Wallpaper groups

1.1 The orbifold wallpaper shop

Our slogan is “ examples first and theory after”. This section is a crash course in associating a plane pattern with one of the 17 different possible symmetry groups. We will understand how and why all this works in a few weeks. A repeating pattern is one with translation symmetries in two independent directions. The symmetry group of such a pattern is often called a *wallpaper group*[1]

1.1.1 Types of symmetry As we will see, there are four types of symmetry of the plane:

- translation by a given vector;
- rotation by a given angle about a given point;
- reflection in a given line;
- glide reflection in a given line by a given non-zero amount.

The one of these that may be unfamiliar is the last, which we will often just call a glide. It is the composition of a reflection in a given line in the plane followed by a translation by a given non-zero distance in the direction parallel to the line. So, for example,

$$(x, y) \mapsto (1 - x, y + 2)$$

is a glide with underlying line $x = 1/2$.

Of the symmetries, the first two types are *direct* whereas the last two are *indirect*, meaning that in the latter case elements of the pattern are mapped to a mirror image of themselves. The identity can be regarded as either a translation by a zero vector or a rotation by zero angle.

1.1.2 Order of precedence In our analysis, we will look for symmetry features in a particular order and for this reason we have some nomenclature.

1. A *mirror* is a line of reflection symmetry and a *kaleidoscope* is an intersecting system of mirrors. A *kaleidoscope point* of order k is a point where k mirrors meet.

[1] There is an added technicality that there should not be arbitrarily small translations or rotations.

2. A *gyration* of order k is a point *not* on a mirror line about which there is rotational symmetry of order k.

3. A *miracle* is a glide symmetry of the pattern which is not "done with mirrors". (See discussion below.)

4. A *wandering* is a pair of independent translation symmetries in a pattern containing none of the three previous features.

We will be lax with our terminology and use e.g. "gyration" to refer to both the particular symmetry of the pattern and also the point in the pattern which is the centre of rotation.

If k mirrors meet at a point in a pattern, then there is k-fold rotational symmetry about that point. We don't count this as a gyration because the rotational symmetry comes from the mirrors.

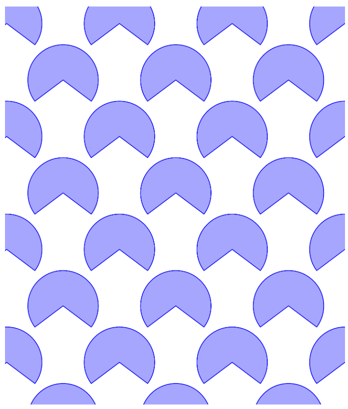
Wanderings are not difficult: if there are no mirrors, gyrations or miracles then for it to be a repeating pattern there must be a wandering.

1.1.3 We count features once We will be counting features only once. We will explain this eventually in terms of counting the features of the orbifold but for now, we will try and understand it intuitively.

Gyrations are easy: two gyrations in the pattern are "the same" feature if there is a symmetry of the pattern taking one centre of symmetry to the other. For two gyrations to be the same, they have to have the same order, but two gyrations of the same degree may be different. (For example, if one is green and the other is yellow they are certainly different.)

Mirrors are a bit more subtle although generally intuition works. We will discuss this more below.

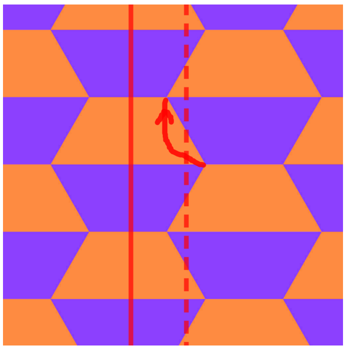
1.1.4 Miracles If a repeating pattern has mirrors then combining reflection in the mirror with a translation will produce a glide symmetry (unless the translation is perpendicular to the mirror). These are *not* miracles because these glides are done with mirrors.



The pattern on the right is a good example. You should see that it has vertical mirror lines. All the mirror lines are "the same". And if we were to reflect in one of these mirrors and then translate upwards by the distance between two of the slices of pie, we would have a glide symmetry, bu it is not a miracle because we did it with the aid of mirrors.

