Lecture 1 : What is space?

Well, that's a tough question. As organisms that have been optimised by evolution to solve complex problems of perception, locomotion and action in a spatial environment, it is not a surprise that we have a rich internal supply of concepts to do with space. In today's lecture we will begin with the elementary formalisations of these intuitions that you are familiar with, for example Euclidean space $\mathbb{R}^n$, and begin the process of pushing outward towards more sophisticated concepts.

Colloquial notions of space : My dictionary says space is

1. a continuous area or expanse which is free, available, or unoccupied

   implicitly refers to objects, i.e.
   "space as a stage for things"

2. the dimensions of height, depth and width within which all things exist and move.

   refers implicitly to measurement
   meaning "degrees of freedom"
   "space as a stage for motion"

3. a blank between printed words, characters, numbers, etc.

   i.e. a stage is a stage because it is empty. To communicate information, a prerequisite is a low entropy (i.e. highly ordered) channel. We would say "space as a channel for information flow"

We have a powerful set of mathematical abstractions capturing the role of space as a stage for things, and for motion. The third conception, of space as a channel, is (I think) more profound, but we do not understand it yet!

Outline of common abstractions

The notion of space we are most familiar with is Euclidean space $\mathbb{R}^n$. The more exotic concepts of spaces are, roughly speaking, obtained by abstracting part of the rich structure possessed by Euclidean space. For example:

- Vector spaces abstract the operations on $\mathbb{R}^n$ of addition and scalar multiplication (used to model e.g. displacement / motion)

- Metric spaces abstract the function $d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ which computes the distance between two points. (measurement)

- Topological spaces abstract the set of open balls (locality)

$$B_\varepsilon(\underline{x}) = \{ \underline{y} \in \mathbb{R}^n \,/\, d(\underline{x},\underline{y}) < \varepsilon \} \subseteq \mathbb{R}^n.$$

- Normed vector spaces abstract addition, scalar multiplication and the norm $\|\underline{v}\| = \left(\sum_{i=1}^{n} v_i^2 \right)^{1/2}$. (motion & measurement)

- Inner product spaces abstract addition, scalar multiplication and the dot product. (motion & angles)

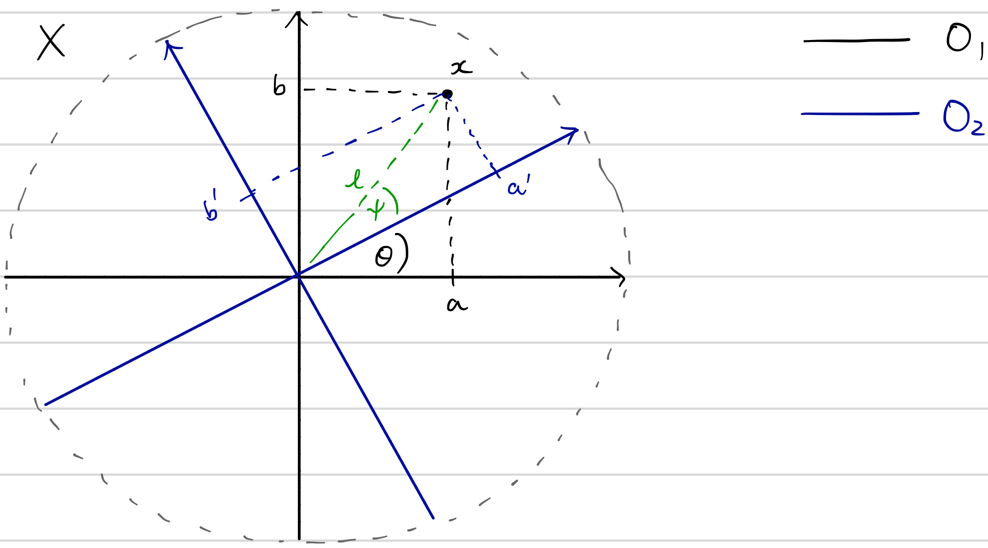
Another familiar example of a space is $\mathbb{C}^n$, which due to the operation of complex conjugation, gives rise to another special abstraction:

- Hilbert spaces abstract addition, scalar multiplication, and the pairing $\langle \underline{v}, \underline{w} \rangle = \sum_{i=1}^{n} \bar{v}_i w_i$ (??).

