series partial sums for f satisfy

$$\lim_{N\to\infty}\left[\int_0^1 |f(x) - S_N(x)|^p\,dx\right]^{1/p} = 0.$$

If f is continuous on [0, 1], then S_N converges uniformly to f on [0, 1].

This theorem is reminiscent of Theorems 10 and 11 for the modified Fourier series partial sums obtained by arithmetic means or by a regular summation kernel. The difference here, however, is that for the Haar series no modifications of the partial sums are needed.

One glaring defect of Haar series is that the partial sums are discontinuous functions. This defect is remedied by the wavelet series discovered by Meyer, Daubechies, and others. The fundamental Haar wavelet is replaced by some new fundamental wavelet Ψ and the set of wavelets $\{\Psi_{j,k}\}$ is then defined by $\Psi_{j,k}(x) = 2^{-j/2}\Psi[2^j x - k]$. (The bracket symbolism $\Psi[2^j x - k]$ means that the value, $2^j x - k \bmod 1$, is evaluated by Ψ. This technicality is needed in order to ensure periodicity of $\Psi_{j,k}$.) For example, in Fig. 13, we show graphs of $\Psi_{4,1}$ and $\Psi_{6,46}$ for one of the Daubechies wavelets (a Coif18 wavelet), which is *continuously differentiable.* For a complete discussion of the definition of these wavelet functions, see Daubechies (1992) or Mallat (1998).

The *wavelet series,* generated by the fundamental wavelet Ψ, is defined by

$$b + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} c_{j,k}\Psi_{j,k}(x) \tag{36}$$

with $b = \int_0^1 f(x)\,dx$ and

$$c_{j,k} = \int_0^1 f(x)\Psi_{j,k}(x)\,dx. \tag{37}$$

This wavelet series has partial sums S_N defined by

$$S_N(x) = b + \sum_{\{j,k\,|\,2^j+k\leq N\}} c_{j,k}\Psi_{j,k}(x).$$

Notice that when Ψ is continuously differentiable, then so is each partial sum S_N. These wavelet series partial sums satisfy the following theorem, which generalizes Theorem 13 for Haar series, for a proof, see Daubechies (1992) or Meyer (1992).

Theorem 14: Suppose that $\int_0^1 |f(x)|^p\,dx$ is finite, for $p \geq 1$. Then the Daubechies wavelet series partial sums for f satisfy

$$\lim_{N\to\infty}\left[\int_0^1 |f(x) - S_N(x)|^p\,dx\right]^{1/p} = 0.$$

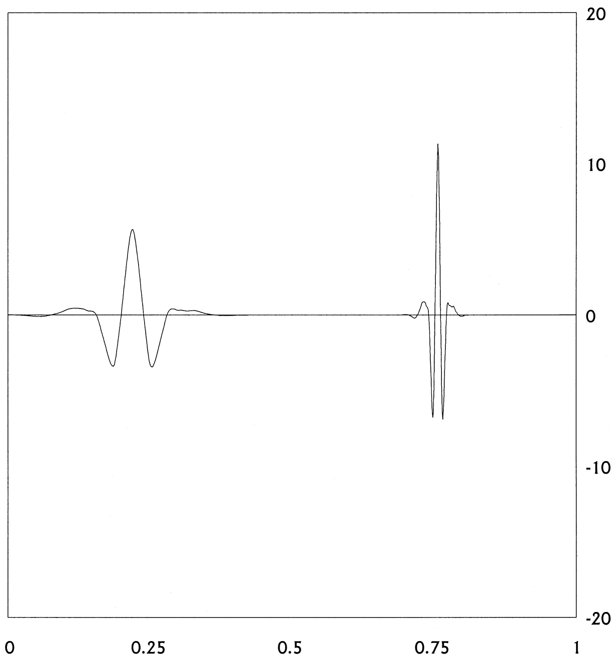

FIGURE 13 Two Daubechies wavelets.

If f is continuous on [0, 1], then S_N converges uniformly to f on [0, 1].

Theorem 14 does not reveal the full power of wavelet series. In almost all cases, it is possible to rearrange the terms in the wavelet series *in any manner whatsoever* and convergence will still hold. One reason for doing a rearrangement is in order to add the terms in the series with coefficients of largest magnitude (thus largest energy) *first* so as to speed up convergence to the function. Here is a convergence theorem for such permuted series.

