MA 350 Number Theory – Spring 2024

Homework 6

Due: April 19, 2024

Submit your written work in Canvas as a single PDF file, and be sure to show your work. Answers without accompanying work will receive zero credit.

1. (5 points) Find $\mu(78)$.
2. (5 points) Explain why it is not possible to exist a positive integer n such that
$$\mu(n) \neq 0, \mu(n+1) \neq 0, \mu(n+2) \neq 0 \text{ and } \mu(n+3) \neq 0.$$
3. (5 points) Prove that a product of two multiplicative arithmetic functions is multiplicative.
4. (30 points) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of n.
   (a) Let F be the arithmetic function defined by
   $$F(n) = \sum_{d|n} \mu(d)\varphi(d)$$
   First, find an expression for $F\left(p_i^{\alpha_i}\right)$, where $i \in \{1, 2, \cdots, k\}$, and then using Theorem 7.8, show that
   $$F(n) = \prod_{i=1}^{k} (2 - p_i).$$
   (b) By using a suitable function instead of φ used in part (a) above, prove that
   $$\sum_{d|n} \mu^2(d) = 2^{\omega(n)}$$
   where $\omega(n)$ is the number of distinct prime factors of n (i.e. $\omega(n) = k$)), and then deduce, using the Möbius inversion formula, that
   $$\mu^2(n) = \sum_{d|n} \mu(d) 2^{\omega\left(\frac{n}{d}\right)}.$$
   Moreover, verify this formula for $n = 12$.
   (c) Let F be the arithmetic function defined by
   $$F(n) = \sum_{d|n} \mu^2(d)\, \varphi(d).$$
   Prove that
   $$F(n) = \begin{cases} 1 & ; \text{ if } n = 1 \\ \prod_{i=1}^{k} p_i & ; \text{otherwise.} \end{cases}$$
   Deduce that
   $$\mu^2(n) = \begin{cases} 1 & ; \text{ if } n = 1 \\ \dfrac{1}{\varphi(n)} \sum_{d|n} \mu(d) F\left(\dfrac{n}{d}\right) & ; \text{otherwise} \end{cases}$$
   and verify this formula for $n = 15$.