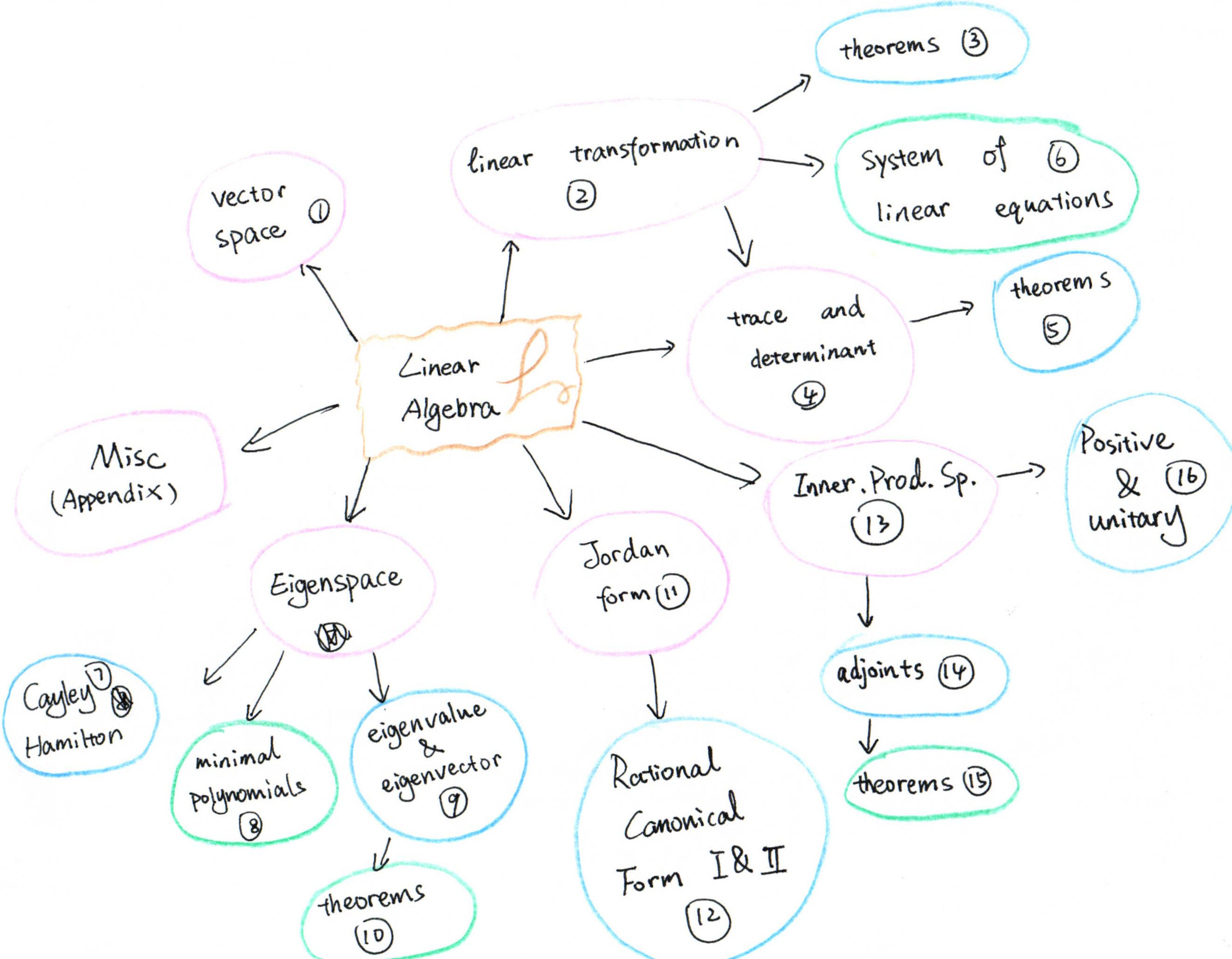
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① vector space V over field K, basis, linear independence, dimension, subspace, complementary subspace, internal direct sum, $(V = U \oplus U')$,

② isomorphism, matrix, row/column rank,

③ (1) rank-nullity theorem: Let $T \in L(V, W)$, then $\dim(\ker T) + \dim(\mathrm{Im}\, T) = \dim(V)$ (V is finite dimensional)

(2) rank theorem: ~~rank~~ For a $m \times n$ matrix over K, its row rank equals its column rank.

