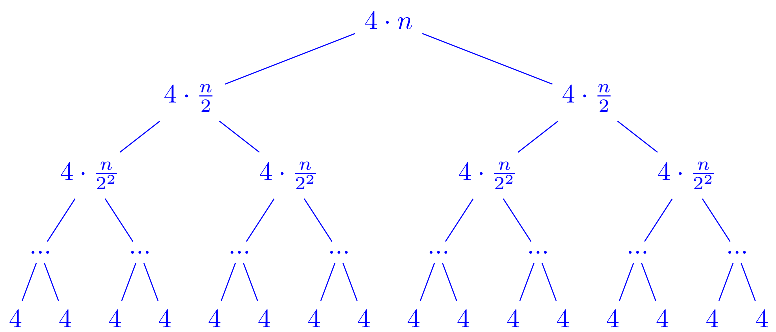
Recitation Problems - Com S 311

Week of Feb 19^{th} - Feb 24^{th}

1. Given the following recurrence:

$$T(n) = \begin{cases} 4 & \text{if } n \leq 1 \\ 2 \cdot T(\frac{n}{2}) + 4n & \text{if } n > 1 \end{cases}$$

Solve this using the recurrence tree method. Determine the cost of each level, the total number of levels, and the total cost by adding the cost of all levels. Give the time using Big-O notation.



Level	Problem Size	# Nodes	Work / Node	Level Total
1	$\frac{n}{2^0}$	$2^0 = 1$	$4n$	$4n$
2	$\frac{n}{2^1}$	2^1	$4(\frac{n}{2^1})$	$4n$
...	...	...	...	...
i	$\frac{n}{2^{i-1}}$	2^{i-1}	$4(\frac{n}{2^{i-1}})$	$4n$

Assuming at level l when problem size $= 1$, then

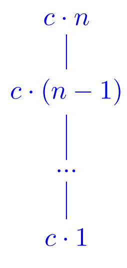
$$\frac{n}{2^{l-1}} = 1 \Rightarrow l = \log(n) + 1$$

So for all i from 1 to $\log(n)$, the work done at level i is $4n$. At level $\log(n) + 1$, the work done at each node is 4, and there are $2^{\log(n)+1-1} = n$ nodes, so total work at level $\log(n) + 1$ is $4n$.

$$\text{Total cost} = \left[\sum_{i=1}^{\log(n)} 4n \right] + 4n$$
$$\in O(n \log n)$$

2. Do the same as in Problem 1 for determining the running time of selection sort, whose recurrence is:

$$S(n) = \begin{cases} 1 & \text{if } n \leq 1 \\ 1 \cdot S(n-1) + O(n) & \text{if } n > 1 \end{cases}$$



In this case, the number of levels is n.

$$\begin{aligned} T(n) &= \sum_{i=1}^{n} c \cdot (i) \\ &= \frac{c \cdot n(n+1)}{2} \\ &\in O(n^2) \end{aligned}$$

3. Consider the following recurrence equation:

$$T(n) = \begin{cases} d, \text{ where } d > 0 & \text{if } n \leq 1 \\ 4 \cdot T(\lfloor \frac{n}{2} \rfloor) + f(n) & \text{if } n > 1 \end{cases}$$

For each of the values of $f(n)$ given below, solve this recurrence equation using the Master Theorem.

(a) $f(n) = n^{\frac{3}{2}}$
(b) $f(n) = n^2$
(c) $f(n) = n^{\frac{5}{2}}$

Provide clear explanations of why a particular case of the Master Theorem applies.

(a) $f(n) = n^{\frac{3}{2}}$, $a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^{\log_2 4} = n^2$.
$f(n) = n^{\frac{3}{2}} \in O(n^{2-\varepsilon})$ for $\varepsilon = \frac{1}{2}$
So case 1 of Master Theorem applies and $T(n) = \Theta(n^2)$.

(b) $f(n) = n^2$, $a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^{\log_2 4} = n^2$.
$f(n) = n^2 \in \Theta(n^{\log_b a} \log^k n)$ for $k = 0$.
So case 2 of Master Theorem applies and $T(n) = \Theta(n^2 \log n)$.

