HOW TO BAKE π

An Edible Exploration of the Mathematics of Mathematics

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To

my parents

and Martin Hyland

In memory of

Christine Pembridge

Contents

They say mathematics is a glorious garden. I know I would certainly lose my way in it without your guidance. Thank you for walking us through the most beautiful entrance pathway.

From a student's letter to the author
University of Chicago, June 2014

Prologue

Here is a recipe for clotted cream.

Ingredients

Cream

Method

1. Pour the cream into a rice cooker.
2. Leave it on the "keep warm" setting with the lid slightly open, for about 8 hours.
3. Cool it in the fridge for about 8 hours.
4. Scoop the top part off: that's the clotted cream.

What on earth does this have to do with math?

Math Myths

Myth: *"Math is all about numbers."*

You might think that rice cookers are for cooking rice. This is true, but the same piece of equipment can be used for other things as well: making clotted cream, cooking vegetables, steaming a chicken. Likewise, math is about numbers, but it's about many other things as well.

Myth: *"Math is all about getting the right answer."*

Cooking is about ways of putting ingredients together to make delicious food. Sometimes it's more about the method than the ingredients, just as in the recipe for clotted cream, which only has one ingredient—the entire recipe is just a method. Math is about ways of putting ideas together to make exciting new ideas. And sometimes it's more about the method than the "ingredients."

Myth: *"Math is all either right or wrong."*

Cooking can go wrong—your eggs can curdle, your soufflé can collapse, your chicken can be undercooked and give everyone food poisoning. But even if it doesn't poison you, some food tastes better than other food. And sometimes when cooking goes "wrong" you have actually accidentally invented a delicious new recipe. Fallen chocolate soufflé is deliciously dark and gooey. If you forget to melt the chocolate for your cookies, you get chocolate chip cookies. Math is like this too. In high school if you write $10 + 4 = 2$ you will be told that is wrong, but actually that's correct in some circumstances, such as telling the time—four hours later than 10:00 is indeed 2:00. The world of math is more weird and wonderful than some people want to tell you.

Myth: *"You're a mathematician? You must be really clever."*

Much as I like the idea that I am very clever, this popular myth shows that people think math is hard. The little-understood truth is that the aim of math is to make things easier. Herein lies the problem—if you need to make things easier, it gives the impression that they were hard in the first place. Math *is* hard, but it makes hard things easier. In fact, since math is a hard thing, math also makes math easier.

Many people are afraid of math, or baffled by it, or both. Or they were completely turned off it by their classes in high school. I understand this—I was completely turned off sports in high school and have never really recovered. I was so bad at sports in high school, my teachers were incredulous that anybody so bad at sports could exist. And yet I'm quite fit now and have even run the New York City Marathon. At least I now appreciate physical exercise, but I still have a horror of any kind of team sports.

Myth: *"How can you do research in math? You can't just discover a new number."*

This book is my answer to that question. It's hard to answer it quickly at a cocktail party without sounding trite, or taking up too much of someone's time, or shocking the gathered company. Yes, one way to shock people at a polite party is to talk about math.

It's true, you can't just discover a new number. So what can we discover that's new in math? In order to explain what this "new math" could possibly be about, I need to clear up some misunderstandings about what math is in the first place. Indeed, not only is math not just about numbers, but the branch of math I'm going to describe is actually not about numbers at all. It's called

CATEGORY THEORY

and it can be thought of as the "mathematics of mathematics." It's about relationships, contexts, processes, principles, structures, cakes, custard.

Yes, even custard. Because mathematics is about drawing analogies, and I'm going to be drawing analogies with all sorts of things to explain how math works, including custard, cake, pie, pastry, donuts, bagels, mayonnaise, yogurt, lasagne, sushi.

Whatever you think math is . . . let go of it now. This is going to be different.

Part I

Math

Chapter 1

What Is Math?

Gluten-Free Chocolate Brownies

Ingredients

4 oz. butter
5 oz. dark chocolate
2 medium eggs
6 oz. sugar
3 oz. potato flour

Method

1. Melt the butter and chocolate, stir together, and allow to cool a little.
2. Whisk the eggs and the sugar together until fluffy.
3. Beat the chocolate into the egg mixture slowly.
4. Fold in the potato flour.
5. Bake in very small individual cupcake liners at 350°F for about 10 minutes.

Math, like recipes, has both ingredients and method. And just as a recipe would be a bit useless if it omitted the method, we can't understand what math is unless we talk about the *way it is done*, not just the *things it studies*. Incidentally the method in the above recipe is quite important—these don't cook very well in a large tray. In math the method is perhaps even more important than the ingredients. Math probably isn't whatever you studied in high school in classes called "math." Yet somehow I always knew that math was more than what we did in high school. So what *is* math?

Recipe Books

What If We Organized Recipes by Equipment?

Cooking often proceeds a bit like this: you decide what you want to cook, you buy the ingredients, and then you cook it. Sometimes it might work the other way round: you go wandering through the store or maybe a market, you see what ingredients look good, and you feel inspired by them to conjure up your meal. Perhaps there's some particularly fresh fish, or a type of mushroom you've never seen before, so you buy it and go home and look up what to do with it afterwards.

Occasionally something completely different happens: you buy a new piece of equipment, and suddenly you want to try making all sorts of different things with that equipment. Perhaps you bought a blender, and suddenly you make soup, smoothies, ice cream. You try making mashed potatoes in it, and it goes horribly wrong (it looks like glue). Maybe you bought a slow cooker. Or a steamer. Or a rice cooker. Perhaps you learn a new technique, like separating eggs or clarifying butter, and suddenly you want to make as many things as possible involving your new technique.

So we might approach cooking in two ways, and one seems much more practical than the other. Most recipe books are divided up according to parts of the meal rather than by techniques. There's a chapter on appetizers, a chapter on soup, a chapter on fish, a chapter on meat, a chapter on dessert, and so on. There might be a whole chapter on an ingredient—say, a chapter on chocolate recipes or vegetable recipes. Sometimes there are whole chapters on particular meals—say, a chapter on Christmas dinner. But it would be quite odd to have a chapter on "recipes that use a rubber spatula" or "recipes that use a balloon whisk." Having said that, kitchen gadgets often come with useful booklets of recipes you can make with your new equipment. A blender will come with blender recipes; likewise a slow cooker or an ice cream maker.

Something similar is true of subjects of research. Usually when you say what a subject is, you describe it according to the thing that you're studying. Maybe you study birds, or plants, or food, or cooking, or how to cut hair, or what happened in the past, or how society works. Once you've decided what you're going to study, you learn the techniques for studying it, or you invent new techniques for studying it, just as you learn how to whisk egg whites or clarify butter.

