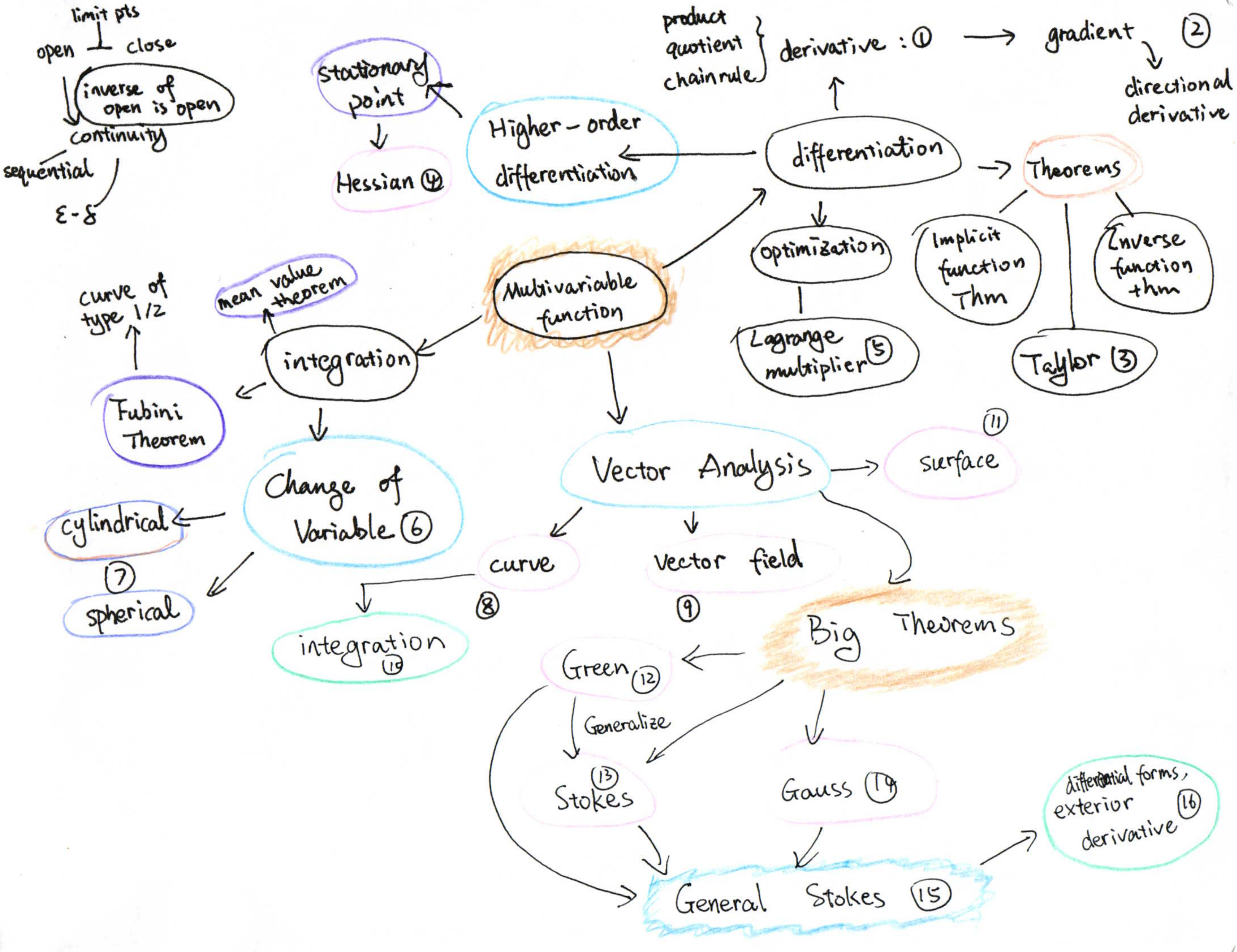
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① For $f: A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, the derivative is given by the Jacobian matrix,

$$J = \frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n} \right] = \begin{bmatrix} \frac{\partial f_1}{\partial x} \\ \vdots \\ \frac{\partial f_m}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

For f to be differentiable, $\frac{\partial f_i}{\partial x_j}$ has to be continuous on a neighborhood of x_0.

