Rademacher Complexities

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26 Rademacher Complexities

1. **Uniform convergence** is a sufficient condition for learnability.
2. **Rademacher complexities** measures the rate of uniform convergence.

26.1 THE RADEMACHER COMPLEXITY

Definition 1. ***(ϵ-Representative Sample).*** *(w.r.t. domain $Z = (\mathcal{X}, \mathcal{Y}) \sim \mathcal{D}$, hypothesis class $\mathcal{H}$, loss function l). A training set S is called ϵ-representative if*

$$\sup_{h \in \mathcal{H}} |L_{\mathcal{D}}(h) - L_S(h)| \leq \epsilon$$

We have $m_{\mathcal{H}}(\epsilon, \delta) \leq m_{\mathcal{H}}^{UC}(\epsilon/2, \delta)$.

Definition 2. ***(The representativeness of S with respect to $\mathcal{F}$).***

$$Rep_{\mathcal{D}}(\mathcal{F}, S) := \sup_{f \in \mathcal{F}} (L_{\mathcal{D}}(f) - L_S(f)) \tag{1}$$

where,

$$\mathcal{F} := l \circ \mathcal{H} := \{z \mapsto l(h, z) : z \in Z, h \in \mathcal{H}\}$$

$$f \in \mathcal{F}, \quad L_{\mathcal{D}}(f) = \mathbb{E}_{z \sim \mathcal{D}}[f(z)], \quad L_S = \frac{1}{m} \sum_{i=1}^{m} f(z_i).$$

Analogizing the concept of validation set which used to estimate the representativeness of S, we define **rademacher complexity**.

Definition 3. ***(The rademacher complexity of $\mathcal{F}$ w.r.t. S).***

$$R(\mathcal{F} \circ S) := \frac{1}{m} \mathbb{E}_{\sigma \sim \{\pm 1\}^m} \left[\sup_{f \in \mathcal{F}} \sum_{i=1}^{m} \sigma_i f(z_i) \right] \tag{2}$$

where,

$$\mathcal{F} \circ S = \{(f(z_1), \dots, f(z_m)) : f \in \mathcal{F}\}$$

$$\sigma = \{\sigma_i : \mathbb{P}[\sigma_i = 1] = \mathbb{P}[\sigma_i = -1] = 0.5\}$$

More generally, given a set of vectors, $A \subset \mathbb{R}^m$, we define

$$R(A) := \frac{1}{m} \mathbb{E}_{\sigma} \left[\sup_{a \in A} \sum_{i=1}^{m} \sigma_i a_i \right]$$

Lemma 1.

$$\mathbb{E}_{S \sim \mathcal{D}^m}[Rep_{\mathcal{D}}(\mathcal{F}, S)] \leq 2 \mathbb{E}_{S \sim \mathcal{D}^m} R(\mathcal{F} \circ S) \tag{3}$$

Proof. Let $S' = \{z'_1, \dots, z'_m\}$ be another i.i.d. sample.Then,

$$L_{\mathcal{D}}(f) - L_S(f) = \mathbb{E}_{S'}[L_{S'}(f)] - L_S(f) = \mathbb{E}_{S'}[L_{S'}(f) - L_S(f)]$$

$$\begin{aligned} Rep_{\mathcal{D}}(\mathcal{F}, S) = \sup_{f \in \mathcal{F}} (L_{\mathcal{D}}(f) - L_S(f)) &= \sup_{f \in \mathcal{F}} (\mathbb{E}_{S'}[L_{S'}(f) - L_S(f)]) \\ &\leq \mathbb{E}_{S'} \left[\sup_{f \in \mathcal{F}} (L_{S'}(f) - L_S(f)) \right] \end{aligned}$$

$$\mathbb{E}_{S\sim\mathcal{D}^m}[Rep_{\mathcal{D}}(\mathcal{F},S)] \leq \mathbb{E}_{S,S'}\left[\sup_{f\in\mathcal{F}}(L_{S'}(f)-L_S(f))\right] \leq \frac{1}{m}\mathbb{E}_{S,S'}\left[\sup_{f\in\mathcal{F}}\sum_{i=1}^{m}(f(z_i')-f(z_i))\right]$$