In math, however, the things we study are also determined by the techniques we use. This is similar to buying a blender and then going round seeing what you can make with it. This is more or less backwards compared with other subjects. Usually the techniques we use are determined by the things we're studying; usually we decide what we want for dinner and then get out the equipment for making it. But when we're really excited about our new blender, we try to make all our dinners with it for a while. (At least I've seen people do this.)

It's a bit of a chicken-and-egg question, but I am going to argue that math is defined by the techniques it uses to study things, and that the things it studies are determined by those techniques.

Cubism

When the Style Affects the Choice of Content

Characterizing math by the techniques it uses is similar to defining styles of art, like cubism or pointillism or impressionism, where the genre is defined by the techniques rather than by the subject matter. Or ballet and opera, where the art form is defined by the methods and the subject matter is duly restricted. Ballet is very powerful at expressing emotion but not so good at expressing dialogue or making demands for political change. Cubism is not that effective for depicting insects. Symphonies are good at expressing tragedy and joy but not very good at saying "Please pass the salt."

In math the technique we use is *logic*. We only want to use sheer logical reasoning. Not experiments, not physical evidence, not blind faith or hope or democracy or violence. Just logic. So what are the things we study? We study *anything that obeys the rules of logic*.

> **Mathematics is the study of anything that obeys the rules of logic, using the rules of logic.**

I will admit immediately that this is a somewhat simplistic definition. But I hope that after reading some more you'll see why this is accurate as far as it goes, not as circular as it sounds at first, and just the sort of thing a category theorist would say.

The Prime Minister

Characterizing Something by What It Does

Imagine if someone asked you "Who's the prime minister?" and you answered "He's the head of the government." This would be correct but annoying, and not really answering the right question: you've characterized the prime minister without telling us who it is. Likewise, my "definition" of mathematics has *characterized* math rather than telling you what it is. This is a little unhelpful, or at least incomplete—but it's just the start.

Instead of describing what math is *like*, can we say what math *is*? What does math actually study? It definitely studies numbers, but also other things like shapes, graphs, and patterns, and then things that you can't see—logical ideas. And more than that: things we don't even know about yet. One of the reasons math keeps growing is that once you have a technique, you can always find more things to study with it, and then you can find more techniques to use to study those things, and then you can find more things to study with the new techniques, and so on, a bit like chickens laying eggs that hatch chickens that lay eggs that hatch chickens...

Mountains

Conquering One Enables You to See the Higher Ones

Do you know that feeling of climbing to the top of a hill, only to find that you can now see all the higher hills beyond it? Math is like that too. The more it progresses, the more things it comes up with to study. There are, broadly, two ways this can happen.

First, there's the process of *abstraction*. We work out how to think logically about something that logic otherwise couldn't handle. For example, you previously only made rice in your rice cooker, and then you work out that you can use it to make cake, it's just a bit different from cake made the normal way in an oven. We take something that wasn't really math before, and look at it differently to turn it into math. This is the reason that x's and y's start appearing—we start by thinking about numbers, but then realize that the things we do with numbers can be done to other things as well. This will be the subject of the next chapter.

Secondly, there's the process of *generalization*: we work out how to build more complicated things out of the things we've already understood. This is like making a cake in your blender, and making the frosting in your blender, and then piling it all up.[†] In math this is how we get things like polynomials and matrices, complicated shapes, four-dimensional space, and so on, out of simpler things like numbers, triangles, and our everyday world. We'll look into this in Chapter 5.

These two processes, abstraction and generalization, will be the subject of the next few chapters, but first I want to draw your attention to something weird and wonderful about how math does these two things.

Birds

They Are Not the Same as the Study of Birds

Imagine for a second that you study birds. You study their behavior, what they eat, how they mate, how they look after their young, how they digest food,

[†] Mathematical generalization isn't the same as the kind where you go round making sweeping statements about things, but we'll come to that later.

and so on. However, you will never be able to build a new bird out of simpler birds—that just isn't how birds are made. So you can't do generalization, at least not in the way that math does it.

Another thing you can't do is take something that isn't a bird and miraculously turn it into a bird. That also isn't how birds are made. So you can't do abstraction either. Sometimes we might realize we've made a mistake of classification—for example, the brontosaurus "became" a form of apatosaurus. However, we didn't turn the brontosaurus into an apatosaurus—we merely realized it had been one all along. We're not magicians, so we can't change something into something it isn't. But in math we can, because math studies ideas of things, rather than real things, so all we have to do to change the thing we're studying is to change the idea in our head. Often this means changing the way we think about something, changing our point of view, or changing how we express it.

A mathematical example is knots.

In the eighteenth and nineteenth centuries Vandermonde, Gauss and others worked out how to think of knots mathematically, so that they could be studied using the rules of logic. The idea is to imagine sticking together the two ends of the piece of string so that becomes a closed loop. This makes the knots impossible to create without glue, but much easier to reason with mathematically. Each one can be expressed as a circle that has been mapped to three-dimensional space. There are many techniques for studying this kind of thing in the field of *topology*, which we'll come back to later. Not only can we then deduce things about real knots in string, but also about the apparently impossible ones arise in nature in molecular structures.

Geometric shapes are another, much older example of this process of turning something from the "real" world into something in the "mathematical" world. We can think of math as developing in the following stages:

1. It started as the study of numbers.
2. Techniques were developed to study those numbers.
3. People started realizing that those techniques could be used to study other things.
4. People went round looking for other things that could be studied like this.

Actually, there's a step 0, before the study of numbers: someone had to come up with the idea of numbers in the first place. We think of them as the most basic things you can study in math, but there was a time before numbers. Perhaps the invention of numbers was the first-ever process of abstraction.

The story I'm going to tell is about abstract mathematics. I'm going to argue that its power and beauty lie not in the answers it provides or the problems it solves, but in the *light* that it sheds. That light enables us to see clearly, and that is the first step to understanding the world around us.

Chapter 2

Abstraction

Mayonnaise or Hollandaise Sauce

Ingredients

2 egg yolks
$1\frac{1}{2}$ cups olive oil
Seasoning

Method

1. Whisk the egg yolks and seasoning using a hand whisk or immersion blender.
2. Drip the olive oil in very slowly while continuing to whisk.

For Hollandaise sauce, use $\frac{1}{2}$ cup melted butter instead of the olive oil.

At some level mayonnaise and hollandaise sauce are the same—they use the same method, but with a different type of fat incorporated into the egg yolk. In both cases, the amazing near-magic properties of egg yolks create something rich and unctuous. It looks so much like magic, I never tire of watching it happen.

