MA 350 Number Theory– Spring 2024

Homework 2

Due: **February 23, 2024**

Submit your written work in Canvas as a single PDF file, and be sure to show your work. Answers without accompanying work will receive zero credit.

1. Prove that $8^n + 125$ is never a prime for $n \in \mathbb{Z}^+$. (Hint: think of factoring.)
2. If $\gcd(a, b) = 1$, then show that $\gcd(a - b, a + b) = 1$ or 2.
3. Using the Euclidean algorithm and showing each step, find $\gcd\,(2394, 846)$. Hence, find $\mathrm{lcm}\,(2394, 846)$.
4. Prove that $\gcd(F_n, F_{n+1}) = 1$ where F_n is the n^{th} Fibonacci number. (Hint: Use induction.)
5. Let $n = 2^2 \times 3^5 \times 5 \times 7^6 \times 13^4$.
   (a) Find the largest perfect cube that divides n. Write the number explicitly.
   (b) Find the smallest positive perfect cube that is a multiple of n. (No need to write the number explicitly.)
6. (a) Let k_i be an integer for each $i \in \mathbb{Z}^+$. Show that, for $n \in \mathbb{Z}^+$, the product
   $$(6k_1 + 1)(6k_2 + 1) \cdots (6k_n + 1)$$
   is of the form $6k + 1$ for some $k \in \mathbb{Z}^+$. (Hint: use induction on n.)

   (b) Explain why any prime greater than 5 should be of the form either $6k + 1$ or $6k + 5$.

   (Hint: use division algorithm.)

   (c) Show that there are infinitely many primes of the form $6k + 5$. (Hint: assume there is a finite number of them, say $5, q_1, q_2, \cdots, q_n$ and then consider the number $6q_1q_2 \cdots q_n + 5$. Claim that this number has a prime divisor of the form $6k + 5$.)