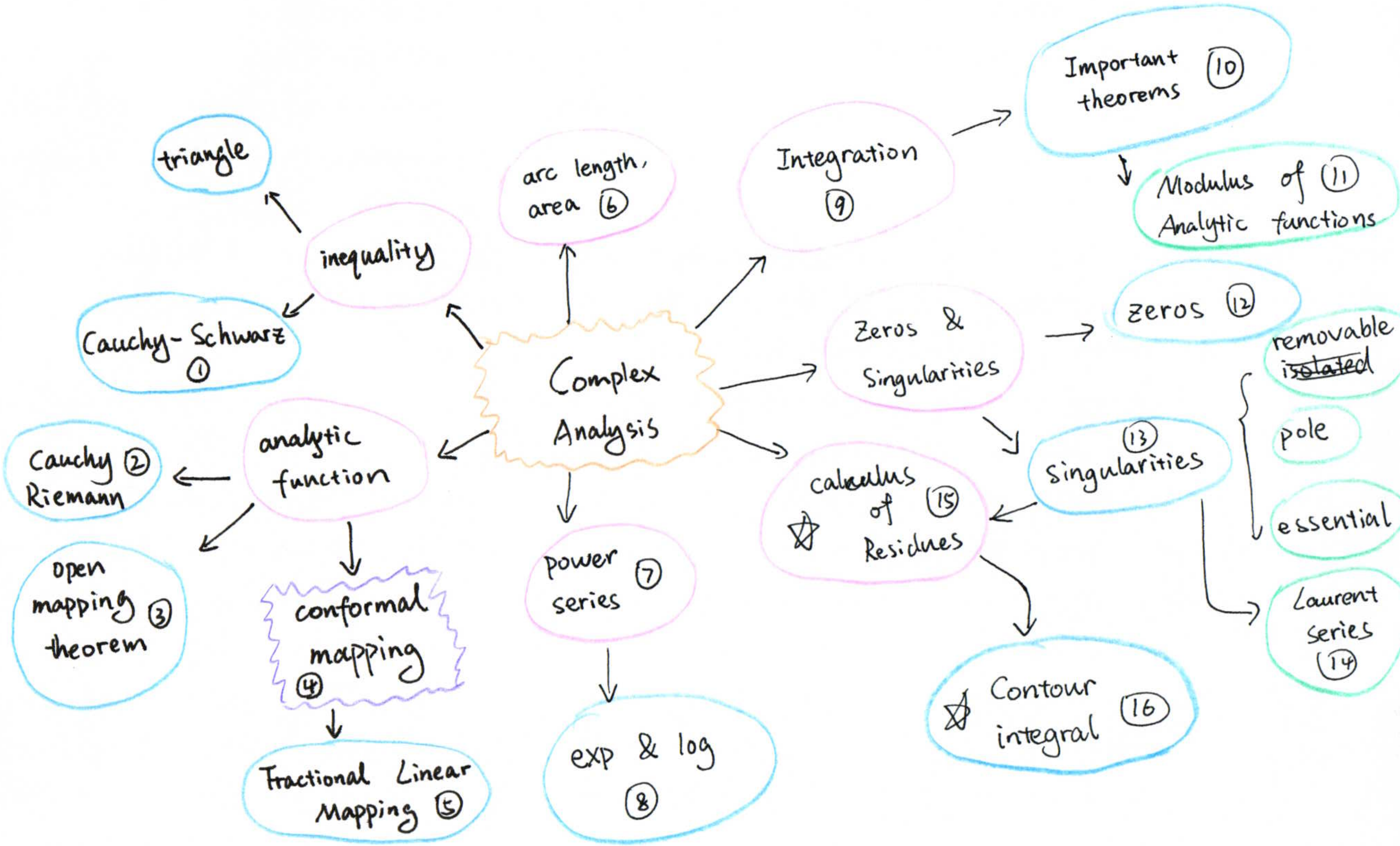

Complex Analysis
inequality
triangle
Cauchy-Schwarz ①
arc length, area ⑥
Integration ⑨
Important theorems ⑩
Modulus of Analytic functions ⑪
Zeros & Singularities
zeros ⑫
⑬ Singularities
removable
~~isolated~~
pole
essential
Laurent series ⑭
calculus of Residues ⑮
Contour integral ⑯
analytic function
Cauchy Riemann ②
open mapping theorem ③
conformal mapping ④
Fractional Linear Mapping ⑤
power series ⑦
exp & log ⑧

