3.2 The Distribution of Primes

Theorem 3.4 (The Prime Number Theorem)

$$\lim_{x \to \infty} \frac{\pi(x)}{\left(x/\log x\right)} = 1$$

This is usually written as

$$\pi(x) \sim \frac{x}{\log x}$$

Proof: Later

* The largest known prime s as of January 26, 2024 is

$$2^{82,589,933} - 1$$

found in 2018. It has 24,862,048 digits.

Corollary: Let p_n be the nth prime, $n \in \mathbb{Z}^+$. Then, $p_n \sim n \log n$. This means

that $\lim_{n \to \infty} \frac{p_n}{n \log n} = 1.$

Proof: Note that

$$\pi(p_n) = n.$$

From Theorem 3.4,

$$\pi(p_n) \sim \frac{p_n}{\log p_n}$$

$$\therefore\ n \sim \frac{p_n}{\log p_n} \quad \text{———} \ (1)$$

$$\therefore\ \log n \sim \log p_n - \log(\log p_n)$$

$$\therefore\ \log n \sim \log p_n \ \text{(why?)} \ - \ (2)$$

$$(1) \Rightarrow p_n = n \log p_n$$

$$\therefore\ p_n \sim n \log n \ \text{(using (2))}$$

* This means " n^{th} prime is as large as $n\log n$".

<u>**Theorem 3.5:**</u> Let $n \in \mathbb{Z}^+$. Then, there are at least n consecutive composite integers.

Proof: Consider the n consecutive integers

$(n+1)!+2,\ (n+1)!+3,\ \cdots,\ (n+1)!+n+1$

Notice that the j^{th} integer is divisible by $j+1$ $(j=1,\cdots,n)$.

Hence the proof!

Bertrand's Postulate

For any $n \in \mathbb{Z}^+$ with $n>1$, $\exists$ a prime p such that $n<p<2n$.

Proof: Omitted.

There are many conjectures on primes. Please read the literature about them.

Some of them are listes below.

1. Twin Prime Conjecture
2. The Erdős Conjecture
3. Goldbach's Conjecture
4. The n^2+1 conjecture
5. The Legendre Conjecture