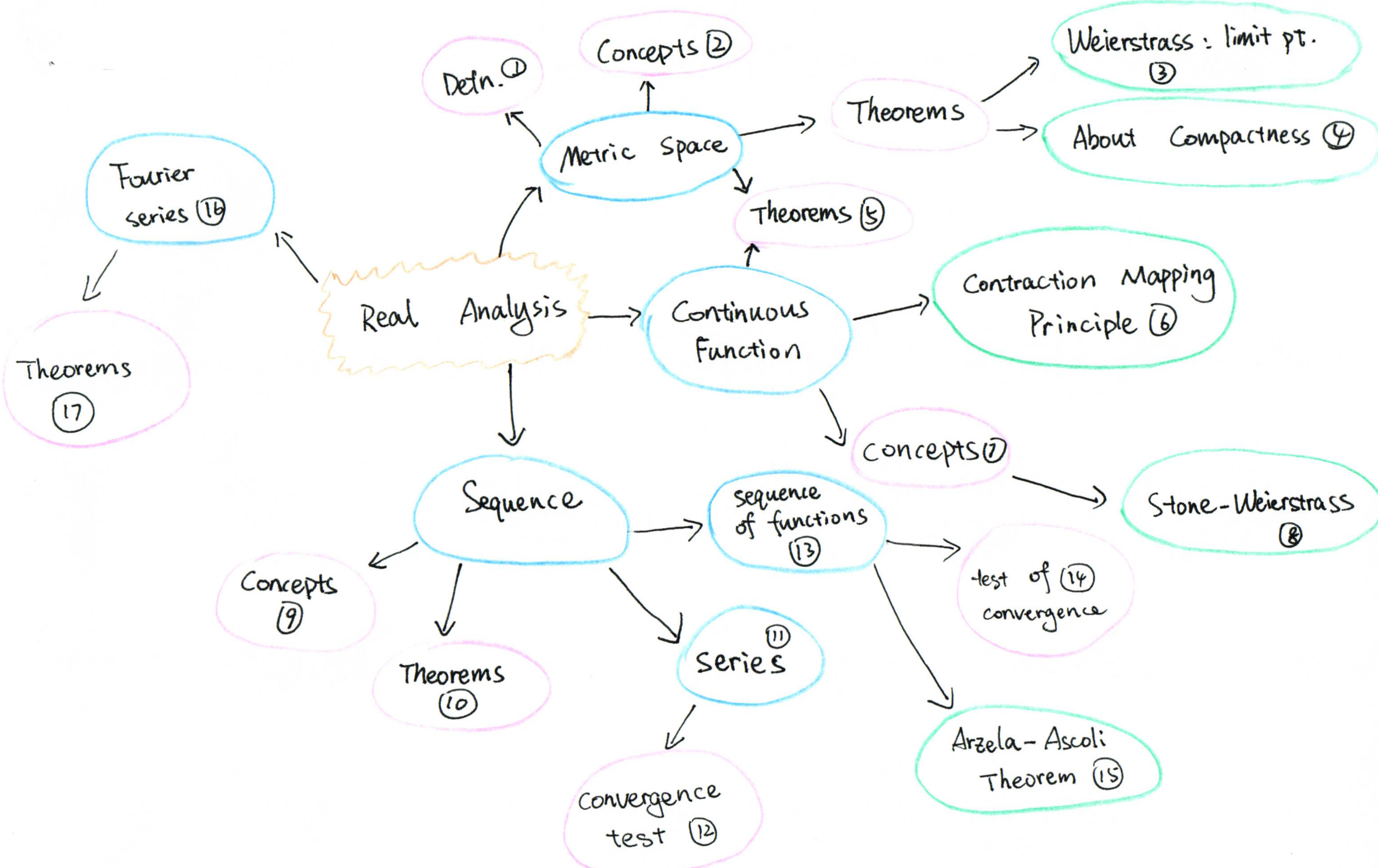
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① (X,d) s.t. $\begin{cases} d(x,y) = d(y,x). \\ d(x,y) = 0 \Leftrightarrow x = y \\ d(x,y) \leq d(x,z) + d(z,y) \\ d(x,y) \geq 0. \end{cases}$

② open, close sets; limit points; neighborhood; open cover; compact sets

connected set (only clopen are U & ϕ); cauchy sequence; complete space;

dense subsets;

③ Every bounded, infinite subset of $\mathbb{R}^n$ has a limit point.

