Stat 330: Module 4 Homework

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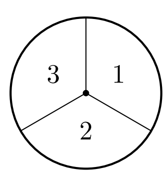
1. Classify each of the following stochastic processes as discrete-time or continuous-time, *and* discrete-space or continuous-space.
   (a) The temperature in downtown Ames throughout the day
   (b) The high temperature in downtown Ames for each day in a year
   (c) The number of customers in line at Café B throughout the day
   (d) The total number of customers served each day at Café B
   (e) The operational state, denoted 1 or 0, of a certain machine recorded at the end of each hour

2. A certain machine used in a manufacturing process can be in one of three states: Fully operational ("full"), partially operational ("part"), or broken ("broken"). If the machine is fully operational today, there is a .7 probability it will be fully operational again tomorrow, a .2 chance it will be partially operational tomorrow, and otherwise tomorrow it will be broken. If the machine is partially operational today, there is a .6 probability it will continue to be partially operational tomorrow and otherwise it will be broken (because the machine is never repaired in its partially operational state). Finally, if the machine is broken today, there is a .8 probability it will be repaired to fully operational status tomorrow; otherwise, it remains broken. Let X = the state of the machine on day n.
   (a) Identify the state space of this chain.
   (b) Determine the one-step transition probability matrix.
   (c) Given the machine is broken today, find the probability that it will stay broken the day after tomorrow.

3. Information bits (0s and 1s) in a binary communication system travel through a long series of relays. At each relay, a "bit-switching" error might occur. Suppose that at each relay, there is a 4% chance of a 0 bit being switched to a 1 bit and a 5% chance of a 1 becoming a 0. Let X_0 = a bit's initial parity (0 or 1), and let X_n = the bit's parity after traversing the nth relay.
   (a) Construct the one-step transition matrix for this chain. [Hint: There are only two states, 0 and 1.]
   (b) Suppose the input stream to this relay system consists of 80% 0s and 20% 1s. Determine the proportions of 0s and 1s exiting the first relay.
   (c) Under the same conditions as the last part, determine the proportions of 0s and 1s exiting the third relay.

4. A hamster is placed into the three-chambered circular habitat shown in the figure below. Sitting in any chamber, the hamster is equally likely to next visit either of the two adjacent chambers. Let X_n = the nth chamber visited by the hamster.



   (a) Construct the one-step transition matrix for this Markov chain.
   (b) Is this a regular Markov chain? [Hint: Look at the 2-step transition matrix]
   (c) Find the steady-state probabilities of this chain.
   (d) Given that the hamster is in Chamber 3, what is the probability that the hamster was in Chamber 1 in the previous step?

5. A Markov chain has 3 possible states: A, B, and C. Every hour, it makes a transition to a *different* state. From state A, transitions to states B and C are equally likely. From state B, transitions to states A and C are equally likely. From state C, it always makes a transition to state A.

   (a) Write down the transition probability matrix.

   (b) If the initial distribution for states A, B, and C is $P_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, find the distribution of state after 2 transitions, i.e., the distribution of X_2.

   (c) Show that this is a regular Markov Chain.

   (d) Find the steady-state distribution for this chain.

6. Radio blackouts are among the most common space weather events to affect Earth. Minor radio blackouts occur, on average, twice per year. Use a Poisson process to model the phenomenon.

   (a) Given that 3 events have occurred in the first half of the year, what is that probability that 3 events will happen in the rest of the year?

   (b) How many radio blackouts do you expect to see in two years?

   (c) Starting from now, how long must we wait so that the probability of seeing the next radio blackout is at least 0.5?

   (d) Find the probability that the time to the 4th blackout is at most 2 years.

7. Suppose that minor errors occur on a computer in a space station, which will require re-calculation. Assume the occurrence of errors follows a Poisson process with a rate of 1/2 per hour.

   (a) Find the probability that no errors occur during a day.

   (b) Suppose that the system cannot correct more than 25 minor errors in a day, in which case a critical error will arise. What is the probability that a critical error occurs since the start of a day? Keep up to the 6th decimal place in your answer.

   (c) Suppose the error correction protocols reset themselves so long as there are no more than five minor errors occurring within a 2 hour window. The system just started up and an error occurred. What is the probability the next reset will occur within 2 hours?

8. During the daily lunch rush, arrivals at the drive-thru at a nearby McDonald follow a Poisson process with a rate of 1 customers per minute.

   (a) What is the expected number of customers in 1 hour, and what is the corresponding standard deviation?

   (b) The drive-thru's workers can't handle more than 10 customers in any 5-minute span. Determine the probability that too many customers arrive for the workers to handle between 12:15 p.m. and 12:20 p.m.

   (c) A customer has just arrived. What is the probability another customer will arrive within the next 30 seconds?

   (d) The 100th lunch customer, starting at 12:00 p.m., gets a free meal. What is the expected value of the arrival time of that lucky customer, and what is the standard deviation of that time?

9. *Extra Credit (2 points)* A variation of a poisson process is a Birth and Death process. In a Birth and Death process, the state of a system can change if something enters the system (a birth) or leaves the system (a death). Suppose that we model the time till a birth as $B \sim \text{Exp}(\lambda)$ and the time till a death as $D \sim \text{Exp}(\mu)$, where B and D are independent of each other. Say we are interested in the time till the system *changes* state. Define a random variable as: Y = time till state changes. We can write Y as: $Y = \text{Min}(B, D)$. Give the distribution of Y and its parameter(s). Hints:

- Start with $\mathbb{P}(Y \geq t)$
- Note that with independent random variables, intersection probabilities can be written as the product of the individual probabilities.
- You will eventually want to get into the form of a CDF. Compare that CDF to one we have seen to identify the distribution of Y.