1. Determine the vale of $\lfloor x \rfloor + \lfloor -x \rfloor$ where $x \in \mathbb{R}$

2 cases x is an integer or has a decimal

Case 1:

x is an integer

then $\lfloor x \rfloor + \lfloor -x \rfloor = x - x = 0$

Case 2:

x has a decimal part, δ and integer part ϕ

This means $\lfloor \phi + \delta \rfloor = \lfloor \phi \rfloor + \lfloor \delta \rfloor = \lfloor \phi \rfloor + 0$

And $\lfloor -(\phi + \delta) \rfloor = \lfloor -\phi - \delta \rfloor = \lfloor -\phi \rfloor + \lfloor -\delta \rfloor$

$= \lfloor -\phi \rfloor - 1$

Therefore $(\lfloor \phi \rfloor + 0) + (\lfloor -\phi \rfloor - 1)$

$= \lfloor \phi \rfloor + \lfloor -\phi \rfloor - 1$, flooring integer yields same integer

$\therefore \phi - \phi - 1 = -1$

2. Let $x \in \mathbb{Z}$, x^2 is odd then x is odd

Proof:

Let's say x^2 is odd, the x is even

$x = 2K_1 \implies x^2 = 4K_1$

This is even though.

$\therefore$ x must be odd

3. Let $c \in \mathbb{Z}$ and be odd. Show that $x^2 + x + c = 0$ has no integer solutions.

Take n a solution to the answer.
Either is odd or even

Case 1, n is odd:

n is a root so we can sub

$n^2 + n + c = 0$

$n = 2k + 1$, because n is odd

$(2k+1)^2 + 2k + 1 + c = 0$

$4k^2 + 4k + 1 + 2k + 1 + c = 0$

$4k^2 + 6k + 2 + c = 0$

$2(2k^2 + 3k + 1) + c = 0$

$-2(2k^2 + 3k + 1) = c$

$\therefore$ c is a multiple of 2 meaning it's even which is a contradiction

Case 2, n is even:

Again n is a root so substitute

$n^2 + n + c = 0$

$n = 2k$, because n is even

$(2k)^2 + 2k + c = 0$

$4k^2 + 2k + c = 0$

$2(2k^2 + k) + c = 0$

$-2(2k^2 + k) = c$

$\therefore$ c is a multiple of 2 meaning it's even which is a contradiction

In both cases where the solution is odd or even c has to be even meaning it can't be odd.

4. $a \mid bc \Rightarrow bc = a \cdot k$

$\gcd(a,b) = 1$, therefore $a \nmid b$

For $bc = a$ to hold up whe $a \nmid b$ must mean that $a \mid c$

Bezout theorem says $\exists x, y \in \mathbb{Z}$ such that $xa + yb = 1$

$\Rightarrow c \cdot xa + c \cdot yb = c$

$\Rightarrow c \cdot xa + y \cdot a \cdot k = c$

$\Rightarrow a(c \cdot x + y \cdot k) = c$

5. Prove $F_m F_n + F_{m+1} F_{n+1} = F_{m+n+1}$ using induction

Base case: $n = 1$

$\Rightarrow F_{m+1+1} = F_{m+2} = F_{m+1} + F_m = F_m \cdot F_{(1)} + F_{m+1} \cdot F_{(2)+1}$

$\Rightarrow F_{m+1} + F_m = F_m \cdot 1 + F_{m+1} \cdot 1$

$n = 2$

$F_{m+2+1} = F_{m+3} = F_{m+2} + F_{m+1} = (F_{m+1} + F_m) + F_{m+1}$

$= F_m \cdot F_2 + F_{m+1} \cdot F_3$

$\Rightarrow 2F_{m+1} + F_m = 1 \cdot F_m + 2 \cdot F_3$

Induction Hypothesis: assume $n = p$ for all $m \in \mathbb{Z}^+$

$F_{m+p+1} = F_m \cdot F_p + F_{m+1} \cdot F_{p+1}$

$F_{m+p} = F_m \cdot F_{p-1} + F_{m+1} \cdot F_p$

Induction Step: Show $n = p+1$ for all $m \in \mathbb{Z}^+$

$F_{m+p+2} = \underline{F_m} F_{p+1} + \underline{F_{m+1}} \cdot F_{p+2}$

substitute

1) $F_{m+p+2} = F_{m+p+1} + F_{m+p} = \underline{F_m} \cdot F_p + \underline{F_{m+1}} \cdot F_{p+1} +$
$+ \underline{F_m} \cdot F_{p-1} + \underline{F_{m+1}} \cdot F_p$

$\Rightarrow F_m (F_p + F_{p-1}) + F_{m+1} (F_{p+1} + F_p)$

$* F_p + F_{p-1} = F_{p+1}$ $* F_{p+1} + F_p = F_{p+2}$

By Definition of Fibinaci series

$\Rightarrow F_m \cdot F_{p+1} + F_{m+1} \cdot F_{p+2} =$

$= F_m \cdot F_{p+1} + F_{m+1} \cdot F_{p+2}$

$\therefore$ By induction this series holds true