HW 3 Stat 330 Joseph Schmidt

1. Suppose a continous random var X has following prob density func

$$f_X(x) = \begin{cases} Kx(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $K \ge 0$ to satisfy

$$1 = \int_0^1 Kx - Kx^2 = K\frac{x^2}{2} - K\frac{x^3}{3}\Big|_0^1$$

$$= \left(K\frac{1}{2} - K\frac{1}{3}\right) - 0 = K\frac{3}{6} - K\frac{2}{6} = K\frac{1}{6} = 1$$

$$K = 6$$

Integrating over whole probability space should equal 1

b) Give the CDF, $F_X(x)$

$$\int_0^x Kx - Kx^2 = K\frac{x^2}{2} - K\frac{x^3}{3} - 0 \overset{K=6}{=} 3x^2 - 2x^3$$

c) Find $P(0.5 \le X \le 1)$ using $f_X(x)$

$$\int_{0.5}^1 6x(1-x) = \int_{0.5}^1 6x - 6x^2 = 3x^2 - 2x^3\Big|_{0.5}^1$$

$$= 1 - (.75 - .25) = 0.5$$

d) Find $P(0 \le X \le .75)$ using $F_X(x)$

$$F_X(0.75) - F_X(0) =$$

$$(1.6875 - .84375) - 0 = .84375$$

e) $$E(X) = \int_0^1 x \cdot 6x(1-x) = \int_0^1 6x^2 - 6x^3 =$$

$$\frac{6x^3}{3} - \frac{6x^4}{4}\Big|_0^1 =$$

$$2x^3 - \frac{3}{2}x^4\Big|_0^1 = 2 - \frac{3}{2} - 0 = \frac{1}{2}$$

f) $$Var(X) = E(X^2) - [E(X)]^2 = \int_0^1 x^2 \cdot 6x(1-x) = \int_0^1 6x^3 - 6x^4$$

$$= \frac{6x^4}{4} - \frac{6x^5}{5}\Big|_0^1 - \left(\frac{1}{2}\right)^2 = \frac{6}{4} - \frac{6}{5} - 0 - \frac{1}{4}$$

$$= \frac{30}{20} - \frac{24}{20} - \frac{5}{20} = \frac{1}{20}$$

2. Suppose a continuous variable X has the following probability density function

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $\int_{-\infty}^{x} \frac{3}{2}x^2 = \frac{3}{2} \cdot \frac{x^3}{3} \Big|_{-\infty}^{x} = \frac{x^3}{2} + \frac{1}{2}$ for $1 \ge x > -1$

0 for $x < -1$

1 for $x > 1$ ← so probability sums to 1

b) $\mathbb{P}(|X - \frac{1}{2}| < \frac{1}{4})$

$= \mathbb{P}(X - \frac{1}{2} < \frac{1}{4})$ or $\mathbb{P}(X - \frac{1}{2} > -\frac{1}{4})$

$\mathbb{P}(X < \frac{3}{4})$ or $\mathbb{P}(X > \frac{1}{4})$

$P(\frac{1}{4} < X < \frac{3}{4})$

$= F(\frac{3}{4}) - F(\frac{1}{4})$

$= .7104 - .5078 = .203125$

c) $P(X \le t) = 0.80$

$F_Y(t) = \frac{t^3}{2} + \frac{1}{2} = .80$

$\frac{t^3}{2} = .3$

$t^3 = .6169$

$t = .8434$

3. $f_X(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & 1 < x \le 1.5 \\ 0 & \text{otherwise} \end{cases}$

a) $F_X(x) = \int_0^x x = \frac{x^2}{2}$ for $0 \le x \le 1$

$\int_1^x 1 = x - 1 + F_X(1)$ for $1 \le x \le 1.5$

$= x - \frac{1}{2}$

0 for $x < 0$ $\quad 1$ for $x \ge 1.5$

b) $P(0.5 \le X \le 1.2)$

$= F(1.2) - F(0.5)$

$= [(1.2 - 1) + 0.5] - .125 = 0.575$

c) $E(X) = \int_0^{1.5} x f_X(x) = \int_1^{1.5} x \cdot 1 + \int_0^1 x \cdot x$

$= \frac{x^2}{2}\Big|_1^{1.5} + \frac{x^3}{3}\Big|_0^1$

$= 1.125 - .5 + 0.333 = .958$

4. 10:00 – 10:30 $\quad X \sim Unif(0, 30)$

a) $E(x) = \frac{a+b}{2} = \frac{0+30}{2} = 15$ mins

b) $F_X(t) = \frac{t-0}{30-0} = \frac{t}{30}$

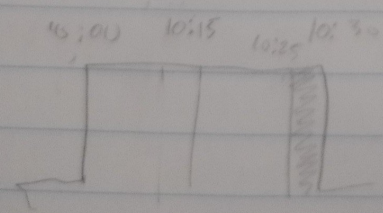
$P(X > 10) = 1 - P(X \le 10) = 1 - F(10) = 1 - \frac{1}{3} = \frac{2}{3}$