In some techniques, we can get:

$$\begin{aligned}\mathbb{E}_{S,S'}\left[\sup_{f\in\mathcal{F}}\sum_{i=1}^{m}(f(z_i')-f(z_i))\right] =&\mathbb{E}_{S,S',\sigma}\left[\sup_{f\in\mathcal{F}}\sum_{i=1}^{m}\sigma_i(f(z_i')-f(z_i))\right]\\ \leq&\mathbb{E}_{S,S',\sigma}\left[\sup_{f\in\mathcal{F}}\sum_{i=1}^{m}\sigma_i(f(z_i'))+\sup_{f\in\mathcal{F}}\sum_{i=1}^{m}(-\sigma_i)f(z_i)\right]\\ =&m\mathbb{E}_{S'}[R(\mathcal{F}\circ S')]+m\mathbb{E}_S[R(\mathcal{F}\circ S)]=2m\mathbb{E}_S[R(\mathcal{F}\circ S)].\end{aligned}$$

□

Theorem 1.

$$\mathbb{E}_{S\sim\mathcal{D}^m}[L_{\mathcal{D}}(ERM_{\mathcal{H}}(S))-L_S(ERM_{\mathcal{H}}(S))] \leq 2\mathbb{E}_{S\sim\mathcal{D}^m}R(l\circ\mathcal{H}\circ S)$$

$\mathbb{E}_{S\sim\mathcal{D}^m}[L_{\mathcal{D}}(ERM_{\mathcal{H}}(S))-L_S(h^*)] \leq 2\mathbb{E}_{S\sim\mathcal{D}^m}R(l\circ\mathcal{H}\circ S),$ *where* $h^*=\arg\min_h L_{\mathcal{D}}(h)$

Because $L_{\mathcal{D}}(ERM_{\mathcal{H}}(S))-L_{\mathcal{D}}(h^*)\geq 0$, *then*

$$\mathbb{P}\left\{L_{\mathcal{D}}(ERM_{\mathcal{H}}(S))-L_{\mathcal{D}}(h^*)\geq\frac{2\mathbb{E}_{S'\sim D^m}R(l\circ\mathcal{H}\circ S')}{\delta}\right\}\leq\delta$$

Lemma 2. ***(McDiarmid's Inequality).***
If

$$|f(x_1,\ldots,x_m)-f(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_m)|\leq c_i.$$

then,

$$\mathbb{P}\{f-\mathbb{E}f\geq\epsilon\}\leq\exp\left(\frac{-2\epsilon^2}{\sum_{i=1}^m c_i}\right)$$
$$\mathbb{P}\{f-\mathbb{E}f\leq-\epsilon\}\leq\exp\left(\frac{-2\epsilon^2}{\sum_{i=1}^m c_i}\right)$$

which also means

$$\mathbb{P}\left\{|f-\mathbb{E}f|\geq c\sqrt{\ln(2/\delta)m/2}\right\}\leq\delta$$

Theorem 2. ***(Data-dependent bound).*** *Assume that for all z and $h\in\mathcal{H}$, we have that $|l(h,z)|\leq c$. Then,*

1. $$\mathbb{P}\left\{\forall h\in\mathcal{H},L_{\mathcal{D}}(h)-L_S(h)\leq 2\mathbb{E}_{S'\sim\mathcal{D}^m}R(l\circ\mathcal{H}\circ S')+c\sqrt{2\ln(1/\delta)/m}\right\}\geq 1-\delta$$

   Proof. $Rep_{\mathcal{D}}(\mathcal{F},S)$ satisfies the preceeding condition with a constant $2c/m$, □

2. $$\mathbb{P}\left\{\forall h\in\mathcal{H},L_{\mathcal{D}}(h)-L_S(h)\leq 2R(l\circ\mathcal{H}\circ S)+3c\sqrt{2\ln(2/\delta)/m}\right\}\geq 1-\delta$$

