Indiscernibles in infinitary languages and Erdős cardinals

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1 Conventions and aims

Almost all conventions in this note are defined in [1], except $\subset$ is replaced by $\subseteq$, including in quotation. Lowercase Greek letters other than ϕ denote ordinals.

For infinite cardinals π, ρ, let $\pi^{\underline{\varrho}}$ denote $\sum_{\kappa<\rho} \pi^\kappa$.

Given a first-order language $\mathcal{L}$, for cardinals π, ρ, we take $\mathcal{L}_{\pi,\rho}$ to be the infinitary extension of $\mathcal{L}$ as used in [1], i.e. permitting conjunctions and disjunctions of length $< \pi$ and quantifications of length $< \rho$, and (in contrast to $\mathcal{L}_{\alpha,\beta}$ as defined in Karp's book [3]) allowing function and relation symbols of infinite arity.

For a well-ordered set X and an ordinal γ, define $[X]^\gamma$ to be the set of subsets of X with order type γ, likewise for $[X]^{<\gamma}$ and subsets of order type $< \gamma$. We define the Erdős-Rado partition calculus as usual except we allow infinite exponents, namely for a cardinal κ and ordinals α, γ, λ, the property $\kappa \to (\alpha)^\gamma_\lambda$ holds if for any function $f : [\kappa]^\gamma \to \lambda$, there is a subset H of κ of order type α such that the image of $[H]^\gamma$ under f is a singleton (such an H is said to be "homogeneous for f"). The property $\kappa \to (\alpha)^{<\gamma}_\lambda$ is defined likewise. For an infinite ordinal α, define the α-Erdős cardinal to be the least κ such that $\kappa \to (\alpha)^{<\omega}_2$. For further reading on the partition calculus, see chapter 7, section 2 of [1].

For a structure $\mathfrak{A} = (A, <, \ldots)$ in a language $\mathcal{L}$, a set $Y \subseteq A$ that is linearly ordered by a relation $<$ is said to be a set of indiscernibles for $\mathfrak{A}$ if for any finite sequences $x_0 < \ldots < x_n$, $y_0 < \ldots < y_n$ from A and any formula ϕ in $\mathcal{L}$ with $n+1$ free variables, we have $\mathfrak{A} \vDash \phi(x_0, \ldots, x_n)$ iff $\mathfrak{A} \vDash \phi(y_0, \ldots, y_n)$.

Theorem 2.1 of chapter 8, section 2 of [1], is that for infinite cardinals κ, λ and ordinals $\alpha < \kappa$, the property $\kappa \to (\alpha)^{<\omega}_{2^\lambda}$ is equivalent to the following condition [which Drake calls $(*)$]:

> For every structure $\mathfrak{A}$ of length λ, which has a subset X of its universe which is ordered in type κ by a relation $<$, there is a subset $Y \subseteq X$ with order type α under $<$ such that $\langle Y, < \rangle$ is a set of indiscernibles for $\mathfrak{A}$.

We will denote this property by $*(\alpha, \lambda)$.

We will present a version of $(*)$ for infinitary languages, show the property inconsistent for some languages, and give an upper bound on its consistency strength for others. Previous work on infinitary versions of $(*)$ has been done in [2], however the author of this note is not aware of work on the properties presented here.

2 Infinitary quantifications

In this section we consider languages allowing infinitely many quantifications.

For a structure $\mathfrak{A} = (A, <, \ldots)$ in a language $\mathcal{L}$, say that a set $Y \subseteq A$ that is linearly ordered by a relation $<$ is a set of $\mathcal{L}_{\pi,\rho}$-indiscernibles for $\mathfrak{A}$ if for any $\beta < \rho$, any $<$-increasing sequences $(x_\xi)_{\xi<\beta}$, $(y_\xi)_{\xi<\beta}$ from A, and any formula ϕ in $\mathcal{L}_{\pi,\rho}$ with $|\beta|$ free variables, we have $\mathfrak{A} \vDash \phi((x_\xi)_{\xi<\beta})$ iff $\mathfrak{A} \vDash \phi((y_\xi)_{\xi<\beta})$.