② $\nabla f = \frac{\partial f}{\partial x_1} \cdot e_1 + \cdots + \frac{\partial f}{\partial x_n} \cdot e_n$ is the gradient.

The directional derivative along v is $\nabla f \cdot v = D_v f$

③ 1-var: $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_k(x)$

and $R_k(x) = \int_{x_0}^{x} \frac{(x-t)^k}{k!} f^{(k+1)}(t)\, dt$. and $\lim_{x \to x_0} \frac{R_k(x)}{(x - x_0)^k} = 0$.

multi-var: $T(x_1, \cdots, x_d) = f(a_1, \cdots, a_d) + \sum_{j=1}^{d} \left(\frac{\partial f}{\partial x_j}(\vec{a}) \right) \cdot (x_j - a_j)$

$$+ \frac{1}{2!} \sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial^2 f}{\partial x_j \partial x_k}(a)\, (x_j - a_j)(x_k - a_k)$$ ← Hessian

$$\boxed{\lim_{\vec{x} \to \vec{x}_0} \frac{|R_k(x)|}{\|x - x_0\|^k} = 0}$$

$$+ \frac{1}{3!} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{\ell=1}^{d} \frac{\partial^3 f}{\partial x_j \partial x_k \partial x_\ell}(a)\, (x_j - a_j)(x_k - a_k)(x_\ell - a_\ell) + \cdots$$

④ If the Hessian at a stationary point is positive definite / negative definite, then the function has a local minimum / maximum there.

Positive definite $\Longleftrightarrow$ all positive eigenvalues.

$$\boxed{H_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}}$$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \text{ is } \begin{cases} \text{Pos. def. if } a>0 \ \& \ ac-b^2>0. \\ \text{Neg. def if } a<0 \ \& \ ac-b^2>0. \end{cases}$$

⑤ For f, g ~~continuously~~ $\in C^1$, on $S = \{g(\vec{x}) = 0\}$, if f has max/min on S,

then $\exists \lambda$ such that $\nabla f = \lambda \nabla g$. on that point. (necessary, but not sufficient)

For multiple constraints, $S = \{\vec{x} : g_i(\vec{x}) = c_i \text{ for } i=1,\dots,n\}$.

if f has a max/min at $\vec{x}_0$ & $\nabla g_i(\vec{x}_0)$ are lin. indep.,

then $$\nabla f + \sum_{i=1}^{n} \lambda_i \nabla g_i(\vec{x}_0) = 0$$ for some λ_i. where $i=1,\dots,n$.

⑥ $$\iint_S f(x,y)\,dx\,dy = \iint_{S^*} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\Longrightarrow \int_{\varphi(U)} f(v)\,dv = \int_U f(\varphi(u))\,|\det(D\varphi)(u)|\,du.$$

⑦ $$\begin{cases} \text{cylindrical}: & x = r\cos\theta,\ y = r\sin\theta,\ z = z \quad \Rightarrow \left|\det \dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r \\ \text{spherical}: & x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi. \Rightarrow \left|\det \dfrac{\partial(x,y,z)}{\partial(r,\phi,\theta)}\right| = r^2\sin\phi \end{cases}$$

⑧ Notes: $\frac{d}{dt}(\phi(t) \times \psi(t)) = \phi(t) \times \psi'(t) + \phi'(t) \times \psi(t)$.

arc length: $ds = \|\phi'(t)\|\, dt$.

⑨ Notes: ~~flow for~~ A vector field is $F: A \to \mathbb{R}^n$, where $A \subseteq \mathbb{R}^n$.

Its flow line is $\phi(t)$ s.t. $\phi'(t) = F(\phi(t))$.