Appendix

① C-S inequality: $|\langle u,v\rangle|^2 \le \langle u,u\rangle \cdot \langle v,v\rangle$

in complex numbers: $\left|\sum_{j=1}^{n} z_j w_j\right|^2 \le \left(\sum_{j=1}^{n}|z_j|^2\right)\left(\sum_{j=1}^{n}|w_j|^2\right)$

② f is analytic (complex differentiable) $\Longleftrightarrow$

$f(x+iy) = u+iv$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

③ If f is non-constant, analytic, then for any U open, $f(U)$ is open.

④ A function f is conformal if $\forall \gamma_1, \gamma_2$ intersecting with angle θ,

$f(\gamma_1)$ and $f(\gamma_2)$ also intersect with θ.

↳ f is conformal on U if $f'(z) \neq 0$ $\forall z \in U$.

⑤ F.L.M.: $a,b,c,d \in \mathbb{C}$ s.t. $ad-bc \neq 0$. A F.L.M. then takes the form $\phi(z) = \frac{az+b}{cz+d}$.

ϕ is conformal, bijective ($\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ ($\mathbb{C} \cup \{\infty\}$)).

Property (1): Any F.L.M. preserves generalized circles (circle + line).

(2): $\phi(z) = \frac{i+z}{1+iz}$ maps the unit disk onto ~~right~~ upper half plane;

thus, $i\phi$, $-\phi$, $-i\phi$ maps the unit disk onto ~~upper~~, left, lower, right half planes.

(3): Let $a \in D$, where D is the unit disk.

Then $\phi(z) = \frac{z-a}{1-\bar{a}z}$ maps D onto D. (can be multiplied by $e^{i\phi}$)

(4): If $ad-bc=1$, then ϕ maps H bijectively to H.

(5): (note for (2)) $\phi(z) = \frac{1+z}{1-z}$: $D \to$ right half plane.

⑥ $\gamma : [a,b] \to \mathcal{U}$,

then length of $\begin{cases} \gamma : & \int_\gamma |dz| = \int_{t=a}^{b} |\gamma'(t)|\, dt \\ f(\gamma) : & \int_{f(\gamma)} |dz| = \int_{t=a}^{b} |f'(\gamma(t))| \cdot |\gamma'(t)|\, dt . \end{cases}$

If f is conformal on $\mathcal{U}$, then area of $f(\mathcal{U})$ is

$$\iint_{\mathcal{U}} |f'(z)|^2 \, dx\, dy$$

⑦ $\{f_n\}$ converges locally uniformly on $\mathcal{U}$ if $\forall D \subset \mathcal{U}$ s.t. D compact,

$\{f_n\}$ converges uniformly on D.

$\hookrightarrow$ $\{f_n\}$ analytic, loc. uni. converges to $f \Rightarrow f$ analytic.

power series at α : $f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$.

it converges locally uniformly in $\{z : |z-\alpha| < R\}$, where $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.

$\inf_{n\geq 0} \sup_{m \geq n} |a_n|^{\frac{1}{n}}$.

If $\sum_{n=0}^{\infty} \alpha_n (z-\alpha)^n = \sum_{n=0}^{\infty} \beta_n (z-\alpha)^n$ then $\alpha_n = \beta_n$ for all $n \geq 0$

⑧ $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow \sin z = \frac{e^{iz} - e^{-iz}}{2i}$; $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$\to \log z = \underbrace{\log|z|}_{\mathbb{R}} + i \underbrace{\arg z}_{\text{unique up to } 2k\pi,\ k \in \mathbb{Z}.}$

$\hookrightarrow$ branch: fix the k in the formula above so that log is analytic in $\mathcal{U}$ (open).