Proof.

$$\mathbb{P}\left\{Rep_{\mathcal{D}}(F,S) \le \mathbb{E}_{S'} Rep_{\mathcal{D}}(l \circ \mathcal{H} \circ S') + c\sqrt{2\ln(2/\delta)/m}\right\} \ge 1-\delta/2$$

$$\mathbb{P}\left\{\mathbb{E} Rep_{\mathcal{D}}(F,S) \le 2\mathbb{E} R(l \circ \mathcal{H} \circ S')\right\} = 1$$

$$\mathbb{P}\left\{\mathbb{E}_{S'} R(l \circ \mathcal{H} \circ S') \le R(l \circ \mathcal{H} \circ S) + c\sqrt{2\ln(2/\delta)/m}\right\} \ge 1-\delta/2$$

□

3.

$$\mathbb{P}\left\{L_{\mathcal{D}}(ERM_{\mathcal{H}}(S)) - L_S(h^*) \le 2R(l \circ \mathcal{H} \circ S) + 4c\sqrt{2\ln(3/\delta)/m}\right\} \ge 1-\delta$$

Proof.

$$\begin{aligned} L_{\mathcal{D}}(h_S) - L_{\mathcal{D}}(h^*) =& L_{\mathcal{D}}(h_S) - L_S(h_S) + L_S(h_S) - L_S(h^*) + L_S(h^*) - L_{\mathcal{D}}(h^*) \\ \le& (L_{\mathcal{D}}(h_S) - L_S(h_S)) + (L_S(h^*) - L_{\mathcal{D}}(h^*)) \end{aligned}$$

$$\mathbb{P}\left\{L_{\mathcal{D}}(h_S) - L_S(h_S) \le 2R(l \circ \mathcal{H} \circ S) + 3c\sqrt{2\ln(3/\delta)/m}\right\} \ge 1-2\delta/3$$

Because $L_{\mathcal{D}}(h^*)$ does not depend on S, so we can use hoeffding's inequality to get

$$\mathbb{P}\left\{L_S(h^*) - L_{\mathcal{D}}(h^*) \le c\sqrt{2\ln(3/\delta)/m}\right\} \ge 1-\delta/3$$

□

26.1.1 Rademacher Calculus

Lemma 3. *$\forall A \subset \mathbb{R}^m, c \in \mathbb{R}, \mathbf{a}_0 \in \mathbb{R}^m$, we have*

$$R(\{c\mathbf{a} + \mathbf{a}_0 : \mathbf{a} \in A\}) = |c|R(A) \tag{4}$$

Lemma 4. *$\forall A \subset \mathbb{R}^m$, if $A' = \left\{\sum_{j=1}^N \alpha_j \mathbf{a}^{(j)} : N \in \mathbb{N}, \forall j, \mathbf{a}^{(j)} \in A, \alpha_j \ge 0, \|\vec{\alpha}\|_1 = 1\right\}$, then $R(A') = R(A)$.*

Proof.

$$\begin{aligned} mR(A') =& \mathbb{E}_\sigma \sup_{\vec{\alpha} \succeq \vec{0} : \|\vec{\alpha}\|_1 = 1} \sup_{\mathbf{a}^{(1)},\ldots,\mathbf{a}^{(N)}} \sum_{i=1}^m \sigma_i \sum_{j=1}^N \alpha_j a_i^{(j)} \\ =& \mathbb{E}_\sigma \sup_{\vec{\alpha} \succeq \vec{0} : \|\vec{\alpha}\|_1 = 1} \sum_{j=1}^N \alpha_j \sup_{\mathbf{a}^{(j)}} \sum_{i=1}^m \sigma_i a_i^{(j)} \\ =& \mathbb{E}_\sigma \sup_{\mathbf{a} \in A} \sum_{i=1}^m \sigma_i a_i \end{aligned}$$

