Individual Assignment 1

Introduction to Computer Graphics

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Abstract

In this report we wish to discuss the problem of rasterization of a straight line by means of an approximation algorithm. By examining the mathematical definition of a straight line we will derive a series of equations with which we can construct an efficient algorithm.

The reader is not expected to have any prior knowledge of the material presented. For the code the reader is expected to have some basic understanding of the C++ language.

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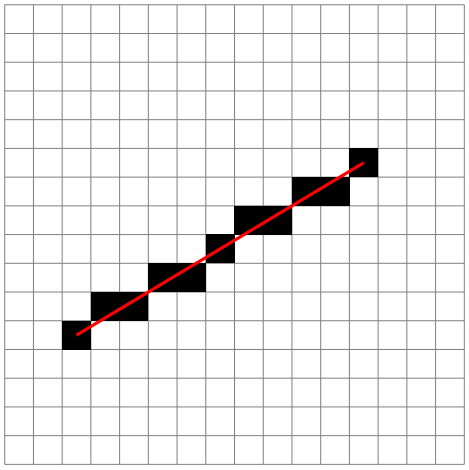
1 The Problem

We wish to construct an algorithm for rendering straight lines on a computer screen. Inorder to do so, one must recognize that a computer screen is incapable of displaying straight lines. A computer screen is made up of a grid of picture elements (pixels).



If we zoom in on a line displayed on a computer screen (as shown in fig. 1), we can see the individual pixels that make up the line. Each pixel is an approximation of the line drawn.

Figure 1: Individual pixels of a line segment

1.1 Theory

A straight line be can described in math by $y = \alpha x + \beta$, where α is the slope of the line, and β is where the line intersects with the y-axis. This equation, however, does not yield a finite line, therefore we turn to the definition of a straight line, containing the start- and end-points of the line $(x, y)|(a, b) \cdot (x - x_0, y - y_0) = 0$, by which we describe a straight line as a dot-product of the line vector $(x - x_0, y - y_0)$ and its normal vector (a, b), which are orthogonal to each other, making the dot-product zero as a consequence of its geometric definition $a \cdot b = \|a\|\|b\| \cos\theta$, for which the angle θ would be $90°$ making $\cos\theta = 0$.

Expanding on the above equation, we can derive the homogeneous equation of a straight line.

$$(a, b) \cdot (x - x_0, y - y_0) = 0 \tag{1}$$
$$a(x - x_0) + b(y - y_0) = 0 \tag{2}$$
$$ax - ax_0 + by - by_0 = 0 \tag{3}$$
$$ax + by \underbrace{-ax_0 - by_0}_{c} = 0 \tag{4}$$

For later convenience we allow for the contraction c. By the above derivation we have that a line l can be defined by the homogeneous equation.

$$l: \quad (x, y)|ax + by + c = 0 \tag{5}$$

Now that we have established a mathematical definition of a straight line, we can turn our attention toward the problem of approximating a given line described by these equations.

For this purpose we will be using the formula for the distance from a point to a line.

$$d(x,y) = \frac{(a,b)}{\sqrt{a^2+b^2}} \cdot (x - x_0, y - y_0) \tag{6}$$

With this equation we can tell how far away a point may be from the line in question, moreover it tells us on which side of the line l a point p resides. That is,

$$\text{Point } p \text{ resides on } \begin{cases} \text{north of line } l & \text{iff. } d(x,y) > 0 \\ \text{on line } l & \text{iff. } d(x,y) = 0 \\ \text{south of line } l & \text{iff. } d(x,y) < 0 \end{cases} \tag{7}$$

By the same expansion as before, we can derive another form of this equation.

$$d(x,y) = \frac{(a,b)}{\sqrt{a^2+b^2}} \cdot (x - x_0, y - y_0) \tag{8}$$

$$= \frac{a(x - x_0) + b(y - y_0)}{\sqrt{a^2+b^2}} \tag{9}$$

$$= \frac{ax + by - ax_0 - by_0)}{\sqrt{a^2+b^2}} \tag{10}$$

$$= \frac{ax + by + c)}{\sqrt{a^2+b^2}} \tag{11}$$

$$\tag{12}$$

Now that we have a way of expressing the distance from an arbitrary point perpendicular to the line, as derived above, we can use this to determine which pixels lies closest to the line. This in effect allows us to perform the approximation necessary to produce a rasterization of a straight line.