④ trace, $\mathrm{tr}\, A = \sum_{i=1}^{n} a_{ii}$; $\det A = \sum_{\sigma} (\mathrm{sgn}\, \sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(n),n}$; $\begin{cases} \det A = \sum_{i=1}^{n} (-1)^{k+i} a_{k,i} \det A_{ki} \\ \text{aka} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}. \end{cases}$

⑤ trace: (1) $\mathrm{tr}\, AB = \mathrm{tr}\, BA$

(2) $\mathrm{tr}\,(A+B) = \mathrm{tr}\, A + \mathrm{tr}\, B$.

(3) If A and B are matrices w.r.t. two bases for the same linear transformation, then $\mathrm{tr}\, A = \mathrm{tr}\, B$ and $\det A = \det B$.

⑥ The solutions to $Ax = b$ takes the form $x_0 + y_0$, where x_0 is the particular solution and y_0 solves $Ax = 0$. The solution to $Ax = 0$ forms a subspace of dimension $n - r$. (recall rank-nullity theorem)

⑦ characteristic polynomial: For a linear transformation from V to V, its char. poly. is $C_T(\lambda) := \det(\lambda I - T)$.

$\Rightarrow$ Cayley-Hamilton theorem: $C_T(T) = 0$, the zero transformation.

$\Rightarrow \begin{cases} (1)\ A \text{ is similar to an upper triangle matrix} \Longleftrightarrow C_A(\lambda) \text{ splits over } \mathbb{K}. \\ (2)\ \det(T) = (-1)^n a_0, \quad \mathrm{tr}\, T = -a_{n-1}. \text{ where } a_j\text{'s are coeffs for } C_T(\lambda)). \end{cases}$

⑧ minimal polynomials: Let $T \in L(V)$. The minimal polynomial m_T generates (by polynomial multiple) all other polymonials p s.t. $p(T) = 0$.

And $m_T(\lambda) \mid c_T(\lambda) \mid m_T^n(\lambda)$.

⑨ $\begin{cases} \text{eigenvalue: } \lambda \text{ s.t. for } T \in L(V): \exists\, v \text{ and } Tv = \lambda v. \\ \text{eigenvector: the } v \text{ in above.} \end{cases}$

$\Rightarrow$ All eigenvalues constitute the spectrum, $\sigma(T)$.

↳ $\lambda \in \sigma(T) \Leftrightarrow c_T(\lambda) = m_T(\lambda) = 0$.

$\begin{cases} \text{algebraic multiplicity: multiplicity of } \lambda \text{ in } c_T(\lambda). \\ \text{geometric multiplicity: dimension of eigenspace w.r.t. } \lambda. \end{cases}$ $\longrightarrow$ G.M. $\leq$ A.M.

⑩ (1) The eigenvectors form a basis $\Leftrightarrow$ the minimal polynomial splits over $\mathbb{K}$ into distinct factor.

$\Updownarrow$

T is diagonalizable.

(2) If $c_T(\lambda)$ splits over $\mathbb{K}$, then $\det T = \prod$ eigenvalues, $\operatorname{tr} T = \sum$ eigenvalues.

⑪ Suppose A has eigenvalues $\lambda_1, \cdots, \lambda_k$, with a.m. $a_1, \cdots, a_k$, and g.m. $g_1, \cdots, g_k$. (Suppose c_A and m_A splits over $\mathbb{K}$):

A is similar to J_A, where

$$J_A = \begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix}$$

and B_j is $a_j \times a_j$ matrix

$$a_j \left\{ \underbrace{\begin{pmatrix} J_{j,1} & & & 0 \\ & J_{j,2} & & \\ & & \ddots & \\ 0 & & & J_{j,g_j} \end{pmatrix}}_{a_j} \right. \longrightarrow J_{j,\ell} = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \lambda_j & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

↖ There are g_j sub-blocks.

Futhermore, (1) $|J_{j,1}| = m_j \times m_j$, where m_j is the multiplicity of λ_j in $m_A(t)$.

(Here we suppose $J_{j,1}$ is the largest sub-block.)

(2)

(12) companion matrix: For $f(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$, its companion matrix is

(Note: $f(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$)

$$C(f) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & \\ 0 & 1 & 0 & & \vdots & \vdots \\ \vdots & \vdots & 1 & \ddots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

$\Longrightarrow$ Thm: $m_{C(f)}(x) = f(t)$; $C_{C(f)}(x) = f(x)$.

Rational Canonical Form I : Given A, $\exists\, p_1, \cdots, p_r$ s.t.

