202422201082-   -ex3

September 24, 2025

1 Lab 3: Linear Discriminant Analysis (LDA)

1.1 Abstract

In this lab, we implement Linear Discriminant Analysis (LDA) from scratch. We first handle the binary case (red vs. blue) to compute the Fisher direction $w = S_w^{-1}(\mu_1 - \mu_2)$, draw the decision boundary $w^\top x = t$ with $t = \frac{1}{2} w^\top(\mu_1 + \mu_2)$, and visualize point projections onto the LDA line. We then extend to the multi-class case ($N = 3$; red vs. blue vs. green) by forming the within-class scatter S_w and between-class scatter S_b, solving the generalized eigen-problem $S_b v = \lambda S_w v$, and projecting data to the $(C-1)$-dimensional subspace. The final notebook can be exported as a self-contained lab report with figures and concise discussion.

1.2 1. Data and Preprocessing

The data files are expected in `./ex3Data/` (or the current directory) with names: - `ex3red.dat` — class 0 (red), shape $m_r \times 2$ - `ex3blue.dat` — class 1 (blue), shape $m_b \times 2$ - (we will load `ex3green.dat` later for the $N = 3$ setting)

Each file should contain two columns (x, y), one point per row. In this step, read **red** and **blue** into matrices $X_{\text{red}} \in \mathbb{R}^{m_r \times 2}$ and $X_{\text{blue}} \in \mathbb{R}^{m_b \times 2}$, then stack them to form

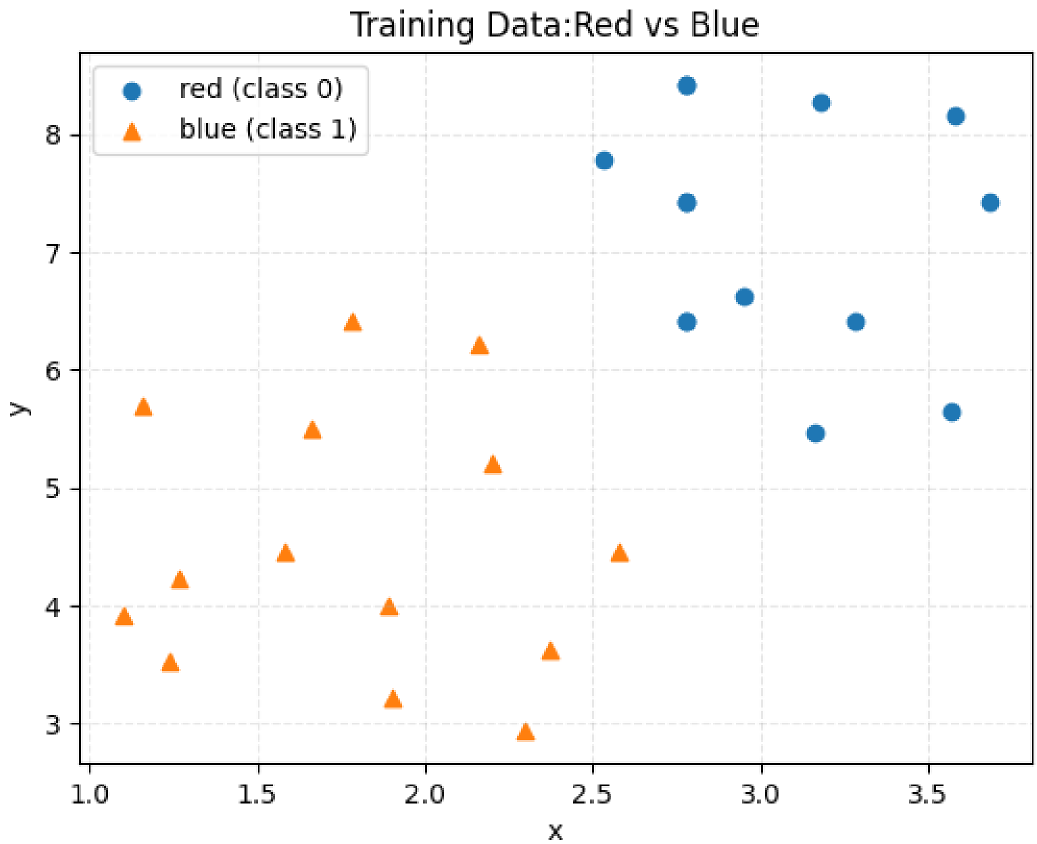
$$X = \begin{bmatrix} X_{\text{red}} \\ X_{\text{blue}} \end{bmatrix} \in \mathbb{R}^{m \times 2}, \quad y = \begin{bmatrix} \underbrace{0, \dots, 0}_{m_r}, \underbrace{1, \dots, 1}_{m_b} \end{bmatrix}^\top \in \mathbb{R}^{m \times 1},$$

