Stat 330: Homework 5 (module 5) Solutions

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1. The following data set represents the number of new computer accounts registered during ten consecutive days:

$$43,\ 37,\ 50,\ 51,\ 58,\ 52,\ 45,\ 45,\ 58,\ 130$$

Answer: The ordered data is: 37, 43, 45, 45, 50, 51, 52, 58, 58, 130

(a) Compute the mean, median, IQR, and standard deviation

Answer:

- mean $= \frac{1}{10}\sum_{i=1}^{10} x_i = 56.9$
- median $= \frac{50+51}{2} = 50.5$
- $Q_1 = \frac{43+45}{2} = 44$; $Q_3 = \frac{58+58}{2} = 58$
  $\rightarrow IQR = Q_3 - Q_1 = 58 - 44 = 14$
- variance $= s^2 = \frac{1}{10-1}\sum_{i=1}^{10}(x_i - \bar{x})^2 = 702.7667$
  $\rightarrow$ standard deviation $= s = \sqrt{s^2} = \sqrt{702.7667} = 26.5097$

(b) Check for outliers using the 1.5(IQR) rule, and indicate which data points are outliers.

Answer:

- $Q_1 - 1.5(IQR) = 44 - 1.5(14) = 23$
- $Q_3 + 1.5(IQR) = 58 + 1.5(14) = 79$
- Any values less than 23 or greater than 79 are outliers.
- Outlier: 130

(c) Remove the detected outliers and compute the new mean, median, IQR, and standard deviation.

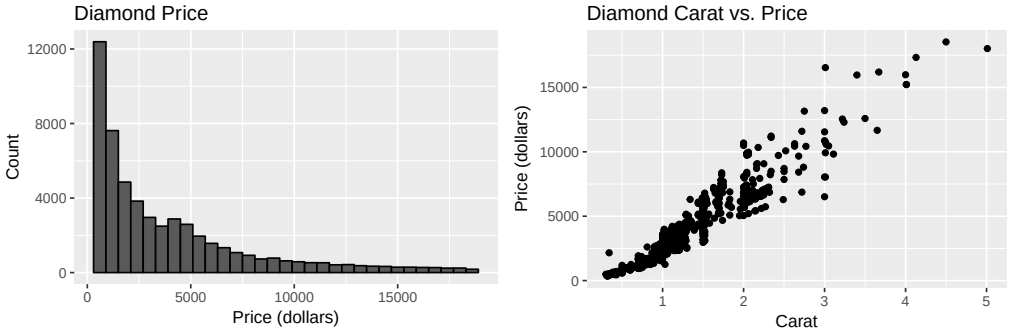
Answer: The new ordered data: 37, 43, 45, 45, 50, 51, 52, 58, 58

- mean $= \frac{1}{9}\sum_{i=1}^{9} x_i = 48.78$
- median $= 50$
- $Q_1 = \frac{43+45}{2} = 44$; $Q_3 = \frac{52+58}{2} = 55$
  $\rightarrow IQR = Q_3 - Q_1 = 55 - 44 = 11$
- variance $= s^2 = \frac{1}{9-1}\sum_{i=1}^{9}(x_i - \bar{x})^2 = 48.4444$
  $\rightarrow$ standard deviation $= s = \sqrt{s^2} = \sqrt{48.4444} = 6.9602$

(d) Make a conclusion about the effect of outliers on the basic descriptive statistics from (a) and (c).

Answer: The outlier increased the mean and variance. The median and IQR *slightly* increased with the outlier but not by much. Thus, the mean and variance seem to be affected greatly by outlier, but the median and IQR were not affected much by the outlier (robust)

2. A histogram of the price of diamonds, and a scatterplot of carat vs. price of diamonds are given below.



(a) Describe the shape of the histogram of price of diamonds. (Where are the majority of diamond prices located? Where are the minority of diamond prices located?)

Answer: The majority of diamond prices are in the lower end of the price spectrum between about 0 and 5,000 dollars. After about 5,000 dollar, the number of diamonds drop off.

(b) Are exponential, normal, or uniform distributions reasonable as the population distribution for the price of diamonds? Justify your answer.

Answer: Since the histogram of our sample has a similar shape to an exponential distribution, an exponential distribution is a reasonable choice for the population distribution.

(c) Describe the relationship between carat and price of diamonds. (What happens to price as number of carats increases? What happens to the variability as number of carats increases?)

Answer: In general, as number of carats increases, the price of the diamond also increases. The data points are more compact for lower (< 2) carats indicating lower variability in prices for lower carats. Conversely, the data points are more spread out for higher (> 2) carats indicating higher variability in prices for larger carats.

3. Suppose $X_i \overset{iid}{\sim} \text{Unif}(0, \theta)$ for $i = 1, \ldots, n.$. Suppose we propose an estimator for θ as $\hat{\theta} = \frac{2}{n}\sum_{i=1}^{n} X_i$.

(a) Is $\hat{\theta}$ an unbiased estimator for θ?