Theorem 15: Suppose that $\int_0^1 |\mathrm{f}(x)|^p\,dx$ is finite, for $p > 1$. If the terms of a Daubechies wavelet series are permuted (in any manner whatsoever), then the partial sums S_N of the permuted series satisfy

$$\lim_{N\to\infty}\left[\int_0^1 |\mathrm{f}(x) - S_N(x)|^p\,dx\right]^{1/p} = 0.$$

If f is uniformly Lipschitz, then the partial sums S_N of the permuted series converge uniformly to f.

This theorem is proved in Daubechies (1992) and Meyer (1992). This type of convergence of wavelet series is called *unconditional convergence.* It is known [see Mallat (1998)] that unconditional convergence of wavelet series ensures an optimality of compression of signals. For details about compression of signals and other applications of wavelet series, see Walker (1999) for a simple introduction and Mallat (1998) for a thorough treatment.

VII. DISCRETE FOURIER SERIES

The digital computer has revolutionized the practice of science in the latter half of the twentieth century. The methods of computerized Fourier series, based upon the *fast Fourier transform* algorithms for digital approximation of Fourier series, have completely transformed the application of Fourier series to scientific problems. In this section, we shall briefly outline the main facts in the theory of discrete Fourier series.

The Fourier series coefficients $\{c_n\}$ can be discretely approximated via Riemann sums for the integrals in Eq. (9). For a (large) positive integer M, let $x_k = -\pi + 2\pi k/M$ for $k = 0, 1, 2, \ldots, M-1$ and let $\Delta x = 2\pi/M$. Then the nth Fourier coefficient c_n for a function f is approximated as follows:

$$\begin{aligned} c_n &\approx \frac{1}{2\pi}\sum_{k=0}^{M-1} \mathrm{f}(x_k)\, e^{-i2\pi n x_k}\Delta x \\ &= \frac{e^{-in\pi}}{M}\sum_{k=0}^{M-1} \mathrm{f}(x_k) e^{-i2\pi kn/M}. \end{aligned}$$

The last sum above is called the *Discrete Fourier Transform* (DFT) of the finite sequence of numbers $\{\mathrm{f}(x_k)\}$. That is, we define the DFT of a sequence $\{g_k\}_{k=0}^{M-1}$ of numbers by

$$G_n = \sum_{k=0}^{M-1} g_k\, e^{-i2\pi kn/M}. \tag{38}$$

The DFT is the set of numbers $\{G_n\}$, and we see from the discussion above that the Fourier coefficients of a function f can be approximated by a DFT (multiplied by the factors $e^{-in\pi}/M$). For example, in Fig. 14 we show a graph of approximations of the Fourier coefficients $\{c_n\}_{n=-50}^{50}$ of the square wave f_1 obtained via a DFT (using $M = 1024$). For all values, these approximate Fourier coefficients differ from the exact coefficients by no more than 10^{-3}. By taking M even larger, the error can be reduced still further.

The two principal properties of DFTs are that they can be inverted and they preserve energy (up to a scale factor). The inversion formula for the DFT is

$$g_k = \sum_{n=0}^{M-1} G_n\, e^{i2\pi kn/M}. \tag{39}$$

And the conservation of energy property is

$$\sum_{k=0}^{M-1} |g_k|^2 = \frac{1}{N}\sum_{n=0}^{M-1} |G_n|^2. \tag{40}$$

Interpreting a sum of squares as energy, Eq. (40) says that, up to multiplication by the factor $1/N$, the energy of the discrete signal $\{g_k\}$ and its DFT $\{G_n\}$ are the same. These

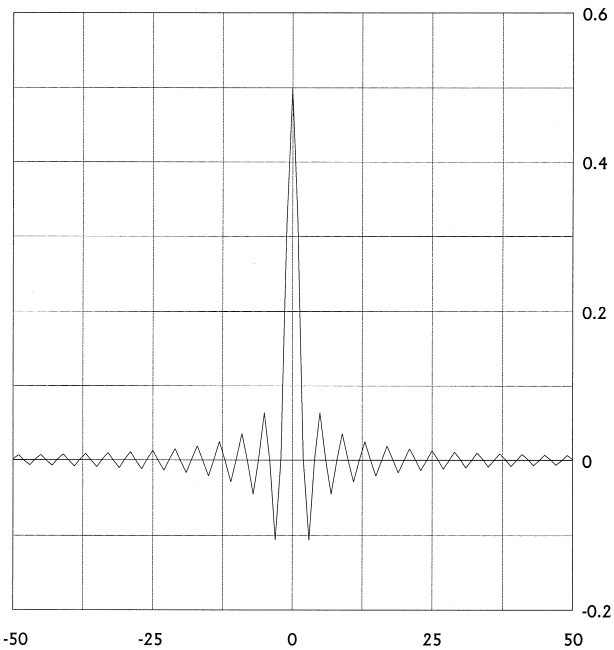

FIGURE 14 Fourier coefficients for square wave, $n = -50$ to 50. Successive values are connected with line segments.

