Friezes, Archimedean and Catalan solids

4.1 Frieze groups

4.1.1 Definition Let F be the subgroup of $E(2)$ that fixes an infinite, constant width, closed strip.

4.1.2 Proposition Taking the strip to by $-1 \leq y \leq 1$ in the (x, y)-plane, F consists of the following types of symmetry.

- Translations in the x direction
- Rotations by π about points on the x-axis.
- Reflections in lines perpendicular to the x-axis.
- Reflections in the x-axis.
- Glides along the x axis.

4.1.3 Definition A *frieze group* is a discrete subgroup of F containing a translation.

4.1.4 Theorem There are seven different types of frieze pattern. They correspond to the seven infinite families of subgroups of $O(3)$ that fix an axis.

The argument here is just that if we take N periods of a frieze pattern we can wrap that around the equator of a sphere and the frieze symmetries become a subgroup of $O(3)$ fixing the axis joining the poles.

4.1.5 Notation Our signature for a frieze group is obtained from the corresponding subgroup of $O(3)$ by replacing instances of N by an infinity symbol of the same colour. The blue and red infinity symbols have a cost of $\$1$ and $\$0.5$ respectively, in which case all frieze patterns cost $\$2$ (which seems reasonable given that they are groups of symmetries of the plane.)

4.1.6 The frieze groups

Vert Ref Lines?	Additions	Signature	Footprints	LaTeX example
No	Miracle	$\infty\times$	walk	pbpbpbpbpbp
No	None	$\infty\infty$	hop	LLLLLLLLLL
No	Horiz. Refl.	$\infty*$	jump	EEEEEEEEEE
No	Rots.	22∞	dizzy hop	NNNNNNNNNN
Yes	None	$*\infty\infty$	sidle	MMMMMMMMMM
Yes	Horiz. Refl.	$*22\infty$	dizzy jump	XXXXXXXXXX
Yes	Rots.	$2*\infty$	dizzy sidle	MWMWMWM

OK – I cheated with the last one and used rotated ‘M’s for the ‘W’s because real ‘w’s have slanting sides in all the fonts I could access.

For the “footprints” classification, see the image on Learn or search online.

4.1.7 Exercise (NL) For each frieze group, define a periodic “function” $\mathbb{R} \longrightarrow [-1, 1]$ whose graph has that frieze symmetry. Don’t be frightened of discontinuous or piecewise-defined examples. The inverted commas are because in some cases your function may need to be of the form $y^2 = ??$ in order to accommodate the symmetries in the group.

How does this relate to the usual analysis of “odd” and “even” periodic functions? You might want to allow the idea of functions being odd or even about points other than just the origin.

4.1.8 Exercise (CS) If you were a wallpaper manufacturer wanting to produce rolls of wall-paper that could be “hung” to produce a given wallpaper pattern, how do you go about this? You need to find a way of cutting the pattern into horizontal strips of fixed width so that all the strips have the same repeating pattern (which would in fact then have the symmetry of a frieze pattern).

4.1.9 Exercise (NL) Show that every element of $O(3)$ that fixes the z-axis is of the form

$$\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \text{where } A \in O(2).$$

Four possible types of matrix arise according to the plus/minus sign and whether or not $A \in SO(2)$. Identify how these correspond with the different frieze transformations listed in 4.1.2.

The identification is the one that takes a number of periods of a frieze pattern and wraps it round the equator of the sphere.

4.2 Archimedean and Catalan solids

4.2.1 Introduction We can talk about vertex transitive polyhedra just as we can for tilings. The five platonic solids are v-transitive because they are flag transitive. The trick of truncation can be used on platonic solids also: imagine sawing off the corners of a cube with planes perpendicular to lines joining opposite vertices. If you do this just right the cut faces will be equilateral triangles and the original square faces will be reduced to regular octagons.

4.2.2 The Wythoff construction The Wythoff construction allows us to construct all the v-transitive convex polyhedra with symmetry group $*2pq$. It works precisely as the plane version does for tilings, although one has to imagine the fundamental triangle as being a spherical triangle on the sphere; it is a fundamental domain for the group. Note from Table 4.1 how the naming completely parallels the tiling case.

4.2.3 Archimedean solids The v-transitive convex polyhedra comprise the following.

- five platonic solids (which are flag-regular)
- Thirteen *Archimedean solids* made up of
  - Five with symmetry $*532$ (from the Wythoff construction)
  - Five with symmetry $*432$ (from the Wythoff construction)
  - The truncated tetrahedron (symmetry group $*332$).
  - The snub cube and snub dodecahedron
- *Prisms* (face code $(4)(4)(N)$ for $N \geq 3$) and *Antiprisms* (face code $(3)(3)(3)(N)$ for $N \geq 3$)

Conway's "The symmetries of things" gives a (very) generalised Wythoff construction that, among other things, proves that this list is complete.

