Stat 330: Module 2 Homework Solution

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1. A concrete beam may fail either by shear (S) or flexure (F). Suppose that three failed beams are randomly selected and the type of failure is determined for each one. Let X = the number of beams among the three selected that failed by shear. List each outcome in the sample space for the original random experiment of selecting three failed beams and denoting the type of failure along with each outcomes associated value of X.

   Answer: X takes the values 0, 1, 2 or 3.

Outcome	$\{X = x\}$
FFF	0
SFF, FSF, FFS	1
SSF, SFS, FSS	2
SSS	3

2. A box contains nine marbles. Four of them are red, three of them are green, and two of them are blue. You reach in and choose three at random without replacement. Define a random variable X as: X = the number of red marbles selected.

   (a) What are the possible values X can take on? (i.e. give $Im(X)$)

   (b) Find $\mathbb{P}(X = x)$ for all x in $Im(X)$.

   (c) Make a table for the probability distribution of X as shown in lecture. (Leave probabilities as fractions)

   Answer:

   (a) $Im(X) = \{0, 1, 2, 3\}$

   (b) $\mathbb{P}(X = 0) = \frac{\binom{4}{0}\binom{5}{3}}{\binom{9}{3}} = 0.119$; $\mathbb{P}(X = 1) = \frac{\binom{4}{1}\binom{5}{2}}{\binom{9}{3}} = 0.476$

   $\mathbb{P}(X = 2) = \frac{\binom{4}{2}\binom{5}{1}}{\binom{9}{3}} = 0.357$; $\mathbb{P}(X = 3) = \frac{\binom{4}{3}\binom{5}{0}}{\binom{9}{3}} = 0.048$

   (c) Table for the probability distribution of X:

x	0	1	2	3
$P_X(x)$	0.119	0.476	0.357	0.048

3. A mail-order computer business has six telephone lines. Let X denote the number of lines in use at a specified time. Suppose the pmf of X is as given in the following table.

x	0	1	2	3	4	5	6
$P_X(x)$	0.1	0.15	0.20	0.25	0.20	0.06	0.04

   (a) Calculate the probability that at most three lines are in use.

   (b) Calculate the probability that between two and five lines (inclusive) are in use.

   (c) Find the cumulative distribution function of X.

   (d) Find the expected value of X.

   (e) Find the variance of X.

   (f) If four or more phone lines are in use at the same time, a supervisor must be present. Find the expected number of supervisors present.

Answer:

(a) $\mathbb{P}(X \leq 3) = 0.7$

(b) $\mathbb{P}(2 \leq X \leq 5) = 0.71$

(c) The cumulative distribution function of X is

x	0	1	2	3	4	5	6
$F_X(x)$	0.1	0.25	0.45	0.7	0.9	0.96	1

(d) $\mathbb{E}(X) = \sum_{x=0}^{6} xP_X(x) = 2.64$

(e) $\mathbb{E}(X^2) = \sum_{x=0}^{6} x^2 P_X(x) = 9.34$, so $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 9.34 - (2.64^2) = 2.37$

(f) Let Y= number of supervisors present
$Im(Y) = \{0, 1\}$
$\mathbb{P}(Y = 0) = \mathbb{P}(X < 4) = 0.7$
$\mathbb{P}(Y = 1) = \mathbb{P}(X \geq 4) = 0.3$
$\mathbb{E}(Y) = 0(0.7) + 1(0.3) = 0.3$

4. Let X be a random variable which equals 1 if the price of bitcoin rises tomorrow, and 0 if otherwise. Suppose the probability for the price of the bitcoin to rise is 0.51.

(a) Write down the distribution of X.

(b) Compute the expected value of X.

(c) Compute the variance of X.

(d) Compute $\mathbb{E}(X^{20})$.

Answer:

(a) $X \sim Bernoulli(0.51)$

x	0	1
$P_X(x)$	0.49	0.51

(b) $\mathbb{E}(X) = 0(0.49) + 1(0.51) = 0.51$

(c) $\mathbb{E}(X^2) = 0.51$ so $Var(X) = 0.51 - (0.51^2) = 0.2499$

(d) $\mathbb{E}(X^{20}) = 0^{20}(0.49) + 1^{20}(0.51) = 0.51$

5. Determine whether each of the following random variables has a binomial distribution. If it does, identify the values of the parameters n and p (if possible).

