Stat 330: Module 1 Homework Solution

Show all of your work, and upload this homework to Canvas.

1. A coin is tossed three times, and the sequence of heads and tails is recorded.

   (a) Determine the sample space, Ω.

   (b) List the elements that make up the following events: i. A = exactly two tails, ii. B = at least two tails, iii. C = the last two tosses are heads

   (c) List the elements of the following events: i. $\overline{A}$, ii. $A \cup B$, iii. $A \cap B$, iv. $A \cap C$

   Answer:

   (a) Let H and T stand for the events of head and tail, respectively. The sample space is

   $$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

   (b) i. $A =$ exactly two tails $= \{TTH, THT, HTT\}$
   ii. $B =$ at least two tails $= \{TTH, THT, HTT, TTT\}$
   iii. $C =$ the last two tosses are heads $= \{HHH, THH\}$

   (c) i. $\bar{A} = \{HHH, HHT, HTH, THH, TTT\} =$ Elements with 0,1, or 3 tails
   ii. $A \cup B = \{TTH, THT, HTT, TTT\} =$ B since $A \subset B$
   iii. $A \cap B = \{TTH, THT, HTT\} = A$ since $A \subset B$
   iv. $A \cap C = \emptyset$ since they have no elements in common

2. Suppose that after 10 years of service, 35% of computers have problems with motherboards (MB), 30% have problems with hard drive (HD), and 20% have problems with both MB and HD.

   (a) What is the probability that a 10-year old computer has a problem with MB or HD? **Answer:**

   $$\begin{aligned}\mathbb{P}(MB \cup HD) &= \mathbb{P}(MB) + \mathbb{P}(HD) - \mathbb{P}(MB \cap HD)\\ &= 0.35 + 0.3 - 0.20\\ &= 0.45\end{aligned}$$

   (b) What is the probability that a 10-year old computer still has a fully functioning MB and HD? **Answer:**

   $$\begin{aligned}\mathbb{P}(\overline{MB} \cap \overline{HD}) &= \mathbb{P}(\overline{MB \cup HD})\\ &= 1 - \mathbb{P}(MB \cup HD)\\ &= 1 - 0.45\\ &= 0.55\end{aligned}$$

3. Twelve athletes compete in an archery event at the Olympics.

   (a) How many ways are there to award the Gold, Silver, and Bronze medals to these athletes?

   (b) How many ways are there to award 3 medals if we do not care about the color of the medal?

   (c) If we know the three individuals who got a medal, how many ways are there to distribute the Gold, Silver, and Bronze to these three individuals?

   Answer:

   (a) This is an ordered sample without replacement. There are

   $$P(12, 3) = \frac{12!}{(12-3)!} = 12 \times 11 \times 10 = 1320$$

   ways.

(b) This is an unordered sample without replacement. There are

$$\binom{12}{3} = \frac{12!}{3!(12-3)!} = 220$$

ways.

(c) This is an ordered sample without replacement. There are

$$P(3,3) = \frac{3!}{(3-3)!} = 3! = 6$$

ways.

4. The AccessPlus system at ISU has the following policy for creating a password:

   - Passwords must be exactly 8 characters in length.
   - Passwords must include at least one letter (a-z, A-Z) or supported special character (@, #, $ only). All letters are case-sensitive.
   - Passwords must include at least one number (0-9).
   - Passwords cannot contain spaces or unsupported special characters.

   According to this policy, how many possible AccessPlus passwords are available? Round to the nearest trillion. (Hint: Count up the number of 8 character passwords that could be made, and then subtract off the number that don't meet the requirement above)

   Answer: A way to do this is to count all the passwords we could make with all the useable characters and then subtract off the number of invalid passwords. There are 52 letters (26 + 26), 3 supported characters, and 10 numbers. So we could make 65^8 passwords using those characters. But some of those include passwords like 12345678 which is invalid because it doesn't contain at least on letter or supported special character. Also abcdef$ is invalid because it doesn't contain at least one number. We have to subtract off the all letter/special character passwords and the all number passwords.

   There are 55^8 passwords that don't contain at least one number and 10^8 passwords that don't include at least one letter/special character. So, the number of valid passwords is $65^8 - 55^8 - 10^8 \approx 235$ trillion passwords.