□

Lemma 5. ***(Massart Lemma)***. *Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ be a finite set of vectors in $\mathbf{R}^m$. Define $\bar{\mathbf{a}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{a}_i$. Then,*

$$R(A) \leq \max_{\mathbf{a}\in A} \|\mathbf{a} - \bar{\mathbf{a}}\| \frac{\sqrt{2\log(N)}}{m} \tag{5}$$

Proof.

$$\begin{aligned}
\forall A, \quad mR(A) =& \mathbb{E}_{\vec{\sigma}}\left[\max_{\mathbf{a}\in A}\langle\vec{\sigma}, \mathbf{a}\rangle\right] = \mathbb{E}_{\vec{\sigma}}\left[\log\left(\max_{\mathbf{a}\in A} e^{\langle\vec{\sigma},\mathbf{a}\rangle}\right)\right] \\
=& \mathbb{E}_{\vec{\sigma}}\left[\log\left(\sum_{\mathbf{a}\in A} e^{\langle\vec{\sigma},\mathbf{a}\rangle}\right)\right] \leq \log\left[\mathbb{E}_{\vec{\sigma}}\left(\sum_{\mathbf{a}\in A} e^{\langle\vec{\sigma},\mathbf{a}\rangle}\right)\right] \\
\leq& \log\left(\sum_{\mathbf{a}\in A}\prod_{i=1}^{m}\mathbb{E}_{\sigma_i}[e^{\sigma_i a_i}]\right) \leq \log\left(\sum_{\mathbf{a}\in A}\prod_{i=1}^{m}[e^{a_i} + e^{-a_i}]/2\right) \\
\leq& \log\left(\sum_{\mathbf{a}\in A}\prod_{i=1}^{m} e^{a_i^2/2}\right) = \log\left(\sum_{a\in A}\exp\left(\|\mathbf{a}\|^2/2\right)\right) \\
\leq& \log\left(|A|\max_{\mathbf{a}\in A}\exp\left(\|\mathbf{a}\|^2/2\right)\right) = \log\left(|A|\right) + \max_{\mathbf{a}\in A}(\|\mathbf{a}\|^2/2)
\end{aligned}$$

Since $R(A) = R(A')/\lambda$ we obtain that

$$R(A) \leq \frac{\log(|A|) + \lambda^2 \max_{\mathbf{a}\in A}(\|\mathbf{a}\|^2/2)}{\lambda m}$$

□

Lemma 6. ***(Contraction Lemma).*** *$\forall i \in [m]$, let $\phi_i : \mathbb{R} \to \mathbb{R}$ be a $\rho -$ Lipschitz function. For $\mathbf{a} \in \mathbb{R}^m$ let $\phi(a) = (\phi_1(a1), \ldots, \phi_m(y_m))$. Let $\phi \circ A = \{\phi(\vec{a}) : a \in A\}$. Then,*

$$R(\phi \circ A) \leq \rho R(A).$$

Proof. First, $\rho = 1$. Let $A_i = \{(a_1, \ldots, a_{i-1}, \phi_i(a_i), a_{i+1}, \ldots, a_m) : \mathbf{a} \in A\}$.

$$\begin{aligned}
mR(A_1) =& \mathbb{E}_{\sigma}\left[\sup_{\mathbf{a}\in A_1}\sum_{i=1}^{m}\sigma_i a_i\right] \\
=& \mathbb{E}_{\sigma}\left[\sup_{\mathbf{a}\in A}\sigma_1\phi(a_1) + \sum_{i=2}^{m}\sigma_i a_i\right] \\
=& \frac{1}{2}\mathbb{E}_{\sigma_2,\ldots,\sigma_m}\left[\sup_{\mathbf{a},\mathbf{a}'\in A}\left(\phi(a_1) - \phi(a_1') + \sum_{i=2}^{m}\sigma_i a_i + \sum_{i=2}^{m}\sigma_i a_i'\right)\right] \\
\leq& \frac{1}{2}\mathbb{E}_{\sigma_2,\ldots,\sigma_m}\left[\sup_{\mathbf{a},\mathbf{a}'\in A}\left(|a_1 - a_1'| + \sum_{i=2}^{m}\sigma_i a_i + \sum_{i=2}^{m}\sigma_i a_i'\right)\right] \\
=& \frac{1}{2}\mathbb{E}_{\sigma_2,\ldots,\sigma_m}\left[\sup_{\mathbf{a},\mathbf{a}'\in A}\left(a_1 - a_1' + \sum_{i=2}^{m}\sigma_i a_i + \sum_{i=2}^{m}\sigma_i a_i'\right)\right] \\
mR(A_1) \leq& mR(A)
\end{aligned}$$