The abstraction we use for given domain (e.g. classical mechanics) is generally dictated by the group of symmetries of $\mathbb{R}^n$ or $\mathbb{C}^n$ which preserve the quantities of interest in that domain. We illustrate this principle with a very simple example.

Observers in the plane

Let X stand for a plane, without any pre-existing coordinate system, in which are embedded two observers O_1, O_2 at the same point. Each observer imposes their own coordinate system on X, and accordingly are able to measure the coordinates of an arbitrary point $x \in X$. Suppose O_2's coordinate system is rotated by θ radians relative to O_1, as in the following diagram:



Observe

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \ell\cos(\theta+\varphi) \\ \ell\sin(\theta+\varphi) \end{pmatrix} = \begin{pmatrix} \ell\cos\theta\cos\varphi - \ell\sin\theta\sin\varphi \\ \ell\cos\theta\sin\varphi + \ell\sin\theta\cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} a'\cos\theta - b'\sin\theta \\ b'\cos\theta + a'\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix}$$

indicates a bijection

If we denote the measurement functions

$$m_1 : X \xrightarrow{\cong} \mathbb{R}^2 \qquad m_2 : X \xrightarrow{\cong} \mathbb{R}^2$$

and denote by $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ the function $R_\theta(\underline{v}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underline{v}$, This calculation says that

$$m_1(x) = \begin{pmatrix} a \\ b \end{pmatrix} = R_\theta \begin{pmatrix} a' \\ b' \end{pmatrix} = R_\theta(m_2(x))$$

We say in this situation that the following diagram <u>commutes</u>

$$\begin{tikzcd} & X \arrow[dl, "m_2"'] \arrow[dr, "m_1"] & \\ \mathbb{R}^2 \arrow[rr, "R_\theta"'] & & \mathbb{R}^2 \end{tikzcd}$$

and so the <u>symmetry</u> R_θ of $\mathbb{R}^2$ (observe R_θ is an isomorphism) converts O_2's measurements into O_1's measurements. Now, note that

- <u>The observers agree about lengths</u> that is, for all $x \in X$

$$\| m_1(x) \| = \| m_2(x) \|.$$

Equivalently, for all $x \in X$,

$$\| R_\theta(m_2(x)) \| = \| m_2(x) \|.$$

But since $m_2 : X \longrightarrow \mathbb{R}^2$ is a bijection, it is actually the same to say $\| R_\theta(\underline{v}) \| = \| \underline{v} \|$ for all $\underline{v} \in \mathbb{R}^2$. That is, the diagram below commutes

$$\begin{tikzcd} & X \arrow[dl, "m_2"'] \arrow[dr, "m_1"] & \\ \mathbb{R}^2 \arrow[rr, "R_\theta"] \arrow[dr, "\|-\|"'] & & \mathbb{R}^2 \arrow[dl, "\|-\|"] \\ & \mathbb{R} & \end{tikzcd}$$

- <u>The observers disagree about $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f(a_1, a_2) = a_1^2$</u>

$$f(m_1(x)) = a^2 = (a' \cos\theta - b' \sin\theta)^2$$

$$f(m_2(x)) = (a')^2$$

which for example disagree when $\theta = \frac{\pi}{2}$ and $\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

<u>Def^n</u> We say a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (i.e. a "measurable quantity") is <u>coordinate-independent</u> if for any observers O_1, O_2 related as above

$$f(m_1(x)) = f(m_2(x)) \qquad \text{for all } x \in X.$$

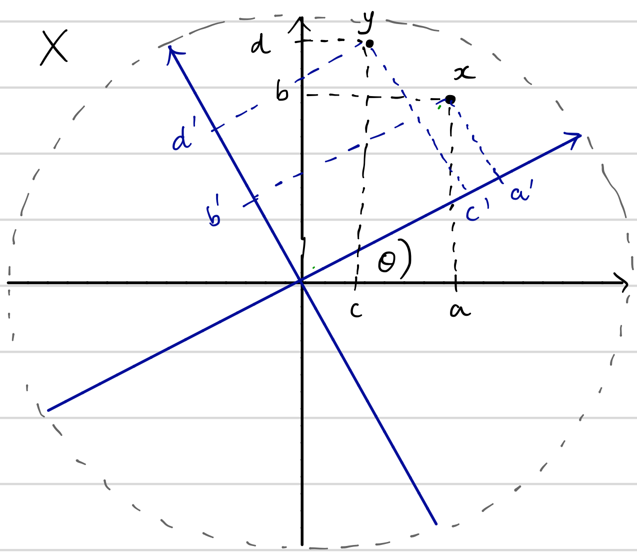
So, $f(\underline{v}) = \|\underline{v}\|$ is coordinate-independent but $f(a_1, a_2) = a_1^2$ is not.