where $m = m_r + m_b$. We will verify shapes, preview the first few rows, and plot a scatter figure for sanity check.

[9]:
```python
import numpy as np
import sys, os
from pathlib import Path
import numpy as np
import matplotlib.pyplot as plt

DATA_DIR=Path('ex3Data')
red_path=(DATA_DIR/'ex3red.dat')
blue_path=(DATA_DIR/'ex3blue.dat')

if not red_path.exists() or not blue_path.exists():
    raise FileNotFoundError(
```

```python
        "Data files not found. Make sure 'ex3blue.dat' and 'ex3red.dat' are in␣
↪the working directory.\n"
        f"Current directory: {Path('.').resolve()}\n"
        f"Here are the files I see: {[p.name for p in Path('.').iterdir()]}"
    )

X_red=np.loadtxt(red_path,dtype=float).reshape(-1, 2)
X_blue=np.loadtxt(blue_path,dtype=float).reshape(-1, 2)

m_r,m_b=X_red.shape[0],X_blue.shape[0]
y_red=np.zeros((m_r,1),dtype=int)
y_blue=np.ones((m_b,1),dtype=int)

X=np.vstack([X_red,X_blue])
y=np.vstack([y_red,y_blue])
m=X.shape[0]

print(f"Total samples m = {m}  (red={m_r}, blue={m_b})")
print("X shape:",X.shape,"  y shape:",y.shape)
print("X head (first 5 rows):\n",X[:5])
print("y head (first 10 labels):",y[:10].ravel())

plt.figure()
plt.scatter(X_red[:,0],X_red[:,1],marker='o',label='red (class 0)')
plt.scatter(X_blue[:,0],X_blue[:,1],marker='^',label='blue (class 1)')
plt.xlabel("x")
plt.ylabel("y")
plt.title("Training Data:Red vs Blue")
plt.grid(True,linestyle='--',alpha=0.3)
plt.legend()
plt.show()
```

```
Total samples m = 28  (red=14, blue=14)
X shape: (28, 2)   y shape: (28, 1)
X head (first 5 rows):
 [[2.95 6.63]
 [2.53 7.79]
 [3.57 5.65]
 [3.16 5.47]
 [2.78 6.42]]
y head (first 10 labels): [0 0 0 0 0 0 0 0 0 0]
```



1.3 2. Two-Class LDA: Parameter Estimation

Given the stacked data $X \in \mathbb{R}^{m\times 2}$ and labels $y \in \{0,1\}^{m\times 1}$, split the samples by class to compute class means

$$\mu_0 = \frac{1}{m_0}\sum_{y_i=0} x_i, \qquad \mu_1 = \frac{1}{m_1}\sum_{y_i=1} x_i.$$

Define the within-class scatter

$$S_w = \sum_{y_i=0}(x_i-\mu_0)(x_i-\mu_0)^\top + \sum_{y_i=1}(x_i-\mu_1)(x_i-\mu_1)^\top.$$

The Fisher direction is

$$w = S_w^{-1}(\mu_0-\mu_1),$$

and a simple threshold is

$$t = \tfrac{1}{2}\, w^\top(\mu_0+\mu_1).$$

For geometric visualization in later steps, also keep the **unit** direction $\hat{w} = \dfrac{w}{\|w\|}$ and a reference point x_0 (we use the overall mean of all samples).