Draw a vertical line half way between two adjacent mirror lines. You should see that the pattern has a glide symmetry formed by reflecting in that line combined with a vertical translation by half the distance used previously. This glide cannot be obtained by combining our known mirrors with translations and so it is a real miracle. Mirrors and miracles alternate across the pattern. Note that the translation part of the miracle is exactly half of the smallest vertical translation symmetry; in fact, the square of the miracle is exactly that translation. For the purposes of classification, this pattern contains one mirror and one miracle.

1.1.5 When is it a miracle? It can be tricky to spot glide symmetries, and once one has one, it can be tricky to decide if it is a miracle.



- If the axis of the glide is a mirror then it is NOT a miracle, BUT the converse is NOT true.
- If you take a small area of the pattern and you can connect it to the area which is its image under the glide by a curve which does no cross a mirror, then it is a miracle.
- Generally for wallpaper groups the cost (see below) will tell you if you need a miracle.

The image on the right above shows a pattern with the same symmetry as the one in the previous paragraph. It is marked up: the solid red line shows the mirror and the dotted red line the axis of the miracle. The curved red arrow demonstrates that one can join a point in the pattern to its image under the glide without crossing a mirror.

1.1.6 Signature At the Orbifold Wallpaper Shop, wallpaper is priced according to its *signature*, which records the symmetry features. The costs appear in Table 1.1.

1.1.7 Conway's "Magic Theorem" The signatures of wallpaper groups are those that cost exactly $\$2$.

Proof. We will understand why the Magic Theorem is true later. □

For now, Conway's theorem tells us exactly which signatures are possible for wallpaper groups. A reference list is provided in Table 1.2.

If you are wondering what exactly constitutes a signature, it is a non-empty set of symbols consisting of:

- Zero or more blue circles, followed by
- Zero or more (blue) natural numbers (all greater than 1), followed by
- Zero or more (red) "stars" each independently followed by zero or more (red) natural numbers (each greater than 1), followed by
- zero or more (red) 'x's.

For the examples we will consider, the ordering of the sets of blue numbers and red numbers is unimportant, so for example $*442$ is the same signature as $*244$. And note that the use of red and blue numbers is redundant, although it can help to keep ones head straight.

1.2 The field guide to wallpaper groups

Kaleidoscopes

1.2.1 First look for mirror lines in your pattern (and it helps to mark some of them on a paper copy, or put a transparency over the top and mark on that to save spoiling it). If there are

Feature	Notation	Cost	Example
Kaleidoscope (a mirror or a system of intersecting mirrors)	$*$ for a single mirror or $*ab\dots c$ for coincident mirrors where $a, b, \dots, c$ correspond to points where that number of mirrors meet	$\$1$ for the $*$ and $\$\frac{n-1}{2n}$ for a red number n, etc	$**$ (two independent mirrors) costs $\$2$ and $*632$ costs $1+\frac{5}{12}+\frac{2}{6}+\frac{1}{4} = \$2$
Gyration (centre of rotation not on a mirror)	a where $a \geq 2$ is the order of the rotational symmetry	$\$\frac{n-1}{n}$ for a blue number n, etc	442 costs $\frac{3}{4} + \frac{3}{4} + \frac{1}{2} = \$2$
Miracle (a glide not generated by other symmetries)	$\times$	$\$1$	$22\times$ costs $\frac{1}{2} + \frac{1}{2} + 1 = \$2$
Wandering (Two independent translations not generated by other symmetries)	$\mathbf{o}$	$\$2$	Can only appear on its own as $\mathbf{o}$.