<u>Lemma</u> The following are equivalent, for a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

(i) f is coordinate-independent
(ii) $f \circ R_\theta = f$ for all $\theta \in \mathbb{R}$.
(iii) there exists a function $g : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}$ s.t. $f = g \circ \|-\|$.

<u>Exercise L1-1</u> : Prove the lemma.

How about pairs of points $x, y \in X$



Question : what functions $g : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are coordinate-independent in the sense that O_1, O_2 will compute the same answer for each pair $(x,y) \in X \times X$, no matter the angle θ?

To simplify the exposition (but in a way which causes no real loss) we answer the question in the case where neither x nor y are equal to $\underline{0} \in X$, the point occupied by both observers. So the question becomes: which functions

$$g : (\mathbb{R}^2 \times \mathbb{R}^2) \setminus Z \longrightarrow \mathbb{R} \qquad Z = \mathbb{R}^2 \times \{\underline{0}\} \cup \{\underline{0}\} \times \mathbb{R}^2$$

make the diagram below commute for all θ?

$$\begin{tikzcd}
(\mathbb{R}^2 \times \mathbb{R}^2) \setminus Z \arrow[rr, "R_\theta \times R_\theta"] \arrow[dr, "g"'] & & (\mathbb{R}^2 \times \mathbb{R}^2) \setminus Z \arrow[dl, "g"] \\
& \mathbb{R} &
\end{tikzcd} \qquad (*)$$

<u>Lemma</u> The diagram (*) commutes for every θ if and only if there is a function $h : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times [0, 2\pi) \longrightarrow \mathbb{R}$ such that for all nonzero $\underline{v}, \underline{w} \in \mathbb{R}^2$ we have $g(\underline{v}, \underline{w}) = h(\|\underline{v}\|, \|\underline{w}\|, \Delta(\underline{v}, \underline{w}))$ where $\Delta(\underline{v}, \underline{w})$ is the angle between $\underline{v}, \underline{w}$ moving anticlockwise from $\underline{v}$ to $\underline{w}$.

<u>Proof</u> Suppose $g = h \circ (\|-\|, \|-\|, \Delta(-,-))$ for some h. Then

$$
\begin{aligned}
g(R_\theta \underline{v}, R_\theta \underline{w}) &= h(\|R_\theta \underline{v}\|, \|R_\theta \underline{w}\|, \Delta(R_\theta \underline{v}, R_\theta \underline{w})) \\
&= h(\|\underline{v}\|, \|\underline{w}\|, \Delta(\underline{v}, \underline{w}))
\end{aligned}
$$

In the other direction, suppose $g \circ (R_\theta \times R_\theta) = g$ for all $\theta \in \mathbb{R}$. Given $\underline{v} = (\alpha \cos\delta, \alpha \sin\delta)^T$ we have

$$
\begin{aligned}
g(\underline{v}, \underline{w}) &= g(R_{-\delta} \underline{v}, R_{-\delta} \underline{w}) \\
&= g\left((\|\underline{v}\|, 0)^T, (\|\underline{w}\| \cos\Delta(\underline{v}, \underline{w}), \|\underline{w}\| \sin\Delta(\underline{v}, \underline{w}))^T\right)
\end{aligned}
$$

so if we set $h(\alpha, \beta, \gamma) = g((\alpha, 0)^T, (\beta\cos\gamma, \beta\sin\gamma)^T)$ then we will have $g(\underline{v}, \underline{w}) = h(\|\underline{v}\|, \|\underline{w}\|, \Delta(\underline{v}, \underline{w}))$ for all nonzero $\underline{v}, \underline{w}$. $\square$

given the setup above

<u>The upshot</u>: the only coordinate-independent (aka "meaningful") quantities associated to an ordered pair of points in the plane are functions of their <u>distances</u> from a fixed origin, and the <u>oriented angle</u> from the first point to the second (moving anticlockwise).