The similarity between mayonnaise and hollandaise sauce is the sort of thing that mathematics goes round looking for—situations where things are somehow the same apart from some small detail. This is a way of saving effort, so that you can understand how to do both things at once. Books might tell you that hollandaise sauce needs to be done differently, but I ignore them to make my life simpler. Math is also there to make things simpler, by finding things that look the same if you ignore some small details.

Pie

Abstractions as Blueprints

Cottage pie, shepherd's pie, and fisherman's pie are all more or less the same—the only difference is the filling that is sitting underneath the mashed potato topping. Fruit crisp is also very similar—you don't really need a different recipe for different types of crisp, you just need to know how to make the topping. Then you put the fruit of your choice in a dish, add the topping, and bake it.

Another favorite of mine is upside-down cake. You put the fruit in the bottom of the cake pan, pour the cake batter on top, and after baking it you turn it out upside-down so that the fruit is on top. For extra effect you can put melted butter and brown sugar on the bottom of the cake tin first, to caramelize the fruit a bit. Of course, this works better with some fruit than others: bananas, apples, pears, and plums work well, grapes less well. Watermelon would be terrible. The same is true for crisp. Watermelon crisp? Probably not.

Savory tarts and quiches also follow a general pattern. You bake an empty pastry shell, put in some filling of your choice, and then top it up with a mixture of egg and milk or cream before baking it again. The filling could be bacon and cheese, or fish, or vegetables—whatever you feel like.

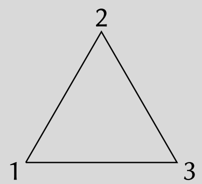
In all these cases the "recipe" is not a full recipe but a blueprint. You can insert your own choice of fruit, meat, or other fillings to make your own variations, within reason.

This is also how math works. The idea of math is to look for similarities between things so that you only need one "recipe" for many different situations. The key is that when you ignore some details, the situations become easier to understand, and you can fill in the variables later. This is the process of abstraction.

As with the watermelon crisp, once you've made the abstract "recipe" you will find that you won't be able to apply it to *everything*. But you are at least in a position to try, and sometimes surprising things turn out to work in the same recipe.

Think about the symmetry of an equilateral triangle. It has two types of symmetry: reflectional and rotational. How can we describe the different symmetries without cutting out the triangle and folding it up or waving it around?

One way is that we could label the corners 1, 2, and 3,



and then just talk about how the numbers get swapped around. For example, if we reflect the triangle in a vertical line, we will swap the numbers 1 and 3. Whereas if we rotate the triangle 120° clockwise we will send 1 to where 2 was, 2 to where 3 was, and 3 to where 1 was.

You can try checking that the six symmetries of the triangle correspond exactly to the six different ways of shuffling the numbers 1, 2, and 3. There are three lines of symmetry, and they correspond to swapping 1 and 3, or 1 and 2, or 2 and 3. There are three types of rotational symmetry: 120° clockwise, 240° clockwise, and the "trivial" one where nothing moves.

This shows that the symmetry of an equilateral triangle is *abstractly* the same as the permutations of the numbers 1, 2, and 3, and the two situations can be studied at the same time.

Kitchen clutter

Abstraction as Tidying away the Things You Don't Need

Abstraction is like preparing to cook something and putting away the equipment and ingredients that you don't need for this recipe, so that your kitchen is less cluttered. It is the process of putting away the ideas you don't need for the present purposes, so that your *brain* is less cluttered.

Are you better at this in your kitchen or in your brain? (I am definitely better at it in my brain.) Abstraction is the important first step of doing mathematics. It's also a step that can make you feel uneasy because you're stepping away from reality a little bit. I never put my blender away because it's such a hassle to move it, and I want to know that I can use it at any time without going through the rigmarole of getting it out of the cabinet. You might feel like that about abstraction in the brain as well.

Try the following problem.

> *I buy two stamps for 36¢ each. How much does it cost?*

When children do this sort of thing in elementary school it sometimes gets called a "word problem," because it has been stated in words, and they're told that the first step in solving this "word problem" is to turn it into numbers and symbols:

$$36 \times 2 = ?$$

This is a process of abstraction. We have thrown away, or ignored, the fact that the thing we are buying is *stamps*, because it doesn't make any difference to the answer. It could be apples, bananas, monkeys... the equation would still be the same, and so the answer would still be the same: 72 of whatever we're buying.

What about this one:

> *My father is three times as old as I am now but in ten years' time he will be twice as old as me. How old am I?*

Or this one:

> *I have a recipe for frosting the top and sides of a 6-inch cake. How much frosting do I need for the top and sides of an 8-inch cake?*

For the question about stamps you probably didn't need to write down an equation, because the answer was immediately obvious to you. However for these last two questions perhaps you would need to perform some abstraction to work out the answer, where you throw away the fact that you're talking about your father, or a cake and frosting, and write down some arithmetic, with numbers and symbols. We'll see what arithmetic we get from these word problems a bit later in this chapter.

Cookies

How Things That Are Too Real Don't Obey Mathematics

If you've ever tried teaching arithmetic to small children, you might have come up with the following problem. You try and get them to think about a real-life situation such as:

> *If Grandma gives you five cookies and Grandpa gives you five cookies, how many cookies will you have?*

And the child answers: "None, because I'll eat them all!"

The trouble here is that cookies do not obey the rules of logic, so using math to study them doesn't quite work. Can we force cookies to obey logic? We could impose an extra rule on the situation by adding "…and you're not allowed to eat the cookies." If you're not allowed to eat them, what's the point of them being cookies? We could treat the cookies as just *things* rather than cookies. We lose some resemblance to reality, but we gain scope and with it efficiency. The point of numbers is that we can reason about "things" without having to change the reasoning depending on what "thing" we are thinking about. Once we know that $2 + 2 = 4$ we know that two things and another two things make four things, whether they are cookies, monkeys, houses, or anything else. That is the process of abstraction: going from cookies, monkeys, houses, or whatever, to numbers.

Numbers are so fundamental, it's difficult to imagine life without them, and difficult to imagine the process of inventing them. We don't even notice that we're making a leap of abstraction when we count things. It's much more noticeable if you watch small children struggling to do it, because they're not yet used to making that leap.

Eeny Meeny Miny Moe

Numbers as an Abstraction

I remember a wonderfully feisty mother at an elementary school I was helping at. She remarked on how frustrating it was when other mothers competitively declared that *their* child could count up to 20 or 30. "My son can count up to three," she said defiantly, "But he knows what three *is*."

