Mathematics 3 **Honours Analysis**

Skeleton Notes on Metric Spaces

These notes provide only the briefest structural outline of the material on metric spaces, and should be read in conjunction with Wade's book, the relevant worksheets, and the amplifications given during the lectures themselves. Starred items are those which are important in the theory but which won't be examinable. *Note the unstarred item at the end of Section 10.* Numbers ((10.1) etc.) refer to sections in Wade's book.

1. **Definition.** (10.1) A metric space is a pair (X, d) where X is a nonempty set and $d : X \times X \to \mathbb{R}$, the metric, satisfies
   - $d(x, y) \geq 0$ for all $x, y \in X$
   - $d(x, y) = 0$ iff $x = y$
   - $d(x, y) = d(y, x)$ for all $x, y \in X$
   - (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
2. **Examples.** (10.1), (8.1); Worksheet 6
   - $\mathbb{R}^n$ with the usual or standard metric $d_2(x, y) = |x - y| = \left(\sum_{i=1}^n (x_i - y_i)^2\right)^{1/2}$. The triangle inequality is proved using the Cauchy–Schwarz inequality
   $$\sum_{i=1}^n |a_j b_j| \leq \left(\sum_{i=1}^n a_i^2\right)^{1/2} \left(\sum_{i=1}^n b_i^2\right)^{1/2}.$$
   - $\mathbb{R}^n$ with the metric $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$.
   - $\mathbb{R}^n$ with the metric $d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$.
   - (X, d) any metric space, $Y \subseteq X$ with $\tilde{d} = d|_{Y \times Y}$. Then $(Y, \tilde{d})$ is a metric space. (e.g. $\mathbb{Q} \subseteq \mathbb{R}$ or $\{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$.)
   - X is any nonempty set; the discrete metric on X is defined by $d(x, y) = 1$ when $x \neq y$ and $d(x, y) = 0$ when $x = y$.
   - $C([0, 1])$ with the uniform metric $d_\infty(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$.
   - $C([0, 1])$ with the metric $d_2(f, g) = (\int_0^1 |f - g|^2)^{1/2}$. The triangle inequality is proved using the Cauchy–Schwarz inequality $\int_0^1 |fg| \leq (\int_0^1 f^2)^{1/2} (\int_0^1 g^2)^{1/2}$.
   - $C([0, 1])$ with the metric $d_1(f, g) = \int_0^1 |f - g|$.
3. **Relations between metrics** – (8.1); Worksheets 6 and 7.
   - On $\mathbb{R}^n$ we have $d_\infty(x, y) \leq d_2(x, y) \leq d_1(x, y)$ and $d_1(x, y) \leq n^{1/2} d_2(x, y)$ and $d_2(x, y) \leq n^{1/2} d_\infty(x, y)$. Pictures illustrating this.
   - On $C([0, 1])$ we have $d_1(f, g) \leq d_2(f, g) \leq d_\infty(f, g)$. Examples to show that for no K do we have reverse inequalities of the form $d_p(f, g) \leq K d_q(f, g)$ for $p > q$ (for $p, q \in \{1, 2, \infty\}$) which are valid for all $f, g \in C([0, 1])$.
   - Two metrics d and ρ on a set are **equivalent** if for all $x \in X$, for all $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(x, y) < \epsilon$ and $\rho(x, y) < \delta$ implies $d(x, y) < \epsilon$.

- Two metrics d and ρ on a set are **uniformly equivalent** if for all $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(x, y) < \epsilon$ and $\rho(x, y) < \delta$ implies $d(x, y) < \epsilon$.
- Two metrics d and ρ on a set are **strongly equivalent** if there exist $A, B > 0$ such that $d(x, y) \leq A\rho(x, y)$ and $\rho(x, y) \leq Bd(x, y)$. So the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ are strongly equivalent.
- Strong equivalence implies uniform equivalence implies equivalence
- On $C([0, 1])$, no pair of metrics from $\{d_1, d_2, d_\infty\}$ is equivalent.