□

26.2 RADEMACHER COMPLEXITY OF LINEAR CLASSES

1. $\mathcal{H}_1 = \{\mathbf{x} \mapsto \langle \mathbf{w}, \mathbf{x} \rangle : \|\mathbf{w}\|_1 \leq 1\}$
2. $\mathcal{H}_2 = \{\mathbf{x} \mapsto \langle \mathbf{w}, \mathbf{x} \rangle : \|\mathbf{w}\|_2 \leq 1\}$

Lemma 7.

$$R(\mathcal{H}_2 \circ S) \leq \frac{\max_i \|\mathbf{x}_i\|_2}{\sqrt{m}} \tag{6}$$

Proof.

$$\begin{aligned}
mR(\mathcal{H}_2 \circ S) =& \mathbb{E}_\sigma \left[\sup_{\mathbf{a} \in \mathcal{H}_2 \circ S} \sum_{i=1}^m \sigma_i a_i \right] = \mathbb{E}_\sigma \left[\sup_{\mathbf{w}:\|\mathbf{w}\|_2 \leq 1} \sum_{i=1}^m \sigma_i \langle \mathbf{w}, \mathbf{x}_i \rangle \right] \\
=& \mathbb{E}_\sigma \left[\sup_{\mathbf{w}:\|\mathbf{w}\|_2 \leq 1} \langle \mathbf{w}, \sum_{i=1}^m \sigma_i \mathbf{x}_i \rangle \right] \leq \mathbb{E}_\sigma \left[\| \sum_{i=1}^m \sigma_i \mathbf{x}_i \|_2 \right] \\
\leq& \left(\mathbb{E}_\sigma \left[\| \sum_{i=1}^m \sigma_i \mathbf{x}_i \|^2 \right] \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_\sigma \left[\| \sum_{i=1}^m \sigma_i \mathbf{x}_i \|^2 \right] &= \sum_{i \neq j} \langle \mathbf{x_i}, \mathbf{x_j} \rangle \mathbb{E}_\sigma [\sigma_i \sigma_j] + \sum_{i=1}^m \langle \mathbf{x}_i, \mathbf{x}_i \rangle \mathbb{E}_\sigma [\sigma_i^2] \\
&= \sum_{i=1}^m \|\mathbf{x}_i\|_2^2 \leq m \max_i \|\mathbf{x}_i\|_2^2
\end{aligned}$$

□

Lemma 8. *Let $S = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$ be the vectors in $\mathbb{R}^n$, then,*

$$R(\mathcal{H}_1 \circ S) \leq \max_i \|\mathbf{x}_i\|_\infty \sqrt{\frac{2\log(2n)}{m}} \tag{7}$$

Proof. Using Holder's inequality, we have $\langle \mathbf{w}, \mathbf{v} \rangle \leq \|\mathbf{w}\|_1 \|\mathbf{v}\|_\infty$. Therefore,

$$mR(\mathcal{H}_1 \circ S) = \mathbb{E}_\sigma \left[\sup_{\mathbf{w}:\|\mathbf{w}\|_1 \leq 1} \langle \mathbf{w}, \sum_{i=1}^m \sigma_i \mathbf{x}_i \rangle \right] \leq \mathbb{E}_\sigma \left[\| \sum_{i=1}^m \sigma_i \mathbf{x}_i \|_\infty \right].$$

□