The Sum and Number of Divisors

Definition: $\sigma(n)$ = sum of all positive divisors of n

Ex: $\sigma(1) = 1$, $\sigma(2) = 1+2 = 3$, $\sigma(3) = 1+3 = 4$
$\sigma(4) = 1+2+4 = 7$, $\sigma(5) = 1+5 = 6$

Definition: $\tau(n)$ = number of positive divisors of n

Ex: $\tau(1) = 1$, $\tau(2) = 2$, $\tau(3) = 2$, $\tau(4) = 3$
$\tau(5) = 2$, $\tau(6) = 4$, $\tau(7) = 2$, $\tau(8) = 4$

* It is easy to see that

$$\sigma(n) = \sum_{d \mid n} d \quad \text{and} \quad \tau(n) = \sum_{d \mid n} 1$$

Theorem 7.8: Let f be a multiplicative function. Then,

$$F(n) = \sum_{d \mid n} f(d)$$

is also multiplicative.

Proof: Let m and n be relatively prime.
We need to show that $F(mn) = F(m)F(n)$.
We've

$$F(mn) = \sum_{d \mid mn} f(d).$$

By Lemma 3.7, if $d \mid mn$, then d can be written uniquely as $d = d_1 d_2$ where $d_1 \mid m$, $d_2 \mid n$ and $\gcd(d_1, d_2) = 1$.

Moreover, each pair (d_1, d_2) corresponds to a divisor of mn.
Hence,

$$F(mn) = \sum_{d_1 d_2 \mid mn} f(d_1 d_2) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2)$$

$$= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1) f(d_2)$$

$$= \sum_{d_1 \mid m} f(d_1) \sum_{d_2 \mid n} f(d_2)$$

$$= F(m) F(n)$$

<u>Corollary 7.8.1:</u> σ and τ are multiplicative functions.

Proof: Both $f(n) = n$ and $g(n) = 1$ are clearly multiplicative functions. Hence, by Theorem 7.8, the result follows.

Next, we derive formulas for σ and τ.

<u>Lemma 7.1</u>: Let p be prime and $\alpha \in \mathbb{Z}^+$. Then,

$$\sigma(p^n) = 1 + p + p^2 + \cdots + p^\alpha = \frac{p^{\alpha+1} - 1}{p-1}$$

and

$$\tau(p^\alpha) = \alpha + 1$$

Proof: Exercise

Ex: $\sigma(125) = \sigma(5^3) = \dfrac{5^4 - 1}{5-1} = \dfrac{624}{4} = 156$

$\tau(125) = \tau(5^3) = 3 + 1 = 4$

<u>Theorem 7.9</u>: Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$. Then,

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \cdots \frac{p_s^{\alpha_s+1} - 1}{p_s - 1}$$

$$= \prod_{j=1}^{s} \frac{p_j^{\alpha_j+1} - 1}{p_j - 1} \quad \text{and}$$

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_s + 1) = \prod_{j=1}^{s} (\alpha_j + 1)$$

Proof: Exercise

Ex: $\sigma(72) = \sigma(2^3 \cdot 3^2)$

$= \frac{2^4 - 1}{2-1} \cdot \frac{3^3 - 1}{3-1} = \frac{15}{1} \cdot \frac{26}{2} = 195$

$\tau(72) = \tau(2^3 \cdot 3^2) = (3+1)(2+1) = 12$