$\Rightarrow$ An analytic branch exists on domain $\mathcal{U}$ if $\mathcal{U}$ is simply connected, containing 0 in its interior.

or more generally i.e., there is no closed curve in $\mathcal{U}$ containing 0 in its interior.

Define $z^\alpha := (e^{\log z})^\alpha$, where $\log z$ can be any well-defined branch.

⑨ For $\gamma : [a,b] \to \mathcal{U}$, $\int_\gamma f(z)\,dz = \int_{t=a}^{b} f(\gamma(t))\,\gamma'(t)\,dt$. (Defn)

$\longrightarrow \left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,|dz|$.

If $\mathcal{U}$ is simply connected, γ is from z_0 to z_1, in $\mathcal{U}$,

then $\int_\gamma f'(z)\,dz = f(z_1) - f(z_0)$.

$= \int_{z_0}^{z_1} f'(z)\,dz$ (since it's path-independent.)

(10) Let γ be a closed curve. The winding number of γ around z_0 ($z_0 \notin \gamma$) is the number of times γ wraps around z_0 counterclockwise, also given by

$\operatorname{Ind}_{\gamma}(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0}\, dz$. If we want to count that of $f(\gamma)$,

$$\Rightarrow \operatorname{Ind}_{f(\gamma)}(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - z_0}\, dz.$$

Theorems :

Cauchy: $\mathcal{U}$ simply connected domain, γ closed rectifiable curve.

If f is analytic, then $\int_{\gamma} f(z)\, dz = 0$.

(converse) Morera: If f is continuous, complex-valued on $\mathcal{U}$, $\int_{\gamma} f(z)\, dz = 0$ for all closed, rectifiable γ, then f is analytic.

(Holds if $\int_{\gamma} f(z)\, dz = 0$ for all triangle γ.).

Cauchy's integral formula: With the same condition in Cauchy's thm, if c is inside γ, then for $n \geq 0$,

$$\cancel{f^{(n)}(z)} \quad f^{(n)}(c) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-c)^{n+1}}\, dz.$$

(Maximum Modulus Principle)

(11) (1) $\mathcal{U}$ open, $K \subset \mathcal{U}$ compact, f analytic in $\mathcal{U}$, then $|f|$ attains max at ∂K.

If $|f|$ has max in $\mathrm{int}(K)$ then f is constant.

(2) (Liouville): A bounded, entire function is constant.

(3) (Schwarz lemma): If f is analytic in unit disk D, $|f(z)| \le 1$, $f(0) = 0$, then

$$|f(z)| \le |z| \quad \forall z \in D, \quad \text{and} \quad |f'(0)| \le 1.$$

Moreover, if $|f(z)| = |z|$ for $z \neq 0$ or if $|f'(0)| = 1$, then $f(z) = \lambda z$, where $|\lambda| = 1$.

(4) (Min modulus principle): same condition, but $f(z) \neq 0$ $\forall z \in K$, then min is attained in ∂K.

(12) Identity Principle: If $\{\alpha \in \mathcal{U} : f(\alpha) = 0\}$ has a limit point in $\mathcal{U}$, then $f \equiv 0$ in $\mathcal{U}$.

$\Rightarrow$ If $\{\alpha \in \mathcal{U} : f(\alpha) = g(\alpha)\}$ has a limit point, then $f \equiv g$.

(f, g analytic, $\mathcal{U}$ a domain)

Theorem: If $f(\alpha) = 0$, $f(z) \not\equiv 0$ in $\mathcal{U}$, then $\exists\, g(\alpha) \neq 0$ s.t. $f(z) = (z-\alpha)^n g(z)$.