For cardinals π, ρ and ordinals α, λ, let $*(\alpha, \lambda)_{\pi,\rho}$ denote the property obtained from $*(\alpha, \lambda)$ by replacing "a set of indiscernibles for $\mathfrak{A}$" with "a set of $\mathcal{L}_{\pi,\rho}$-indiscernibles for $\mathfrak{A}$". If this property holds, we say that κ has property $*(\alpha, \lambda)_{\pi,\rho}$. We will only consider the case where $\alpha \geq \rho$ holds, otherwise the property becomes trivial as there would be no $<$-increasing sequences $(x_\xi)_{\xi<\beta}$ from Y for any β such that $\alpha < \beta < \rho$.

The following theorem is adapted from one way of theorem 2.1 of chapter 8, section 2 of [1].

Theorem 1: For cardinals κ and ordinals $\alpha \geq \omega_1$, if κ has property $*(\alpha, \omega_1)_{\omega,\omega_1}$, then $\kappa \to (\alpha)^{<\omega_1}_{\omega_1}$.

Proof: Let κ be as given. Let $f : [\kappa]^{<\omega_1} \to \omega_1$ be arbitrary. For each $\beta < \omega_1$ and $\gamma < \omega_1$, define a predicate on $[\kappa]^{<\omega_1}$ by $R_{\beta,\gamma}((x_\xi)_{\xi<\beta}) \iff \bigwedge_{\xi<\nu<\beta}(x_\xi < x_\nu) \wedge f(\{x_\xi \mid \xi < \beta\}) = \gamma$. Let $\mathfrak{A}$ be the structure $(\kappa, <, (R_{\beta,\gamma})_{\beta<\omega_1,\gamma<\omega_1})$, with length ω_1 by using a pairing function from $\omega_1 \times \omega_1$ to ω_1. By property $*(\alpha, \omega_1)_{\omega,\omega_1}$, there is a set $Y \subseteq \kappa$ of $\mathcal{L}_{\omega,\omega_1}$-indiscernibles for $\mathfrak{A}$ of order type α. Choose any $\beta < \omega_1$ and choose arbitrary $<$-increasing sequences $(x_\xi)_{\xi<\beta}$ and $(y_\xi)_{\xi<\beta}$ from Y. By $\mathcal{L}_{\omega,\omega_1}$-indiscernibility, for any $\gamma < \omega_1$, we have $\mathfrak{A} \vDash R_{\beta,\gamma}((x_\xi)_{\xi<\beta})$ iff $\mathfrak{A} \vDash R_{\beta,\gamma}((y_\xi)_{\xi<\beta})$, so $f(\{x_\xi \mid \xi < \beta\}) = f(\{y_\xi \mid \xi < \beta\})$. As $(x_\xi)_{\xi<\beta}$ and $(y_\xi)_{\xi<\beta}$ were arbitrary sequences from $Y^{<\omega_1}$, $\{x_\xi \mid \xi < \beta\}$ and $\{y_\xi \mid \xi < \beta\}$ are arbitrary members of $[Y]^{<\omega_1}$, so Y is homogeneous for f. $\square$

Corollary 2: Assuming the axiom of choice, there can be no infinite cardinals κ, λ and infinite ordinal $\alpha \geq \omega_1$ such that κ has property $*(\alpha, \lambda)_{\omega,\omega_1}$.

Proof: By theorem 4.1 of chapter 7, section 4 of [1], assuming choice, there are no cardinals κ such that $\kappa \to (\omega)^\omega_2$. $\square$

Corollary 3: Assuming the axiom of choice, there can be no infinite cardinals $\kappa, \lambda, \pi, \rho$ with $\rho \geq \omega_1$ and ordinal $\alpha \geq \rho$ such that κ has property $*(\alpha, \lambda)_{\pi,\rho}$.

As we do not consider properties $*(\alpha, \lambda)_{\pi,\rho}$ where $\alpha < \rho$, this is all that can be said for $\rho \geq \omega_1$.

3 Infinitary conjunctions and disjunctions

Corollary 3 above rules out the consistency of properties $*(\alpha,\lambda)_{\pi,\rho}$ with $\rho \geq \omega_1$. However, it does not rule out consistency of the $\rho = \omega$ case. In fact, $*(\alpha,\lambda)_{\pi,\omega}$ is consistent relative to an α-Erdős cardinal. The following lemma is a generalization of one way of theorem 2.1 of chapter 8, section 2 of [1].

