Chapter 2

Integer Representations and Operations.

We know that

$$257 = 2\times 10^2 + 5\times 10^1 + 7\times 10^0$$

This is decimal notation.

We may generalize this positional number system.

Theorem 2.1: Let b be a positive integer with $b \geq 2$. Then, every $n \in \mathbb{Z}^+$ can be written uniquely in the form

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b_1 + a_0$$

where, k is a nonnegative integer, $a_j \in \mathbb{Z}$ with $0 \leq a_j \leq b-1$ for all j and $a_k \neq 0$.

Proof: First divide n by b to obtain, by division algorithm,

$$n = bq_0 + a_0 \; ; \quad 0 \leq a_0 \leq b-1$$

If $q_0 \neq 0$, divide q_0 by b to obtain

$$q_0 = bq_1 + a_1 \; ; \quad 0 \leq a_1 \leq b-1$$

Continue this way to get

$$q_1 = bq_2 + a_2 \; ; \; 0 \le a_2 \le b-1$$

$$q_2 = bq_3 + a_3 \; ; \; 0 \le a_3 \le b-1$$

$$\vdots$$

$$q_{k-2} = bq_{k-1} + a_{k-1} \; ; \; 0 \le a_{k-1} \le b-1$$

$$q_{k-1} = b\underbrace{q_k}_{=0} + a_k \; ; \; 0 \le a_k \le b-1$$

The last step occurs when $q_k = 0$. This should happen because

$$n > q_0 > q_1 > q_2 > \cdots \geq 0.$$

(You may justify these inequalities from $q_{i-1} = bq_i + a_i$.)

From the above equations, we get

$$n = bq_0 + a_0$$

$$= b(bq_1 + a_1) + a_0 = b^2 q_1 + ba_1 + a_0$$

$$= b^2(bq_2 + a_2) + ba_1 + a_0$$

$$= b^3 q_2 + b^2 a_2 + ba_1 + a_0$$

$$\vdots$$

$$= b^k q_{k-1} + b^{k-1} a_{k-1} + \cdots + b a_1 + a_0$$

$$= b^k a_k + b^{k-1} a_{k-1} + \cdots + b a_1 + a_0$$

$$= a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$$

with $0 \leq a_j \leq b-1 \;\; \forall j$ and $a_k \neq 0$

To prove the uniqueness, assume

$$n = c_k b^k + c_{k-1} b^{k-1} + \cdots + c_1 b + c_0$$

is another representation. Then,

$$a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0 = c_k b^k + c_{k-1} b^{k-1} + \cdots + c_1 b + c_0$$

$$\Rightarrow (a_k - c_k) b^k + (a_{k-1} - c_{k-1}) b^{k-1} + \cdots + (a_1 - c_1) b + (a_0 - c_0) = 0$$

If the two representations are different, then $a_i \neq c_i$ for at least one i.

Let j be the smallest such i.

Then, $a_i - c_i = 0 \;\; \forall \; i < j$

$$\therefore (a_k - c_k) b^k + (a_{k-1} - c_{k-1}) b^{k-1} + \cdots + (a_j - c_j) b^j = 0$$

Divide by b^j to get

$$(a_k - c_k) b^{k-j} + (a_{k-1} - c_{k-1}) b^{k-1-j} + \cdots + (a_{j+1} - c_{j+1}) b$$

$+(a_j - c_j) = 0$

It follows that $b \mid a_j - c_j$ (because the above equation can be written in the form $a_j - c_j = b \times (\text{an integer})$)

But $0 \leq a_j, c_j \leq b-1$.

Hence, $a_j - c_j = 0$, a contradiction to the assumption that $a_j \neq c_j$.

Hence, $a_i = c_i \ \forall \ i = 0, 1, 2, \ldots, k$ and the uniqueness follow.

Hence the proof.

In the theorem, b is called the base (or radix) of the expansion.

When $b = 2$, it is called the binary expansion. When $b = 10$, it is called the decimal expansion.

$a_0, a_1, a_2, \ldots$ are called the digits of the expansion.

ex: For $b = 2$,

$$25 = 1 \times 2^4 + 1 \times 2^3 + 1$$

$$\therefore (25)_{10} = (11001)_2$$