facts are proved in Briggs and Henson (1995) and Walker (1996).

An application of inversion of DFTs is to the calculation of Fourier series partial sums. If we substitute $x_k = -\pi + 2\pi k/M$ into the Fourier series partial sum $S_N(x)$ we obtain (assuming that $N < M/2$ and after making a change of indices $m = n + N$):

$$\begin{aligned} S_N(x_k) &= \sum_{n=-N}^{N} c_n\, e^{in(-\pi+2\pi k/M)} \\ &= \sum_{n=-N}^{N} c_n\,(-1)^n\, e^{i2\pi nk/M} \\ &= \sum_{m=0}^{2N} c_{m-N}\,(-1)^{m-N}\, e^{-i2\pi kN/M}\, e^{i2\pi km/M}. \end{aligned}$$

Thus, if we let $g_m = c_{m-N}$ for $m = 0, 1, \ldots, 2N$ and $g_m = 0$ for $m = 2N+1, \ldots, M-1$, we have

$$S_M(x_k) = e^{-i2\pi kN/M} \sum_{m=0}^{M-1} g_m\,(-1)^{m-N}\, e^{i2\pi km/M}.$$

This equation shows that $S_M(x_k)$ can be computed using a DFT inversion (along with multiplications by exponential factors). By combining DFT approximations of Fourier coefficients with this last equation, it is also possible to approximate Fourier series partial sums, or arithmetic means, or other modified partial sums. See Briggs and Henson (1995) or Walker (1996) for further details.

These calculations with DFTs are facilitated on a computer using various algorithms which are all referred to as *fast Fourier transforms* (FFTs). Using FFTs, the process of computing DFTs, and hence Fourier coefficients and Fourier series, is now practically instantaneous. This allows for rapid, so-called *real-time,* calculation of the frequency content of signals. One of the most widely used applications is in calculating *spectrograms.* A spectrogram is calculated by dividing a signal (typically a recorded, digitally sampled, audio signal) into a successive series of short duration subsignals, and performing an FFT on each subsignal. This gives a portrait of the main frequencies present in the signal as time proceeds. For example, in Fig. 15a we analyze discrete samples of the function

$$\sin(2\nu_1\pi x)e^{-100\pi(x-0.2)^2} + [\sin(2\nu_1\pi x) + \cos(2\nu_2\pi x)] \times e^{-50\pi(x-0.5)^2} + \sin(2\nu_2\pi x)e^{-100\pi(x-0.8)^2} \qquad (41)$$

where the frequencies ν_1 and ν_2 of the sinusoidal factors are 128 and 256, respectively. The signal is graphed at the bottom of Fig. 15a and the magnitudes of the values of its spectrogram are graphed at the top. The more intense spectrogram magnitudes are shaded more darkly, while white regions indicate magnitudes that are essentially zero. The dark blobs in the graph of the spectrogram magnitudes clearly correspond to the regions of highest energy in the signal and are centered on the frequencies 128 and 256, the two frequencies used in Eq. (41).

As a second example, we show in Fig. 15b the spectrogram magnitudes for the signal

$$e^{-5\pi[(x-0.5)/0.4]^{10}}[\sin(400\pi x^2) + \sin(200\pi x^2)\,]. \qquad (42)$$

This signal is a combination of two tones with sharply increasing frequency of oscillations. When run through a sound generator, it produces a sharply rising pitch. Signals like this bear some similarity to certain bird calls, and are also used in radar. The spectrogram magnitudes for this signal are shown in Fig. 15b. We can see two, somewhat blurred, line segments corresponding to the factors $400\pi x$ and $200\pi x$ multiplying x in the two sine factors in Eq. (42).