(Frobenius canonical form) $p_1 \mid p_2 \mid \cdots \mid p_r$ and $\prod p_j = C_A$, $p_r = m_A$.

and A is similar to

$$\begin{pmatrix} C(p_1) & & & O \\ & C(p_2) & & \\ & & \ddots & \\ O & & & C(p_r) \end{pmatrix}.$$

Rational Canonical Form II : For A, suppose $\begin{cases} C_A(x) = f_1^{n_1}(x) \cdots f_r^{n_r}(x) \\ m_A(x) = f_1^{m_1}(x) \cdots f_r^{m_r}(x), \end{cases}$

where f_j's are irreducible. Then A is similar to a unique A' : (unique up to ordering of blocks)

$$A' = \begin{pmatrix} B(f_1) & & & 0 \\ & B(f_2) & & \\ & & \ddots & \\ 0 & & & B(f_r) \end{pmatrix} \quad \text{where } B(f_j) = \begin{pmatrix} C(f_j^{s_1}) & & 0 \\ & \ddots & \\ 0 & & C(f_j^{s_t}) \end{pmatrix}$$

and $\begin{cases} s_t \le s_{t-1} \le \cdots \le s_1 = m_j \\ \sum s_\ell = n_j \end{cases}$.

(13) I.P. Space : (1) $\langle u,v\rangle \geq 0$ & $\langle u,u\rangle = 0 \Leftrightarrow u=0$ (2) $\langle u,v\rangle = \overline{\langle v,u\rangle}$ (3) $\langle au+bv, w\rangle = a\langle u,w\rangle + b\langle v,w\rangle$.

Norm : $\|u\| = \langle u,u\rangle$

Cauchy-Schwarz In.eq. : $|\langle u,v\rangle| \le \|u\| \cdot \|v\|$

Gram-Schmidt : Given a basis $\{v_1, \cdots, v_n\}$ construct orthonormal $\{e_1, \cdots, e_n\}$ by

$e_1 = \frac{v_1}{\|v_1\|}$. $w_j = v_j - \sum_{\ell=1}^{j-1} \cancel{\langle v_\ell, w_j, v_\ell\rangle v_\ell}$ $\langle v_j, e_\ell\rangle \cdot e_\ell$, $\Rightarrow e_j = \frac{w_j}{\|w_j\|}$.

(14) Let T be a linear map in V, its adjoint operator is T^* s.t.

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall x, y \in V.$$

If V is complex then T^* is $\overline{T^t}$ (when V is bounded)

(1) $(S+T)^* = S^* + T^*$; (2) $(T^*)^* = T$; (3) $(ST)^* = T^*S^*$. (4) $\operatorname{tr} T^* = \overline{\operatorname{tr} T}$, $\det T^* = \overline{\det T}$

More special :

(1) T is self-adjoint / Hermitian if $T^* = T$. (In $\mathbb{R}$, it's called symmetric)

(2) T is skew-Hermitian if $T^* = -T$.

(3) $\mathbb{R}$: ~~unitary~~ / orthogonal if $T^*T = I$. $\xrightarrow{\mathbb{C}}$ unitary if $T^*T = TT^* = I$.

↳ (i.e. $\langle x, y\rangle = \langle Tx, Ty\rangle$.)

(4) normal if $TT^* = T^*T$.

(15) (1) If T is $\begin{cases} \text{self-adjoint} \\ \text{skew-symmetric} \\ \text{unitary (orthogonal)} \end{cases}$, then $\forall \lambda \in \sigma(T)$, $\begin{cases} \lambda \in \mathbb{R}. \\ \lambda \text{ is purely imaginary} \\ |\lambda| = 1 \quad (\lambda = \pm 1) \end{cases}$

(2) Rayleigh : For a Hermitian T, $\forall v \neq 0$, $\lambda_{min} \leq \dfrac{\langle Tv, v\rangle}{\langle v, v\rangle} \leq \lambda_{max}$.

Equality attained only when v is eigenvector of $\lambda_{min} / \lambda_{max}$.

(16) For a self-adjoint T, it is $\begin{cases} \text{positive semidefinite} \\ \text{positive definite} \end{cases}$ if $\begin{cases} \cancel{\langle Tx,y\rangle \geq 0} \quad \langle Tx,x\rangle \geq 0 \\ \cancel{\langle Tx,y\rangle \geq 0} \quad \langle Tx,x\rangle > 0 \end{cases}$ $\forall x \neq 0$.

