Hw 5

7. 43, 37, 50, 51, 58, 52, 45, 45, 58, 130

37, 43, 45, 45, 50, 51, 52, 58, 58, 130

a) Mean $= \frac{569}{10} = 56.9$ median $= \frac{50+51}{2} = 50.5$

$IQR = \frac{58+58}{2} - \frac{43+45}{2} = 14$

Standard Deviation $= s^2 = \frac{1}{10-1} \sum_{i=1}^{10} (x_i - \bar{x})^2 = 5343.61 + 1.21$

$+ (141.61)2 + 24.01 + 1.21 + 34.81$

$+ 47.61 + 396.01 + 193.21$

9

$= 702.7\bar{6} = \sqrt{s^2} = \sqrt{702.7\bar{6}} = 26.5097$

b) $44 - 1.5 \cdot 14 = 23 = Q_1 - 1.5(IQR)$

$58 + 1.5 \cdot 14 = 79 = Q_2 + 1.5(IQR)$

$23 \leq x_i \leq 79$

$130 > 79$, therefore 130 outlier

c) 37, 43, 45, 45, 50, 51, 52, 58, 58, ~~130~~

Mean: $\frac{434}{9} = 48.\overline{7}$ Median $= 50$

$Q_1 = \frac{43+45}{2} = 44$ $Q_2 = \frac{52+58}{2} = 55$

$IQR = Q_3 - Q_1 = 55 - 44 = 11$

variance $= s^2 = \frac{1}{9-1} \sum_{i=1}^{9} (x_i - \bar{x})^2 = \sqrt{46.44} = 6.96$

d) The IQR and median are not affected much at all by outliers. Mean and variance are affected greatly by outliers

2

a) the majority of diamonds are of smaller carat size which means the majority are of lower price. they are located from price 0 to 5000 and drops off after that

b) Exponential, it is the shape of our histogram

c) As the carat increases the price increases. The points are precise for carrots less than 2 meaning lower variability. For carrots greater than 2 it becomes more sparse meaning a greater variability in the price for larger carats.

3.a) $\hat{\theta} = \frac{2}{n}\sum_{i=1}^{n} X_i$, $E(\hat{\theta}) = \frac{2}{n}\sum_{i=1}^{n} E(X_i)$, $E(X_i) = \frac{\theta - 0}{2}$ for uniform$(0,\theta)$

n times in summation

$= \frac{2}{\cancel{n}} \cdot \cancel{n} \cdot \frac{\theta}{2} = \frac{2}{2}\theta = \theta$, unbiased

b) $se(\hat{\theta}) = \sqrt{var(\hat{\theta})}$ $\quad Var(KX) = K^2 \cdot Var(X)$

$Var(\hat{\theta}) = \left(\frac{2}{n}\right)^2 \cdot Var(X_i) = \frac{4}{n^2} \cdot \sum_{i=1}^{n} \frac{(\theta - 0)^2}{12} = \frac{4}{n^2} \cdot n \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n}$

$se(\hat{\theta}) = \sqrt{\frac{\theta^2}{3n}} = \frac{\theta}{\sqrt{3n}}$

$\hat{p}_1 = \frac{x_1 + x_2 + x_3 + x_4}{4}$ $\hat{p}_2 = \frac{x_1 + 2x_2 + x_4}{4}$

4 a) $E(\hat{p}_1) = \frac{1}{4} \cdot E(x_1 + x_2 + x_3 + x_4) = \frac{1}{4} \cdot 4p = p$

$E(\hat{p}_2) = \frac{1}{4} \cdot E(x_1 + 2x_2 + x_4) = \frac{1}{4} \cdot 4p = p$

$\text{Bias}(\hat{p}_1) = E(\hat{p}_1 - p) = E(\hat{p}_1) - p = p - p = 0$
$\text{Bias}(\hat{p}_2) = E(\hat{p}_2 - p) = E(\hat{p}_2) - p = p - p = 0$ } Both unbiased

b) $MSE(\hat{p}_1) = \text{Bias}^2(\hat{p}_1) + Var(\hat{p}_1)$

$Var(\hat{p}_1) = \left(\frac{1}{4}\right)^2 \cdot Var(x_1 + x_2 + x_3 + x_4)$
$= \frac{1}{16} \cdot 4 \cdot Var(x) = \frac{1}{4} \cdot p(1-p)$

