$e = [\,0\ 0\ 0\ 1\ 0\ 0\ 0\,]_{1\times 7}$

$e(x) = x^3$

$g(x) = x^3 + x + 1$

} single-term poly
can not be divided by $g(x)$

$\Rightarrow$ detectable

$e = [\,0\ 0\ 0\ 0\ 1\ 1\ 1\,]_{1\times 7}$

$e(x) = x^2 + x + 1$

$g(x) = x^3 + x + 1$

} $deg(e(x)) < deg(g(x))$

$\Rightarrow$ $e(x)$ cannot be divided by $g(x)$

$\Rightarrow$ detectable

Error Burst

$$e = [\,0 \cdots 0 \mid \overbrace{?\,? \cdots ?}^{L} \mid 0 \cdots 0\,]_{1\times n}$$

$\uparrow$ $i+L-1$ $\qquad$ $\uparrow$ i

$$e(x) = 1 * x^{i+L-1} + ? * x^{i+L-2} + \cdots + ? * x^{i+1} + 1 * x^{i}$$

$$= x^{i}\left(x^{L-1} + ???+ 1\right)$$

$$FUE(L) = \frac{\text{total \# undetectable error burst of } L}{\text{total \# error burst of } L}$$

$\Downarrow$

- $i: \underline{0, \cdots, n-L}$ $\Rightarrow$ $n-L+1$
- "?": $L-2$ $\Rightarrow$ 2^{L-2}

$\Rightarrow (n-L+1) * 2^{L-2}$

undetectable error bursts of L $\Rightarrow (n-L+1) * 2^{(L-1)-(n-k)-1}$

$$\begin{cases} e(x) = x^i (x^{L-1} + ??? + 1) \\ e(x) = \text{multiple of } g(x) = g(x) \cdot C(x) \end{cases}$$

$$x^i (x^{L-1} + ??? + 1) = \underbrace{(x^{n-k} + \cdots + 1)}_{\text{given}} \cdot C(x)$$

For $C(x)$:

① has x^i as its factor : $i = 0, \cdots, n-L \Rightarrow n-L+1$

$$C(x) = x^i (x^{(L-1)-(n-k)} + \cdots + 1)$$

$$x^{L-1} + ??? + 1 = (x^{n-k} + \cdots + 1)(x^{(L-1)-(n-k)} + \cdots + 1)$$

$\Downarrow$ $((L-1)-(n-k)-1)$ terms

② $\deg(C(x)) = (L-1)-(n-k)$

③ has a constant term

$$FUE(L) = \frac{(n-\cancel{L}+1) * 2^{(L-\cancel{1})-(n-k)-\cancel{1}}}{(n-\cancel{L}+1) * 2^{L-2}}$$

$$= \frac{1}{2^{n-k}}$$

Eg. $g(x) = x^3 + x + 1 \qquad k=4 \qquad n=7$

$$FUE(L=6) = ? = \frac{1}{2^{n-k}} = \frac{1}{8} = \frac{4}{2\cdot 2^{L-2}} = \frac{4}{32}$$

$L=6$: $e(x) = x^i\,(x^5 + ??? + 1)$ $\qquad \underline{i = 0,1}$

$$= g(x)\cdot c(x) = (x^3+x+1)\cdot c(x)$$

$$\Rightarrow c(x) = x^i\,(x^2 + \underbrace{?}_{0/1}x^1 + 1) \quad \Rightarrow \text{total \# } c(x):\ 2\cdot 2^1 = 4$$

$i=0,\ ?=0 \Rightarrow x^0(x^2+1) = \underline{x^2+1} \Rightarrow g(x)\cdot c(x)$

$$= (x^3+x+1)\cdot(x^2+1)$$

$$= \underline{x^5+x^2+x+1} = e(x)$$

$$\Rightarrow e = [0\,1\,0\,0\,1\,1\,1]$$

$i=1,\ ?=1 \Rightarrow x\,(x^2+x+1) = x^3+x^2+x \Rightarrow g(x)\cdot c(x)$

$$= x^6+x^5+x = e(x)$$

$$\Rightarrow e = [1\,1\,0\,0\,0\,1\,0]$$

- $(L-1)-(n-k) > 0$ "long"

$$FUE(L) = \frac{1}{2^{n-k}}$$

- $(L-1)-(n-k) < 0$ "short"

$$\left.\begin{cases} e(x) = x^i\,(x^{L-1} + ?? + 1) \\ g(x) = x^{n-k} + \cdots + 1 \end{cases}\right\} \Rightarrow FUE(L) = 0$$

- $(L-1)-(n-k) = 0$ "special"

$$\begin{cases} e(x) = x^i\,(x^{L-1} + ??+1) = x^i\,(x^{n-k} + ?? + 1) \\ g(x) = x^{n-k} + \cdots + 1 \end{cases}$$

$$\Rightarrow c(x) = \underline{x^i\cdot 1} \Rightarrow \frac{n-\cancel{L}+1}{(n-\cancel{L}+1) * 2^{L-2}} = \frac{1}{2^{L-2}} = FUE$$

$$= \frac{1}{2^{n-k-1}}$$

Error Detection Capabilities

- For Error Bursts of Length L:
  - For error burst starting at bit location i and ending at bit location (i + L - 1)
    - $e(x) = x^{i+L-1} + \dots\dots + x^i = x^i\, d(x)$ where $d(x) = x^{L-1} + \dots\dots + 1$

L

0000**1**???... ... ... ?????**1**000000

d(x)

  - g(x) has degree (n – k)
    - L < (n – k + 1)
      - g(x) cannot divide d(x) because deg(d(x)) < deg(g(x))
      - Can detect all such error bursts
    - L = (n – k + 1)
      - d(x) is divisible by g(x) if and only if d(x) = g(x)
      - Fraction of such error bursts that are undetectable is $(½)^{(n-k-1)}$
    - L > (n – k + 1)
      - Fraction of such error bursts that are undetectable is $(½)^{(n-k)}$

Overall $FUE = \dfrac{2^k - 1}{2^n - 1} < \dfrac{1}{2^{n-k}}$

E.g. $n-k = 32$ $\quad FUE < \dfrac{1}{2^{32}} \approx 2.33 * 10^{-10}$

Shift-Register Circuit

Given $g(x)$, $\deg(g(x)) = n-k$

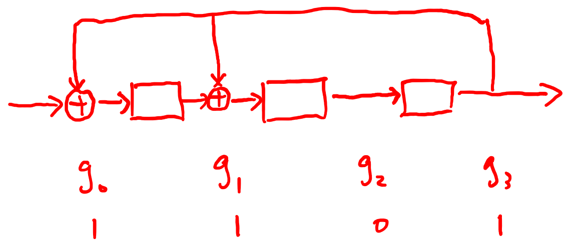
① # registers $= n-k$

② coefficients of $g(x)$

$\Rightarrow$ how to connect, feedback

E.g. $g(x) = x^3 + x + 1$

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ g_3 & g_2 & g_1 & g_0 \end{array}$$



③ Input from left

④ After n clock ticks, check bits in registers