5. Harry Potter's closet contains 12 brooms. 7 brooms are *Comet 260*s, 4 brooms are *Nimbus 2000*s, and 1 broom is a *Firebolt*. Harry, Ron, George and Fred want to sneak out in the middle of the night for a game of Quidditch. They are afraid to turn on the light in case they get caught. Harry reaches into the closet and randomly pulls 4 brooms out at once without looking.

   (a) What is the probability that all 4 chosen brooms are *Comet 260*s?

   (b) What is the probability that Harry pulls out 1 *Comet 260*, 2 *Nimbus 2000*s, and 1 *Firebolt* broom?

   (c) What is the probability that at least 1 of the 4 chosen brooms is a *Comet 260*?

   Answer: First, this is sampling without replacement and order does not matter. $|\Omega|$ = number of ways to select 4 objects from 12 which is $\binom{12}{4} = 495$.

   (a) We need to choose 4 of the 7 *Comet 260*s and 0 from the others. The number of ways to do this is $\binom{7}{4} \cdot \binom{4}{0} \cdot \binom{1}{0} = 35$. So the probability is $\frac{35}{495} = 0.0707$

   (b) This is done in $\binom{7}{1} \cdot \binom{4}{2} \cdot \binom{1}{1} = 42$ ways. So the probability is $\frac{42}{495} = 0.085$.

   (c) You could do $\mathbb{P}(1\textit{Comet 260}) + ... + \mathbb{P}(4\textit{Comet 260s})$ and proceed as above. But, it is quicker to find the compliment and subtract from 1. To get no *Comet 260*s we need to choose 0 from the 7

*Comet 260*s and then 4 brooms from the 5 non *Comet 260*s.

$$\begin{aligned}
\mathbb{P}(\text{at least one C}) &= 1 - \mathbb{P}(\text{no } C) \\
&= 1 - \left[\frac{\binom{7}{0} \cdot \binom{5}{4}}{\binom{12}{4}}\right] \\
&= 1 - \frac{5}{495} \\
&= 0.9899
\end{aligned}$$

6. Suppose you have two urns with poker chips in them. Urn I contains two red chips and four white chips. Urn II contains three red chips and one white chip. You randomly select one chip from urn I and put it into urn II. Then you randomly select a chip from urn II. What is the probability that the chip you select from urn II is white?

   Answer: Let W_i = white chip chosen from urn i, R_i = red chip chosen from urn i
   We have:

$$\begin{aligned}
\mathbb{P}(W_2) &= \mathbb{P}(W_1 \cap W_2) + \mathbb{P}(R_1 \cap W_2) \\
&= \mathbb{P}(W_1)\mathbb{P}(W_2|W_1) + \mathbb{P}(R_1)\mathbb{P}(W_2|R_1) \\
&= \frac{4}{6} \times \frac{2}{5} + \frac{2}{6} \times \frac{1}{5} \\
&= \frac{1}{3}
\end{aligned}$$

7. When dealing with more than two events, we talk about the events being *Mutually Independent*. In general, the independence of n events requires that the probabilities of *all possible* intersections equal the products of the corresponding individual probabilities. For three events A, B, C, this would require **both**:

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \tag{1}$$

and

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B), \quad \mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C), \quad \mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C) \tag{2}$$

to hold.

Suppose that two fair dice (one red and one green) are rolled. Define events:
A: a 1 or 2 shows on the red die
B: a 3, 4, or 5 shows on the green die
C: the sum of the two dice is 4, 11, or 12

Show that these events satisfy equation 1 but not equation 2 and are thus not Mutually Independent.

Answer:

The sample space is the 36 pairs of the form (R_i, G_i). We have $\mathbb{P}(A) = \frac{1}{3}, \mathbb{P}(B) = \frac{1}{2}, \mathbb{P}(C) = \frac{1}{6}$

<u>Equation 1</u>:
$(A \cap B \cap C) = \{(1,3)\} \Rightarrow \mathbb{P}(A \cap B \cap C) = \frac{1}{36}$

$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = (\frac{1}{3})(\frac{1}{2})(\frac{1}{6}) = \frac{1}{36}$. Equation 1 is satisfied.

