Recitation Week 9 Extra Problem

April 1, 2024

Problem statement

Byteland has n cities, and m roads between them. The goal is to construct new roads so that there is a route between any two cities. Your task is to find out the minimum number of roads required, and also determine which roads should be built.

Example 1

There are 4 nodes and 2 edges, as shown below on the left. One way to connect them would be adding edge between nodes 1 and 4.

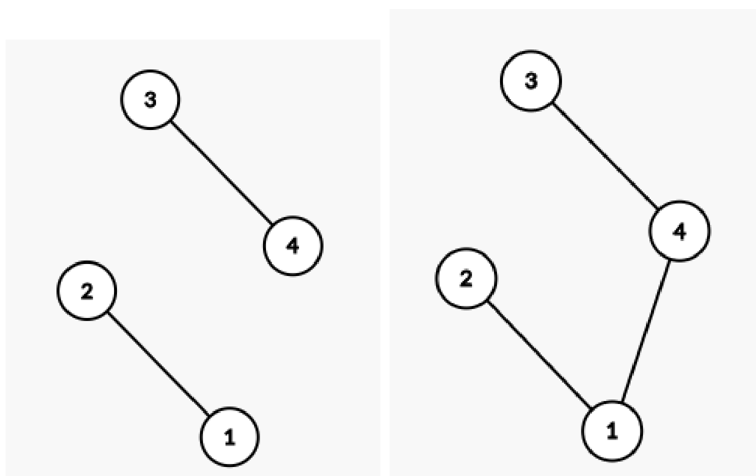

Figure 1: Adding the edge $(4, 1)$

Example 2

There are 7 nodes and 5 edges among them, as shown below on the left. One way to connect them would be adding edges $(2, 7)$ and $(4, 5)$.

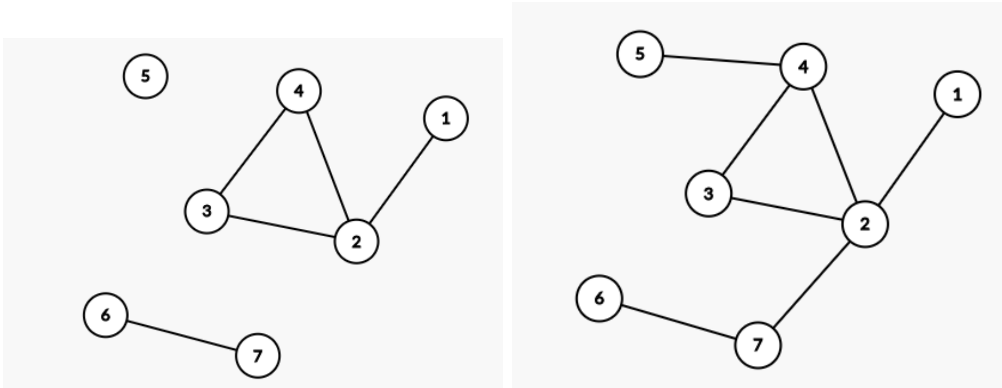

Figure 2: Adding the edges $(4, 5)$ and $(2, 7)$

Solution

Since the roads are bi-directional, adding a single edge between two connected components allows transportation from any node to any other node in those components. Therefore, all we need to do is just link different connected components. Let the number of such components be k. The number of necessary edges to connect all nodes is then exactly $k-1$. To connect one component A to another component B, we can choose arbitrary nodes $u \in A$ and $v \in B$, and add the edge (u, v).

We can find the number of connected components by initiating BFS calls from unexplored nodes and exploring as much as possible. Each time we call BFS from an unexplored node u we discover a new connected component C. To report the edges added to the graph, we can let u be the arbitrary node of A. If we let $U = \{u_1, u_2, \ldots, u_k\}$ be the candidate nodes of k components, the edges added can be $E' = \{(u_1, u_2), (u_2, u_3), \ldots, (u_{k-1}, u_k)\}$ and $|E'| = k - 1$.

```
1  BuildingRoads(G = (V, E))
2      for u ∈ V do
3          u.explored = false
4      Let U be an empty list
5      for u ∈ V do
6          if u.explored == false then
7              BFS(G, u)
8              Add u to the list U
9      Let NewEdges be an empty set
10     for u_i ∈ U \ {u_0} do
11         Add the edge (u_{i-1}, u_i) to the set NewEdges
12     return NewEdges
```

Runtime:

The line 2 for loop does $O(V)$ (or $O(n)$) operations. So does the line 10 for loop, because the size of U is at most $|V|$. The lines 5-8 take $O(V + E)$ time due to BFS: since BFS is only called from unexplored nodes, and each node is explored only once, the running time is linear in terms of $|V|$

and $|E|$. The overall runtime is then also $O(V+E)$ or equivalently, $O(n+m)$.