$Var(\hat{p}_2) = \left(\frac{1}{4}\right)^2 \cdot Var(x_1 + 2x_2 + x_3)$
$= \frac{1}{16} \cdot [Var(x_1) + 4 \cdot Var(x_2) + Var(x_3)]$
$= \frac{6 \cdot Var(x)}{16} = \frac{3p(1-p)}{8}$

$MSE(\hat{p}_1) = 0^2 + \frac{1}{4} p(1-p) = \frac{1}{4} p(1-p)$
$MSE(\hat{p}_2) = 0^2 + \frac{3}{8} \cdot p(1-p) = \frac{3}{8} \cdot p(1-p)$

$MSE(\hat{p}_1) < MSE(\hat{p}_2)$, $\hat{p}_1$ is best estimator

5. $\mu_1 = E(X) = \sum_{i=1}^{N} i \cdot p_X(i) = \sum_{i=1}^{N} \frac{i}{N} = \frac{1}{N} \cdot \frac{N(N+1)}{2}$

$= \frac{N+1}{2}$

$\bar{X} = \frac{N+1}{2}, \quad N = 2\bar{X} - 1$

6. a) $E(X_i) = \lambda = \bar{X}$

b) $L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$

$l(\lambda) = \log L(\lambda) = -n\lambda + \sum x_i \log(\lambda) - \sum_{i=1}^{n} \log(x_i!)$

$\frac{d}{d\lambda} l(\lambda) = -n + \frac{\sum x_i}{\lambda} = 0$

$\Rightarrow \lambda = \frac{\sum x_i}{n} = \bar{X}$

$\frac{d^2}{d\lambda^2} l(\lambda) = \frac{-\sum x_i}{\lambda^2} < 0$, so max at $\lambda = \bar{X}$

c) $\bar{\lambda} = \frac{7+6+7+2+4}{5} = 5.2 =$ MOM & MLE

Method of Moments

7. a) $\mu_1 = E(X) = \int_{\mathbb{R}} x f(x)dx = \int_0^1 x\theta x^{\theta-1}dx = \frac{\theta}{\theta+2}$

$\frac{\hat{\theta}}{\hat{\theta}+1} = \bar{x} \Rightarrow \hat{\theta} = \frac{\bar{x}}{1-\bar{x}}$ for MOM

$\bar{x} = \frac{2}{3} \Rightarrow \hat{\theta} = \frac{\frac{2}{3}}{1-\frac{2}{3}} = 2$

b) $L(\theta) = \prod_{i=1}^{3} \theta t_i^{\theta-1} = \theta^3 \left(\prod_{i=1}^{3} t_i\right)^{\theta-1}$

$\ell(\theta) = 3\log\theta + (\theta-1)\sum_{i=1}^{3} \log t_i$

$\frac{d}{d\theta}\ell(\theta) = \frac{3}{\theta} + \sum_{i=1}^{3} \log t_i = 0$

$\hat{\theta} = \frac{-3}{\sum_{i=1}^{3} \log t_i} = \frac{-3}{-1.378} = 2.176$

$\frac{d^2}{d\theta^2}\ell(\theta) = \frac{-3}{\theta^2} < 0$ max at $\hat{\theta} = 2.176$

8. $E(X) = \frac{\alpha}{\lambda} = \frac{3}{\lambda}$ For gamma

a)

$$\mu_1 = E(X) = \frac{3}{\lambda}$$

$$\frac{3}{\hat{\lambda}} = \bar{x} \Rightarrow \hat{\lambda} = \frac{3}{\bar{x}} \text{ for MOM}$$

b) $$L(\lambda) = \prod_{i=1}^{n} f(x_i) = \frac{\lambda^{3n}}{2^n}\left(\prod_{i=1}^{n} x_i^2\right) e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\ell(\lambda) = \log L(\lambda) = 3n\log(\lambda) - n\log 2 + 2\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n} x_i$$

$$\frac{d}{d\lambda}\ell(\lambda) = \frac{3n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{3n} = \frac{3}{\bar{x}}$$

$$\frac{d^2}{d\lambda^2}\ell(\lambda) = \frac{-3n}{\lambda^2} < 0 \text{ max at } \hat{\lambda} = \frac{-3n}{\lambda^2}$$