<u>Equation 2</u>:
$(A \cap B) = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\} \Rightarrow \mathbb{P}(A \cap B) = \frac{1}{6} = \mathbb{P}(A)\mathbb{P}(B)$

$(A \cap C) = \{(1,3),(2,2)\} \Rightarrow \mathbb{P}(A \cap C) = \frac{1}{18} = \mathbb{P}(A)\mathbb{P}(C)$

$(B \cap C) = \{(1,3),(6,5)\} \Rightarrow \mathbb{P}(B \cap C) = \frac{1}{18} \neq \mathbb{P}(B)\mathbb{P}(C) = \frac{1}{12}$

Equation 2 is not satisfied for all pairs and thus the three events are not mutually independent.

8. A diagnostic test has a 95% probability of giving a positive result when given to a person who has a certain disease. It has a 10% probability of giving a (false) positive result when given to a person who doesn't have the disease. It is estimated that 15% of the population suffers from this disease.

Answer: Let P = positive test result, $\overline{P}$ = negative test result, D = has disease, $\overline{D}$ = doesn't have disease

Given:

$$\mathbb{P}(P|D) = 0.95 \to \mathbb{P}(\overline{P}|D) = 1 - 0.95 = 0.05$$
$$\mathbb{P}(P|\overline{D}) = 0.10 \to \mathbb{P}(\overline{P}|\overline{D}) = 1 - 0.10 = 0.90$$
$$\mathbb{P}(D) = 0.15 \to \mathbb{P}(\overline{D}) = 1 - 0.15 = 0.85$$

(a) What is the probability that a test result is positive? **Answer:**

$$\begin{aligned}\mathbb{P}(P) &= \mathbb{P}(P \cap D) + \mathbb{P}(P \cap \overline{D}) \\ &= \mathbb{P}(P|D)\mathbb{P}(D) + \mathbb{P}(P|\overline{D})\mathbb{P}(\overline{D}) \\ &= (0.95)(0.15) + (0.10)(0.85) \\ &= 0.2275\end{aligned}$$

(b) A person receives a positive test result. What is the probability that this person actually has the disease?

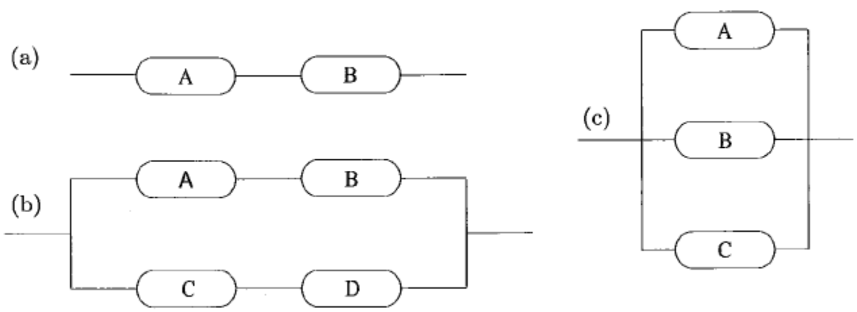
Answer:

$$\begin{aligned}
\mathbb{P}(D|P) &= \frac{\mathbb{P}(P \cap D)}{\mathbb{P}(P)} \\
&= \frac{\mathbb{P}(P|D)\mathbb{P}(D)}{\mathbb{P}(P)} \\
&= \frac{(0.95)(0.15)}{(0.95)(0.15) + (0.10)(0.85)} \\
&= \frac{0.1425}{0.2275} \\
&= 0.6264
\end{aligned}$$

(c) A person receives a positive test result. What is the probability that this person doesn't actually have the disease? **Answer:**

$$\begin{aligned}
\mathbb{P}(\overline{D}|P) &= \frac{\mathbb{P}(P \cap \overline{D})}{\mathbb{P}(P)} \\
&= \frac{\mathbb{P}(P|\overline{D})\mathbb{P}(\overline{D})}{\mathbb{P}(P)} \\
&= \frac{(0.10)(0.85)}{(0.95)(0.15) + (0.10)(0.85)} \\
&= \frac{0.085}{0.2275} \\
&= 0.3736
\end{aligned}$$

9. Calculate the reliability of each system show below, if components A, B, C, and D function properly (independently of each other) with probabilities 0.95, 0.9, 0.8, and 0.7 respectively.



Answer:

(a) $(0.95)(0.9) = 0.855$

(b) $1 - [(1 - (0.95)(0.9))(1 - (0.8)(0.7))] = 1 - [(1 - 0.855)(1 - 0.56)] = 0.9362$

(c) $1 - [(1 - 0.95)(1 - 0.9)(1 - 0.8)] = 1 - 0.001 = 0.999$