4. **Open balls and open sets** (10.1), (10.3)

   - For a metric space (X, d), $a \in X$ and $r > 0$, $B(a, r) := \{x \in X : d(x, a) < r\}$ is the **open ball** centred at a with radius r.
   - $U \subseteq X$ is **open** iff for every $a \in U$ there is an $r > 0$ such that $B(a, r) \subseteq U$.
   - The open ball $B(a, r)$ is itself open.
   - In any metric space, $\emptyset$ and X are open; if U_α is open for each $\alpha \in \mathcal{A}$ then $\bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is open; if U_j is open for $1 \leq j \leq n$ then $\bigcap_{j=1}^{n} U_j$ is open. Infinite intersections of open sets need not be open, by example.
   - In a discrete metric space (X, d), *every* subset of X is open.
   - Examples of open and non-open sets in $\mathbb{R}^n$ with usual metric.
   - If two metrics d and ρ on a set are equivalent, then every d-open set is ρ- open and vice-versa; conversely if the classes of d- and ρ-open sets coincide, then the metrics are equivalent.

5. **Closed sets** (10.1), (10.3)

   - $F \subseteq X$ is **closed** iff its complement $X \setminus F$ is open.
   - In any metric space, $\emptyset$ and X are closed; if F_α is closed for each $\alpha \in \mathcal{A}$ then $\bigcap_{\alpha \in \mathcal{A}} F_\alpha$ is closed; if F_j is closed for $1 \leq j \leq n$ then $\bigcup_{j=1}^{n} F_j$ is closed. Infinite unions of closed sets need not be closed, by example.
   - Examples and non-examples of closed sets in $\mathbb{R}^n$ with the usual metric; every subset of a discrete metric space is closed.

6. **Interior, closure and boundary** (10.3)

   - For $A \subseteq X$, $\mathrm{int}A := \bigcup_{U \subseteq A, U \text{ open}} U$ is the **interior** of A: it is the largest subset of A which is open.
   - For $A \subseteq X$, $\overline{A} := \bigcap_{A \subseteq F, F \text{ closed}} F$ is the **closure** of A: it is the smallest set containing A which is closed.
   - For $A \subseteq X$, $\partial A := \overline{A} \setminus \mathrm{int}A$ is the **boundary** of A. In some cases this agrees with an intuitive notion of "boundary", in other cases not.
   - A set is open iff it equals its interior; a set is closed iff it equals its closure.

7. **Convergence, Cauchy sequences and completeness** (10.1)

   - In a metric space (X, d), a sequence (x_n) of members of X **converges** to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$. A sequence can converge to at most one limit.

- In $C([0,1])$, $d_\infty(f_n, f) \to 0$ if and only if (f_n) converges to f uniformly on $[0,1]$.
- In a metric space (X,d), a sequence (x_n) of members of X is **bounded** if there exists some ball $B(a,r)$ such that $x_n \in B(a,r)$ for all n.
- Every convergent sequence is bounded.
- In a metric space (X,d), a sequence (x_n) of members of X is a **Cauchy sequence** iff for every $\epsilon > 0$ there is some N such that $m, n \geq N$ implies $d(x_m, x_n) < \epsilon$.
- Every convergent sequence is Cauchy, but not in general the other way round by examples.
- Every Cauchy sequence is bounded.
- A metric space (X,d) is **complete** iff every Cauchy sequence in X converges.
- If a Cauchy sequence (x_n) has a convergent subsequence, then (x_n) is convergent. Consequently, if X is a metric space such that every bounded sequence has a convergent subsequence, then X is complete.
- $\mathbb{R}^n$ with any of the metrics d_1, d_2, d_∞ is complete. $\mathbb{R}$ with the metric $\rho(x,y) = |\arctan x - \arctan y|$ is not complete, although this ρ is equivalent to the usual metric on $\mathbb{R}$.
- $C([0,1])$ with d_∞ is complete by the uniform Cauchy criterion. But with the metrics d_1 and d_2 there are examples of Cauchy sequences (f_n) for which there is *no* continuous f such that $d_i(f_n, f) \to 0$ as $n \to \infty$ $(i = 1, 2)$.