Before doing so, we would like to reduce unnecessary overhead. Squareroots are very CPU intensive operations, and so by recognizing that $\sqrt{a^2+b^2} > 0$, we can substitute it by multiplying by it, effectively removing it from the equation. We declare a new function $F(x,y)$ for this purpose.

$$F(x,y) = \frac{ax + by + c}{\sqrt{a^2+b^2}} \sqrt{a^2+b^2} = ax + by + c \tag{13}$$

The reason we can expect the properties to be preserved is that we ensured that since the previous denominator $\sqrt{a^2+b^2}$ was ensured to be positive at all times, removing it does not affect the sign, as it was purely determined by the numerator $ax + by + c$. This proves that the properties given in (7) holds for $F(x,y)$ as well.

Under considerations of simplification and maintaining simplicity of the algorithm itself, assume that $|\frac{\Delta y}{\Delta x}| \leq 1$. That is, the slope of line α can only take on values in the interval $[-1;1]$ — should it exceed, then $|\frac{\Delta x}{\Delta y}| \leq 1$, which we will use to our advantage in constructing the algorithm.

Lastly, we need to state that one can substitute the normal vector (a,b) by $(\Delta y, -\Delta x)$. This can be derived from the linear function $y = \alpha x + \beta$, which can be rewritten as $x\Delta y - y\Delta x + \beta\Delta x = 0$.

With all of these derivations of equations and their properties stated, we may proceed and construct the algorithm.

2 The Solution

2.1 Choosing a point

Let p_i be the ith point with coordinates (x_i, y_i). Under the assumption that point p_i has been drawn, let $d_p = F(x_i + 1, y_i + \frac{1}{2}$ be a decision variable that we know the value of. Then by the assumption that $\alpha \in [-1; 1]$ we must decide from the known point p_i whether or not p_{i+1} lies above or below the line. From the properties given in (7) we have that

$$(x_{i+1}, y_{i+1}) = \begin{cases} (x_i + 1, y_i + 1) & \text{iff. } d_p \geq 0 \\ (x_i + 1, y_i) & \text{iff. } d_p < 0 \end{cases} \tag{14}$$

2.2 The decision variable

Choosing either we must update the decision variable correspondingly for d_{p+1}. In the case of $d_p \geq 0$ (above the line), we have that

$$d_{i+1} = F(x_i + 2, y_i + \frac{3}{2}) \tag{15}$$

$$= a(x_i + 2) + b(y_i + \frac{3}{2}) + c \tag{16}$$

$$= \underbrace{a(x_i + 1) + b(y_i + \frac{1}{2}) + c}_{d_p} + \underbrace{a + b}_{\Delta d} \tag{17}$$

That is, the difference $\Delta d = d_{p+1} - d_p$ is simply $a + b$. Correspondingly, if $d_p < 0$ (below the line), we have that

$$d_{i+1} = F(x_i + 2, y_i + \frac{1}{2}) \tag{18}$$

$$= a(x_i + 2) + b(y_i + \frac{1}{2}) + c \tag{19}$$

$$= \underbrace{a(x_i + 1) + b(y_i + \frac{1}{2}) + c}_{d_p} + \underbrace{a}_{\Delta d} \tag{20}$$

In the same way as before Δd is in this case simply a.

2.3 Initializing the algorithm

Now that we have a way of deciding how to choose a point p and update the decision variable d_p correspondingly, we can provide the initial values for these.

We remark that we have at least two definite points beknownst to us; a starting- and ending point. For the initialization of the point p_0, we simply choose the starting point.

Doing so, we can calculate the initial value of d_p.

$$d_0 = F(x_0 + 1, y_0 + \frac{1}{2}) \tag{21}$$

$$= a(x_0 + 1) + b(y_0 + \frac{1}{2}) + c \tag{22}$$

$$= \underbrace{ax_0 + by_0 + c}_{0} + \underbrace{a + \frac{1}{2}b}_{\Delta d_0} \tag{23}$$

Under the premise that a point $p \in \mathbb{Z}^2$, then by multiplying each constituent just discussed — d_p and Δd_p — by 2 we ensure that all values calculated are integers, which yields better performance.

We now have all that we need to perform the initialization of the algorithm, let us briefly list them, for convience and later reference.

1. $\Delta x = x_n - x_0$ and $\Delta y = y_n - y_0$
2. $d = 2\Delta y - \Delta x$
3. $\Delta d = \begin{cases} 2(\Delta y + \Delta x) & \text{iff. } d \geq 0 \\ 2\Delta y & \text{iff. } d < 0 \end{cases}$
4. $step_x = \begin{cases} 1 & \text{iff. } \Delta x \geq 0 \\ -1 & \text{iff. } \Delta x < 0 \end{cases}$ and $step_y = \begin{cases} 1 & \text{iff. } \Delta y \geq 0 \\ -1 & \text{iff. } \Delta y < 0 \end{cases}$

All of which are set on lines 1–11 in the code excerpt (see 2.5 on page 2.5).