(c) $f(n) = n^{\frac{5}{2}}$, $a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^{\log_2 4} = n^2$.
$f(n) = n^{\frac{5}{2}} \in \Omega(n^{2+\varepsilon})$ for $\varepsilon = \frac{1}{2}$

Regularity condition check:

$$\begin{aligned} a \cdot f(\lfloor \frac{n}{b} \rfloor) &= 4 \cdot (\lfloor \frac{n}{2} \rfloor^{\frac{5}{2}}) \\ &\leq 4 \cdot (\frac{n}{2})^{\frac{5}{2}} \\ &= 4 \cdot \frac{n^{\frac{5}{2}}}{2^{\frac{5}{2}}} \\ &= \frac{4}{2^{\frac{5}{2}}} \cdot n^{\frac{5}{2}} \\ &= c \cdot n^{\frac{5}{2}} \quad \text{for } c = \frac{4}{2^{\frac{5}{2}}} < 1 \end{aligned}$$

So case 3 of Master Theorem applies and $T(n) = \Theta(n^{\frac{5}{2}})$

4. Consider the following recurrence equation:

$$T(n) = \begin{cases} 1 & \text{if } n \leq 2 \\ 2 \cdot T(\frac{n}{2}) + \sqrt[4]{n} & \text{if } n > 2 \end{cases}$$

(a) Assuming $n = 2^k$, solve this recurrence equation using the recurrence tree method. Show all the steps in your derivation of $T(n)$.
(b) Solve this recurrence equation using the Master Theorem.

(a)

Level	Problem Size	# Nodes	Work / Node	Level Total
i	$\frac{n}{2^{i-1}}$	2^{i-1}	$\sqrt[4]{\frac{n}{2^{i-1}}}$	$2^{i-1}(\frac{n}{2^{i-1}})^{\frac{1}{4}}$

Assuming at level l when problem size $= 2$, then

$$\frac{n}{2^{l-1}} = 2 \Rightarrow l = \log(n)$$

So for all i from 1 to $\log(n) - 1$, the work done at level i is $2^{i-1}(\frac{n}{2^{i-1}})^{\frac{1}{4}}$. At level $\log(n)$, the work done at each node is 1, and there are $2^{\log(n)-1} = \frac{n}{2}$ nodes, so total work at level $\log(n)$ is $\frac{n}{2}$.

$$\begin{aligned}
\text{Total cost} &= \left[\sum_{i=1}^{\log(n)-1} 2^{i-1}(\frac{n}{2^{i-1}})^{\frac{1}{4}}\right] + \frac{n}{2} \\
&= n^{\frac{1}{4}}\left[\sum_{i=1}^{\log(n)-1} (2^{i-1})^{\frac{3}{4}}\right] + \frac{n}{2} \\
&= n^{\frac{1}{4}}\left[\frac{(\frac{n}{2})^{\frac{3}{4}} - 1}{2^{\frac{3}{4}} - 1}\right] + \frac{n}{2} \\
&= \frac{1}{2^{\frac{3}{4}} - 1}\left[\frac{n}{2^{\frac{3}{4}}} - n^{\frac{1}{4}}\right] + \frac{n}{2} \\
&\le \frac{1}{2^{\frac{3}{4}} - 1}(n + n) + n \\
&= \frac{2n}{2^{\frac{3}{4}} - 1} + \frac{(2^{\frac{3}{4}} - 1)n}{2^{\frac{3}{4}} - 1} \\
&= \frac{2^{\frac{3}{4}} + 1}{2^{\frac{3}{4}} - 1} \cdot n \\
&\in O(n)
\end{aligned}$$

(b) $f(n) = n^{\frac{1}{4}}$, $a = 2$, $b = 2 \Rightarrow n^{\log_b a} = n^{\log_2 2} = n$.
$f(n) = n^{\frac{1}{4}} \in O(n^{1-\varepsilon})$ for $\varepsilon = \frac{3}{4}$
So case 1 of Master Theorem applies and $T(n) = \Theta(n)$.