DM MT23 : Exercices 12 et 16 du Chp. 3

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Exercice 12

1. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j \end{pmatrix}$

Calcul de l'inverse par la méthode des cofacteurs :

$A = (a_{ij})_{(i,j)\in\{1,2,3\}}$

$cof(a_{11}) = \begin{vmatrix} j & j^2 \\ j^2 & j \end{vmatrix} = (j^2 - j^4) = (j - j^2)(j^2 + j) = j^2 - j$

$cof(a_{12}) = -\begin{vmatrix} 1 & j^2 \\ 1 & j \end{vmatrix} = -(j - j^2) = j^2 - j$

$cof(a_{13}) = \begin{vmatrix} 1 & j \\ 1 & j^2 \end{vmatrix} = j^2 - j$

$cof(a_{21}) = -\begin{vmatrix} 1 & 1 \\ j^2 & j \end{vmatrix} = j^2 - j$

$cof(a_{31}) = \begin{vmatrix} 1 & 1 \\ j & j^2 \end{vmatrix} = j^2 - j$

$cof(a_{22}) = \begin{vmatrix} 1 & 1 \\ 1 & j \end{vmatrix} = j - 1$

$cof(a_{23}) = -\begin{vmatrix} 1 & 1 \\ 1 & j^2 \end{vmatrix} = 1 - j^2$

$cof(a_{32}) = -\begin{vmatrix} 1 & 1 \\ 1 & j^2 \end{vmatrix} = 1 - j^2$

$cof(a_{33}) = \begin{vmatrix} 1 & 1 \\ 1 & j \end{vmatrix} = j - 1$

$co(U) = \begin{pmatrix} j^2 - j & j^2 - j & j^2 - j \\ j^2 - j & j - 1 & 1 - j^2 \\ j^2 - j & 1 - j^2 & j - 1 \end{pmatrix}$

$$
\begin{aligned}
det(U) &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j \end{vmatrix} \\
&= a_{11}cof(a11) + a_{12}cof(a12) + a_{13}cof(a13) \\
&= 3(j^2 - 1)
\end{aligned}
$$

$$
U^{-1} = \frac{1}{det(U)} co(U) = \frac{1}{3}\frac{1}{j^2-1}\begin{pmatrix} j^2-j & j^2-j & j^2-j \\ j^2-j & j-1 & 1-j^2 \\ j^2-j & 1-j^2 & j-1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \frac{j-1}{j^2-1} & -1 \\ 1 & -1 & \frac{j-1}{j^2-1} \end{pmatrix}
$$

Exercice 16

2. $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 2 & -3 \\ 1 & 2 & -1 \\ 4 & 0 & -5 \end{pmatrix}$

$$
\begin{pmatrix} 1 & -1 & -1 \\ 2 & 2 & -3 \\ 1 & 2 & -1 \\ 4 & 0 & -5 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \\ L_4 \end{array}
$$

$$
\begin{pmatrix} 1 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & 3 & 0 \\ 0 & 4 & -1 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - L_1 \\ L_4 \leftarrow L_4 - 4L_1 \end{array}
$$

$$
\begin{pmatrix} 1 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \leftarrow 4L_3 - 3L_2 \\ L_4 \leftarrow L_4 - L_2 \end{array}
$$

$$\begin{cases} x_1 - x_2 - x_3 = 0 \\ 4x_2 - x_3 = 0 \\ 3x_3 = 0 \end{cases}$$ Il vient $x_1 = x_2 = x_3 = 0$.

Avec $b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$:

$\begin{pmatrix} 1 \\ -2 \\ -1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \end{pmatrix}$ Et le système qui s'en suit est :

$$\begin{cases} x_1 - x_2 - x_3 = 1 \\ 4x_2 - x_3 = -2 \\ 3x_3 = 2 \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{4}{3} \\ x_2 = -\frac{1}{3} \\ x_3 = \frac{2}{3} \end{cases}$$

Avec $b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$:

$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -3 \\ -1 \end{pmatrix}$ ce qui est impossible car $0 \neq -1$. Pas de solution.

3. $A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \leftarrow L_2 - L_1 \\ L_3 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & -2 & 1 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \leftarrow L_3 \\ L_3 \leftarrow L_2 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -6 & 1 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \leftarrow L_3 - 2L_2 \end{array}$$

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ x_2 + 2x_3 = 0 \\ -6x_3 + x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_4 = 6x_3 \\ x_2 = -2x_3 \\ x_1 = -9x_3 \end{cases}$$

On a donc : $x = \alpha \begin{pmatrix} -9 \\ -2 \\ 1 \\ 6 \end{pmatrix}$

Avec $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$:

$$\begin{pmatrix} b_1 \\ b_2 - b_1 \\ b_3 \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ b_3 \\ b_2 - b_1 \end{pmatrix}$$

$$\begin{pmatrix} b_1 \\ b_3 \\ b_2 - b_1 - 2b_3 \end{pmatrix}$$

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = b_1 \\ x_2 + 2x_3 = b_3 \\ -6x_3 + x_4 = b_2 - b_1 - 2b_3 \end{cases} \Leftrightarrow \begin{cases} x_4 = 6x_3 + b_2 - b_1 - 2b_3 \\ x_2 = -2x_3 + b_3 \\ x_1 = -9x_3 + 2b_1 - b_2 + 3b_3 \end{cases}$$

On a donc : $x = \alpha \begin{pmatrix} -9 \\ -2 \\ 1 \\ 6 \end{pmatrix} + \begin{pmatrix} 2b_1 - b_2 + 3b_3 \\ b_3 \\ 0 \\ -b_1 + b_2 - 2b_3 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 0 & 2 & -2 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 0 & 2 & -2 & 1 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 2 & -2 & 1 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \leftarrow L_2 - L_1 \\ L_3 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 1 & 1 \\ 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} L_1 \\ L_2 \\ L_3 \leftarrow L_3 - L_2 \end{array}$$

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ -2x_2 + 2x_3 + x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_4 = 2(x_2 - x_3) \\ x_1 = x_2 - x_3 - 2x_2 + 2x_3 = x_3 - x_2 \end{cases}$$

On a donc :

$$\begin{aligned} x &= (x_3 - x_2; x_2; x_3; 2x_2 - 2x_3) \\ &= x_2(-1;1;0;2) + x_3(1;0;1;-2) \\ &= \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} \end{aligned}$$

Avec $b = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$:

$$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 - x_2 + x_3 + x_4 = 0 \\ -2x_2 + 2x_3 + x_4 = 2 \end{cases} \Leftrightarrow \begin{cases} x_4 = 2(x_2 - x_3) + 2 \\ x_1 = x_2 - x_3 - 2x_2 + 2x_3 - 2 = x_3 - x_2 - 2 \end{cases}$$

On a donc $x = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Avec $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ Ce qui est impossible car $0 \neq 1$. Pas de solution.