Table 1.1: The cost of wallpaper features

	Kaleidoscopic
$*632$	Fundamental domain is a 30-60-90 right-angled triangle (1/12 of a regular hexagon).
$*442$	Fundamental domain is a 45-45-90 right-angled triangle (1/8 of a square).
$*333$	Fundamental domain is an equilateral triangle.
$*2222$	Fundamental domain is a rectangle.
$**$	Two different types of mirror. The two types must be parallel to each other. It is a "kaleidoscope with the mirrors meeting at infinity".)
	Gyratory
632	Gyrations of orders $2, 3, 6$ arranged at the vertices of a 30-60-90 triangle. (Rarely seen in the wild.)
442	Gyrations of orders $2, 4, 4$ arranged at the vertices of a 45-45-90 triangle.
333	Gyrations of orders 3 (three different types) arranged at the vertices of an equilateral triangle.
2222	Gyrations of orders 2 (four different types) arranged at the vertices of a parallelogram. (NB: the corresponding kaleidoscope has to be rectangular).
	Gyroscopic (kaleidoscopic with central gyrations)
$3*3$	Gyration of order 3 at centre of an equilateral triangle kaleidoscope.
$2*22$	Gyration of order 2 at centre of a rectangular (possibly square) kaleidoscope.
$4*2$	Gyration of order 4 at centre of a square kaleidoscope.
$22*$	Mirrors all in the same direction and of the same type with two different types of order-2 gyrations between adjacent ones
	Miraculous (and trivial)
$*\times$	Mirrors all in the same direction and of the same type and a miracle. The glide line of the miracle is half way between adjacent mirrors.
$\times\times$	Two miracles of different types. The glide lines of the two have to be parallel.
$22\times$	As the previous case except there are two types of order-2 gyrations half way between adjacent glide lines.
o	A "wandering": there are no mirrors, gyrations or miracles – just a lattice of translations.

Table 1.2: List of the 17 wallpaper groups

mirrors in more than one direction, then find a polygon, as small as you can and bounded by mirrors. The possibilities are:

- a rectangle bounded by four mirrors;
- an equilateral triangle bounded by three mirrors;
- a right-angled isosceles triangle (45-45-90 — one eighth of square) bounded by three mirrors;
- a 30-60-90 right-angled triangle (one twelfth of a regular hexagon) bounded by three mirrors.

There is also the degenerate case of $**$ where there are two different sorts mirror (with all mirrors in the same direction).

If you are in the rectangular or equilateral triangle cases, don't forget to check for gyrations at the centre of the polygon, the existence of which means you have one of the "gyroscopic" cases: see discussion below. Also beware of gyrations half way between parallel mirrors in the degenerate case (which leads to $22*$).

1.2.2 In eg $*333$, the three segments of mirror bounding the triangle are different. There is no symmetry taking a point on one side to a point on one of the other sides. Furthermore, given a point on a mirror in the pattern there is a symmetry taking that point to one on the edges of the triangle.

On the other hand, adding a 3-fold gyration point in the centre makes all three mirror segments and all three kaleidoscope points equivalent and so we should count them only once. Hence the result is $3*3$ (and not $3*333$, which cannot be right because it is too expensive to buy).

1.2.3 In terms of thinking about combinations of features whose costs add up to $\$2$, it is worth noting that a red number costs half as much as the corresponding blue number. So, for example, since $*442$ costs exactly $\$2$ then so must $4*2$. The gyroscopic cases all come from this process from kaleidoscopes.

The gyratory groups

We next come to *gyrations* which are rotational symmetries that are not products of reflections.

1.2.4 By arithmetic, if $*abc$ costs $\$2$ then so does abc. Therefore the following signatures cost $\$2$ and so define wallpaper groups:

$$632, \quad 442, \quad 333, \quad 2222.$$

1.2.5 But more is true because the direct isometry (rotations and translations, remember) subgroups of the corresponding kaleidoscope do give examples of these wallpaper groups, with the gyration points being at the kaleidoscope points.

In fact, the gyrations have to be arranged in exactly the same way as the corresponding kaleidoscope points in every case except 2222 where the four gyrations points can be in an arbitrary parallelogram and not just in a rectangle.

1.2.6 There is one other group in the "true blue" category (ie which has only direct isometries), which is where there are no symmetries but the lattice of translations. This has signature **o** but do not diagnose that until you have checked for miracles below.