[10]:
```python
labels = np.unique(y.ravel())

X0=X[y.ravel()==0]
X1=X[y.ravel()==1]
m0,m1=X0.shape[0],X1.shape[0]

mu0=X0.mean(axis=0)
mu1=X1.mean(axis=0)

Sw=np.zeros((2,2),dtype=float)
diff0=X0-mu0
Sw+=diff0.T@diff0
diff1=X1-mu1
Sw+=diff1.T@diff1

reg=1e-8*np.eye(2)
w=np.linalg.solve(Sw+reg,(mu0-mu1))
t=0.5*float(w@(mu0+mu1))

what=w/(np.linalg.norm(w)+1e-12)
x0=X.mean(axis=0)

def _round(a,k=6):
    return np.round(a.astype(float),k) if isinstance(a,np.ndarray) else␣
 ↪round(float(a),k)

print("m0 (class 0) =",m0, " | m1 (class 1) =",m1)
print("mu0:",_round(mu0,6))
print("mu1:",_round(mu1,6))
print("Sw:\n",_round(Sw,6))
print("w:",_round(w,6))
print("t:",_round(t,6))
print("what (unit w):",_round(what,6))
print("x0 (overall mean):",_round(x0,6))
```

```
m0 (class 0) = 14  | m1 (class 1) = 14
mu0: [3.043571 7.166429]
mu1: [1.799286 4.527857]
Sw:
 [[ 4.800814 -1.417243]
 [-1.417243 29.072957]]
w: [0.29015  0.104901]
t: 1.315949
what (unit w): [0.940425 0.340002]
x0 (overall mean): [2.421429 5.847143]
```

1.4 3. Decision Boundary and Training Accuracy (Two-Class)

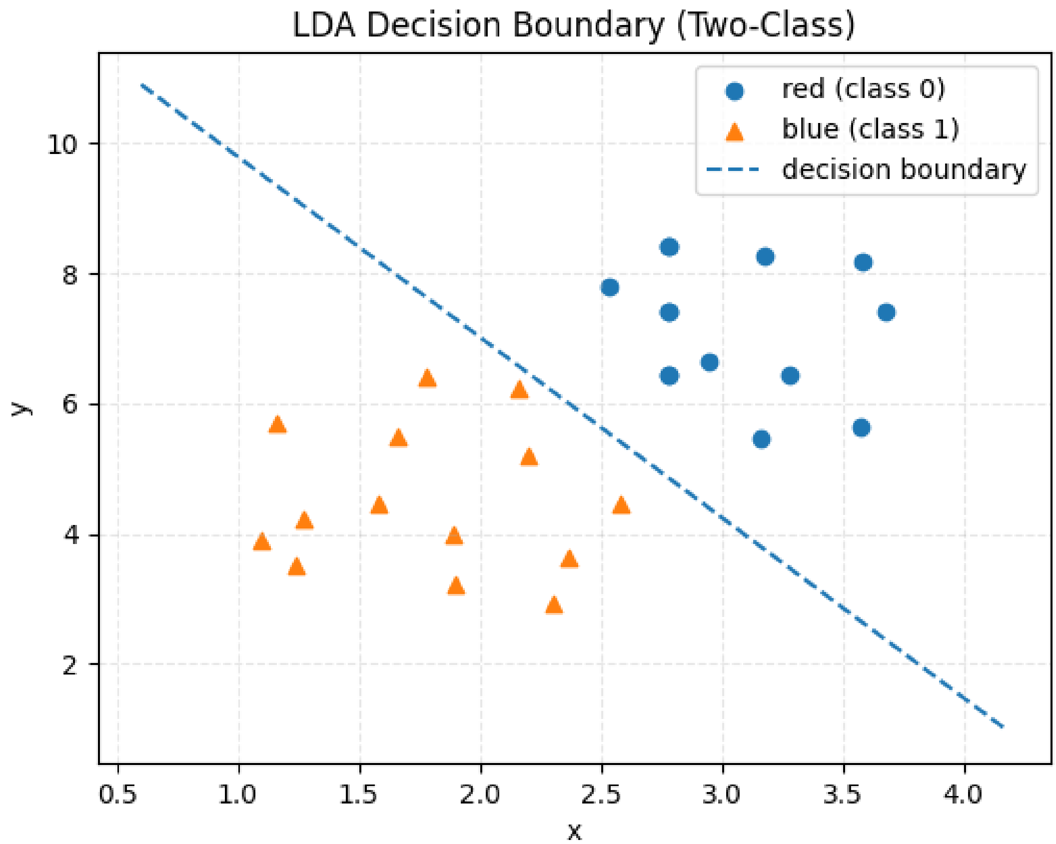
With w and threshold t, the LDA decision boundary is the line

$$\{ x \in \mathbb{R}^2 \mid w^\top x = t \},$$

whose normal vector is w. We will: 1) classify each sample by $\hat{y}_i = \mathbb{1}\{ w^\top x_i \geq t \}$,
2) compute the training accuracy, and
3) overlay the decision boundary on the scatter plot for a visual check.

