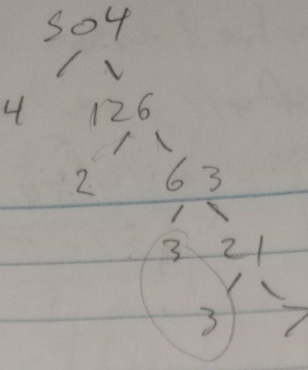
HW5



1. $\varphi(504)$

$= \varphi(4)\varphi(9)\varphi(2)\varphi(7)$

$= \varphi(2^2)\varphi(3^2)\varphi(2)\varphi(7)$ $\quad (2^2-2)(3^2-3)(1)(6)$

$\to \varphi(2^3)\varphi(3^2)\varphi(7)$

$= (2^3-2^2)(3^2-3)\cdot(7-1)$

$= 4\cdot 6\cdot 6 = 144$

2. multiplicative : $\gcd(p,n)=1$

completly : $\gcd(p,n)=d$

$f(7\cdot 14) = \underset{7}{\gcd(7,7)}\cdot \underset{7}{\gcd(7,14)} = \gcd(7,$

$f(5\cdot 25) = \gcd(5,5)\cdot\gcd(5,25) = \gcd(5,125)$

$5\cdot 5 = 25 \neq 5$

Suppose that $f(n)=\gcd(p,n)$ is multiplicative and completly multiplicative

That means $f(n\cdot m) = \gcd(p,n)\cdot\gcd(p,m) = \gcd(p,nm)$ where $n,m\in\mathbb{Z}$ s.t. $p\mid n$ and $p\mid m$

$\therefore \gcd(p,n)=p$ since, p divides n and m

$\gcd(p,m)=p$

$\therefore f(n\cdot m) = p\cdot p = \gcd(p,nm) = p$

since $p\mid n$

$p\cdot p \neq p$

$\therefore$ Contradiction, cannot be completely multiplicative

To prove that it is multiplicative

You have n and m s.t. $\gcd(n,m)=1$

$f(n \cdot m) = \gcd(p,n) \cdot \gcd(p \cdot m) = \gcd(p, n \cdot m)$

Case 1: $p \mid n \Rightarrow p \nmid m$ since $\gcd(n,m)=1$

$f(n \cdot m) = \gcd(p,n) \cdot \gcd(p \cdot m)$
$= p \cdot 1$ (Because m is not a factor of p and p is prime)
$= \gcd(p, nm)$ since $p \mid n \Rightarrow p \mid nm$

Case 2: $p \mid m \Rightarrow p \nmid n$

By symmetry this should hold

Case 3: $p \nmid n$ and $p \nmid m$

$f(n \cdot m) = \gcd(p,n) \cdot \gcd(p,m) = \gcd(p, n \cdot m)$

Since $p \nmid n$ and $p \nmid m$

$f(n \cdot m) = 1 \cdot 1 = \gcd(p, n \cdot m) = 1$
since $p \nmid n$ and $p \nmid m$
p won't divide $n \cdot m$

$\therefore$ By proof of cases all cases hold meaning the function is multiplicative

3. $\phi(n) = 10 \quad p-1 \mid \phi(n)$

$p-1 = 1, 2, 5, 10$ - All factors of 10

$p = 2, 3, \cancel{6}, 11$ - 6 not prime

$11 \nmid \phi(n)$, multiplicity of 11 can be at most 1 in n

$\therefore n = 11$ one solution

$n = 11m, (m, 11) = 1$

$\therefore \phi(n) = \phi(11)\cdot\phi(m) \Rightarrow 10 = 10 \cdot \phi(m) =$

$\Rightarrow \phi(m) = 1$ only if $m = 1$ or $m = 2$

$\Rightarrow n = 22$ is another solution

$n = 2^a \cdot 3^b$, $\phi(2^a \cdot 3^b) = 2^{a-1}(2-1)3^{b-1}(3-1) = 10$

$5 \mid 10$, but $5 \nmid 2^{a-1}(2-1)3^{b-1}(3-1)$

Prime factor Not a factor of 2 or 3 which are prime

$\therefore n = 11$ or 22

4. $\sigma(70)$ & $\tau(60)$ Multiplicative

$\sigma(7\cdot5\cdot2)$ & $\tau(6\cdot5\cdot4)$

$$\sigma(7)\cdot\sigma(5)\cdot\sigma(2) = (1+7)(1+5)(1+2) = 8\cdot6\cdot3 = 144$$

$$\tau(6)\cdot\tau(5)\cdot\tau(4) = (2)(2)(3) = 12$$

5. $\tau(n) = \sum_{d|n} 1$

$4 = \sum_{d|n} 1$

$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

$\tau(n) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$

$4 = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_k+1)$

1×4 1 4 1

2×2 2 2

$\cancel{4\times1}$ 4 1

$n = \cancel{p^2}\, p^3$

$n = pq$

$n = p^3$ or $p\cdot q$

$4 = \underbrace{(\alpha_1+1)}_{4}\underbrace{(\alpha_2+1)}_{1}\cdots\underbrace{(\alpha_k+1)}_{1}$

$\alpha_2 - \alpha_k = 0$

Case 1:

$4 = \alpha_1 + 1$

$\alpha_1 = 3$

By symmetry, it can be generalized that $\alpha_k = 3$, for any such k

Case 2:

$4 = 2\cdot2$

$4 = (\alpha_1+1)(\alpha_2+1)$

α_1 & $\alpha_2 = 1$