c) $P(X > 25) = 1 - P(X \le 25) = 1 - F(25) = 1 - \frac{25}{30} = \frac{5}{30} = \frac{1}{6}$

d) $F(x) = .1$

$\frac{t}{30} = .1 \Rightarrow t = 3$ mins $\quad$ 10:03

e)



$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(X > 25)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}$

A – Probability of bus not arriving ~~by~~ 10:15

B – You have to wait additional 10 mins

5. $X \sim Exp(20)$ $\qquad F_X(t) = \begin{cases} 1-e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

a) $\lambda = 20$

b) $E(X) = \frac{1}{\lambda} = \frac{1}{20} = 3$ minutes

c) $P(X \leq \frac{1}{3}) = 1 - e^{-20t}$ $\qquad$ mins to hrs

$= F_X(\frac{1}{3}) = 1 - e^{-20/3} = .9987$

d) Memoryless property, The distribution of time until the second hit doesn't depend on first hit

$X \sim Gamma(2, 20)$

e) wait for 5 hits $X \sim Gamma(5, 20)$

f) $5 * 3$ minutes $= 15$ minutes

g) $X \sim pois(20)$

$P(X < 5) = 0.0001$

6. $P(Y \leq s + t \mid Y \geq s) = P(Y \leq t)$

$= \dfrac{F_Y(s+t) - F_Y(s)}{e^{-\lambda s}}$

$= \dfrac{(1 - e^{-\lambda(s+t)}) - (1 - e^{-\lambda s})}{e^{-\lambda s}}$

$= (e^{-\lambda s}) - (e^{-\lambda(s+t)})$

$= 1 - e^{-\lambda t} = P(Y \leq t)$

7. a) $X \sim exp(0.2)$ $\quad \frac{1}{\lambda} = 5$ mins $\quad \lambda = 0.2$ customers per min

b) $f_X(x) = \begin{cases} 0.2e^{-0.2x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$F_X(t) = \begin{cases} 1 - e^{-0.2t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

c) $F_X(5) = 1 - e^{-0.2(5)} = .632$

d) $P(X<5 \mid X>2)$ memoryless property

$= P(X<3) = F_X(3) = 1 - e^{-2(.3)} = 0.451$

8. a) $X \sim gamma(25, 6)$ $\quad \lambda$ = customers per hour

b) $E(T) = 25/\lambda = 25/6 \approx 4.17$ hours

c) $P(X \geq 25) = 1 - P(X \leq 24) = .843$

~~6 people per hour~~

~~total 25th 24 people have to come before you~~

~~so probability 24 people come in 5 hrs~~

$\mu = E(x) = -3$
$\sigma = Var(x) = 4$

9. $X \sim N(\mu, \sigma^2)$

a) $P(x \le 2.39) = \frac{2.39-(-3)}{\sqrt{4}} = P(z \le 2.695)$
$= .9964$

b) $P(\frac{x+3}{2} \le -.99) = P(x \le -4.98) = P(z \le \frac{-4.98+3}{2})$
$= P(-.99) = .1611$

c) $P(|x| \ge 2.39)$
$P(-2.39 \le x$ OR $-2.39 \ge x)$

-2.39 $\quad$ 2.39

$1 - .9964 + P(z \le \frac{-2.39+3}{2}) = .6217$

d) $P(|x+3| \ge 2.39) = P(x+3 \ge 2.39)$ OR $P(x+3 \le -2.39)$
$P(x \ge -.61)$ $\quad$ $P(x \le -5.39)$
$1 - P(z \le \frac{-.61+3}{2}) + P(z \le \frac{-5.39+3}{2})$
$1 - P(z \le 1.195) + P(z \le \frac{-2.39}{2})$
$1 - .8849$ $\quad$ $P(z \le -1.195)$
$.1151$