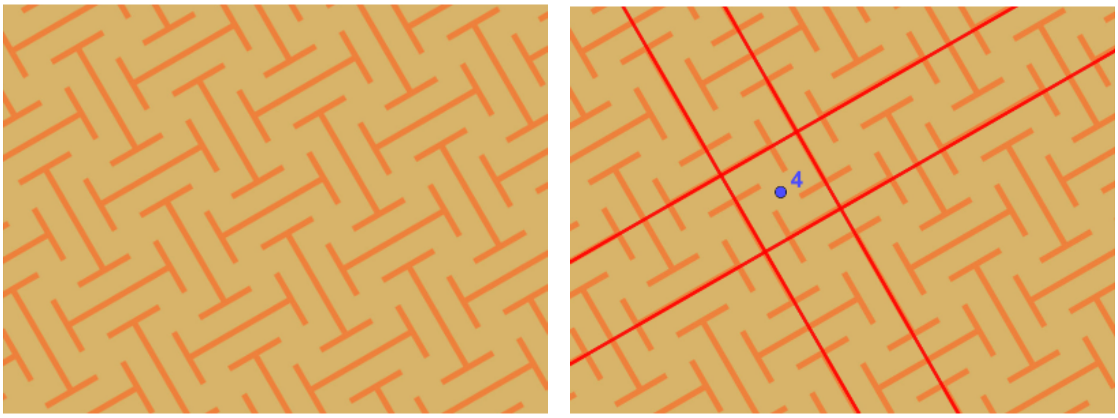


Figure 1.1: Marking up a pattern

Gyroscopic groups

1.2.7 Etymology "Gyroscopic" is a cross of "kaleidoscopic" and "gyratory". The idea is that you have a kaleidoscope with a gyration in the centre of what would be its fundamental domain. You have to watch out for these. There are four possibilities listed in Table 1.2.

Miracles and odds and ends

1.2.8 Miracles The final thing you need to check for are *miracles*. A miracle is a glide reflection symmetry that does not come from mirrors and rotations and we denote it by $\times$. A give away is if you find a piece of pattern and its mirror image without there being a mirror dividing the two. This leads to the following signatures: $*\times$, $\times\times$, $22\times$. In all these, all mirror lines and glide lines denoted in the signature are parallel to each other.

1.2.9 Wanderings And if there are no mirrors, gyrations or miracles, then it must just be a *wandering*: a lattice of translations: *o* which should probably be in the "true blue" category.

1.2.10 Marking up We "mark up" a pattern to indicate its symmetry features and hence justify the identification of its signatures. We indicate mirrors by solid red lines and gyrations by blue points labelled with the degree. We can mark the axes of miracles by dashed red lines (my preference) and/or by a red, usually curly, arrow linking a piece of pattern to its mirror image.

1.3 Isometries of the plane

By a symmetry of the plane we mean an *isometry*: a distance preserving bijection $\mathbb{R}^2 \to \mathbb{R}^2$. Obvious examples include: translation $T_{\mathbf{a}}$ by a vector displacement $\mathbf{a}$; rotation $R_{p,\theta}$ by an angle θ anticlockwise about the point p and reflection M_l in a line l (not necessarily through the origin). The first two are called *direct isometries* and the final one is *indirect* since it sends shapes or patterns to their mirror images.

1.3.1 Definition The *Euclidean group* $E(2)$ is the group of all transformations of the plane of the form

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + \mathbf{a}$$

where $A \in O(2)$ (i.e. A is a 2×2 orthogonal matrix) and $\mathbf{a}$ is a vector in $\mathbb{R}^2$. An element of $E(2)$ is called an *isometry*. It is a *direct isometry* if A is a rotation ($\det A = 1$) and *indirect* if A is a reflection ($\det A = -1$).

Note that when $A = 1$ an isometry is a translation and when $\mathbf{b} = \mathbf{0}$ an isometry is an orthogonal linear transformation.

1.3.2 Exercise (BS) Write $(A, \mathbf{a})$ and $(B, \mathbf{b})$ for two elements of $E(2)$ as above. Compute the product of group elements $(A, \mathbf{a})(B, \mathbf{b})$ and express it in this notation. Hence find a formula for the inverse $(A, \mathbf{a})^{-1}$.