One important area of application of spectrograms is in *speech coding.* As an example, in Fig. 16 we show spectrogram magnitudes for two audio recordings. The spectrogram magnitudes in Fig. 16a come from a recording of a four-year-old girl singing the phrase "twinkle, twinkle, little star," and the spectrogram magnitudes in Fig. 16b come from a recording of the author of this article singing the same phrase. The main frequencies are seen to be in harmonic progression (integer multiples of a lowest, fundamental frequency) in both cases, but the young girl's main frequencies are higher (higher in pitch) than the adult male's. The slightly curved ribbons of frequency content are known as *formants* in linguistics. For more

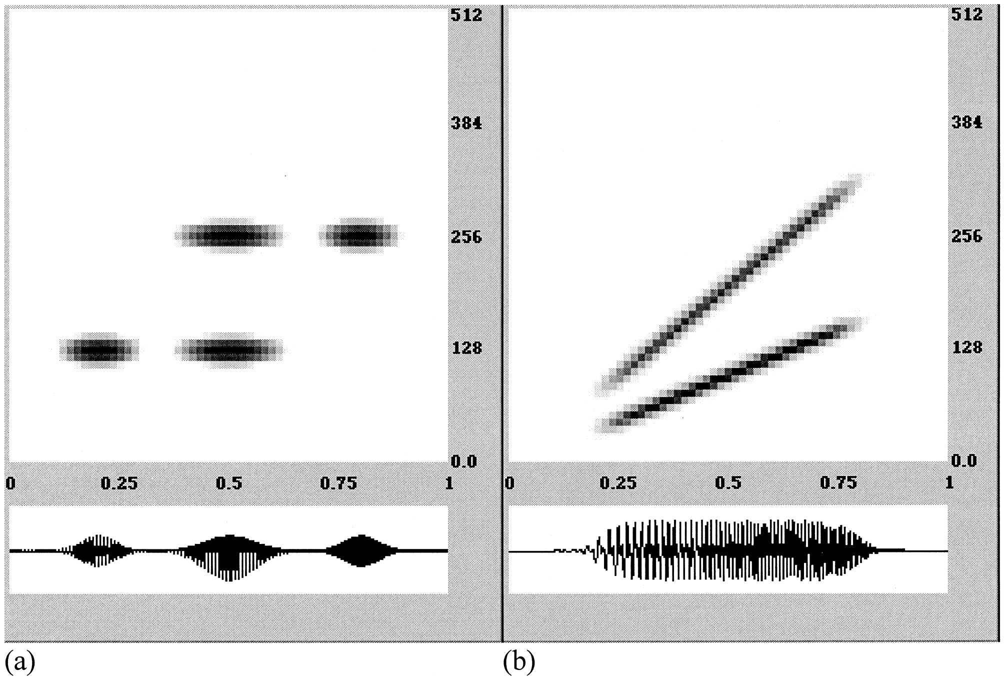

FIGURE 15 Spectrograms of test signals. (a) Bottom graph is the signal in Eq. (41). Top graph is the spectrogram magnitudes for this signal. (b) Signal and spectrogram magnitudes for the signal in (42). Horizontal axes are time values (in sec); vertical axes are frequency values (in Hz). Darker pixels denote larger magnitudes, white pixels are near zero in magnitude.