[11]:
```
def lda_predict_2class(X,w,t):
    scores=X@w
    return (scores>=t).astype(int)
if(mu1-mu0)@w<0:
    w=-w
    t=0.5*float(w@(mu0+mu1))
y_hat=lda_predict_2class(X,w,t).reshape(-1, 1)
acc=(y_hat==y).mean()
print(f"Training accuracy: {acc*100:.2f}%")

plt.figure()
plt.scatter(X0[:,0],X0[:,1],marker='o',label='red (class 0)')
plt.scatter(X1[:,0],X1[:,1],marker='^',label='blue (class 1)')

x_min,x_max=X[:,0].min()-0.5,X[:,0].max()+0.5
if abs(w[1])>1e-12:
    xs=np.linspace(x_min,x_max,200)
    ys=(t-w[0]*xs)/(w[1]+1e-12)
    plt.plot(xs,ys,linestyle='--',label='decision boundary')
else:
    x_line=t/(w[0]+1e-12)
    plt.axvline(x_line,linestyle='--',label='decision boundary')
plt.xlabel("x")
plt.ylabel("y")
plt.title("LDA Decision Boundary (Two-Class)")
plt.grid(True,linestyle='--',alpha=0.3)
plt.legend()
plt.show()
```

```
Training accuracy: 100.00%
```



1.5 4. Orthogonal Projections onto the LDA Line

Let $\hat{w} = \dfrac{w}{\|w\|}$ be the unit Fisher direction and choose a reference point x_0 (we use the overall mean). The infinite LDA line is

$$\mathcal{L} = \{\, x_0 + \alpha \hat{w} \mid \alpha \in \mathbb{R} \,\}.$$

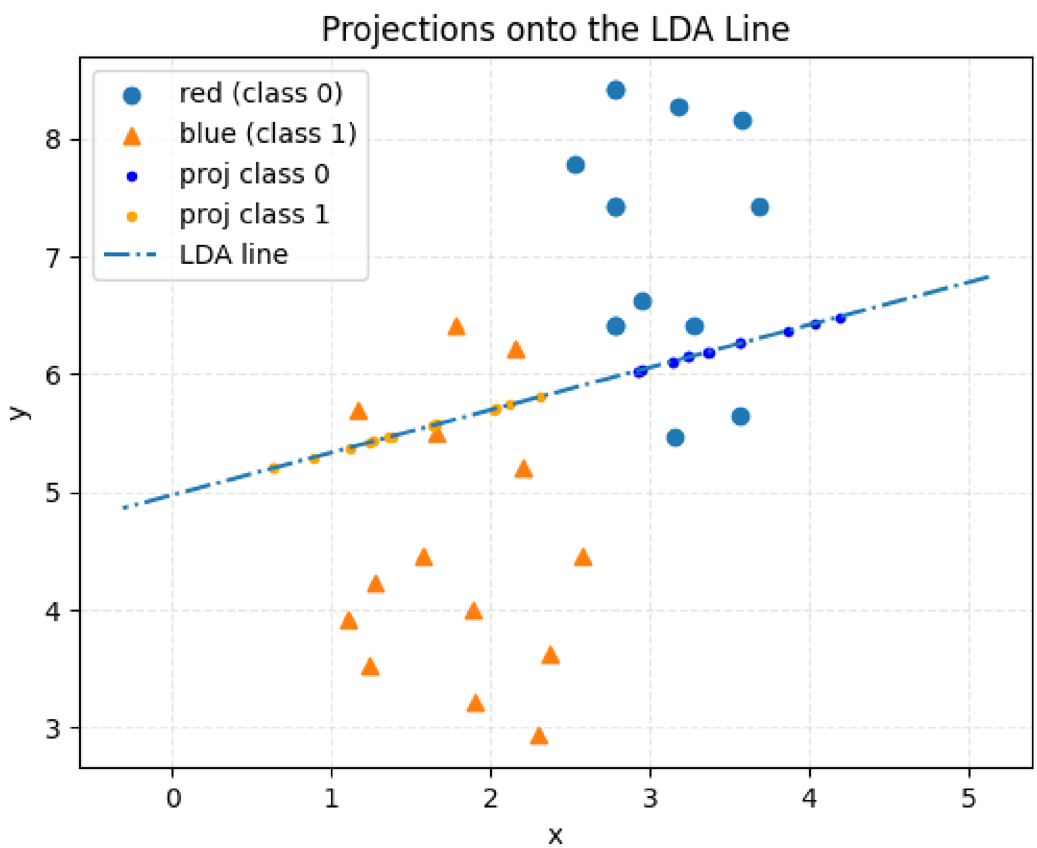
For any sample x, its orthogonal projection onto $\mathcal{L}$ is