(C doesn't have to be self-adjoint)

(1) Theorem: T is positive semidefinite $\Leftrightarrow$ $\exists$ self-adjoint B & C such that $T = B^2 = C^*C$.

And, $\begin{cases} \text{if we require } B \text{ to be positive semidefinite, } B \text{ is unique.} \\ \text{且 } B \,\&\, C \text{ are invertible} \Leftrightarrow T \text{ is positive definite.} \end{cases}$

(2) Theorem: T is positive-semidefinite $\Leftrightarrow$ $\forall \lambda \in \sigma(T)$, $\lambda \geq 0$. (definite $\Leftrightarrow$ >0)

(3) T is pos-def. 且 $\Leftrightarrow$ $\det T_k > 0$, where T_k is the first $k \times k$ submatrix of T.

$\hookrightarrow$ then $\langle u,v\rangle_{\text{新}} = \langle Tu,v\rangle$ defines a new inner product.

Polar decomposition: For any $T \in L(V)$, $T = UP$,

where P is unique, positive semidefinite, and U is unitary.

If T is invertible, then P is pos-def and U is also unique.

$P = \sqrt{T^*T}$.

Unitary : $\|Tv\| = \|v\|$

$\Leftrightarrow$ rows of T ~~are~~ form orthonormal basis of $\mathbb{K}^n$.

$\Leftrightarrow$ columns — — — — — - .

Spectral theorem:

(1) If A is normal, then A is diagonalizable and $\exists$ unitary U s.t. UAU^* is diagonal.

(2) If A is symmetric real matrix, then A is diagonalizable and $\exists$ orthogonal U s.t. UAU^* is diagonal.

Orthogonal matrix can be represented as

$$\begin{bmatrix} R_1 & & & & 0 \\ & \ddots & & & \\ & & R_{\cancel{k}} & & \\ & & & \ddots & \\ 0 & & & & R_k \end{bmatrix} \quad \text{if } n \text{ is even}$$

and

$$\begin{bmatrix} R_1 & & & 0 \\ & \ddots & & \\ & & R_k & \\ 0 & & & 1 \end{bmatrix} \quad \text{if } n \text{ is odd.}$$

Here $R_j = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Appendix

① Vandermonde matrix : $\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}$ has determinant $\prod_{i>j} (x_i - x_j)$.

② Change of basis: Suppose we want to change from basis B_1 to B_2,

let $W \in V$, and assume $B_1 = \{u_1, \cdots, u_n\}$, $B_2 = \{v_1, \cdots, v_n\}$.

$\Rightarrow$ $v_1, \cdots, v_n$ are lin. comb. of $u_1, \cdots, u_n$.

Suppose $[v_1]_{B_1} = \begin{pmatrix} m_{11} \\ \vdots \\ m_{1n} \end{pmatrix}$, $\cdots$, $[v_n]_{B_1} = \begin{pmatrix} m_{n1} \\ \vdots \\ m_{nn} \end{pmatrix}$.

Let $A = \begin{pmatrix} [v_1]_{B_1} & \cdots & [v_n]_{B_1} \end{pmatrix}$, then $A[v]_{B_2} = [v]_{B_1}$.

matrix representation: Let $\overset{\beta=}{\{v_1, \cdots, v_n\}}$ and $\overset{\gamma=}{\{w_1, \cdots, w_m\}}$ be bases for V & W.

Then $\left([T]_\beta^\gamma\right)_{ij}$ can be obtained by $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$. and ~~repto~~ from a_{ij}.

Then, the change of coordinate basis is $[I]_{B_2}^{B_1}$, i.e., $[I]_{B_2}^{B_1} \cdot [v]_{B_2} = [v]_{B_1}$.

③ diagonalizable : A is d-able $\Leftrightarrow$ $\exists P$ s.t. PAP^{-1} is diagonal

$\Leftrightarrow$ sum of dimensions of eigenspace is n.

$\Leftrightarrow$ $\exists$ a basis consisting of A's eigenvectors.

④ inverse $(A^{-1})_{ij} = \cancel{(-1)^{i+j} \det(A^{ij})}$, where A^{ij} is A without ith row & jth column.

$$= \frac{(-1)^{i+j} \det(A^{ji})}{\det A} = \frac{\left[(-1)^{i+j} \det(A^{ij})\right]^T}{\det A}.$$

↳ Cramer's rule : If $Ax = b$, and A is nonsingular,

then if we denote A_i by <u>A with ith column replaced by b</u>,

then $x_i = \dfrac{\det A_i}{\det A}$.