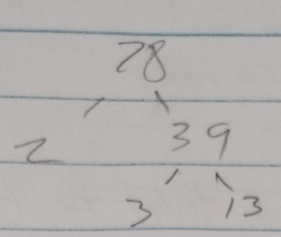
HW6

1. Find $\mu(78) =$



$78 = 2 \cdot 3 \cdot 13$

$= p_1 \cdot p_2 \cdot p_3$ then $r = 3$

so, $\mu(n) = (-1)^3 = -1$

$\therefore \mu(78) = -1$

2. $\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 \cdot p_2 \cdot \ldots p_r \\ 0 & \text{otherwise} \end{cases}$

$\therefore$ If there exists a duplicate p, $\mu(n)$ will be 0

$\mu(n) \neq 0$, $\mu(n+1) \neq 0$, $\mu(n+2) \neq 0$, $\mu(n+3) \neq 0$ means that n, $n+1$, $n+2$, and $n+3$ has no duplicate primes

$\underset{\text{nodiss}}{78}$ $\underset{\text{Prime}}{79}$ $\underset{\substack{8 \\ 4=2\cdot 2}}{80}$

$\underbrace{n \quad n+1 \quad n+2 \quad n+3}$

This ensures that one of these numbers will be a factor of 4

If a number is a factor of 4 it's $\mu(n) = 0$ because $4 = 2^2$, duplicate primes

$\therefore$ Either $\mu(n)$, $\mu(n+1)$, $\mu(n+2)$, or $\mu(n+3)$ has to equal 0

3. Prove that the product of two multiplicative arithmetic functions is multiplicative.

arthemetic function means that the function is defined for all positive integers

It is multiplicative if $f(mn) = f(m)f(n)$

Prove $fg(mn) = f(mn) \cdot g(mn)$

Since f & g are multiplicative $\Rightarrow$

$f(mn) = f(m) \cdot f(n)$ $\quad$ $g(mn) = g(m) \cdot g(n)$

$$f(mn) \cdot g(mn) = f(m)g(m) \cdot f(n)g(n)$$

$$= fg(m)\, fg(n)$$

4. $n = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot p_3^{\alpha_3} \cdots p_k^{\alpha_k}$

a) $F(n) = \sum_{d|n} \mu(d)\varphi(d)$

First, find an expression for $F(p_i^{\alpha_i})$, where $i \in \{1, 2, \ldots, k\}$

Let f be a multiplicative function. Then, using theorem 7.8: $F(n) = \sum_{d|n} f(d)$ is also multiplicative

Show that, $F(n) = \prod_{i=1}^{k} (2 - p_i)$

$F(p_i^{\alpha_i}) = \sum_{d|p_i^{\alpha_i}} \mu(d)\varphi(d)$

only divisor is 1 & $p_i^{\alpha_i}$ so,

$\mu(1)\cdot\varphi(1) + \mu(p_i^{\alpha_i})\varphi(p_i^{\alpha_i})$

$= 1 \cdot 1 + (-1)(p_i - 1)$

$= 1 - 1(p_i - 1) = 1 - p_i + 1 = 2 - p_i$

$F(n) = F(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_k^{\alpha_k})$

need this function to be multiplicative so we can break it up and use our generalization

well if $f(d)$ is multiplicative $\longrightarrow$ then $F(n)$ is

$f(d) = \mu(d) \cdot \varphi(d)$

multiplicative functions

From question #3 we know 2 multiplicative functions multiplied is also multiplicative

$\therefore$ $F(n)$ is multiplicative by theorem 7.8

$\therefore F(n) = F(p_1^{\alpha_1}) \cdot F(p_2^{\alpha_2}) \cdot \ldots \cdot F(p_k^{\alpha_k})$

$\therefore F(n) = \prod_{i=1}^{k} (2 - p_i)$

b) Prove that

$$\sum_{d|n} \mu^2(d) = 2^{\omega(n)}$$

NTS $\sum_{d|n} \mu^2(d)$ is multiplicative and further more each iteration multiplies by 2

$$\prod_{i=1}^{k} 2 = 2^{\omega(n)}$$

$$\sum_{d|n} \mu(d) \cdot \mu(d)$$

$$\sum_{d|n} \mu(d) \cdot \sum_{d|n} \mu(d)$$

$$F(p^\alpha) = \sum_{d|p^\alpha} \mu^2(d) = \mu^2(1) + \mu^2(p) + \mu^2(p^2) + \dots + \mu^2(p^\alpha)$$

$$= 1 + 1 + 0 + 0 + 0$$

$$= 2$$

$$n = \prod p_i^{\alpha_i}$$

$$\cancel{\sum_n \mu^2(n)} \qquad n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

$$F(n) = F\left(\prod^{k} p_i^{\alpha_i}\right)$$

$$= \prod \underbrace{F(p_i^{\alpha_i})}_{2}$$

$$= \prod_{\omega(n)} 2$$

$$= 2^{\omega(n)}$$

If $F(n) = \sum_{d|n} f(d)$, $n \in \mathbb{Z}^+$

then $f(n) = \sum_{d|n} \mu(d) F(\frac{n}{d})$, $F(n) = 2^{w(n)}$ from earlier

$\therefore \mu^2(n) = \sum_{d|n} \mu(d) \cdot 2^{w(\frac{n}{d})}$ ← Illy m

From Möbius inversion formula

c) $F(n) = \sum_{d|n} \mu^2(d) \phi(d)$

Prove $F(n) = \begin{cases} 1 ; \text{ if } n=1 \\ \prod_{i=1}^{k} p_i ; \text{ otherwise} \end{cases}$

$F(p^q) = \sum_{d|p^q} \mu^2(d) \phi(d) = \mu^2(1)\phi(1) + \mu^2(p)\phi(p) + \mu^2(p^2) + \ldots$ $\mu^2(p^q)$, $\phi(p^q)$

$= 1 \cdot 1 + 1(p_i - 1)$

$= 1 + p - 1$

$= p$

$F(n) = F(\prod_{i=1}^{k} p_i^{\alpha_i}) = \prod_{i=1}^{k} F(p_i^{\alpha_i}) = \prod_{i=1}^{k} p_i$ because F multiplicative

↑ multiplicative because f(n) is multiplicative.

$\mu^2(d) \cdot \phi(d)$

Both multiplicative

$\therefore F(n)$ multiplicative

$f(n) = \mu^2(d) \cdot \phi(d)$, also $f(n) = \sum_{d|n} \mu(d) \cdot F(\frac{n}{d})$

$\Rightarrow \mu^2 \cdot \phi(d) = \sum_{d|n} \mu(d) \cdot F(\frac{n}{d})$ from Möbius inversion formula

$\Rightarrow \mu^2 = \frac{1}{\phi(d)} \sum_{d|n} \mu(d) \cdot F(\frac{n}{d})$