Lemma 4: Let κ, λ be infinite cardinals and $\alpha < \kappa$ be an ordinal. If $\kappa \to (\alpha)^{<\omega}_{2^{\lambda^{\underline{\pi}}}}$ holds, then κ has property $*(\alpha,\lambda)_{\pi,\omega}$.

Proof: Assume $\kappa \to (\alpha)^{<\omega}_{2^{\lambda^{\underline{\pi}}}}$, and let $\mathfrak{A} = (A, <, \ldots)$ be a structure of length λ such that there is a subset X of A which is ordered in order type κ by a relation $<$ of the structure. Define an equivalence relation $\sim$ on $[X]^{<\omega}$ by $\{x_0,\ldots,x_n\} \sim \{y_0,\ldots,y_n\}$ iff $\mathfrak{A} \vDash \phi(x_0,\ldots,x_n) \iff \mathfrak{A} \vDash \phi(y_0,\ldots,y_n)$ for all $\mathcal{L}_{\pi,\omega}$ formulas ϕ with the displayed free variables, where $x_0,\ldots,x_n$ and $y_0,\ldots,y_n$ are enumerated in $<$-increasing order. As there are λ-many nonlogical symbols of the language of $\mathcal{A}$ and formulas are of length $< \pi$, there are $\lambda^{\underline{\pi}}$ formulas ϕ in $\mathcal{L}_{\pi,\omega}$. For each of these, $\phi(x_0,\ldots,x_n)$ may either be true or false, so $\sim$ partitions X into at most $2^{\lambda^{\underline{\pi}}}$ pieces. By $\kappa \to (\alpha)^{<\omega}_{2^{\lambda^{\underline{\pi}}}}$, there is an $H \subseteq X$ of order type α under $<$ which is homogeneous for the partition $\sim$. Then $\langle H, <\rangle$ is a set of $\mathcal{L}_{\pi,\omega}$-indiscernibles for $\mathfrak{A}$ of order type α. $\Box$

Lemma 5: Let α be an ordinal and κ be the α-Erdős cardinal. For any ordinals $\lambda, \pi < \kappa$, we have $2^{\lambda^{\underline{\pi}}} < \kappa$.

Proof: By corollary 4.7 of chapter 7, section 4 of [1], κ is strongly inaccessible. Thus for any $\nu < \kappa$, we have $\lambda^\nu < \kappa$. As $\lambda^{\underline{\pi}} = \sum_{\nu<\pi} \lambda^\nu$ is a sum of ν-many cardinals that are less than κ, we have $\lambda^{\underline{\pi}} < \kappa$. Finally, again by strong inaccessibility $2^{\lambda^{\underline{\pi}}}$ must be less than κ. $\Box$

Corollary 6: Let α be an ordinal. Assuming the α-Erdős cardinal exists, for all ordinals $\lambda, \pi, \omega < \kappa$, it is consistent that there is a cardinal with property $*(\alpha,\lambda)_{\pi,\omega}$.

Proof: Let κ be the α-Erdős cardinal. By lemma 5, for any $\lambda, \pi < \kappa$ we have $2^{\lambda^{\underline{\pi}}} < \kappa$. By corollary 2.2 of chapter 8, section 2 of [1], we have $\kappa \to (\alpha)^{<\omega}_\nu$ for all $\nu < \kappa$, so $\kappa \to (\alpha)^{<\omega}_{2^{\lambda^{\underline{\pi}}}}$. By lemma 4, κ has property $*(\alpha,\lambda)_{\pi,\omega}$. $\Box$

References

[1] F. R. Drake. *Set Theory: An Introduction to Large Cardinals*, volume 76 of *Studies in Logic and the Foundations of Mathematics*. 1974.

[2] Q. Feng. A hierarchy of Ramsey cardinals. *Annals of Pure and Applied Logic*, 49, 1990.

[3] C. R. Karp. *Languages with Expressions of Infinite Length*, volume 36 of *Studies in Logic and the Foundations of Mathematics*. 1964.