④ (1) Compactness $\Rightarrow$ Bounded and closed.

In $\mathbb{R}^n$, the inverse is also true.

(2) $\{K_n\}$ compact & $K_{n+1} \subseteq K_n \Rightarrow \cap K_n$ is nonempty.

⑤ (1) Continuous functions map compact sets to compact sets.

(2) It also preserves connectedness.

(3) Continuous function on compact set is uniformly continuous,

i.e., ~~$\forall x, y$~~ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y$ with $d_1(x,y) < \delta \Rightarrow d_2(f(x), f(y)) < \varepsilon$.

⑥ $f: X \to X$ & (1) X is complete & f is strict contraction or (2) X is compact & f is contraction $\Longrightarrow \exists x$ s.t. $f(x) = x$.

⑦ (1) $\mathcal{F}$ is an algebra if $\forall f, g \in \mathcal{F}$, and $c \in \mathbb{C}$, then $f+g$, $f \cdot g$, cf are in $\mathcal{F}$.

If the complex conjugate $\bar{f}$ is in $\mathcal{F}$, then $\mathcal{F}$ is a self-adjoint algebra.

Restricting to only real-valued f, and $c \in \mathbb{R}$, then $\mathcal{F}$ is a real algebra.

(2) $\mathcal{F}$ separates points in (X, d), ~~wh~~ ~~the~~ $\Leftrightarrow$ $\exists f \in \mathcal{F}$ s.t. $f(x) \neq f(y)$ $(\forall x, y \in X)$

(3) $\mathcal{F}$ vanishes at x if $f(x) = 0$ $\forall f \in \mathcal{F}$.

⑧ Weierstrass: Let $C(X)$ be continuous, complex-valued functions on X.

Let $X = [a, b]$, then the set of polynomials is dense in $C(X)$.

$\Rightarrow$ Stone-Weierstrass: If $\mathcal{F}$ separates points in X and does not vanish anywhere, then (1) if $\mathcal{F}$ is self-adjont algebra, it must be dense in $C(X)$; (2) if $\mathcal{F}$ is real algebra, it is dense in real-valued functions of $C(X)$.

⑨ convergence ; cauchy sequence ; monotonic sequence

⑩ (1) If X is compact, then $\{x_n\}$ admits $\{x_{n_k}\}$ that is convergent. (Consider ③)

(2) Bounded seq in $\mathbb{R}$ or $\mathbb{C}$ admits convergent subseq.

(3) Bounded, monotonic seq in $\mathbb{R}$ converges.

(4) Convergent $\Rightarrow$ cauchy.

In $\mathbb{R}/\mathbb{C}$, cauchy $\Rightarrow$ convergent.

(11) For a sequence $\{a_n\}$, the series is $S_n = \sum_{k=0}^{n} a_k$.

(12) 1) Integral test: If $a_n = f(n)$ and f is decreasing on $x \in [0, \infty)$, then

$S_n = \sum_{k=0}^{n} f(k)$ converges, i.e., $S = \sum_{k=0}^{\infty} f(k)$ exists iff $\int_0^\infty f(x)\,dx$ exists.

2) Ratio test: consider $\left|\frac{a_{n+1}}{a_n}\right|$. If $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists and $L > 1$, then S_n diverges,

if $L < 1$, then S_n converges. If L does not exist, but $R = \limsup \left|\frac{a_{n+1}}{a_n}\right|$ exists and $R < 1$,

then S_n converges absolutely; if $L = \liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then S_n diverges; if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$

for all $n \geq N$, then S_n diverges.

3) Root test: let $\alpha = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$. If $\alpha < 1$, then S_n converges; if $\alpha > 1$, then S_n diverges.