$= .2302$

e) $P(x<5) = P(z < \frac{5+3}{2}) = P(z<2) = .9772$

f) $P(|x|<5) = P(x<5) + P(x \ge -5)$
$= P(-5 < x < 5)$
$= P(\frac{-5+3}{2} < z < \frac{5+3}{2})$
$= P(-1 < z < 4)$
$= \phi(4) - \phi(-1) =$
$= 1 - .1587 = .8413$

g) $P(x<X) = .33$
$P(x<X) = 1 - P(X<x) = 1 - P(z < \frac{X+3}{2}) = .33$
$P(z < \frac{x+3}{2}) = .66$
$z < \frac{x+3}{2} = .43$
$x+3 = .86$
$x = -2.14$

10. $X \sim N(500, 225)$

a) $p(x < 490) = p\left(z < \frac{490-500}{15}\right) = p(z < -.666) = .2546$

b) $p(x > 530) = 1 - p(x \le 530) = 1 - p\left(z \le \frac{530-500}{15}\right) = 1 - p(z \le 2) =$
$= .0228$

c) $p(490 < x < 530) = .9722 - .2546 = .7226$

d) $.9 = p(z < 1.28) \Rightarrow 1.28 = \frac{x-500}{15} \Rightarrow x = 519.2$

e) $\overline{X}_{25} \approx N\left(500, \frac{225}{25}\right) = N(500, 9)$

$P(\overline{X}_{25} \ge 510) = P\left(z < \frac{510-500}{3}\right) = P(z < 3.333) = 1 - .9996 = 0.0004$

11. $f(x) = \frac{4}{3}(1-x^3),\ 0 \le x \le 1$

a) $\frac{4}{3}\int_{0.3}^{0.5} 1 - x^3\, dx = \frac{4}{3}\left[x - \frac{x^4}{4}\right]\Big|_{0.3}^{0.5} = \frac{4}{3}[0.5 - 0.015625] - \frac{4}{3}[0.3 -$
$.6483 - .3973 = .2485$

b) $\overline{X}_n = N\left(\mu, \frac{\sigma^2}{n}\right)$

$\overline{X}_n \approx N\left(\frac{2}{5}, \frac{.284}{36}\right)$

$P(0.3 \le x \le 0.5)$

$P\left(\frac{0.3 - \frac{2}{5}}{0.047} \le z \le \frac{0.5 - \frac{2}{5}}{.047}\right)$

$-1.02 \le z \le 1.0277$

$\phi(1.027) - \phi(-1.027)$

$= .84 - .15 = .69$

$\mu = \int_0^1 x \cdot \frac{4}{3}(1-x^3)dx$

$= \frac{4}{3}\frac{x^2}{2} - \frac{4}{3}\cdot\frac{x^5}{5}\Big|_0^1$

$= \frac{4x^2}{6} - \frac{4x^5}{15}$

$= \frac{4}{3}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{4}{3}\cdot\frac{3}{10} = \frac{4}{10} = \frac{2}{5}$

$\sigma^2 = \int_0^1 x^2 \cdot \frac{4}{3}(1-x^3)dx$

$E(x^2) = \int_0^1 \frac{4}{3}[x^2 - x^5]dx = \frac{4}{3}\left[\frac{x^3}{3} - \frac{x^6}{6}\right]\Big|_0^1$

$= \frac{4}{3}\left[\frac{1}{3} - \frac{1}{6}\right]$

$= \frac{4}{18} = \frac{2}{9}$

$\frac{2}{9} - [E(x)]^2 = \frac{2}{9} - \frac{4}{25}$

$= .284$

c) $\overline{X}_n = N\left(\frac{2}{5}, \frac{.284}{60}\right)$

$P(0.3 \le x \le 0.5)$

$P\left(\frac{0.3 - \frac{2}{5}}{.0688} \le z \le \frac{0.5 - \frac{2}{5}}{.0688}\right)$

$P(-1.4535 \le z \le 1.4535)$

$\phi(1.453) - \phi(-1.4535) = .9236 - .0735 = .8591$

12. $S = N(1200, 1312)$

$P(S < 1200) = P\left(Z \leq \frac{1200 - 1230}{\sqrt{1312}}\right) = \phi(-.83) = .203$

13. $F_Y(t) = P(Y \leq t) = P(F_X^{-1}(U) \leq t) = P(U \leq F_X(t)) = F_X(t)$