And she had a point. When children first "learn to count to ten" they aren't really doing more than learning to recite a little poem, like "The itsy-bitsy spider climbed up the water spout. . . ." It just so happens that the "poem" goes:

"One, two, three, four, five, six, . . . "

Then they learn that this has something to do with pointing at things, so they start pointing while reciting the "poem," a bit haphazardly. Only later do they learn that they're supposed to point at one thing per word in the poem, but they have trouble making sure they have only pointed at each thing once, so they will get rather variable answers if you ask them, "How many ducks are in this picture?" Or they might latch on to a particular number—say, six—and somehow manage to count everything as being six, no matter how many ducks there really are.

Finally they'll get the idea that they're supposed to match up the items rather precisely with the words in the poem, one item per word, no more and no less. That is when they *really* know how to count. This is a process of abstraction, and a surprisingly profound one.

Imagine trying to engage in trade without knowing how to count. "Hey, I'll trade you one sack of grain for each of your sheep," you say, and then you go and line up sacks of grain against sheep to make sure you really have one per sheep. Then you work out that it's more practical to recite a little poem while pointing at the sheep in rhythm, and do the same thing with the sacks of grain. The poem could be anything as long as you recite it exactly the same way for the sheep and for the grain. It could be "Eeny meeny miny moe."

Finally you make up a poem once and for all to use for all your trades, and you stick to it. And suddenly you've invented numbers. That is the process of abstraction that we don't even notice when we learn to count. So we see that there is a crucial difference between simply learning the poem "One, two, three, four. . . " and understanding how to use it.

The Baby and the Bathwater

Being Careful Not to Throw Away Too Much

It is important, as everyone knows, not to throw the baby out with the bathwater. When we go round simplifying or idealizing our situations, we must be careful not to *oversimplify*—we must not simplify our objects to the point that they've lost *all* of their useful characteristics. If we're thinking about stacking Lego blocks, for example, we can ignore what color they are, but we shouldn't ignore what size they are, as that affects how we can stack them. But in another situation we might be using Lego bricks merely as counting blocks, in which case we can ignore their size as well.

Choosing what features to ignore should depend heavily on what context we're thinking about. This is a theme that will come back importantly later. Category theory brings context to the forefront.

Suppose you're organizing an outing for 100 people, and you're renting minibuses that can hold 15 people each. How many minibuses do you need? Basically you need to calculate

$$100 \div 15 \approx 6.7$$

But then you have to take the context into account: you can't book 0.7 of a minibus, so you have to round up to 7 minibuses.

Now consider a different context. You want to send a friend some chocolates in the mail, and a first-class stamp is valid for up to 100 g. The chocolates weigh 15 g each, so how many chocolates can you send? You still need to start with the same calculation

$$100 \div 15 \approx 6.7$$

But this time the context gives a different answer: since you can't send 0.7 of a chocolate, you'll need to round *down* to 6 chocolates.

Heartbreak

Abstraction as Simplification

After one major episode of heartbreak I was getting tired of well-meaning friends asking me for more and more details of exactly what happened, in an attempt to "understand" it. Finally one wise friend said to me, "It's very simple, really. You've lost something you loved." That was all anyone needed to know of the situation. She then successfully distracted me into a long discussion about how it's really more intelligent to be able to simplify things than to complicate them, even if some people think it makes you look stupid. There's a subtle difference between something that's "simple" and something that's "simplistic"; the latter indicates that you've missed the point and ignored a complication that is crucial.

My friend's wisdom was a type of abstraction, abstracting heartbreak down to its very essence. Abstraction can appear to take you further and further away from reality, but really you're getting closer and closer to the heart of the matter. To get to the heart, you have to strip away clothes and skin and flesh and bone.

Road Signs

Abstraction as the Study of Ideal Versions of Things

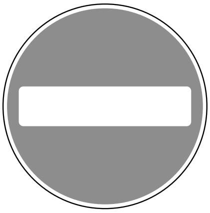
Road signs are a form of abstraction. They don't precisely depict what is going on in the road but represent some idealized form of it, where just the essence is captured. Not every humpbacked bridge looks exactly like this:†

† Road sign images are Crown Copyright and reproduced under the Open Government Licence.

but this captures the essence of humpbacked-bridge-ness. Similarly, not all children crossing the road look exactly like this:

Nevertheless, the benefits of this system are clear. It's much quicker to take in a symbol than read some words while you are driving. Also it's much easier for foreigners to understand. The disadvantage is that when you first start driving you have to learn what all these funny symbols mean. Some of them, such as this one,

are much closer to reality than others, like this:



This "No entry" sign is entirely abstract: it doesn't look like the thing it is representing at all. (What does "No entry" look like?) But it's also more important—you will probably encounter more of those in your driving life than the one warning you there might be deer crossing the road.

One side effect of the abstraction of math is that a variety of funny symbols gets used as well, for the same sorts of reasons: once you know what they mean, the symbols are much quicker to take in, and you can reserve your mathematical brainpower for the more complicated parts of the math you're supposed to be focusing on. It also makes the math easier to understand across different languages—it's surprisingly easy to read a math book in a language you don't know.

The most basic "funny symbols" used in math are the ones for normal arithmetic: $+, -, \times, \div, =$. Once you're comfortable with these symbols, it's much quicker and easier to read

$$2 + 2 = 4$$

than "two plus two equals four." As math gets more and more complicated, the symbols get more and more complicated as well, with things like

$$\sum, \int, \oint, \otimes, \Leftrightarrow, \models, \ldots$$

I'm not going to explain what the more esoteric symbols mean here—this is just to give an idea of some of the symbols that get used. As with road signs, they make math look a bit incomprehensible at first, but they make it easier in the long run.

Google Maps

The Difficulty of Relating the Map to the Reality

What's difficult about reading a map? It's not the actual reading of the map that's hard, but matching that up with reality in order to put the map to practical use. A map is an abstraction of reality. It depicts certain aspects of reality that are supposed to help you find your way around. The difficulty, in practice, is in translating between the abstraction and the reality—that is, making the link between the map and the place you're actually trying to find your way around.

Google maps gives us a brilliant way of moving from the abstract to the concrete, via Street View, and GPS. Often the hardest part about using a map is working out (a) where you are in the first place, and (b) which way you're facing. Those are the crucial pivot points between the map and the reality. GPS has sorted out the business of working out where you are, and Street View has sorted out the business of which way you're facing by giving us a very realistic representation of reality in the form of an actual picture of it.