(a) X = the number of fours that turn up in 10 rolls of a fair die

(b) X = the number of multiple-choice questions a student gets right on a 10-question test, when each question has four choices and the student is completely guessing

(c) X = the same as in the last part, but half the questions have four choices and the other half have three

(d) X = the number of apples, out of a random sample of 15, that weigh more than 150 g

Answer:

(a) Yes, $X \sim Bin(n = 10, p = \frac{1}{6})$

(b) Yes, Assume the students responds all of the questions, and each option per each question is equally likely to be marked by the student. $X \sim Bin(n = 10, p = \frac{1}{4})$

(c) No.

(d) Yes, $X \sim Bin(n = 15, p = \frac{1}{2})$

6. Some iPhones produced within a time period have faulty battery issues. Thirty percent of such iPhones will need battery service while under warranty. A family owns 5 iPhones produced within the time frame of potential faulty production.

   (a) What is the probability that exactly two iPhones will end up being serviced under warranty?

   (b) What is the probability that at least two iPhones will end up being serviced under warranty?

   (c) What is the expected number of iPhones in the family that require service?

   (d) What is the standard deviation of the number of iPhones in the family that requires service?

   Answer: Let X be the number of iphones (out of 5) that need service. Therefore, $X \sim Bin(5, 0.3)$

   (a) $\mathbb{P}(X = 2) = \binom{5}{2} 0.3^2 0.7^3 = 0.3087$

   (b) $\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X \leq 1) = 1 - 0.5282 = 0.4718$ Using Bin CDF table

   (c) $\mathbb{E}(X) = np = 5(0.3) = 1.5$

   (d) $Var(X) = np(1 - p) = 5(0.3)(0.7) = 1.05$ so std. Dev.$=\sqrt{Var(X)} = \sqrt{1.05} = 1.025$

7. Bit transmission errors between computers sometimes occur, where one computer sends a 0 but the other computer receives a 1 (or vice versa). Because of this, the computer sending a message repeats each bit three times, so a 0 is sent as 000 and a 1 as 111. The receiving computer "decodes" each triplet by majority rule: Whichever number, 0 or 1, appears more often in a triplet is declared to be the intended bit. For example, both 000 and 100 are decoded as 0, while 101 and 011 are decoded as 1. Suppose that 6% of bits are switched (0 to 1, or 1 to 0) during transmission between two particular computers, and that these errors occur independently during transmission.

   (a) Find the probability that a triplet is decoded incorrectly by the receiving computer.

   (b) How does your answer to part (a) change if each bit is repeated five times (instead of three)?

   (c) Imagine a 25 kilobit message (i.e., one requiring 25,000 bits to be sent). What is the expected number of errors if there is no bit repetition implemented? If each bit is repeated three times?

   Answer: Let X be the number of bit switches in a triplet $X \sim Bin(3, 0.06)$.

   (a) $\mathbb{P}(\text{Receive Incorrect}) = \mathbb{P}(X \geq 2) = 0.0104$

   (b) Now $X \sim Bin(5, 0.06)$
   $\mathbb{P}(\text{Receive Incorrect}) = P(X \geq 3) = 0.00197$

   (c) Let Y=number of errors with no repetition $Y \sim Bin(25000, 0.06)$
   $\mathbb{E}(Y) = 25000(0.06) = 1500$
   with triplets $Y \sim Bin(25000, 0.0104)$
   $\mathbb{E}(Y) = 25000(0.0104) = 260$

8. An NBA team has a 60% chance to win over any opponent teams. Let X the the total number of games played until (and including) the first loss since the start of the season.

   (a) What is the probability that exactly 2 games are played until (and including) the first loss since the start of the season?

   (b) What is the probability that the winning streak is at least 2 games since the start of the season? (Notice that the winning streak is one less than the number of games played until the first loss.)

   (c) What is the expected number of the winning streak at the start of the season?

   (d) What is the standard deviation of the winning streak at the start of the season?

   Answer: Let X be the number of games played until first loss $X \sim Geo(0.4)$ $\quad 0.4 = \mathbb{P}(\text{loss game})$.

(a) $\mathbb{P}(X = 2) = 0.6 \times 0.4 = 0.24$

(b) $\mathbb{P}(X \geq 3) = 1 - \mathbb{P}(X \leq 2) = 1 - (1 - 0.6^2) = 0.6^2 = 0.36$

(c) $\mathbb{E}(X - 1) = \mathbb{E}(X) - 1 = \frac{1}{0.4} - 1 = 1.5$

(d) $Var(X - 1) = Var(X) = \frac{0.6}{0.4^2} = 3.75$ so std. dev.$=\sqrt{3.75} = 1.936$

9. Suppose the number of goals scored in a game by your soccer team follows a Poisson distribution. Based on past records of your team, they score an average of 0.8 goals per game.