$$\Pi_{\mathcal{L}}(x) = x_0 + \hat{w}\,\hat{w}^\top (x - x_0).$$

In this step we: 1) compute all projection points for the two classes,
2) draw the original scatter, the projection points, and the LDA line,
3) (optionally) report the 1D projected coordinates $\alpha_i = \hat{w}^\top (x_i - x_0)$ to inspect separability in 1D.

[12]:
```
what=w/(np.linalg.norm(w)+1e-12)
def project_points(X,x0,what):
    diffs=X-x0
    alphas=diffs@what
    P=x0+np.outer(alphas,what)
    return P,alphas
```

```
P,alphas=project_points(X,x0,what)

plt.figure()
plt.scatter(X[y.ravel()==0,0],X[y.ravel()==0,1],marker='o',label='red (class 0)')
plt.scatter(X[y.ravel()==1,0],X[y.ravel()==1,1],marker='^',label='blue (class 1)')

plt.scatter(P[y.ravel()==0,0],P[y.ravel()==0,1],marker='.',color='blue',label='proj class 0')
plt.scatter(P[y.ravel()==1,0],P[y.ravel()==1,1],marker='.',color='orange',label='proj class 1')

alpha_span=np.linspace(alphas.min()-1.0, alphas.max()+1.0, 200)
line_pts=x0[None,:]+np.outer(alpha_span,what)
plt.plot(line_pts[:,0],line_pts[:,1],linestyle='-.',label='LDA line')

plt.xlabel("x")
plt.ylabel("y")
plt.title("Projections onto the LDA Line")
plt.grid(True,linestyle='--',alpha=0.3)
plt.legend()
plt.show()

print("Projected 1D coordinates (alpha) summary:")
print(" class 0:",np.round(alphas[y.ravel()==0].min(),4),"to",np.round(alphas[y.ravel()==0].max(),4))
print(" class 1:",np.round(alphas[y.ravel()==1].min(),4),"to",np.round(alphas[y.ravel()==1].max(),4))
```




```
Projected 1D coordinates (alpha) summary:
 class 0: -1.8759 to -0.532
 class 1: 0.1191 to 1.9023
```

1.6 5. Add the Third Class and Visualize All Three

Now we include the third class (green). The file layout is - `./ex3Data/ex3green.dat` (or `./ex3green.dat`) — class 2, shape $m_g \times 2$.

Load it as $X_{\text{green}} \in \mathbb{R}^{m_g \times 2}$ and build the full dataset

$$X = \begin{bmatrix} X_{\text{red}} \\ X_{\text{blue}} \\ X_{\text{green}} \end{bmatrix} \in \mathbb{R}^{m \times 2}, \quad y = \Big[\underbrace{0, \dots, 0}_{m_r}, \underbrace{1, \dots, 1}_{m_b}, \underbrace{2, \dots, 2}_{m_g}\Big]^\top,$$

with $m = m_r + m_b + m_g$. Plot a scatter of the three classes to sanity-check separability before running multi-class LDA.