4.2.4 Example How can one calculate how many faces of different types a Wythoffian Archimedean solid has? Take the rhomboocosidodecahedron for example. The group has size 120 and so the sphere is tiled by 120 Wyhoff triangles. The vertex is on an edge of the fundamental domain and so counts for two fundamental domains. Thus it has $V = 60$. From the face code, each vertex is on a single triangle but triangles have three vertices. So overall there are $60/3 = 20$ triangles. Similarly there are $60/5 = 12$ pentagons and $2(60/4) = 30$ squares. Thus $F = 62$. Euler's formula then tells us that there are 120 edges. (We can double check that last figure e.g. by noticing that the Wythoff triangle contains two half-edges.)

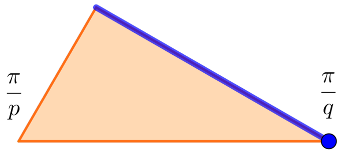
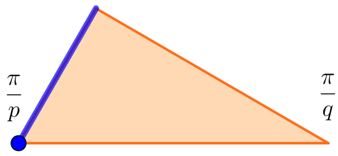
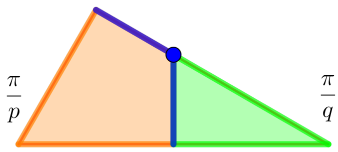
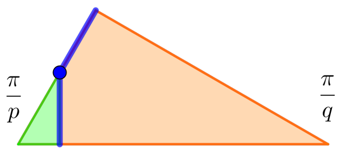
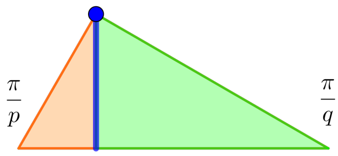
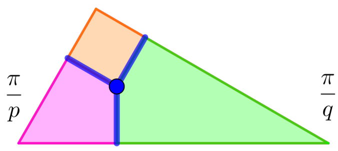
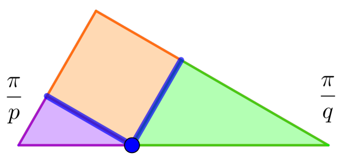
Construction	$*235$	$*234$
	3^5 Icosahedron	3^4 Octahedron
	5^3 Dodecahedron	4^3 Cube
	$(6)^2 5$ Truncated icosahedron	$(6)^2 4$ Truncated octahedron
	$(10)^2 3$ Truncated dodecahedron	$(8)^2 3$ Truncated cube
	$(3)(5)(3)(5)$ Icosidodecahedron	$(3)(4)(3)(4)$ Cuboctahedron
	$(4)(6)(10)$ Truncated icosidodecahedron	$(4)(6)(8)$ Truncated cuboctahedron
	$(4)(3)(4)(5)$ Rhombicosidodecahedron	$(4)(3)(4)(4)$ Rhombicuboctahedron

Table 4.1: The Wythoff construction for $*235$ and $*235$

4.2.5 Catalan solids Just as for tilings, one can take the dual of the Archimedean solids. These *Catalan solids* have the same symmetry group as the solid from which they are derived, and the symmetry group acts transitively on the faces. They have two or three different types of vertex (corresponding to the different types of face of the corresponding Archimedean solid) but each vertex is regular in the sense that points unit distance along the emerging edges form a regular polygon.

4.2.6 Exercise (NS) How many faces of what sort does a truncated icosidodecahedron have? And how many vertices and edges?

4.2.7 Exercise (NL) Investigate the Wythoff construction for $*332$. Of the seven possibilities, you should discover that the tetrahedron and truncated tetrahedron appear twice owing to symmetry. The remaining three lead to "coloured versions" of Archimedean solids with a larger symmetry group where $*332$ is the symmetry group of the coloured object which still acts transitively on its vertices. Identify the three coloured objects.

Conway refers to these as "relative Archimedean solids".

4.2.8 Exercise (NM) Which cases of the Wythoff construction results in polyhedra that are also edge-transitive? Hence identify the edge-transitive Archimedian solids.

What characterises the Catalan solid dual to an edge-regular Archimedean solid?

4.2.9 Exercise (NM) For each case of the Wythoff construction, draw on the Wythoff triangle the edges of the faces of the corresponding Catalan solid.