MA 350 Number Theory – Spring 2024

Homework 3

Due: March 08, 2024

Submit your written work in Canvas as a single PDF file, and be sure to show your work. Answers without accompanying work will receive zero credit.

1. Show that the linear diophantine equation $15x + 6y = 8$ has no solutions.
2. Consider the diophantine equation $24x + 35y = 94$.
   (a) Show that there exists a solution.
   (b) Using the Euclidean Algorithm, find a particular solution.
   (c) Find all solutions.
   (d) Show that there is only one positive solution and then find it.
3. Use mathematical induction to prove that the last digit of 2^{2^n} is 6 where $n \geq 2$. Hence, prove that, if $n \geq 2$, the last digit of the Fermat number F_n is 7.
4. Congruence can be used to find the remainder of some large numbers in a very efficient way. For example, suppose we need to find the remainder of $47^{18} + 26^{12}$ when divided by 23. A sketch of the major steps is given below. I want you to identify the reason for each step in the process. You may mention the theorem numbers in the textbook (lecture notes) where necessary. You should rewrite the proof expanding and giving reason for each congruence in the calculations.

   $47 \equiv 1 \ (mod\ 23)$
   $47^{18} \equiv 1^{18} \equiv 1 \ (mod\ 23)$
   $26 \equiv 3 \ (mod\ 23)$
   $26^{12} \equiv 3^{12} \equiv \ 27^4 \equiv 4^4 \equiv 3 \ (mod\ 23)$
   $47^{18} + 26^{12} \equiv 4 \ (mod\ 23)$

5. Find all incongruent solutions of $12x \equiv 8 \ (mod\ 20)$.
6. Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$. By the division algorithm, we know that $n = mq + r$ with $0 \leq r < m$. An equivalent way to say this is, $n \equiv r \ (mod\ m)$ for some $0 \leq r < m$. Thus, when $m = 4$, we can say that, for any $n \in \mathbb{Z}$, $n \equiv r \ (mod\ 4)$ where $r = 0, 1, 2$ or 3.
   Show that, for all $n \in \mathbb{Z}$ and for all $k \geq 2$, $n^k \not\equiv 2 \ (mod\ 4)$. In other words, prove that if $a \equiv 2 \ (mod\ 4)$, then a cannot equal the k^{th} power of an integer for any $k \geq 2$.