4) Leibnitz rule: Let $\{a_n\}$ be positive & decreasing monotonically to 0.

Then $\sum_{n=0}^{\infty} (-1)^n a_n$ converges to s and $(s - s_n) \leq |a_{n+1}|$.

5) Comparison test: if $|a_n| \leq b_n$ for big enough n, then

① if $\sum a_n$ diverges, then $\sum b_n$ diverges

② if $\sum b_n$ converges, then $\sum a_n$ converges.

(13) $\{f_n\}$ converges to f { pointwise if $\forall x$, $f_n(x) \to f(x)$

uniformly if $\forall \varepsilon > 0$, $\exists N$ s.t. if $n > N$ then $|f_n(x) - f(x)| < \varepsilon$ for any x.

(14) Weierstrass M-test: if $|f_n(x)| \leq M_n$ for all x, in all n, then if $\sum M_n$ converges,

then $\sum f_n$ converges absolutely and uniformly.

(15) ({uniformly} equicontinuous : For $\{f_n\}$, for any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $\forall x, y \in X$, if $d(x,y) < \delta$ then $|f_n(x) - f_n(y)| < \varepsilon$ for any n).

Arzela - Ascoli : let (X, d) be compact. Suppose $\{f_n\}$ is {uniformly} equicontinuous & pointwise bounded, then $\{f_n\}$ is uniformly bounded and admits uniformly convergent subsequence.

⇒ (In short, if $\{f_n\}$ is uniformly bounded & equicontinuous, then $\exists \{f_{n_k}\}$ uniformly convergent.)

(uniformly bounded : $\exists\ M$ s.t. $|f_n(x)| \leq M\ \forall n, x$)

↳ cor. 1 : if $\{f_n\}$ are continuous and uniformly converging to f, then f is continuous.

↳ cor. 2 : if $\{f_n\}$ are integrable, and uniformly convergenging to f, then

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx \qquad (\text{domain } [a,b])$$

↳ Cor. 3 : if $\{f_n\}$ are differentiable and at x_0, $\{f_n(x_0)\}$ is convergent, then if $\{f_n'\}$ are uniformly convergent, ⇒ $\{f_n\}$ uniformly converges to f and $\{f_n'\}$ to f'. (f_n on $[a,b]$)

(16) Let f be periodic with period L on $\mathbb{R}$. Then, the Fourier coefficients of f are

$$\hat{f}(n) = f_n = \frac{1}{L} \int_{x=-\frac{L}{2}}^{\frac{L}{2}} f(x)\, e^{-inx \cdot \left(\frac{2\pi}{L}\right)}\, dx$$

and

$$f \sim \sum_{n=-\infty}^{\infty} f_n \cdot e^{inx \cdot \left(\frac{2\pi}{L}\right)}$$

↑

not necessarily equal.

(17) (1) Fourier coefficients are unique. (All $0 \to f \equiv 0$; All 0 except $\hat{f}(0) \to f \equiv C$)

(2) If f is integrable, then $\hat{f}_n \to 0$ as $|n| \to \infty$. (integrable in $\mathcal{L}^1$, i.e., $\int |f| dx$ is finite.)

That is $\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{inx(\frac{2\pi}{L})} dx \to 0$.

(3) Parseval : If f, g are square integrable, complex-valued, on $\mathbb{R}$, with period 2π,

and $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$, $g(x) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{inx}$,

then $\sum_{n=-\infty}^{\infty} \cancel{a_n b_n} \hat{f}(n) \overline{\hat{g}(n)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$.

If $f = g \Rightarrow \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$.

(4) Let f be integrable & real-valued, then $\hat{f}(-n) = \overline{\hat{f}(n)}$ and if $f \in \mathcal{C}^1$, $\hat{f'}(n) = in\hat{f}(n)$.

(5) Dirichlet - Jordan : If f is periodic (w/ 2π) with bounded variation,

then its Fouries series converges to $\frac{1}{2}[f(x^+) + f(x^-)]$ on each x.

If f is also continuous on some $[a,b]$ then the convergence is uniform.

on $[a,b]$.