8. **Closedness, limit points, cluster points and completeness** (10.1), (10.2)

   - $x \in X$ is a **limit point** for $A \subseteq X$ iff there is a sequence $(x_n) \subseteq A$ such that $x_n \to x$ as $n \to \infty$.
   - $x \in X$ is a **cluster point** for $A \subseteq X$ iff every open ball centred at x contains infinitely many points of A.
   - $x \in X$ is a cluster point for $A \subseteq X$ $\iff$ for all $r > 0$, $B(x,r) \setminus \{x\}$ contains a point of A $\iff$ there is a sequence $(x_n) \subseteq A$, *with* $x_n \neq x$ *for all* n, such that $x_n \to x$ as $n \to \infty$.
   - So every cluster point for A is a limit point for A. But x can be a limit point for A without being a cluster point.
   - The limit points of $E = E \cup$ cluster points of E.
   - For every set $A \subseteq X$, $\overline{A} = A \cup \{\text{cluster points of } A\} = A \cup \{\text{limit points of } A\}$. So a set F is closed $\iff$ F contains all its cluster points $\iff$ F contains all its limit points.
   - A closed subset of a complete metric space is complete.
   - A complete subset of any metric space is closed.

9. **Continuity and limits of functions** (10.2),(9.3),(10.6)

   - Let $f : X \to Y$ where (X,d) and (Y,ρ) are metric spaces. Then $\lim_{x\to a} f(x) = b$ iff for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 \neq d(x,a) < \delta$ implies $\rho(f(x), b) < \epsilon$.
   - Let $f : X \to Y$ where (X,d) and (Y,ρ) are metric spaces. Then f is **continuous at** $a \in X$ iff for every $\epsilon > 0$ there is a $\delta > 0$ such that $d(x,a) < \delta$ implies $\rho(f(x), f(a)) < \epsilon$. So f is continuous at a $\iff$ $\lim_{x\to a} f(x) = f(a)$. The function f is **continuous on** X if it is continuous at every $a \in X$.

- Let $f : X \to Y$ where (X, d) and (Y, ρ) are metric spaces. Then f is continuous at $a \in X$ $\iff$ for every sequence $(x_n) \subseteq X$ with $x_n \to a$, $f(x_n) \to f(a)$. There is s similar result for limits.
- For continuous functions $f, g : \mathbb{R}^n \to \mathbb{R}$ (with the usual metrics) the following hold: $\alpha f + \beta g$ is continuous for all $\alpha, \beta \in \mathbb{R}$; fg is continuous; f/g is continuous at all points such that $g(x) \neq 0$. Thus if $p, q : \mathbb{R}^n \to \mathbb{R}$ are polynomials, the ratio p/q is continuous at all points where $q(x) \neq 0$. Nevertheless there are some interesting pathologies involving ratios of polynomials of small degrees. See pp. 315 ff. of Wade.
- A function $f : X \to Y$ is continuous $\iff$ for every open set $U \subseteq Y$, $f^{-1}(U)$ is an open subset of X.
- If d and ρ are two metrics on a set X, then they are equivalent $\iff$ the identity maps $(X, \rho) \to (X, d)$ and $(X, d) \to (X, \rho)$ are both continuous.

10. **Compactness** (10.4 – up to middle of p.363.), (10.6); Worksheet 8

- Let (X, d) be a metric space. A subset $E \subseteq X$ is **compact** iff for every open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ of E, there is a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$.
- Non-examples and easy examples.
- Any compact set must be closed and bounded; converse false by example.
- If X is compact and $A \subseteq X$ is closed, then A is compact.
- **Theorem.** In $\mathbb{R}^n$ with the usual metric, the closed rectangle $[a_1, b_1] \times \ldots \times [a_n, b_n]$ is compact. *This is the content of Worksheet 8.*
- **Heine-Borel Theorem.** In $\mathbb{R}^n$ with the usual metric, a set $E \subseteq \mathbb{R}^n$ is compact $\iff$ it is closed and bounded. The $\Longrightarrow$ half is true in any metric space; for the $\Longleftarrow$ half note that if E is bounded it is contained in some closed rectangle, which is compact, and as a closed subset of this compact space it must be compact.
- *The proof of the compactness of products of closed intervals derived in the workshop generalises as follows: if (X, d) is a complete metric space and it also **precompact** – i.e. for every $r > 0$ there is a cover of X by finitely many closed balls of the form $B[a, r] := \{x \in X : d(x, a) \leq r\}$ – then X is compact.
- *It is therefore natural to ask about the converse. Compactness implies precompactness since for all $r > 0$ the open cover $\{B(a, 2r) : a \in X\}$ of X has a finite subcover. That compactness implies completeness follows from the next item (together with the fact that a Cauchy sequence with a convergent subsequence is necessarily convergent).
- ***Proposition.** If X is compact then every sequence in X has a convergent subsequence. The property that "every sequence in X has a convergent subsequence" is called **sequential compactness**. *Sketch Proof.* Suppose for a contradiction that $(x_n) \subseteq X$ has no convergent subsequence. We may assume that all the terms of the sequence are distinct. Then for no $x \in X$ is x a cluster point of (x_n). So for all $x \in X$ there is an $r_x > 0$ such that $B(x, r_x)$ contains only finitely many members of the sequence. But then $\{B(x, r_x)\}$ forms an open cover of X which must therefore have a finite subcover. But this finite subcover can contain only finitely many terms of the sequence, contradiction.
- *It's now natural to ask if the converse of this last Proposition is also true, i.e. if sequential compactness implies compactness. By our previous characterisation of compactness,