2.4 Direction Issue

While we would like to simply draw on either side of the line based upon whether or not $d \geq 0$, we would also like the algorithm to produce reliable results. Following this naive approach can yield different results depending on the direction of the line. That is, when $d = 0$ for a given point, when $step_x = 1$ the algorithm would choose the pixel above, while choosing the pixel below if $step_x = -1$. Although the lines should have been the exact same rendition, interchanging the starting- and ending points of a line yields different rasterized results, and so this must be accounted for.

A straight-forward solution is to simply check in which direction we are following the line whenever $d = 0$, and if we are going in the positive direction, that is $step_x = 1$, then we would want to choose the pixel above, if not, we stay on the x-axis. In the solution code excerpt provided this is evident on lines 16 and 30, on which we declare the check variable, and are used on lines 20 and 35.

2.5 Algorithm

We present an excerpt from the code, as provided (see the `/src` directory), containing the algorithm discussed.

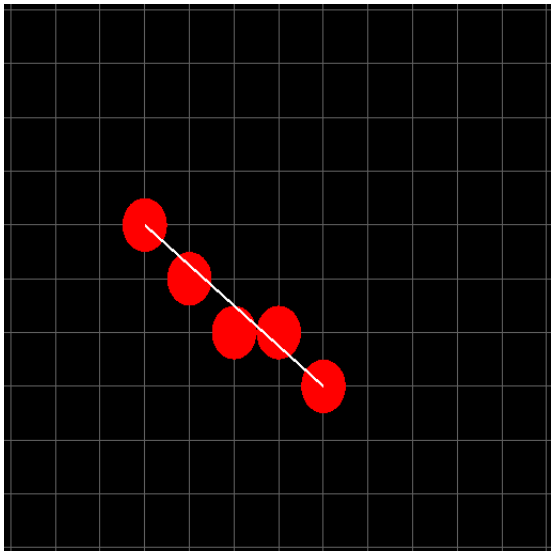
```
1  // initialization
2  int dx = (b.x - a.x);
3  int abs_2dx = (int)std::abs(dx) << 1;
4  int x_step = (dx < 0) ? -1 : 1;
5
6  int dy = (b.y - a.y);
7  int abs_2dy = (int)std::abs(dy) << 1;
8  int y_step = (dy < 0) ? -1 : 1;
9
10 Point2D pixel = a;
11 bool dom = abs_2dx > abs_2dy;
12
13 // x-dominant loop
14 if (dom) {
15     bool left_right = (x_step > 0);
16     int d = abs_2dy - (abs_2dx >> 1);
17     while (true) {
18         draw->point(pixel);
19         if (pixel.x == b.x) return;
20         if ( (d > 0) || ((d == 0) && left_right) ) {
21             pixel.y += y_step;
22             d -= abs_2dx;
23         }
24         pixel.x += x_step;
25         d += abs_2dy;
26     }
27 }
28 // y-dominant loop
29 else {
30     bool left_right = (y_step > 0);
31     int d = abs_2dx - (abs_2dy >> 1);
32     while (true) {
33         draw->point(pixel);
34         if (pixel.y == b.y) return;
35         if ( (d > 0) || ((d == 0) && left_right) ) {
36             pixel.x += x_step;
37             d -= abs_2dy;
38         }
39         pixel.y += y_step;
40         d += abs_2dx;
41     }
42 }
```

Figure 2: Except code containing the actual algorithm

The excerpt can be found in the code provided under `/src/Primitives/Line.cpp`. I will remark that the type `Point2D` is declared in `/src/Types.h`.

2.6 Tests

I let my program generate an interesting example, highlighting the direction issue, and showing that its solution works as expected.



In the example to the left the coords are $(-4, -1)$ to $(0, -4)$, which gives us that $\Delta x = 4$ and $\Delta y = 3$, making x the dominant axis. And since $\Delta x \geq 0$, then $step_x = 1$, making the algorithm go from left to right, which for $x = -2$ hits the case of $d = 0$, and since we have established that it is indeed going left to right, we should follow $step_y$, which it indeed does.

Figure 3: Test showing an x-dominant left-to-right solution

I have deliberately organized my code in such a way, that it is easy to generate a random line within certain constraints on the coordinates, such that if need be, then one need not write any code to verify the correctness of the algorithm, but simply run the program. Hitting the ENTER key will generate a new random line whilst displaying the starting- and ending points of the line in the standard output console. Hitting the ESCAPE key quits the program gracefully.