Mathematical Induction

Theorem (Strong Induction)

Consider the proposition $P(n)$.

If $P(1)$ is true and

$$P(1), P(2), \dots, P(n) \Rightarrow P(n+1)$$

is true for any positive integer n, then

$P(n)$ is true for all $n \in \mathbb{Z}^+$.

Proof: Can be proved using the well-ordering principle.

Ex: Prove that $n! \le n^n$ for all $n \in \mathbb{Z}^+$.

Let $P(n)$ be the proposition $n! \le n^n$.

$P(1)$ is true ($\because 1! \le 1^1$)

Suppose $P(1), P(2), \dots, P(n)$ are true.

We'll show $P(n+1)$ is true.

$$\begin{aligned}
(n+1)! &= (n+1)\,n! \\
&\le (n+1)\,n^n \quad (\because P(n) \text{ is true}) \\
&< (n+1)(n+1)^n \quad (\because n < n+1) \\
&= (n+1)^{n+1}
\end{aligned}$$

$\therefore P(n+1)$ is true.

* In weak induction, the difference is that we assume $P(n) \Longrightarrow P(n+1)$.