Analysis Hand in Three

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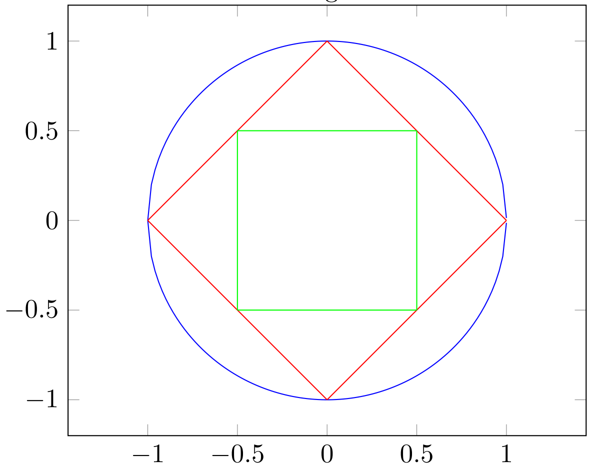
Question One - Workshop 6, Q6

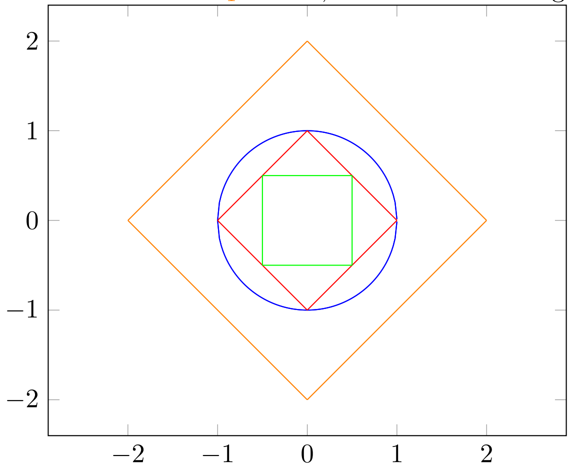
Part A

So we have the three metrics on $\mathbb{R}^2$ given by:

$$d_1(\mathbf{x},\mathbf{y}) := \sum_{k=1}^{n} |x_k - y_k|, \quad d_2(\mathbf{x},\mathbf{y}) := |\mathbf{x}-\mathbf{y}|, \quad d_\infty(\mathbf{x},\mathbf{y}) := \max_{1\le k\le n} |x_k - y_k|.$$

Below we have plots of the unit balls centered at the origin with the d_1 metric, the d_2 metric and the d_∞ metric on the left. On the right I've added a ball of radius two in the d_1 metric, centered at the origin.





Part B

We must show first that $d_\infty(\mathbf{x},\mathbf{y}) \le d_2(\mathbf{x},\mathbf{y})$, that is, that $\max_{1\le k\le 2}|x_k - y_k| < |\mathbf{x}-\mathbf{y}|$. So for any $\mathbf{x}$ and $\mathbf{y}$ we have that $d_\infty(\mathbf{x},\mathbf{y}) = \max_{1\le k\le 2}|x_k - y_k| = |x_j - y_j|$ for some j, and that $d_2(\mathbf{x},\mathbf{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$. Note that $(d_2(\mathbf{x},\mathbf{y}))^2 = (x_1-y_1)^2 + (x_2-y_2)^2 \ge (x_i - y_i)^2$ for any i by the triangle inequality, so for any i, j we have that

$$\begin{aligned} d_2(\mathbf{x},\mathbf{y})^2 &\ge |x_j - y_j|^2 \\ \Rightarrow d_2(\mathbf{x},\mathbf{y}) &\ge |x_j - y_j| \end{aligned}$$

and so $d_2(\mathbf{x},\mathbf{y}) \ge d_\infty(\mathbf{x},\mathbf{y})$

Now we need to show that $d_2(\mathbf{x},\mathbf{y}) \le d_1(\mathbf{x},\mathbf{y})$, in other words we need to show that $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} \le \sum_{k=0}^{n}|x_k - y_k| = |x_1 - y_1| + |x_2 - y_2|$. Note that $a^2 + b^2 \le (a+b)^2 = a^2 + b^2 + 2ab$ in general (for $a, b \ge 0$), so letting $a = |x_1 - y_1| > 0$ and $b = |x_2 - y_2| > 0$ we get

$$\begin{aligned} |x_1-y_1|^2 + |x_2-y_2|^2 &\le (|x_1-y_1| + |x_2-y_2|)^2 \\ \Rightarrow \sqrt{|x_1-y_1|^2 + |x_2-y_2|^2} &\le |x_1-y_1| + |x_2-y_2| \\ \Rightarrow d_2(\mathbf{x},\mathbf{y}) &\le d_1(\mathbf{x},\mathbf{y}) \end{aligned}$$

which is what we wanted, though we've only proved this for $\mathbb{R}^2$, but that's all that's required.