to see that this is true it suffices to see that X sequentially compact implies X is complete and precompact.

- *If X is sequentially compact and (x_n) is a Cauchy sequence, then it has a convergent subsequence, hence it converges; so X is complete.
- *Suppose X is not precompact. We'll consruct a sequence with no convergent subsequence. Indeed, there exists an $r > 0$ such that X is not coverable by finitely many closed balls of radius r. Pick $x_1 \in X$. Then $B[x_1, r]$ does not cover X so there is an $x_2 \in X$ with $d(x_1, x_2) > r$. Then $\{B[x_1, r], B[x_2, r]\}$ does not cover X so there exists an $x_2 \in X$ with $d(x_1, x_3) > r$ and $d(x_2, x_3) > r$. Continuing in this way, this process constructs a sequence with no convergent subsequence.
- *In summary we have established the following **Theorem:** A metric space is compact $\iff$ it is complete and precompact $\iff$ it is sequentially compact.
- If $f : X \to Y$ is continuous and $E \subseteq X$ is compact, then $f(E)$ is a compact subset of Y. In particular, $f(E)$ is closed and bounded. If $Y = \mathbb{R}$ with the usual metric, then there exist $a, b \in E$ such that $f(a) = \min_{x \in E} f(x)$ and $f(b) = \max_{x \in E} f(x)$.

11. **Connectedness** (10.5), (10.6)

- A subset E of a metric space X is **disconnected** iff there exist open subsets U and V of X such that $E \subseteq U \cup V$, $E \cap U \neq \emptyset$, $E \cap V \neq \emptyset$ and $(E \cap U) \cap (E \cap V) = \emptyset$.
- A subset E of a metric space X is **connected** if it is not disconnected, i.e. there do not exist open subsets U and V of X such that $E \subseteq U \cup V$, $E \cap U \neq \emptyset$, $E \cap V \neq \emptyset$ and $(E \cap U) \cap (E \cap V) = \emptyset$.
- Examples of disconnectedness, easy examples of connectedness.
- In $\mathbb{R}$ with the usual metric, a subset $E \subseteq \mathbb{R}$ is connected $\iff$ E is an interval.
- If $f : X \to Y$ is continuous and $E \subseteq X$ is connected, then $f(E)$ is a connected subset of Y. In particular, if $Y = \mathbb{R}$ with the usual metric, $f(E)$ is an interval, and so for all $a, b \in E$ and all y between $f(a)$ and $f(b)$ there is some $x \in E$ with $f(x) = y$.
- A subset E of a metric space X is **path-connected** if for every $a, b \in E$ there is a continuous function (path) $\phi : [0, 1] \to E$ such that $\phi(0) = a$ and $\phi(1) = b$.
- **Proposition.** If E is path-connected then E is connected. *Sketch Proof.* Suppose E is disconnected. Then there are open subsets U and V of X such that $E \subseteq U \cup V$, $E \cap U \neq \emptyset$, $E \cap V \neq \emptyset$ and $(E \cap U) \cap (E \cap V) = \emptyset$. Let $a \in E \cap U$ and $b \in E \cap V$. Let $\phi : [0, 1] \to E$ be a path joining a to b. Then $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are nonempty disjoint open subsets of $[0, 1]$ whose union is $[0, 1]$, contradicting the fact that the interval $[0, 1]$ is connected.
- This last proposition is usually the easiest way to see that sets are connected in practice.