Argument Principle: If f is meromorphic in K, then

$$\oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi i\,(N - P),$$

where $\begin{cases} C \text{ is } \partial K, \\ N \text{ is number of zeros with multiplicity} \\ P \text{ is number of } \cancel{\text{zeros}} \text{ poles with multiplicity.} \end{cases}$

Rouché's theorem: f, g analytic on ∂K & K, $|g| \le |f|$ on ∂K, then f and $f+g$ have the same number of zeros (counting multiplicities)

(13) f has an isolated singularity at α if f is analytic on punctured disk $\{z : 0 < |z-\alpha| < k\}$ but not on α.

(1) removable: If $f(z) \to c$ as $z \to \alpha$, then by defining $f(\alpha) = c$ f becomes analytic in the whole disk.

(2) pole: If $\exists k$ s.t. $(z-\alpha)^k f(z)$ exists, is bounded as $z \to \alpha$, and $(z-\alpha)^j f(z)$ is unbounded $\forall j < k$, then f has a pole of order k at α.

(3) essential: If none of the above.

Note:

(1) If f is bounded in $N_\varepsilon(\alpha)$ then $f(\alpha)$ is removable singularity.

(2) If $f(\alpha)$ is essential then $\forall c \in \mathbb{C}$, $\exists \{z_j\} \to \alpha$ s.t. $\{f(z_j)\} \to c$.
(Casorati - Weierstrass)

(3) An entire function with a pole of degree k at ∞ is a polynomial of degree k.

(14) If f is analytic on $\{z : r < |z-\alpha| < R\}$, then $\exists\, C_n$'s s.t. $\forall z$ in the annulus,

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-\alpha)^n .$$

Let γ be any simple closed curve in the annulus with α inside it,

$$C_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\, dz .$$

(i.e., Cauchy integral formula divided by $n!$)

$$f^{(n)}(c) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-c)^{n+1}}\, dz$$

(1) If $C_n = 0$ $\forall n < 0$, then α is removable.

(2) If $\exists k > 0$, s.t. $C_n = 0$ $\forall n < -k$, then α is a pole of degree k.

(3) If there exist infinite $C_n \neq 0$ $\forall n < 0$, then α is essential.

If f is analytic in some $|z| > R$, then f has a Laurent series about ∞, it is $G(\frac{1}{z})$, where $G(z)$ is the Laurent series of $g(z) = f(\frac{1}{z})$ at the origin.

(15) If f is analytic on $0<|z-\alpha|<r$, then the residue of f at α is

$\text{Res}_f(\alpha) = c_{-1}$, the first negative term's coefficient.

$\boxed{\text{Residue Theorem (interior)}}$: γ simple closed curve, enclosing $\alpha_1, \cdots, \alpha_k$ singularities

inside it, then $\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{j=1}^{k} \text{Res}_f(\alpha_j)$.

Calculate the residue:

(1) essential : calculate the Laurent series

(2) pole: (k-th order) $\text{Res}_f(\alpha) = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left[(z-\alpha)^k f(z)\right]\Big|_{z=\alpha}$.

In particular, for a simple pole, it's $\lim_{z\to\alpha}(z-\alpha)\cdot f(z)$.

special case: if $f=\frac{g}{h}$, $g(\alpha)\neq 0$ and h has a simple zero at α,

$\text{Res}_f(\alpha) = \frac{g(\alpha)}{h'(\alpha)}$.

Residue at ∞: If f is analytic in a neighborhood U of ∞, γ is a simple closed curve in U, then

$$\text{Res}_f(\infty) = \frac{-1}{2\pi i}\int_\gamma f(z)dz.$$

Let $g(z) = \frac{1}{z^2}f(\frac{1}{z})$, then $\text{Res}_f(\infty) = \text{Res}_g(0)$. If $f(z)\to 0$ as $z\to\infty$, then $\text{Res}_f(\infty) = -\lim_{z\to\infty} zf(z)$.