Math has to go through these steps as well. First you have to turn the reality into an abstraction. Then you do your logical reasoning in the abstract world. Then finally you have to turn that back into reality again. Different people are good at different parts of this process. But really the key part is being able to move back and forth between the abstract and the real. Still, *someone had to draw the map.*

For example, suppose you have a recipe for an 8-inch-square cake, but you want to make it round instead. What size of round cake pan should you use? First you perform an abstraction to turn this "real-life" question into a piece of math. We want to find a circle whose area is the same as the area of the given square, which is 8^2 or 64. Now we have to remember that the area of a circle is πr^2 where r is the radius. If we write d for the diameter of the circle (because cake pans are measured by their diameter not their radius), this means we need

$$\pi \left(\frac{d}{2}\right)^2 = 64.$$

Now we actually do the logical reasoning, manipulating the algebra to find out what the diameter d needs to be. This is the only part that's actually math.

$$\begin{aligned}
\left(\frac{d}{2}\right)^2 &= \frac{64}{\pi} \\
\frac{d}{2} &= \sqrt{\frac{64}{\pi}} \\
d &= 2 \times \sqrt{\frac{64}{\pi}} \\
&\approx \pm 9.027
\end{aligned}$$

Finally we take the context into account and turn this back into reality. First of all, we don't want the negative answer because we're talking about cake pans here, so the answer needs to be a positive number. Secondly, we don't need all those decimal places—cake tins are usually only measured to the nearest inch. So the answer in reality is that we need a 9-inch round pan for our cake.

∞ ∞ ∞

The key in math, and with maps, is to find the most appropriate level of abstraction for the given moment. Do you need little pictures of all the buildings on a street when you're looking at a street map? Do you need to know where there is grass and where there isn't? It depends on what you're using the map for, and you'll need different maps for different situations. If you're driving, then you'll want to know which streets are one-way, but that's not very relevant if you're on foot. The same is true of math. There are different levels of abstraction available for different situations.

What is the number 1? Here are two different ways of answering that question, at different levels of abstraction.

First answer: 1 is the basic building block of counting.

Second answer: 1 is the only number with the property that multiplying by it does nothing.

Each of these answers is useful in different contexts. The first is for when we are most interested in adding numbers up; this characterizes numbers as something called a "group"—a world in which we can do addition. The second is for when we are also interested in multiplying; this characterizes numbers as something called a "ring"—a world in which we can do addition *and* multiplication. The study of groups is related to the symmetry of shapes, and the study of rings is related to other aspects of the geometry of shapes. We'll come back to this later.

If you use an inappropriate map for the situation you're in, you'll get frustrated, whether it's too realistic or not realistic enough. (I dislike those street maps with pictures of buildings in three dimensions, so that they actually obscure the lines telling you where the street goes.)

The same is true of math—if you try and use complicated math for a situation that doesn't call for it, you'll think the math is pointless. It's a bit like using the Dewey decimal system if you only own twenty books.

High Jump

Leaps of Abstraction

I was terrible at the high jump in high school. I already said I was terrible at all sports, but with the high jump I failed before I started—I couldn't jump over the bar even at its lowest. The trouble is that nobody tried to teach me what I needed to do to get myself over that bar. Other people in my class just seemed to be able to do it, as if by magic, and the rest of us were simply told to do it again. And again. And again. There are only so many times you can knock down a high jump bar, with an audience, without feeling disillusioned and eager to give up.

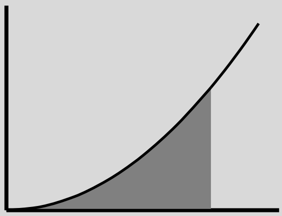
Thinking about more and more abstract concepts is a bit like the high jump. You have to get yourself over a progressively higher and higher bar, and if nobody explains how to do it, you will keep knocking the bar off and want to give up. Different people reach their limit of abstraction at different moments and, just as with the high jump, people drop out at each round.

Most people are able to make the abstraction from *objects* to *numbers* and don't even notice that is a process of abstraction at all. One popular moment where many people find they can't get over the bar any more is where the numbers turn into x's and y's. They can't do it, and they also can't see the point of doing it, so they get disillusioned and give up. (I never saw the point of the high jump either, but now I see that the Fosbury flop is a satisfyingly elegant way of getting your body over a bar as efficiently as possible. If someone had explained to me back then that your center of gravity *doesn't even have to go over the bar,* I'd have been much more interested.)

Another popular moment where people reach their abstract limit is calculus, which involves a completely new and strange—and, frankly, a bit sneaky—way of manipulating and reasoning with "infinitesimally small" things. Some people get through rigorous calculus but unfortunately reach their limit half way through their undergraduate math degree, or in the middle of their PhD.

Rigorous calculus is something most people only meet if they do math in college. People find it hard because it doesn't fit with their idea of what mathematics is—pinning things down and getting answers to things with great certainty.

Calculus in high school usually consists of answering specific questions such as "If you draw the graph of $y = x^2$ and shade in the space under the curve from $x = 0$ up to $x = 2$, what is the area that you have filled in?"



In high school we are taught to answer this by "integrating" x^2, which gives $\frac{x^3}{3}$ and then evaluating this at $x = 2$, to give the answer $\frac{8}{3}$.

In college we actually prove that this argument is valid. In high school you might see it justified somewhat experimentally, by drawing the curve on graph paper and then counting the squares under the line. Some of the squares will only be partial squares, so you will only get a truly accurate answer if you use infinitesimally small squares. But these don't exist.

Rigorous calculus makes this argument into something logically watertight but baffles people because it doesn't pin down an answer in the way that people are expecting. Instead it says something like: There's no such thing as graph paper with infinitesimally small squares, so we use progressively smaller and smaller squares and observe that the answer gets closer and closer to $\frac{8}{3}$ as the squares get smaller. Then we prove that no matter how close we wanted it to get to $\frac{8}{3}$, there is a size of square that would get us that close.

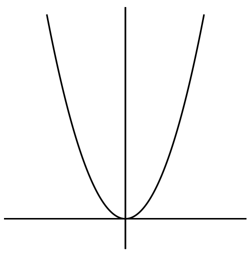
A moment where advanced mathematicians sometimes reach their abstract limit is category theory. They react in much the same way that teenagers do when they meet x's and y's—they say they don't see the point, and resist any further abstraction. I am always reminded of Prof. John Baez, who said the following during an argument about abstraction on the worldwide category theory email list:

> *If you do not like abstraction, why are you in mathematics? Perhaps you should be in finance, where all the numbers have dollar signs in front of them.*

I haven't yet met my abstract limit, but I do remember various key moments in my life where I was pushing a boundary and felt I had to make a conscious effort to get over the next bar.

