1. Prove $[\frac{n(n+1)}{2}]^2 - \frac{n^2(n^2+1)}{4} + 78 \in O(n^3)$

   Take $[\frac{n(n+1)}{2}]^2 - \frac{n^2(n^2+1)}{4} + 78 = f(n)$

   Take $n^3 = g(n)$

   => $[\frac{n^2+n}{2}]^2 - \frac{n^4+n^2}{4} + 78$

   => $\frac{n^4+2n^3+n^2}{4} - \frac{n^4+n^2}{4} + 78$

   => $\frac{2n^3}{4} + 78 \le \frac{2n^3}{4} + 78n^3 = 78.5n^3 = c * n^3$

   Therefore, $with\ c = 78.5\ and\ n_0 = 1,\ f(n) \le c * g(n)\ for\ all\ n \ge n_0$

   Which means $f(n) \in O(n^3)$

2. Prove or disprove $2^{2^n} \in O(2^{2n})$

   Take $f(n) = 2^{2^n}$

   Take $g(n) = 2^{2n}$

   Take $\lim_{n \to \infty} \frac{f(n)}{g(n)}$

   => $\lim_{n \to \infty} \frac{2^{2^n}}{2^{2n}} = \lim_{n \to \infty} 2^{2^n - 2n}$, $2^n\ dominates\ 2n$

   Therefore there is no such c and n_0 you can pick to keep g(n) greater for all $n \ge n_0$

3. Prove that any function that is in O(log2n) is also in O(log3n)

   Take

   $f(n) \in O(log_2 n),\ then\ f(n) \le c_1 * log_2 n\ for\ some\ c_1\ and\ for\ some\ n_{0a} where\ all\ n \ge n_{0a}$

   Notice, $log_3 n = \frac{log_2 n}{log_2 3},\ by\ base\ change\ formula$

   => $log_2 n = log_3 n * log_2 3 = c_0 * log_3 n\ where\ c_0 = log_2 3$

   => $f(n) \le c_1 * c_0 log_3 n,\ substituting\ in\ for\ log_2 n$

   => $f(n) \in O(log_3 n)\ for\ some\ c,\ where\ c = c_0 * c_1\ and\ some\ n_{0b}\ where\ n_{0b} \ge n_{0a}$

   Therefore, $f(n) \in O(log_3 n)$

4. Prove that if $f_1(n) \in O(g_1(n))\ and\ f_2(n) \in O(g_2(n))$, then

   $f_1(n) + f_2(n) \in O(g_1(n) + g_2(n))$

   $f_1(n) \in O(g_1(n)) => f_1(n) \le c_1 g_1(n)\ for\ all\ n \ge n_{0a}$

   $f_2(n) \in O(g_2(n)) => f_2(n) \le c_2 g_2(n)\ for\ all\ n \ge n_{0b}$

   => $f_1(n) + f_2(n) = c_1 g_1(n) + c_2 g_2(n)$

$\Rightarrow f_1(n) + f_2(n) = c_{1+2}(g_1(n) + g_2(n))$

Therefore, for all $n \geq max(n_{0a}, n_{0b})$ and $c = c_{1+2}$, $f_1(n) + f_2(n) \in O(g_1(n) + g_2(n))$

5. $T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{i+j} 1$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} (i + j)$

$= \sum_{i=1}^{n} (\sum_{j=1}^{n} i + \sum_{j=1}^{n} j)$

$= \sum_{i=1}^{n} (in + \frac{n(n+1)}{2})$

$= n \sum_{i=1}^{n} i + \frac{n(n+1)}{2} \sum_{i=1}^{n} 1$

$= n * \frac{n(n+1)}{2} + \frac{n(n+1)}{2} * n$

$= \frac{2n^2(n+1)}{2}$

$= n^2(n + 1)$

$= n^3 + n^2 \in O(n^3)$

6. For each iteration of the outer loop, the inner loop will do i iterations. So if we consider R iterations of the outer loop:

Outer loop Iteration	i Value at Beginning of Iteration	Number of Times Inner Loop Iterates
1	n	n
2	n/2	n/2
3	n/4	n/4
…	…	…
R	$n/2^{R-1}$	$n/2^{R-1}$

The summation of runtime will be: $n + n/2 + n/4 + \ldots + n/2^{R-1}$

Using the geometric series we can say: $S = n\frac{a}{1-r} = n\frac{1}{1/2} = 2n \in O(n)$