1.3.3 Proposition The map $f : E(2) \to O(2)$ defined by $f : (A, \mathbf{a}) \mapsto A$ is a group homomorphism. Its kernel is the normal subgroup $T \cong \mathbb{R}^2$ consisting of all translations.

Proof. The fact that it is a homomorphism follows from the previous exercise. Its kernel is clearly $\{(1, \mathbf{a}) \mid \mathbf{a} \in \mathbb{R}^2\}$, which is all translations of the plane. Kernels always are normal subgroups. □

1.3.4 Proposition We will write $R_{p,\theta}$ for rotation anticlockwise by θ about a point p. Then for all p, θ it is the case that $R_{p,\theta} \in E(2)$.

Proof. Let p have position vector $\mathbf{p}$ with respect to the origin and let

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

the orthogonal matrix that rotates by θ about the origin. Now write $\mathbf{x} = \mathbf{p} + \mathbf{y}$ and observe that

$$R_{P,\theta}(\mathbf{x}) = \mathbf{p} + A(\mathbf{y}) = \mathbf{p} + A(\mathbf{x} - \mathbf{p}) = A\mathbf{x} + (\mathbf{p} - A\mathbf{p})$$

which is of the required form. □

1.3.5 Exercise (SN) Show that given points p, q in the plane

$$R_{p,\theta} = T\, R_{q,\theta}\, T^{-1}$$

where T is the translation that takes q to p. Thus we observe that rotations by θ about different points are conjugate in $E(2)$. Explain this formula by imagining what happens to the plane under the sequence of three transformations. (By this I mean, think of the composition as being “shift the plane, rotate about q then shift back“.)

1.3.6 Proposition Every direct isometry of the plane is either a translation (including the zero translation) or rotation about some point.
Proof. Follows immediately from the above. □

1.3.7 Exercise (NL) Consider the following compositions and understand what you can about them:

1. The composition of two reflections through lines that meet at a point p;
2. The composition of two reflections in parallel lines;
3. The composition of reflection in a line and a translation perpendicular to that line;

1.3.8 Definition A *glide reflection* (just “glide” to its friends) is the composition of reflection in a line and a non-zero translation parallel to that line.

1.3.9 Proposition Every indirect isometry is either a reflection in a line or a glide along a line.
Proof. Consider an indirect isometry $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ where A is a reflection in a line through the origin Decompose $\mathbf{b} = \mathbf{p} + \mathbf{q}$ parallel and perpendicular to that line respectively. Identify $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{q}$ and hence the original transformation. □

Lattices

1.3.10 Definition A (2-dimensional) *lattice* in $\mathbb{R}^2$ is a subgroup of the translations in $E(2)$ of the form

$$\{k\mathbf{u} + l\mathbf{v} \mid k, l \in \mathbb{Z}\}$$

where $\mathbf{u}, \mathbf{v}$ are a pair of linearly independent vectors. We call $\mathbf{u}, \mathbf{v}$ *generators* of the lattice.

1.3.11 Proposition Let L be a discrete subgroup of the translations in $E(2)$ containing translations in two independent directions. Then L is a lattice.
Proof. For the proof, choose a non-zero translation $\mathbf{u}$ of minimum size in L and another non-zero translation $\mathbf{v}$ of minimum size subject to its not being a scalar multiple of $\mathbf{u}$. Clearly, all translations by elements of $\{k\mathbf{u} + l\mathbf{v} \mid k, l \in \mathbb{Z}\}$ are in L. Now show that no other translations can be in L. □

1.3.12 Exercise (MN) Complete the above proof.

1.3.13 Warning It is tempting and often useful when one has a lattice to draw a lattice of parallelograms based on $\mathbf{u}, \mathbf{v}$ but one must remember that the lattice is just the points in the picture, not the lines. In particular, different shaped parallelograms can generate the same lattice as for example in Fig 1.2.