From Numbers to Pictures

My mother taught me how you can draw a graph of x^2 like this:



I distinctly remember my bafflement at the fact that you could turn the process of squaring numbers into a *picture* of a curve. I sat in our big green armchair at home thinking and thinking about this until my brain felt like it was popping out of my head. And in my memory this is the exact same feeling I've had every time I've thought about a difficult mathematical concept in my research.

From Numbers to Letters

I was perfectly comfortable solving equations with x's, say

$$2x + 3 = 7.$$

I knew this would turn into

$$\begin{aligned} 2x &= 7-3 \\ &= 4 \\ x &= \frac{4}{2} \\ &= 2. \end{aligned}$$

But then I met one with a's, b's, and c's instead of the numbers, something like

$$ax + b = c$$

and I vividly remember feeling completely at a loss as to how on earth to find out what x was in this case, without knowing a, b, and c. I think I knew that I should start by subtracting b from both sides, but I had no idea what that would give on the right-hand side. I do remember that when someone explained to me that it would be $c - b$ I felt extremely stupid. Why couldn't I have worked that out myself? The answer is then

$$x = \frac{c-b}{a}.$$

Well, as I say to my students, feeling stupid for not having understood something before just shows that you are now cleverer than you were then.

From Numbers to Relationships

This is the last big leap of abstraction I remember having to make, and it happened when I was first learning category theory. For the sake of completeness and perhaps amusement value I'll include here what it was: the idea that *a one-object category is exactly a monoid.* Laugh as much as you like; there it is. I sat for days thinking about it and feeling like my brain was popping out of my head, just like when I was a child and thinking about a graph for the first time in my life. And the fact that a one-object category is exactly a monoid is now so obvious to me that I know I am definitely cleverer now than I was then. It's a bit early to explain this example now, but I'll come back to it in the second part of the book.

We will see that category theory studies relationships between objects. A *category* is a mathematical context for studying these relationships. A *monoid* is a mathematical context for studying something much more concrete: multiplication of things like numbers. The fact that a one-object category is a monoid corresponds to viewing numbers as relationships between the world and itself. This sounds quite strange, but it is remarkably powerful.

The Goose That Laid the Golden Eggs

Making Machines for Solving Problems

It would be lovely to find a way of making golden eggs. But it would be even better to find a way of making a goose that lays golden eggs: a goose-that-lays-golden-eggs machine. But wouldn't it be even better to make a machine that makes these machines? A "goose-that-lays-golden-eggs machine" machine. This is a form of abstraction. It's the idea of building a machine to do something, rather than directly doing the thing yourself. So really it's just a form of conservation of energy, or of reserving human brainpower for the things machines can't do.

In order to build a machine to do something rather than doing it yourself, you have to understand that thing at a different level. It's like giving someone directions. When you walk somewhere you know well, you don't really think about exactly what streets you're walking on, or which way you're turning and when. You probably go somewhat instinctively. But when you're telling someone else how to get there, you have to analyze more carefully how you do it, in order to explain it. You might have noticed that if you ask a local person where a certain street is, they will often not be very sure, as you don't really think about street names when you're wandering around your own town.

Something similar happens when learning a language. When you learn it as your mother tongue, you don't really think about how it works—you pick it up from the adults around you instinctively. Then when you're an adult and a foreigner asks you to explain some aspect of the language that is confusing them, you have to go back and analyze how you speak, in a way you might never have done before.

If you're building a machine to make a cake, you'll have to analyze each step rather carefully in order to work out how to get a machine to do it. Even cracking an egg would require careful thought—how do we know how hard to tap the egg against the bowl?

The previous example of solving equations is an example of this type of machine. We start by understanding how to solve equations such as

$$2x + 3 = 7.$$

Then we make a "machine" for solving all such equations, that is, we solve the equation

$$ax + b = c$$

because then a, b, and c can be any numbers at all.

We can then try it for quadratic equations

$$ax^2 + bx + c = 0$$

and we learn that the "machine" for solving these gives the famous solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

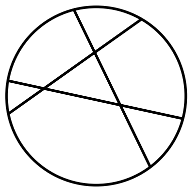
As a further level of building a machine that makes these machines, there is the *Fundamental Theorem of Algebra* which tells us that every polynomial equation has at least one solution, as long as we allow complex numbers, which we'll come to later.

Cake Cutting

An Example of Abstraction

I remember the first independent math investigation I had to do in high school. It was about cutting a cake into as many pieces as possible while making a fixed number of cuts. Obviously, if you can only make one cut (in a straight line), you'll only get two pieces of cake, and if you can only make two cuts, you'll get at most four pieces. But what about three cuts? Four cuts? And so on?

The best answer for three cuts is seven pieces of cake, like this.



Your first thought about this might be the same as mine: that this is a stupid question, because who would ever cut a cake like that? You end up with pieces of all sorts of different sizes. What matters more in cutting a cake—efficiency, or the sizes of your pieces of cake?

Setting aside the question of size for a second, the point of the investigation was to get us to try it experimentally for three cuts, four cuts, and so on, and then to get us to find a *formula* for the maximum possible number of pieces, in terms of the number of cuts you're allowed to make. That is, the aim is not just to solve the problem in any particular case but to build a machine for solving the problem in *every* case. That is what a formula involving x's and y's and things really is—a machine. So you can feed in, say, the number of cuts you're allowed to make, and the machine will spew out the answer: the number of pieces of cake you get. A formula is even better than a machine, because it actually tells you *how the machine works*—it's not just a mysterious black box. So if the formula says the answer is

$$\frac{x^2 + x + 2}{2}$$

this is a machine telling us that we can feed in the number of allowed cuts in the place of x, and the result will be the number of pieces of cake. This is a form of abstraction, because instead of dealing with actual problems, you're dealing with *hypothetical* problems. You're not solving the problem; you're solving the problem of solving the problem. Instead of writing the formula you could make a table of answers like this:

No. of cuts	No. of pieces of cake
1	2
2	4
3	7
4	11
5	16
$\vdots$	$\vdots$

You can't make this table go on *forever*—it will have to stop somewhere, just because you'll run out of paper, not to mention years of your life. The formula, however, doesn't stop anywhere—it is a machine for giving you the answer for *any* number of cuts.

Perhaps you didn't have to do math investigations, but perhaps you had children doing them, and you were helping them. But you were trying to help them without actually doing the investigation for them. That is the meta-problem—instead of solving a problem, you're trying to solve the problem of getting someone else to solve the problem. Teaching is a bit like that, because you're not just telling people answers but trying to get them to find the answers. It's one level removed from answering the question yourself. Teaching teachers is another level of abstraction. And who teaches the people who teach teachers?