Residue Theorem (exterior): Let f be analytic in a neighborhood of ∞, except for singularities $\alpha_1, \cdots, \alpha_K$ (including ∞). Then if γ is a simple closed curve with $\alpha_1, \cdots, \alpha_K$ in its exterior,

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = -\sum_{j=1}^{k} \operatorname{Res}_f(\alpha_j)$$

(16) Motivation: integrating in an easier way.

For example $\displaystyle\int_0^{2\pi} \frac{\cos x}{1+\sin x}\,dx = \operatorname{Re}\int_0^{2\pi} \frac{e^{ix}}{1+\sin x}\,dx.$

Useful theorems:

(1) If f has a simple pole at α, C_r is a semicircle about α,

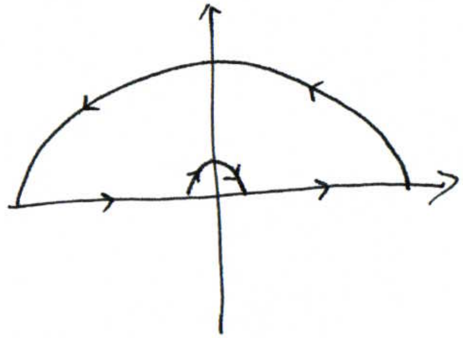
$$\Rightarrow \quad \lim_{r\to 0} \frac{1}{2\pi i}\int_{C_r} f(z)\,dz = \frac{1}{2}\operatorname{Res}_f(\alpha).$$

(2) (Jordan) Let $a>0$. $C_R = \{Re^{i\theta} \mid 0 \le \theta \le \pi\}$.

then, $\displaystyle\left|\int_{C_R} g(z)\cdot e^{iaz}\,dz\right| \le \frac{\pi}{a}\, g_{max}$, where $\displaystyle g_{max} = \max_{\theta\in[0,\pi]} |g(Re^{i\theta})|$.

Various methods:

1. half-annulus:

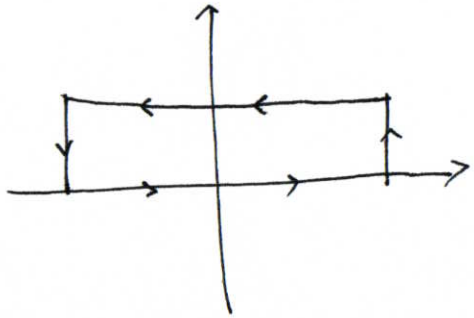


useful for

$$\int_{-\infty}^{\infty} f(x)\,dx,$$

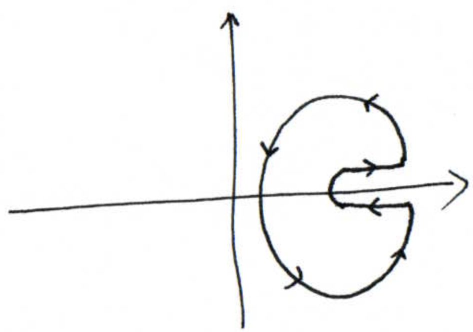
where $f(z)$ has finite singularities in upper/lower half planes.

2. box:



useful when $\exists\, \infty$ singularities.

3. keyhole:



useful when there is logarithm.

Appendix

① Laurent series: $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$

$$-\ln(1-z) = z + \frac{z^2}{2} + \cdots$$

② automorphism of D: $f(z) = e^{i\varphi}\frac{z-a}{1-\bar{a}z}$.

③ Harmonic function: $\cancel{u}f$: $\nabla^2 f = \Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j} = 0$.

In $\mathbb{C}$, if $f = u+iv$ is analytic, then both u and v are harmonic,

If u is real-valued, harmonic, then $\exists!\, v$ (up to a real constant) s.t.

$u+iv$ is analytic.

Mean value: $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$.

$$u(x) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u\, dy = \frac{1}{\cancel{n}\,\omega_n r^n}\int_{B(x,r)} u\, dy.$$

Max/min: In compact K, a harmonic, real valued u ~~attai~~ attains max/min in ∂K. If u attains max/min in $\mathrm{int}(K)$ then $u \equiv C$.