Chapter 1: The Integers

Numbers and Sequences

Notation:

$\mathbb{Z}$ = integers $= \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$\mathbb{Z}^+$ = positive integers $= \{1, 2, 3, \dots\}$

$\mathbb{Q}$ = rational numbers
$= \{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z}^+ \}$

$\mathbb{R}$ = real numbers $= \mathbb{Q} \cup \mathbb{Q}^c$ ← irrational numbers

The Well-Ordering Property

Every nonempty set of nonnegative integers has a least element.

* The Well-Ordering Property can be taken as an axiom or it can be proved from some other axioms (ex: mathematical induction).

Ex: The set

$$\{ n^2 - 6n + 2 : n \in \mathbb{Z}^+, n^2 - 6n + 2 \geq 0 \}$$

has a least element.

<u>Theorem 1.1</u>: $\sqrt{2}$ is irrational

Definition: A real number is algebraic if it is the root of a polynomial with integer coefficients.

ex: $\sqrt{2}$ is algebraic.
It is a root of the polynomial x^2-2.

Greatest Integer Function (Floor Function)

$[x]$ = largest integer less than or equal to x.

Another notation: $\lfloor x \rfloor$
Note that $[x] \leq x < [x]+1$

Ex: $[2.45] = 2$, $[3] = 3$, $[-1.8] = -2$

$[\pi] = 3$, $[e] = 2$

Theorem 1.2 (The Pigeonhole Principle)

If $k+1$ or more objects are placed into k boxes, then at least one box contains two or more of the objects.

Proof: We use the method of contradiction.
If none of the boxes contain more

than 1 object, then the total number of objects in all the boxes should be less than or equal to k, a contradiction to the given assumption that there are at least k+1 elements. Hence, at least one box should contain 2 or more elements.

Definition: A set A is countable if A is finite or if there is a one-to-one and onto function from A to $\mathbb{Z}^+$. Otherwise A is uncountable.

Ex: $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Q}$ are countable.

$\mathbb{Q}^c$, $\mathbb{R}$ are uncountable.