Finally we must show that $d_1(\mathbf{x},\mathbf{y}) \le n d_\infty(\mathbf{x},\mathbf{y})$. By definition of $d_\infty(\mathbf{x},\mathbf{y})$, for any i we have that if

$d_\infty(\mathbf{x},\mathbf{y}) = |x_a - y_a|$ then $|x_i - y_i| \leq |x_a - y_a|$. So we have that

$$\begin{aligned} d_1(\mathbf{x},\mathbf{y}) := \sum_{i=1}^{n} |x_i - y_i| &\leq \sum_{i=1}^{n} |x_a - y_a| \\ &= n|x_a - y_a| \\ &= n d_\infty(\mathbf{x},\mathbf{y}) \end{aligned}$$

and so we have finally that

$$d_\infty(\mathbf{x},\mathbf{y}) \leq d_2(\mathbf{x},\mathbf{y}) \leq d_1(\mathbf{x},\mathbf{y}) \leq n d_\infty(\mathbf{x},\mathbf{y}).$$

Looking at our plots in part a we see that this is true at least in the case $n = 2$.

Part C

This is just an application of the Cauchy-Schwarz inequality:

$$\begin{aligned} \sum_{k=1}^{n} a_k b_k &\leq \sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} b_k^2} \\ \Rightarrow \left(\sum_{k=1}^{n} a_k\right)^2 &\leq n \sum_{k=1}^{n} a_k^2 \end{aligned}$$

by letting $b_i = 1$ for all i and squaring both sides. This gives us that $\sum_{k=1}^{n} |x_k - y_k| \leq \sqrt{n}\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, and so $d_1(\mathbf{x},\mathbf{y}) \leq \sqrt{n} d_2(\mathbf{x},\mathbf{y})$, and we are done.

Let $d_\infty(\mathbf{x},\mathbf{y}) := \max_{1 \leq i \leq n} |x_i - y_i| = |x_k - y_k|$ for some $1 \leq k \leq n$, then for any $i \in \{1,\ldots,n\}$ we have that $|x_i - y_i|^2 \leq |x_k - y_k|^2$, which gives us that $\sum_{i=1}^{n} |x_i - y_i|^2 \leq n|x_k - y_k|^2$ and so $\sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \leq \sqrt{n}\sqrt{|x_k - y_k|^2}$, but this is just saying that $d_2(\mathbf{x},\mathbf{y}) \leq \sqrt{n} d_\infty(\mathbf{x},\mathbf{y})$ so we are done.

Question Two Workshop 6, Q7a

If $f : \mathbb{R} \to \mathbb{R}$ then the form $d(x,y) := |f(x) - f(y)|$ is a metric if

1. $d(x,x) = |f(x) - f(x)| = 0$
2. $d(x,y) = |f(x) - f(y)| > 0$ if $x \neq y$
3. $d(x,y) = d(y,x)$
4. $d(x,z) \leq d(x,y) + d(y,z)$

Criteria one and three are satisfied automatically if f is well-defined, since $|f(x) - f(x)| = 0$ and $|f(x) - f(y)| = |f(y) - f(x)|$, so we need only find criteria for when two and four are satisfied. $|f(x) - f(y)| = 0 \Rightarrow x = y$, is satisfied iff $f(x) = f(y) \Rightarrow x = y$, ie, when f is injective.

We wish to find out when $|f(x) - f(z)| \leq |f(x) - f(y)| + |f(y) - f(z)|$. This is simply inherited from $\mathbb{R}$ for suitable f, so we only have to worry about the values a for which $f(a) \notin \mathbb{R}$. This only happens if $\lim_{x \to a} f(x) = \pm\infty$ at a. Here it might be the case that $|f(a) - f(y)| > |f(a) - f(z)| + |f(z) - f(y)|$, so we require that f is never infinite, that is; f bounded. Thus f being bounded, well-defined and injective are necessary and sufficient conditions for $d(x,y) := |f(x) - f(y)|$ to be a metric.