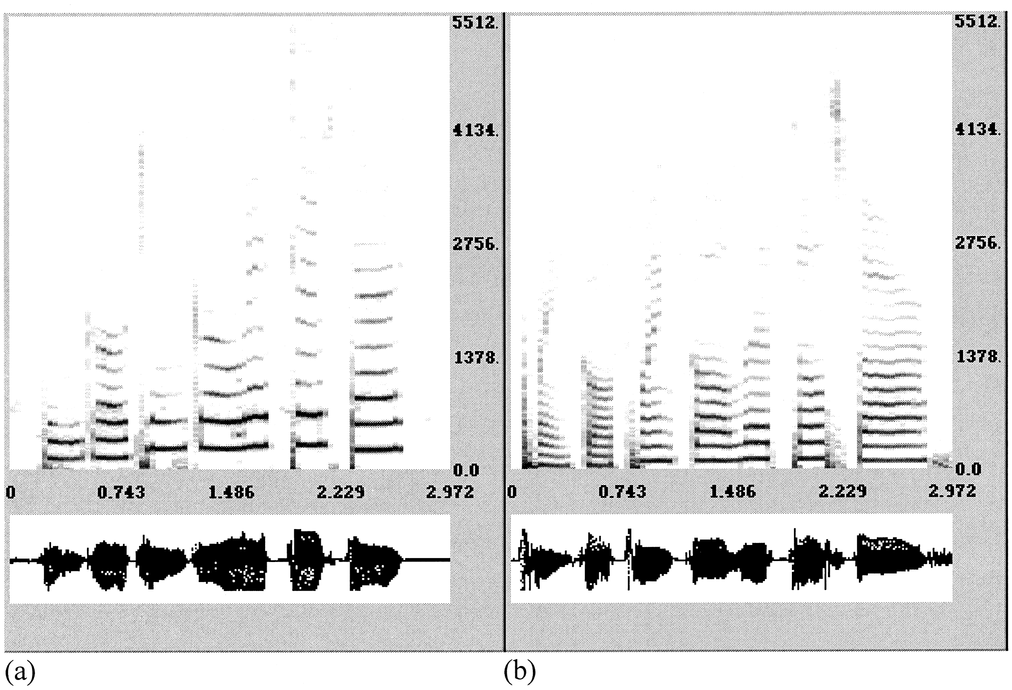

FIGURE 16 Spectrograms of audio signals. (a) Bottom graph displays data from a recording of a young girl singing "twinkle, twinkle, little star." Top graph displays the spectrogram magnitudes for this recording. (b) Similar graphs for the author's rendition of "twinkle, twinkle, little star."

details on the use of spectrograms in signal analysis, see Mallat (1998).

It is possible to invert spectrograms. In other words, we can recover the original signal by inverting the succession of DFTs that make up its spectrogram. One application of this inverse procedure is to the *compression of audio signals.* After discarding (setting to zero) all the values in the spectrogram with magnitudes below a threshold value, the inverse procedure creates an approximation to the signal which uses significantly less data than the original signal. For example, by discarding all of the spectrogram values having magnitudes less than $1/320$ times the largest magnitude spectrogram value, the young girl's version of "twinkle, twinkle, little star" can be approximated, *without noticeable degradation of quality,* using about one-eighth the amount of data as the original recording. Some of the best results in audio compression are based on sophisticated generalizations of this spectrogram technique—referred to either as *lapped transforms or as local cosine expansions*, see Malvar (1992) and Mallat (1998).

VIII. CONCLUSION

In this article, we have outlined the main features of the theory and application of one-variable Fourier series. Much additional information, however, can be found in the references. In particular, we did not have sufficient space to discuss the intricacies of multivariable Fourier series which, for example, have important applications in crystallography and molecular structure determination. For a mathematical introduction to multivariable Fourier series, see Krantz (1999), and for an introduction to their applications, see Walker (1988).

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BIBLIOGRAPHY

Briggs, W. L., and Henson, V. E. (1995). "The DFT. An Owner's Manual," SIAM, Philadelphia.

Daubechies, I. (1992). "Ten Lectures on Wavelets," SIAM, Philadelphia.

Davis, P. J. (1975). "Interpolation and Approximation," Dover, New York.

Davis, P. J., and Hersh, R. (1982). "The Mathematical Experience," Houghton Mifflin, Boston.

Fourier, J. (1955). "The Analytical Theory of Heat," Dover, New York.

Jackson, D. (1941). "Fourier Series and Orthogonal Polynomials," Math. Assoc. of America, Washington, DC.

Krantz, S. G. (1999). "A Panorama of Harmonic Analysis," Math. Assoc. of America, Washington, DC.

Mallat, S. (1998). "A Wavelet Tour of Signal Processing," Academic Press, New York.

Malvar, H. S. (1992). "Signal Processing with Lapped Transforms," Artech House, Norwood.

Meyer, Y. (1992). "Wavelets and Operators," Cambridge Univ. Press, Cambridge.

Rudin, W. (1986). "Real and Complex Analysis," 3rd edition, McGraw-Hill, New York.

Walker, J. S. (1988). "Fourier Analysis," Oxford Univ. Press, Oxford.

Walker, J. S. (1996). "Fast Fourier Transforms," 2nd edition, CRC Press, Boca Raton.

Walker, J. S. (1999). "A Primer on Wavelets and their Scientific Applications," CRC Press, Boca Raton.

Walter, G. G. (1994). "Wavelets and Other Orthogonal Systems with Applications," CRC Press, Boca Raton.

Zygmund, A. (1968). "Trigonometric Series," Cambridge Univ. Press, Cambridge.