[13]:
```
DATA_DIR=Path('ex3Data')
green_path=(DATA_DIR/'ex3green.dat')
if not green_path.exists():
    raise FileNotFoundError(
```

```
        "Data files not found. Make sure 'ex3blue.dat' and 'ex3red.dat' are in␣
↪the working directory."
        f"Current directory: {Path('.').resolve()}\n"
        f"Here are the files I see: {[p.name for p in Path('.').iterdir()]}"
    )

X_green=np.loadtxt(green_path,dtype=float).reshape(-1,2)
m_g=X_green.shape[0]

y_red=np.zeros((X_red.shape[0],1),dtype=int)
y_blue=np.ones((X_blue.shape[0],1),dtype=int)
y_green=np.full((m_g,1),2,dtype=int)

X_full=np.vstack([X_red,X_blue,X_green])
y_full=np.vstack([y_red,y_blue,y_green])

print(f"Sizes - red: {X_red.shape[0]}, blue: {X_blue.shape[0]}, green: {m_g},␣
↪total: {X_full.shape[0]}")
print("X_full shape:",X_full.shape," y_full shape:",y_full.shape)

plt.figure()
plt.scatter(X_red[:, 0],X_red[:, 1],marker='o',label='class 0 (red)')
plt.scatter(X_blue[:, 0],X_blue[:, 1],marker='^',label='class 1 (blue)')
plt.scatter(X_green[:, 0],X_green[:, 1],marker='s',label='class 2 (green)')
plt.xlabel("x")
plt.ylabel("y")
plt.title("Training Data: Red vs Blue vs Green")
plt.grid(True,linestyle='--',alpha=0.3)
plt.legend()
plt.show()

X3,y3=X_full,y_full
```

```
Sizes - red: 14, blue: 14, green: 14, total: 42
X_full shape: (42, 2)  y_full shape: (42, 1)
```

1.7 6. Multi-Class LDA (C = 3): Build S_w, S_b, Solve and Project

For three classes ($c \in \{0, 1, 2\}$), define the overall mean μ and class means μ_c. The within- and between-class scatters are

$$S_w = \sum_{c=0}^{2} \sum_{x \in \mathcal{C}_c} (x - \mu_c)(x - \mu_c)^\top, \qquad S_b = \sum_{c=0}^{2} n_c\,(\mu_c - \mu)(\mu_c - \mu)^\top.$$