An appropriate structure on $\mathbb{R}^2$ for this context is therefore its structure as an <u>inner product space</u>. More precisely: if two observers O_1, O_2 are presented with a finite number of points in the plane, and they restrict their discourse to quantities computed from this data using addition, scalar multiplication, lengths and oriented angles then they are guaranteed to <u>agree</u> on the answers! e.g.

$$\begin{aligned}(3m_1(x) + 2m_1(y)) \cdot m_1(y) &= (3R_\theta(m_2(x)) + 2R_\theta(m_2(y))) \cdot R_\theta(m_2(y)) \\ &= R_\theta(3m_2(x) + 2m_2(y)) \cdot R_\theta(m_2(y)) \\ &= (3m_2(x) + 2m_2(y)) \cdot m_2(y).\end{aligned}$$

Actually, more is true: the observers agree not only on dot products, which depend only on the <u>cosine</u> of angles between vectors, they agree on the angles themselves. The relevant structure is therefore $\mathbb{R}^2$ as an inner product space <u>plus</u> a choice of an orientation (this corresponds to the fact that for both observers "clockwise" has the same meaning).

The relevant symmetry group was $SO(2) = \{ R_\theta \mid \theta \in \mathbb{R} \}$, the <u>rotation group</u>, which is the group of all linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ preserving the inner product <u>and</u> the chosen orientation.

The structure of $\mathbb{R}^2$ as an inner product space (without a chosen orientation) corresponds to the larger group $O(2)$, see the exercises in Lecture 3.

Remark This connection between symmetry and the structures we choose to organise our abstractions (e.g. $SO(2)$ vs. inner product spaces) is not an accident. Here is the briefest sketch of why that is. This is not strictly relevant to the course, but will be useful for some of you.

No classical measurement is atomic, and all measurements are subject to error, so any measurement involves a kind of integration across space and time. This is the fundamental point of thermodynamics, see e.g. Callen "Thermodynamics and an introduction to thermostatistics" 2nd edition, esp. Part III. It is not a surprise our perceptual systems, and the mathematics grounded in them, is organised around quantities invariant to various natural symmetries, because those are the only quantities stable enough in space and time to emerge as macroscopically obsevvable.

More concretely: no matter how hard O_1 tries to keep their coordinate system fixed, there will inevitably be small perturbations, which take the form, say, of infinitesimal rotations or translations. Let us ignore translations (but see the exercises) and O_2 stand for the same observer but at some later time, after one of these infinitesimal rotations. If the observer tries to make measurements and compute a quantity which is not $SO(2)$-invariant (or more generally, which transforms as a representation of $SO(2)$) the answer is not stable over time, under the unavoidable pertubations affecting the observer. Such a quantity is not observable, so naturally we do not invent an abstraction to make it convenient to talk about it.

(Of course there are many other relevant symmetries, like translations, or boosts in relativity)

<u>Exercise L1-2</u> Extend the above to the case of <u>three</u> points in the plane, by stating and proving a characterisation of all coordinate-independent functions $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \setminus Z \longrightarrow \mathbb{R}$ where

$$Z = \{\underline{0}\} \times \mathbb{R}^2 \times \mathbb{R}^2 \cup \mathbb{R}^2 \times \{\underline{0}\} \times \mathbb{R} \cup \mathbb{R}^2 \times \mathbb{R}^2 \times \{\underline{0}\}.$$

<u>Exercise L1-3</u> Do the general case, i.e. n points in the plane for $n \geqslant 2$.

<u>Exercise L1-4</u> What is the relevant group of functions $\mathbb{R}^2 \to \mathbb{R}^2$ if the observers are <u>not</u> located at the same position of X.? Does this change the set of coordinate-independent functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$?

<u>Exercise L1-5</u> Prove that the set $SO(2)$ of rotation matrices may be equivalently described as the set of linear transformations

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

which have determinant $+1$ and satisfy

$$F(\underline{v}) \cdot F(\underline{w}) = \underline{v} \cdot \underline{w} \qquad \text{for all } \underline{v}, \underline{w} \in \mathbb{R}^2$$

Thus, the connection between $SO(2)$ and the structure of inner product spaces runs in both directions