(b) Calculate $se(\hat{\theta})$ (Recall the standard error of an estimator is the square root of the variance of an estimator).

Answer:

(a)

$$E(\hat{\theta}) = \frac{2}{n}\sum_{i=1}^{n} E(X_i) = \frac{2}{n} \cdot n \cdot \frac{\theta}{2} = \theta$$

since X_i are i.i.d. So $\hat{\theta}$ is unbiased.

(b)

$$\text{Var}(\hat{\theta}) = \frac{4}{n^2}\text{Var}(X_i) = \frac{\theta^2}{3n}.$$

So

$$se(\hat{\theta}) = \text{Var}(\hat{\theta})^{1/2} = \theta/\sqrt{3n}.$$

4. Let $X_1, \ldots, X_4 \overset{iid}{\sim} \text{Bern}(p)$. Suppose we propose two estimators for p:

$\hat{p}_1 = \frac{X_1+X_2+X_3+X_4}{4}$

$\hat{p}_2 = \frac{X_1+2X_2+X_3}{4}$

(a) Show that both estimators are unbiased estimators of p.

(b) Which estimator is "best" in terms of having a smaller MSE? Calculate MSE($\hat{p}_1$) and MSE($\hat{p}_2$) (Recall that if an estimator $\hat{\theta}$ is unbiased, MSE($\hat{\theta}$) = $Var(\hat{\theta})$).

Answer:

(a)

$$\mathbb{E}(\hat{p}_1) = \frac{1}{4}\mathbb{E}(X_1 + X_2 + X_3 + X_4) = \frac{1}{4}4\mathbb{E}(X) = p \implies Bias(\hat{p}_1) = E(\hat{p}_1 - p) = E(\hat{p}_1) - p = p - p = 0$$

$$\mathbb{E}(\hat{p}_2) = \frac{1}{4}\mathbb{E}(X_1 + 2X_2 + X_3) = \frac{1}{4}(p + 2p + p) = p \implies Bias(\hat{p}_2) = E(\hat{p}_2 - p) = E(\hat{p}_2) - p = p - p = 0$$

So both estimators are unbiased for p.

(b)

$$Var(\hat{p}_1) = \frac{1}{16}Var(X_1 + X_2 + X_3 + X_4) = \frac{4}{16}Var(X) = \frac{4p(1-p)}{16} = \frac{p(1-p)}{4}$$
$$Var(\hat{p}_1) = \frac{1}{16}Var(X_1 + 2X_2 + X_3) = \frac{1}{16}\left[Var(X_1) + 4Var(X_2) + Var(X_3)\right] = \frac{6Var(X)}{16} = \frac{6p(1-p)}{16}$$
$$MSE(\hat{p}_1) = Bias^2(\hat{p}_1) + Var(\hat{p}_1) = 0^2 + \frac{p(1-p)}{4} = \frac{p(1-p)}{4}$$
$$MSE(\hat{p}_2) = Bias^2(\hat{p}_2) + Var(\hat{p}_2) = 0^2 + \frac{6p(1-p)}{16} = \frac{6p(1-p)}{16}$$

We see that $MSE(\hat{p}_1) < MSE(\hat{p}_2)$, so $\hat{p}_1$ is the better estimator.

5. Suppose $X_i \overset{iid}{\sim} p_X(x)$, where $p_X(x) = \frac{1}{N}$ for $x \in \{1, \ldots, N\}$. Here, N is the parameter. Derive the method of moments estimator for N.

Answer:

$$\mu_1 = E(X) = \sum_{i=1}^{N} \frac{i}{N} = \frac{N+1}{2}$$
$$\bar{X} = \frac{1}{N}\sum_{i=1}^{n} X_i.$$
$$\text{Set } \bar{X} = \mu_1$$
$$\implies N = 2\bar{X} - 1.$$

6. Suppose $X_i \overset{iid}{\sim} Pois(\lambda)$ for $i = 1, \ldots, n$.

   (a) Give the method of moments estimator for λ.

   (b) Next we will give the maximum likelihood estimator by going through the steps:

      i. Write down the likelihood function.
      ii. Give the log-likelihood function.
      iii. Give the derivative of the log-likelihood function with respect to λ.
      iv. Set the derivative equal to zero and solve for λ in terms of the data.
      v. Report the maximum likelihood estimator for λ.

   (c) If we observe the data, 7, 6, 7, 2, and 4, what are the numerical estimates of the method of moments and maximum likelihood for λ?

Answer:

(a) The first population moment is $E[X_i] = \lambda$ and the first sample moment is $\bar{y}$. Thus the method of moments estimator is $\hat{\lambda}_{MOM} = \bar{x}$.

