HW2

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1. Prove that $8^n + 125$ is never a prime

$$8^n + 125 = (2^3)^n + 5^3 = 2^{3n} + 5^3$$

$$2^{3n} + 5^3 = (2^n + 5)(2^{2n} - 5 \cdot 2^n + 25)$$

$2^n + 5$ can never be less than 5, so can't be 1

$$(2^{2n} - 5 \cdot 2^n + 25) \neq 1$$

Will it ever equal 1?

$$2^{2n} - 5 \cdot 2^n + 25 = 1$$
$$\Rightarrow 2^{2n} - 5 \cdot 2^n = -24$$
$$\Rightarrow 2^n(2^n - 5) = -24$$

$n \in \mathbb{Z}^+$, once $n \geq 3$ positive values since $2^3 > -5$

Try $n = 1$, $2(2-5) = -6 \neq -24$

Try $n = 2$, $2^2(2^2 - 5) = 4(-1) = -4 \neq -24$

$\therefore$ Both factors do not equal 1 ever, meaning it cannot be prime

2. If $\gcd(a,b)=1$ show that $\gcd(a-b, a+b) = 1$ or 2

$\gcd(a+b, a-b)$

$\Rightarrow d \mid a+b$ & $d \mid a-b$

Then, $d \mid a+b+(a-b)$

$\Rightarrow d \mid 2a$ (1)

Also, $d \mid a+b-(a-b)$

$\Rightarrow d \mid a+b-a+b$

$\Rightarrow d \mid 2b$ (1)

Given, $\gcd(a,b)=1$, $\exists m, n \in \mathbb{Z}$ s.t. $am + bn = 1$

$\therefore 2a + 2bn = 2$ (2)

Because (1) & (2) $\Rightarrow d \mid 2$

Since $d > 0$, $d=1$ or $d=2$

3. $\gcd(2394, 846)$

$\gcd(8, 6)$

$2^3 \quad 2 \cdot 3$

2

$2394 = 2 \cdot 1692 + 702$

$1692 = 2 \cdot 702 + 288$

$702 = 2 \cdot 288 + 126$

$288 = 2 \cdot 126 + 36$

$126 = 3 \cdot 36 + 18$

$36 = 2 \cdot 18 + 0$

$\gcd(2394, 846) = 18$

$$\mathrm{Lcm}(a,b) = \frac{a \times b}{\gcd(a,b)}$$

$$\mathrm{Lcm}(2394, 846) = \frac{2025324}{18} = 112{,}518$$

4. Prove that $\gcd(F_n, F_{n+1}) = 1$

Base case:

$\gcd(F_1, F_2) = 1$
$= (2, 3) = 1$

Inductive step

$\gcd(F_{n+1}, F_{n+2}) = 1$

$\gcd(F_{n+2}, F_n + F_{n+1}) = d$

$\Rightarrow d \mid F_{n+2} \wedge d \mid F_n + F_{n+1}$

$\Rightarrow d \mid F_n + F_{n+2} - F_{n+1}$

$\Rightarrow d \mid F_n$

$\therefore d \mid F_n \wedge d \mid F_{n+1}$

$\Rightarrow d \mid 1$

$\Rightarrow d = 1$

5. $n = 2^2 \times 3^5 \times 5 \times 7^6 \times 13^4$

a) $3^3 \times 7^3 \times 7^3 \times 13^3$

* All terms need to be able to be cubed

b) $2 \times 3 \times 5^2 \times 13^2 (n)$

* This makes all of n's terms perfect cubes

Base case

6a. $(6 \cdot 1 + 1) = 6k+1$

Inductive step

$(6k_1+1)(6k_2+1)\cdots(6k_n+1)\cdot(6k_{n+1}+1)$

$= 6k+1(6k_{n+1}+1)$

$= 36k \cdot k_{n+1} + 6k + 6k_{n+1} + 1$

$= 6(6k \cdot k_{n+1} + k + k_{n+1}) + 1$

b) Division Algorithm

All numbers can be written as:

$n=6k \wedge n=6k+1 \wedge n=6k+2 \wedge n=6k+3 \wedge n=6k+4 \wedge n=6k+5$

Case 1: $n=6k$, divisible by 6, not prime

Case 2: $n=6k+2$, $n=2(3k+1)$, divisible by 2

Case 3: $n=6k+3$, $n=3(2k+1)$, divisible by 3

Case 4: $n=6k+4$, $n=2(2k+1)$, divisible by 2

$\therefore$ Primes have to take the form

$6k+1$ or $6k+5$

C. Suppose there were finetly many primes of $6k+5$

Let $N = 6(q_1 \cdot q_2 \cdots q_n) + 5$

Assume every prime factor $6k+2$

Then $N = 6l+1$, from part 1

Contradiction because if all products of $6k+1$ then product should be of form $6k+1$. so there is atleast one prime of the form $6u+5$.