1.3.14 Definition A *wallpaper group* is a discrete subgroup of $E(2)$ whose translations form a lattice.

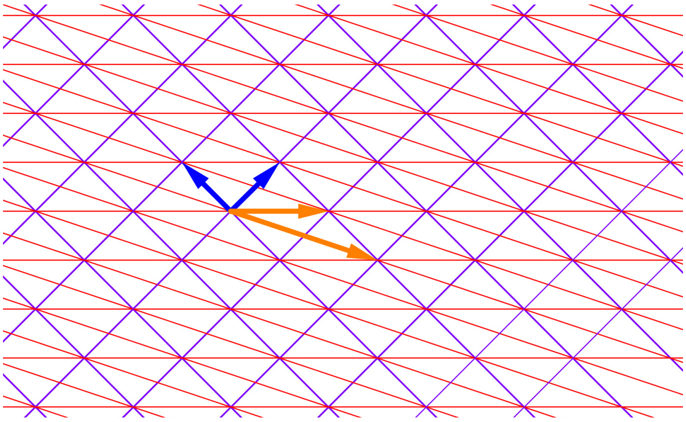
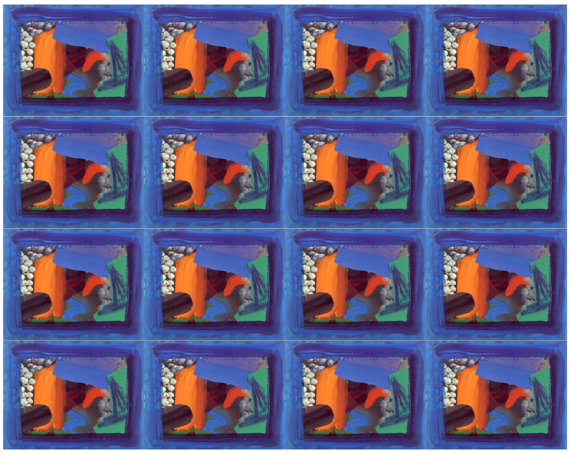


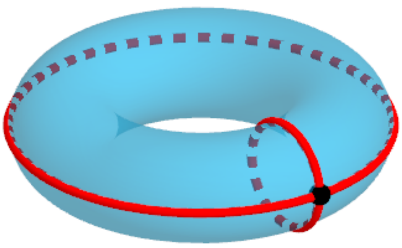
Figure 1.2: Two generating sets for the same lattice

1.3.15 A diversion: a wallpaper orbifold Take any lattice and into each panel put a copy of your favourite picture (not too symmetric please – we do not want extra symmetries). It does not have to be a rectangular lattice like mine below. Imagine it continued over the whole plane.



The symmetries of this are just a lattice of translations, the wallpaper group with signature **o**.

A fundamental domain for this pattern is (or rather, can be chosen to be) one copy of the picture, as below, left.



There is a map from the whole of $\mathbb{R}^2$ covered with this pattern to the fundamental domain but as in previous situations we have seen, it is not continuous. In the picture above left I have added some borders. To make the map continuous we need to zip together the two horizontal edges (or, more formally, *identify* them) to make a tube. And we also need to zip together the two vertical edges (which our previous operation has turned into circles).

The result is a *torus* as shown above on the right. The two zipped pairs of edges have become the two red circles and the vertices of the original confining rectangle have become a single point. You might (indeed you should) worry that we would have to distort Howard Hodgkin's painting as we carry out the second zipping above to produce the pictured torus. The answer to this is that the torus we really have as the orbifold is a mathematical construct where (essentially) we just take the rectangular painting and identify the edges so that, for example, if a 2-dimensional creature crawls off the right edge they magically appear on the left edge (at the same height). Find a simulation of the old computer game "Asteroids" online or install the "Torus Games" app from `www.geometrygames.org/TorusGames` to get a feeling for this.

So the picture of the torus correctly conveys the "topology" of the situation but not quite the correct "geometry". To get a proper "flat" torus in Euclidean space, we have to use four dimensions.

One point of this diversion is to see that we will need to understand the geometry and topology of surfaces quite well in order to understand wallpaper groups via their orbifolds.

1.3.16 Classification of lattices To return to our understanding of the Euclidean group, we need to classify lattices in the plane.