∇: $\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$.

$\begin{cases} \text{divergence of } F \text{ is } \nabla \cdot F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}. \\ \text{curl of } F \text{ is } \nabla \times F = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{pmatrix} \end{cases}$

↳ ($\mathbb{R}^3$ only)

$F: \mathbb{R}^n \to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R} \Longrightarrow \operatorname{div}(f \cdot F) = f \cdot \operatorname{div} F + \nabla f \cdot F$ inn. prod.

Thm: $\begin{cases} \operatorname{div}(F \times G) = G \cdot \operatorname{curl} F - F \cdot \operatorname{curl} G \\ \operatorname{curl}(fF) = f \operatorname{curl} F - F \times \nabla f \end{cases}$

$\left.\begin{array}{l} \operatorname{div}(\operatorname{curl} F) = 0. \\ \operatorname{curl}(\nabla f) = 0 \end{array}\right\}$ if $\begin{cases} F \in C^2. \\ f \in C^2. \end{cases}$

and $\operatorname{div}(\nabla f \times \nabla g) = 0$.

(10) Let $\phi : \mathbb{R} \to \mathbb{R}^n$ be a curve, $f : \mathbb{R}^n \to \mathbb{R}$ be cont., $F : \mathbb{R}^n \to \mathbb{R}^n$

then $\int_\phi f\, ds = \int_A^B f \circ \phi(t) \cdot \|\phi'(t)\|\, dt$. and $\int_\phi F \cdot ds = \int_A^B F(\phi(t)) \cdot \phi'(t)\, dt$.

Theorem $\int_\phi F \cdot ds = f(\phi(B)) - f(\phi(A))$, where $\nabla f = F$.

If $\phi = \psi \circ h$, then $\int_\phi f\, ds = \int_\psi f ds$ and $\int_\phi F \cdot ds = \pm \int_\psi F \cdot ds$,

where the sign depends on whether h is orientation preserving or not.

(11) A surface is a map $f : \mathbb{R}^2 \to \mathbb{R}^3$. It's smooth if its tangent surface exists.

Its t.s. is determined by f_u & f_v by $f_u \times f_v$.

Its area is given by $\iint |f_u \times f_v|\, du\, dv$.

$$\int_\phi f\, ds = \int_u \int_v f(\phi(u,v)) \cdot \|\phi_u \times \phi_v\|\, du \cdot dv$$

$$\int_\phi F \cdot ds = \int_u \int_v F(\phi(u,v)) \cdot (\phi_u \times \phi_v)\, du \cdot dv$$

(12) If C is a simple close curve, with D bounded by C.

and L, M are continuous fns, then

$$\int_C L\, dx + M dy = \int_D \left(-\frac{\partial L}{\partial y} + \frac{\partial M}{\partial x} \right) dx\, dy.$$

(13) For $F:\mathbb{R}^3 \to \mathbb{R}^3$, $\displaystyle\int_C \underset{\substack{\uparrow \\ \text{1-form}}}{F\cdot ds} = \int_S \underset{\substack{\uparrow \\ \text{2-form.}}}{\text{curl}F\cdot dS}$ where S is a surface and C its boundary

(14) $\displaystyle\int_S \underset{\substack{\uparrow \\ \text{2-form}}}{F\cdot dS} = \int_V \underset{\substack{\uparrow \\ \text{3-form}}}{\text{div}F\cdot dU}$

(15) $$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

(16) (1) If f is k-form, df is a $k+1$-form.

(2) $d(df) = 0$

(3) If α is a p-form, $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^p \alpha\wedge d\beta$.

(4) If $\omega = f\,dx^I$, then $d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i \wedge dx^I$.

(5) $\alpha\wedge\beta$ represents the "paralleogram" spanned by α & β.

(6) $\alpha\wedge\beta = (-1)^{kl}\beta\wedge\alpha$ if α is a k-form and β is a l-form.