Making a cake isn't that clever, but inventing a new recipe for making cakes is a bit more clever. Discovering a new number wouldn't really count as "interesting" because we already know the method for producing all new numbers. If you worked out how to cure cancer, it would be somewhat immoral if you merely went round curing individual people's cancer instead of teaching the world how to cure cancer.

All of these examples of abstraction take us arguably one step further from reality, but they have a broader scope as a result. If you shine a flashlight from further away, you will illuminate a larger area. But be careful not to shine it from *too* far away, as the light will then be too dim.

Abstract Mathematics

Abstraction is the key to understanding what mathematics is. Abstraction is also at the heart of why mathematics can seem removed from "real life." That detachment from reality is where math derives its strength, but also its limitations. Every level of abstraction takes it further from real life, and harder to explain what the relevance to real life is, because the relevance comes from a domino effect—abstract mathematics might not be directly applicable to real life, but rather, applicable to something else which is applicable to real life, or via an even longer chain of applications, for example:

$$\text{Category theory} \longrightarrow \text{Topology} \longrightarrow \text{Physics} \longrightarrow \text{Chemistry} \longrightarrow \text{Medicine}$$

Abstraction is the key to understanding why mathematics is different from science at large. Evidence-based science proceeds with, obviously, evidence at its heart. You start with a hypothesis—something you believe might be true, whether because of general observation, gut feeling, suspicion, anecdotes, or whatever. Now you need to test the hypothesis rigorously by finding evidence that holds up to scientific standards. Such standards include things like these:

- You must have a large enough sample size. Three or four cases is anecdotal and could have been a fluke.
- The evidence must be controlled. You must be sure that you have accounted for other factors that might have affected the evidence, such as the placebo effect, socioeconomic factors, the ages of people involved, and so on.
- The evidence must be unbiased. For example, drug tests must be double blind—neither the person taking the drug nor the person administering it can know whether it's a real drug or placebo.

In the end, the result is statistical. You come up with a large body of very convincing evidence, but your conclusion always has a percentage certainty attached to it.

Mathematics is different. The first step is the same—you start with a hypothesis that you think might be true for some reason. But instead of testing it rigorously using evidence, you test it rigorously using *logic*. The meaning of "rigor" is now completely different. It has nothing to do with sample sizes, because you don't ever use any samples—you only use thought processes. Bias doesn't come into it either, because all you're doing is applying logic.

For example, to find out how much frosting you need to cover a cake, you could do it experimentally—you could get a cake, frost it, and see how much you used. Or you could do it logically—you could do a calculation involving the surface area of a cake. To do this calculation you have to make an approximation of the shape of a cake. Perhaps you assume it's perfectly round and perfectly flat on top. Of course, no cake is ever *perfectly* circular and flat. But the advantage of this method is that you don't have to make any frosting in order to find out how much frosting you need.

Using logic instead of experiments has many different sorts of advantages.

Experiments Can Be Impractical

Suppose you want to find out how many bricks you need to build a house. It's not very practical to build an entire house just to find out how many bricks you'll need. Or what if you want to work out how changing a road layout will affect traffic flow?

Experiments Can Be Dangerous

What if you want to find out how much traffic a bridge can carry? You can't just get loads of traffic to drive across it and see when it collapses.

Experiments Can Be Impossible

What if you're trying to work out why the sun rises every day, or why the planets behave the way they do? You can't just change the conditions of outer space and then see how the planets behave differently.

Experiments Can Be Undesirable

Suppose you're trying to work out how an infectious disease can spread across the country. You can't just unleash the disease and see how it spreads, because that's exactly the thing you're trying to avoid.

Experiments Can Be Immoral

At the time of writing, there is a suggestion that culling the badger population will reduce instances of tuberculosis in cows. How can this be tested? Is it morally right to kill a whole lot of badgers to see what happens?

In all these cases, there is an important advantage to working theoretically rather than experimentally, an advantage to using logic rather than evidence. The final crucial advantage is that with logic, the conclusion is not just "almost certainly true": it is irrefutable.

How Does Logic Work?

A logical argument is a series of statements, each of which follows from the previous one using only logic. That's all very well, but where does it start? You always have to start with a basic set of assumptions. For example, you might assume your cake is perfectly circular. You might assume that an infectious disease has a 50 percent chance of being passed from one person to another if they meet. These basic assumptions are part of the process of abstraction. They usually involve turning your real-life objects into something theoretical, so that you can reason with them using logic. The downside is that your theoretical situation won't be *exactly* the same as your real one. But the upside is that you will now be able to apply logical process to work things out about them. The inaccuracy of your final answer will now come from the information you threw away when you performed the initial abstraction. This is very different from statistical results, where the inaccuracy of the final answer comes from a small possibility that your hypothesis was wrong despite the evidence.

The *mathematical method* (as opposed to the more talked-about scientific method) involves making very clear what your assumptions are. People can then disagree with your assumptions, but they aren't entitled to disagree with your overall conclusion, which is:

> *If* we make these assumptions, *then* this conclusion is true.

For example: if one chicken can feed ten people, then two chickens can feed twenty people. You can argue about how many people one chicken can really feed (probably not ten people unless it's a scary genetically modified giant chicken), but you can't argue with the fact that:

> *If* one chicken feeds ten people, *then* two chickens feed twenty.

But there's still a possible flaw here: are all the chickens the same size? We probably need to add an assumption saying "All chickens are about the same size" to ensure that the situation behaves mathematically.

Is this an unrealistic assumption? If you're going to order whole roast chickens for a party with forty people, you're probably going to do a calculation somewhat like this, even though chickens aren't all *exactly* the same size. But on the other hand, you might proceed experimentally instead: you might rely on the experience of the caterer, who has probably held enough parties to have experimental evidence of how many chickens to get for forty people.

Abstraction can be difficult because it takes us out of the realm of physical objects and into the realm of "ideas" that we manipulate only in our head. But there are some abstract ideas we're so used to that we don't even notice how abstract they are any more. If we think about the size of an average chicken, that's an abstraction right there: an "average chicken" isn't a real chicken we're considering, it's just an idea of a chicken. As I mentioned before, numbers are abstract. The numbers 1, 2, 3, 4, and so on are only *ideas.* Because they are ideas, we can manipulate them just using logic.