1.3.17 Definition

1. A lattice is *rectangular* if it has generators that are perpendicular.
2. A lattice is *rhombic* if it has generators of equal length.
3. A lattice is *square* if it has perpendicular generators of equal length.
4. A lattice is *hexagonal* if it has equal length generators at an angle of 60 degrees to each other.
5. We will call our lattice *general* or *generic* if it is none of the above.

In each case, it is not whether the generators one has been given (or first notices) that have to obey the conditions. The question is whether such generators exist.

1.3.18 Exercise (BM) Draw examples of all the above. Make sure your example fits only one category, don't e.g. make your rectangular lattice square. (You will need these below.)

1.3.19 Proposition The *symmetry group of a lattice*, is the subgroup of $O(2)$ consisting of elements that preserve the lattice. The symmetry groups are as follows.

Hexagonal D_6

Square D_4

Rhombic but not square or hexagonal D_2

Rectangular but not square D_2

General C_2

A two element group of rotations (half turn and the identity) is denoted C_2.

1.3.20 Exercise (NM) For each type of lattice, indicate the symmetries on an example diagram.

1.3.21 Exercise (NM) Draw a diagram that illustrates how some of the categories of lattices are special cases of others. (Perhaps five blobs for the types and arrows to indicate which are special cases of other.)

1.3.22 Exercise (NM) For each class of lattice, draw a picture of the corresponding lattice of parallelograms (which will specialise to be squares rhombuses, etc except for the general lattice) and determine the wallpaper group signature for that pattern.

Note that this is different from the symmetry group of the lattice as discussed above. (The symmetry group above is the local group of the wallpaper group at on eof the vertices.)

Three definitions and three theorems

Throughout, G is a wallpaper group. Restricting the group homomorphism $f : E(2) \longrightarrow O(2)$ to G we obtain a group homomorphism $G \longrightarrow O(2)$.

1.3.23 Definition The image of $G \to O(2)$ is called the *point group* of G and is often denoted by P.

So P is obtained as follows: write every element of G in the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$. Then take all the "$A$"s that arise – they form a subgroup P of $O(2)$.

1.3.24 Images in P The images of elements of a wallpaper group in the point group are as follows.

- Translations map to the identity $\mathbb{I}$ in P.
- A rotation by θ about a point maps to a rotation by θ about the origin.
- A reflection in a line l and a glide based on the line l both map to a reflection about the line through the origin parallel to l.

The point group of a wallpaper group G is a sort of reduced picture of the group and it often allows us to deduce things about G itself.

1.3.25 Examples

- In the wallpaper group $*\times$ every element that is not a translation is a reflection or glide and the lines of all these are parallel. Thus P is the two element group consisting of the identity and the reflection in the line through the origin parallel to all the mirrors and glide lines.

- In the group $22\times$ we have order two gyrations that give rise to a half-turn in P and the miracles that one finds are all glides along parallel lines giving rise to a single reflection in P. Also of course translations give rise to the identity in P. But these three elements of P do not form a group: the product of a half-turn and a reflection in $O(2)$ is a reflection in a line perpendicular to that of the first reflection. We deduce then that this new reflection must also be in P.

  This reflection in the point group must come from some symmetries in the pattern: it cannot mirrors so it must be glides and so there must also be glides along lines perpendicular to the ones we first identified in the features. In fact, we always have a choice when identifying the glide lines for the miracle in $22\times$ and those choices have perpendicular glide lines.

1.3.26 Exercise (NS) Use the point group to deduce that if a wallpaper group is such that there are points p, q in the pattern about which there is a rotational symmetry of orders 2 and 3 respectively, then somewhere in the pattern there is a point about which there is rotational symmetry of order 6.

1.3.27 Exercise (NM) Let g, g' be glides in a wallpaper group with the lines of the two glides being orthogonal. What can you deduce about their product gg' by considering images in the point group.

1.3.28 Exercise (NM) Take three wallpaper groups of different character and identify the point group for each.

1.3.29 Definition Let p be a point in $\mathbb{R}^2$. The *local group* at p is the subgroup of G that fixes p. (In group action terms, it is the stabilizer of C.)