(a) Define X as the number of goals scored in a single game by your team. Give the distribution of X and value(s) of any parameter(s).

(b) What is the probability that your team scores less than two goals in a game?

(c) What is the probability that your team scores at least one goal in a game?

(d) What is the expected number of goals scored by you team in the next five games together?

(e) During the season, your team plays several games, where each game is independent and identically distributed. Start at a fresh point, and define a new random variable, Y, as the number of games until your team starts scoring goals (i.e. get at least one goal). Give the distribution of Y and value(s) of any parameter(s).

(f) What is the probability that you team only starts scoring goals after the 3$^{\text{rd}}$ game in a sequence?

Answer:

(a) $X \sim Pois(0.8)$

(b) $\mathbb{P}(X < 2) = \mathbb{P}(X \leq 1) = 0.8087$ Using Pois CDF table

(c) $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X = 0) = 1 - 0.4493 = 0.5507$

(d) Let Y= number of goals in 5 games
$Y \sim Pois(5(0.8)) \Rightarrow Y \sim Pois(4)$
$\mathbb{E}(Y) = 4$

(e) Y is the number of games until 1st success (success is scoring at least 1 goal). $\mathbb{P}(success) = 0.5507$ (from part c). $Y \sim Geo(0.5507)$

(f) $\mathbb{P}(Y > 3) = 1 - \mathbb{P}(Y \leq 3) = 1 - (1 - (1 - 0.5507)^3) = 0.4493^3 = 0.0907$

10. Consider the following joint distribution for the weather in two consecutive days. Let X and Y be the random variables for the weather in the first and the second days, with the weather coded as 0 for sunny, 1 for cloudy, and 2 for rainy.

	Y		
X	0	1	2
0	0.3	0.1	0.1
1	0.2	0.1	0
2	0.1	0.1	0

(a) Find the marginal probability mass functions for X and Y.

(b) Calculate the expectation and variance for X and Y.

(c) Calculate the covariance and correlation between X and Y. Are they correlated?

(d) Are the weather in two consecutive days independent?

Answer:

(a) The marginal probabilities are as follows

x	0	1	2
$P_X(x)$	0.5	0.3	0.2

y	0	1	2
$P_Y(y)$	0.6	0.3	0.1

(b) $\mathbb{E}(X) = 0(0.5) + 1(0.3) + 2(0.2) = 0.7$; $\mathbb{E}(X^2) = 0^2(0.5) + 1^2(0.3) + 2^2(0.2) = 1.1$
$Var(X) = 1.1 - (0.7^2) = 0.61$; $\mathbb{E}(Y) = 0.5$; $Var(Y) = 0.45$

(c)

$$\mathbb{E}(XY) = (0)(0)(0.3) + (0)(1)(0.1) + (0)(2)(0.1) + ... + (2)(2)(0) = 0.3$$
$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0.3 - (0.7)(0.5) = -0.05$$
$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-0.05}{\sqrt{(0.61)(0.45)}} = -0.095$$

Yes, they are correlated, but it is very weak.

(d) No, Since $cov(X,Y) \neq 0$ or we have $P_{XY}(2,2) = 0 \neq P_X(2)P_Y(2) = (0.2)(0.1) = 0.02$ for example.

11. Using the joint distribution table given in problem 10, calculate the following probabilities:

(a) $\mathbb{P}(X = Y)$

(b) $\mathbb{P}(X < Y)$

(c) $\mathbb{P}(X > Y)$

(d) $\mathbb{P}(X = 2|Y = 1)$

(e) Probability that the weather is sunny on two consecutive days.

(f) Probability that the weather is cloudy on the first day, and rainy on the second day.

Answer:

(a) $\mathbb{P}(X = Y) = 0.3 + 0.1 + 0 = 0.4$

(b) $\mathbb{P}(X < Y) = 0.1 + 0.1 = 0.2$

(c) $\mathbb{P}(X > Y) = 0.2 + 0.1 + 0.1 = 0.4$

(d) $\mathbb{P}(X = 2|Y = 1) = \frac{\mathbb{P}(X=2,Y=1)}{\mathbb{P}(Y=1)} = \frac{0.1}{0.3} = 0.33$

(e) $\mathbb{P}(X = 0, Y = 0) = 0.3$

(f) $\mathbb{P}(X = 1, Y = 2) = 0$