The wonderful thing about abstraction is that when you get very used to an abstract idea, it starts to *feel* like an actual object instead of just being a made-up idea. You're probably quite comfortable with "2" as a concept. That means you're comfortable with that level of abstraction. Perhaps you're less comfortable with exactly what "-2" is. What about the square root of 2? It's a number such that when you multiply it by itself, the answer is 2. But what actually is it? You might think it's $1.414\ldots$, but that is a decimal that goes on forever without recurring—you can't write the whole thing down, so how do you know what it is? What about the square root of -1? We'll investigate these questions in more depth later, and look at why rigorous mathematics has much more trouble with the square root of 2 than with -2 or even the square root of -1, even though intuitively the square root of -1 is much harder to think about because nothing like it ever appears in "real life."

Part of the process of abstraction is like using your imagination. Mathematical abstraction takes us into an imaginary world where anything is possible as long as it's not contradictory. Can you imagine transparent Lego blocks? That's not so difficult, but what about squashy Lego blocks? That's a bit more strange. What about Lego blocks that spontaneously change color when you touch them? Four-dimensional Lego blocks? Invisible Lego blocks? Lego blocks that can make coffee for you in the morning? Obviously in the real world, just because you can imagine something doesn't mean it actually exists—particularly if you have a very vivid imagination. The amazing thing about the world of math is that mathematical things exist as soon as you imagine them. The more vivid your imagination, the more math you have access to.

Another abstract concept that we're quite used to is shapes. What is a square? It's a shape with four equal sides and four equal angles. But are there actually any *perfect* squares in the world? No, any physical shape in the real world is not going to be an absolutely microscopically pedantically perfect square. Likewise circles. What about straight lines? Are there really any perfectly straight lines? Not really. And yet we're comfortable with the idea of a straight line, although the things in the real world are only approximations to this ideal.

Abstraction at Work

Here I will give the abstract approach to the two example questions I posed earlier on, so you can see what it looks like.

> *My father is three times as old as I am now but in ten years' time he will be twice as old as me. How old am I?*

I'll write x for my age, and y for my father's age. "My father is three times as old as I am now" becomes

$$y = 3x.$$

So far so good. "In ten years' time he will be twice as old as me" is a bit trickier. The key is that in ten years' time my age will be $x + 10$ and his age will be $y + 10$, and we know that his age will be twice mine at that point, so this turns into:

$$y + 10 = 2(x + 10).$$

We can now substitute $3x$ into the second equation where y is, so we get:

$$\begin{aligned} 3x + 10 &= 2(x + 10) && \\ &= 2x + 20 && \text{multiplying out the parentheses} \\ \text{so } x + 10 &= 20 && \text{subtracting } 2x \text{ from both sides} \\ \text{so } x &= 10 && \text{subtracting 10 from both sides} \end{aligned}$$

So we can conclude that I am 10 years old.

Note that we went through the following steps.

1. We started with a "real-life" situation expressed in words.
2. We performed an *abstraction* to turn it into logical concepts.
3. We manipulated the abstract concepts using logic.
4. We undid the abstraction to get back to the real-life situation.

There's a further level of abstraction we can do here. The step we did helped us solve the problem stated in words above, but if we do another step, we can solve *all similar problems*.

In that problem we started with two specific equations

$$\begin{aligned} y &= 3x \\ y + 10 &= 2(x + 10) \end{aligned}$$

but we could replace all those numbers with letters so that we can solve any pairs of equations involving any numbers:

$$\begin{aligned} y &= a_1 x + b_1 \\ y &= a_2 x + b_2 \end{aligned}$$

The second equation of our original equations might not look like this to you, but when you rearrange it to get y by itself on the left, it turns into

$$y = 2x + 10$$

Now we can solve the general pair of equations by equating the respective right-hand sides, since they're both equal to y on the left:

$$a_1x + b_1 = a_2x + b_2$$

And now if we put all the x terms on one side we get

$$\begin{aligned} a_1x - a_2x &= b_2 - b_1 \\ (a_1 - a_2)x &= b_2 - b_1 \\ x &= \frac{b_2 - b_1}{a_1 - a_2} \end{aligned}$$

This last step is valid unless $a_1 = a_2$; in this case we are forced to have $b_1 = b_2$ as well, which means the two equations are the same, and we don't have enough information to pin down what x and y have to be—there will be infinitely many solutions.

Let's try the other example.

> *I have a recipe for frosting the top and sides of a 6-inch cake.*
> *How much frosting do I need for the top and sides of an 8-inch cake?*

We *assume* that both cakes are round and 2 inches deep. We need to find the area of frosting used in the 6-inch cake, and the area used in the 8-inch cake, and see how much bigger the latter is. Because both cakes are round, we can save some effort by calculating the area of frosting on a cake of radius r, and then we can use $r = 3$ or $r = 4$ afterwards (the radius being half the diameter).

- The top of the cake is a circle, so the area is πr^2.
- The side of the cake has an area that is the height times the circumference. The circumference is $2\pi r$, so the area is $2 \times 2\pi r = 4\pi r$.
- Thus the total frosting for radius r is $\pi r^2 + 4\pi r$.

We can now use this formula to work out the area covered by frosting in each of the two cakes.

- For the 6-inch cake the radius is 3, so the total area covered by frosting is
$$\begin{aligned}(\pi \times 3^2) + (4\pi \times 3) &= 9\pi + 12\pi \\ &= 21\pi.\end{aligned}$$
- For the 8-inch cake the radius is 4, so the total area covered by frosting is
$$\begin{aligned}(\pi \times 4^2) + (4\pi \times 4) &= 16\pi + 16\pi \\ &= 32\pi.\end{aligned}$$

Finally we need to translate this into something we can use for our cake. We want to know how much to scale up the original recipe to make enough frosting for the bigger cake, so we need to know how much bigger the second area is than the first. So we take the area we found for the 8-inch cake and divide it by the area we found for the 6-inch cake:

- The ratio of 8-inch frosting to 6-inch frosting is
$$\frac{32\pi}{21\pi} = \frac{32}{21}.$$

Now because this is only frosting for a cake, and not something extremely critical like a dose of medicine, an approximate answer will do: $\frac{32}{21}$ is about 1.5, so you need to multiply your original recipe by 1.5 to have enough frosting for the bigger cake.

The important thing to notice here is that we made an *assumption* that the cake is 2 inches high. So the final answer might be inaccurate, but only because of this assumption. So our final, irrefutable conclusion is:

If all the cakes are 2 inches high,
then we need to multiply the original recipe by 1.5.

This cake example is somewhat more useful than the example with my father's age. Where the question of age was just a silly brainteaser, the question about frosting was a genuine situation where the abstract thought processes helped us. We could have worked out the answer experimentally, by making a whole load of frosting and seeing how much we needed for the bigger cake, but that would have been a waste of frosting. The abstract approach used more brainpower but wasted less frosting.