Choosing the origin to be at the point we are interested in, the local group can be regarded as a subgroup of $O(2)$.

1.3.30 Exercise (NS) Explain why the local group for a wallpaper groups is trivial except at points at gyrations or on mirrors.

1.3.31 Theorem – the "crystallographic restriction" Rotations appearing in a wallpaper group have order $2, 3, 4$ or 6.

Proof. This follows from the Magic Theorem because there are no signatures with cost \$ 2 containing numbers other than $2, 3, 4, 6$. Alternatively (and non-examinably), we can argue as follows.

Translations and rotations in the group must take centres of n-fold rotational symmetry to centres of n-fold rotational symmetry, and so if there is one centre of rotation then there are lots of them. So let p, q be two centres of rotational symmetry as close together as possible. If $n > 6$, rotate q about p by one n-th of a turn to obtain another centre of n-fold rotational symmetry q'. This is closer to q than p was, giving a contradiction.

For $n = 5$, we find that q' is further from q than p is. But rotating p about q' we do obtain a centre of rotation q'' that gives the contradiction. $\square$

1.3.32 Exercise (BM) Draw some pictures to illustrate the above proof.

1.3.33 Corollary Nontrivial local groups can only be D_1, D_2, D_3, D_4, D_6 or their rotational subgroups.

1.3.34 Theorem The point group P of G is contained in the symmetry group of the lattice L of translations in G. (In other words, if $A \in P$ and $\mathbf{x} \mapsto \mathbf{x} + \mathbf{b}$ is a translation symmetry of our pattern then so is $\mathbf{x} \mapsto \mathbf{x} + A\mathbf{b}$
Proof. Let $A \in P$. Then there exists $\mathbf{a}$ such that $(A, \mathbf{a}) \in G$. Let $\mathbf{t}$ be a vector in the lattice L so that $(1, \mathbf{t}) \in L$. Now

$$(A, \mathbf{a})(1, \mathbf{t})(A, \mathbf{a})^{-1} = (1, A\mathbf{t})$$

and so $A\mathbf{t} \in L$. □

1.3.35 Corollary Nontrivial point groups can only be D_1, D_2, D_3, D_4, D_6 or their rotational subgroups.

1.3.36 Exercise (NM) Prove the Corollary. Bear in mind that the theorem says that the point group is a subgroup of the symmetry group of the lattice, not necessarily the whole thing!

1.3.37 Exercise (NL) Which point groups can occur with which lattice types?

1.3.38 Note By the way, if we believe it is clear that there are no lattices with rotational symmetry of order 5 or greater than 6, then the last theorem provides an alternative proof of the crystallographic restriction. But the contradiction argument we gave is appealingly free of technology.

1.3.39 Definition A *cell* in a wallpaper pattern is a parallelogram whose edges form two translations that generate the lattice of translations. (It is thus a fundamental domain for the subgroup of the wallpaper group consisting of all its translations.)

1.3.40 Theorem Fix a cell C in a wallpaper pattern. The point group P acts on C. Fixing a fundamental domain $F \subseteq C$, the images $\{p \cdot F \mid p \in P\}$ "tile" C with $|P|$ fundamental domains.

1.3.41 Corollary The size of the point group is equal to the area of a cell divided by the area of a fundamental domain.
Proof. (not examinable) We will not give a full proof of the theorem: we will just define the action of P. So let $A \in P$ and suppose $\mathbf{y} \in C$. Then by definition there exists $\mathbf{b}$ such that $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ is in the group. Now let $\mathbf{z}$ be the unique point in C that is obtained by a translation symmetry from $A\mathbf{y} + \mathbf{b}$. Then define $A \cdot \mathbf{y} = \mathbf{z}$; this is well defined because different choices of representing $A \in P$ by an element of the wallpaper group differ by a translation. □

1.3.42 Reference The Wikipedia wallpaper groups page `https://en.wikipedia.org/wiki/Wallpaper_group` gives ways (not unique) of tiling a cell with fundamental domains.