(b)

$$L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$$
$$\ell(\lambda) = logL(\lambda) = -n\lambda + \sum x_i \log(\lambda) - \sum_{i=1}^{n} \log(x_i!)$$
$$\frac{\partial}{\partial\lambda}\ell(\lambda) = -n + \frac{\sum x_i}{\lambda} \overset{set}{=} 0$$

Setting the above equal to zero and solving for λ, we obtain $\hat{\lambda}_{MLE} = \frac{\sum x_i}{n} = \overline{x}$.
2^{nd} derivative test:
$\frac{\partial^2}{\partial \lambda^2}\ell(\lambda)|_{\lambda=\hat{\lambda}} = \frac{\partial}{\partial \lambda}\left(-n + \frac{\sum x_i}{\lambda}\right)|_{\lambda=\hat{\lambda}} = \frac{-\sum x_i}{\lambda^2}|_{\lambda=\hat{\lambda}} = \frac{-\sum x_i}{\hat{\lambda}^2} < 0$. We have a maximum at $\hat{\lambda}_{MLE}$.

(c) The average of 7, 6, 7, 2, and 4 is 5.2. Thus $\hat{\lambda}_{MOM} = \hat{\lambda}_{MLE} = 5.2$.

7. A sample of 3 observations of waiting time to access an internet server is $x_1 = 0.4, x_2 = 0.7, x_3 = 0.9$ seconds. It is believed that the waiting time has the continuous distribution

$$f(t) = \begin{cases} \theta t^{\theta-1}, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find an estimate of the parameter θ using the method of moments. (Give a numerical value)

(b) Find the maximum likelihood estimate of θ. (Give a numerical value)

Answer:

(a) Using method of moments,

$$\mu_1 = E(X) = \int_{\mathbb{R}} x f(x) dx = \int_0^1 x\theta x^{\theta-1} dx = \frac{\theta}{\theta+1}$$

and $m_1 = \bar{x}$ so that

$$\frac{\hat{\theta}}{\hat{\theta}+1} = \bar{x} \implies \hat{\theta}_{MoM} = \frac{\bar{x}}{1-\bar{x}}.$$

For the given data, $\bar{x} = (0.4 + 0.7 + 0.9)/3 = 2/3$ so that

$$\hat{\theta}_{MoM} = \frac{2/3}{1-2/3} = 2$$

(b) Using MLE,

$$L(\theta) = \prod_{i=1}^{3} \theta t_i^{\theta-1} = \theta^3 (\prod_{i=1}^{3} t_i)^{\theta-1}$$

$$\ell(\theta) = 3\log\theta + (\theta-1)\sum_{i=1}^{3} \log t_i$$

$$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{3}{\theta} + \sum_{i=1}^{3} \log t_i \overset{set}{=} 0$$

$$\hat{\theta}_{MLE} = -3/\sum_{i=1}^{3} \log t_i = -3/-1.378326 = 2.17655$$

2^{nd} derivative test:
$\frac{\partial^2}{\partial \theta^2}\ell(\theta)|_{\theta=\hat{\theta}} = \frac{\partial}{\partial \theta}\left(\frac{3}{\theta} + \sum_{i=1}^{3} \log t_i\right)|_{\theta=\hat{\theta}} = \frac{-3}{\theta^2}|_{\theta=\hat{\theta}} = \frac{-3}{\hat{\theta}^2} < 0$. We have a maximum at $\hat{\theta}_{MLE}$.

8. Let $X_1, \ldots, X_n$ be a random sample from the Gamma distribution with $\alpha = 3$. The pdf is shown as follows.

$$f(x) = \frac{\lambda^3}{2} x^2 e^{-\lambda x}$$

for $x \geq 0$.

(a) Find an estimate of the parameter λ using the method of moments.

(b) Find the maximum likelihood estimate of λ.

Answer:

(a) Since it is the gamma distribution, we have

$$E(X) = \frac{\alpha}{\lambda} = \frac{3}{\lambda}$$

Using method of moments,

$$\mu_1 = E(X) = \frac{3}{\lambda}$$

and $m_1 = \bar{x}$ so that

$$\hat{\lambda} = \frac{3}{\bar{x}}.$$

(b)

$$L(\lambda) = \prod_{i=1}^{n} f(x_i) = \frac{\lambda^{3n}}{2^n} \left(\prod_{i=1}^{n} x_i^2 \right) e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\ell(\lambda) = \log L(\lambda) = 3n \log \lambda - n \log 2 + 2 \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i$$

$$\frac{\partial \ell(\lambda)}{\partial \lambda} = \frac{3n}{\lambda} - \sum_{i=1}^{n} x_i \overset{set}{=} 0$$

$$\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{3n} = \frac{3}{\bar{x}}$$

2^{nd} derivative test:
$\frac{\partial^2}{\partial \lambda^2} \ell(\lambda)|_{\lambda=\hat{\lambda}} = \frac{\partial}{\partial \lambda} \left(\frac{3n}{\lambda} - \sum_{i=1}^{n} x_i \right) |_{\lambda=\hat{\lambda}} = \frac{-3n}{\lambda^2}|_{\lambda=\hat{\lambda}} = \frac{-3n}{\hat{\lambda}^2} < 0$. We have a maximum at $\hat{\lambda}_{MLE}$.