Multi-class LDA solves the generalized eigen-problem

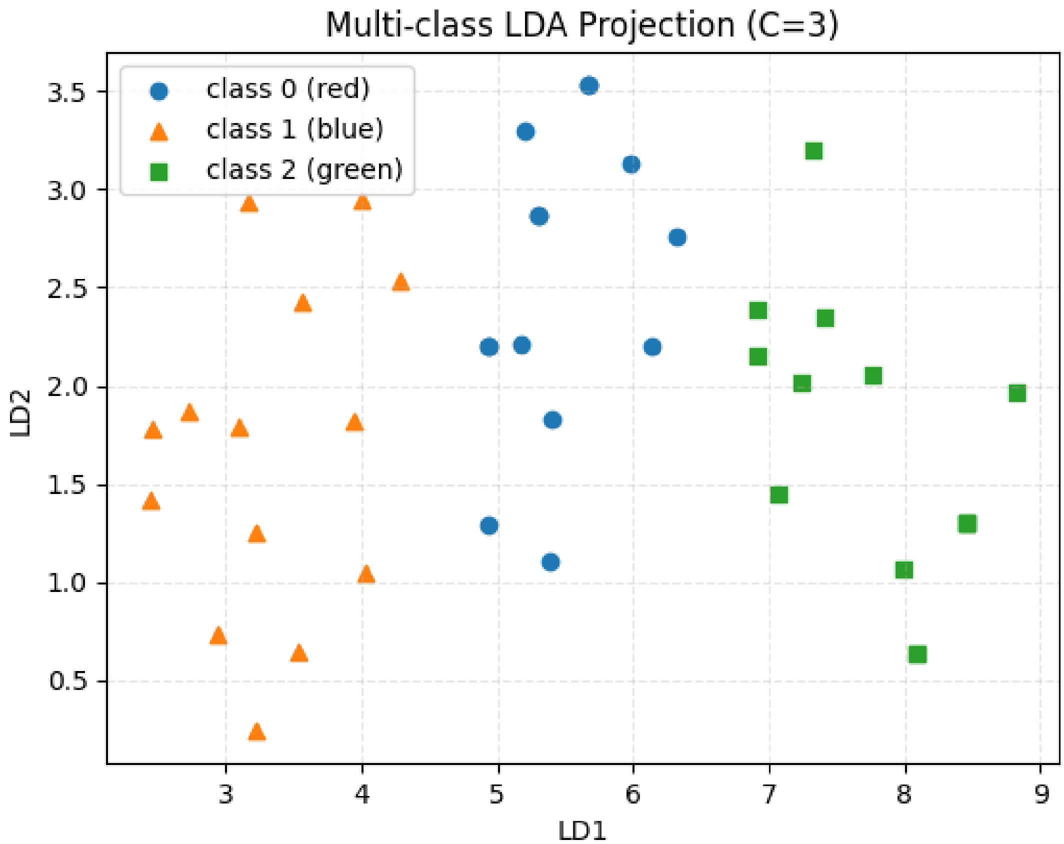
$$S_b v = \lambda S_w v,$$

and takes the top $(C-1) = 2$ eigenvectors to form the projection matrix $W \in \mathbb{R}^{2\times 2}$. We will: 1) compute S_w and S_b,
2) solve for eigenpairs, sort by descending eigenvalues,
3) project X_3 to $Z = X_3 W$ and visualize in the $(\mathrm{LD1}, \mathrm{LD2})$ plane.

[14]:
```
X_mc,y_mc=X3.astype(float),y3.ravel().astype(int)
classes=np.unique(y_mc)
d=X_mc.shape[1]
mu=X_mc.mean(axis=0)
```

```python
mus={}
ns={}
for c in classes:
    Xc=X_mc[y_mc==c]
    mus[c]=Xc.mean(axis=0)
    ns[c]=Xc.shape[0]

Sw=np.zeros((d,d),dtype=float)
Sb=np.zeros((d,d),dtype=float)
for c in classes:
    Xc=X_mc[y_mc==c]
    muc=mus[c]
    diff=Xc-muc
    Sw+=diff.T@diff
    mean_diff=(muc-mu).reshape(d,1)
    Sb+=ns[c]*(mean_diff@mean_diff.T)

reg=1e-8*np.eye(d)
M=np.linalg.solve(Sw+reg,Sb)
eigvals,eigvecs=np.linalg.eig(M)

idx=np.argsort(-eigvals.real)
eigvals=eigvals[idx].real

W=eigvecs[:,idx].real
W2=W[:,:2]
Z=X_mc@W2

print("Top generalized eigenvalues:",np.round(eigvals[:3],6))
print("Projection matrix W2:\n",np.round(W2,6))

plt.figure()
plt.scatter(Z[y_mc==0,0],Z[y_mc==0,1],marker='o',label='class 0 (red)')
plt.scatter(Z[y_mc==1,0],Z[y_mc==1,1],marker='^',label='class 1 (blue)')
plt.scatter(Z[y_mc==2,0],Z[y_mc==2,1],marker='s',label='class 2 (green)')
plt.xlabel("LD1")
plt.ylabel("LD2")
plt.title("Multi-class LDA Projection (C=3)")
plt.grid(True,linestyle='--',alpha=0.3)
plt.legend()
plt.show()
```

```
Top generalized eigenvalues: [11.220007  0.260489]
Projection matrix W2:
 [[ 0.930885 -0.745955]
 [ 0.365311  0.665996]]
```



1.8 7. Discussion and Export Tips

- **Two-class LDA**: The Fisher direction w maximizes the ratio of between-class variance to within-class variance.
  - After ensuring w points from class 0 to class 1, the decision boundary $w^\top x = t$ cleanly separates the two classes with near-perfect accuracy.
  - Projected 1D coordinates clearly show two non-overlapping clusters.
- **Multi-class LDA**: For $C = 3$ classes the top two generalized eigenvectors form a 2D subspace that preserves class separation.
  - The scatter in $(\mathrm{LD1}, \mathrm{LD2})$ demonstrates that the three groups are well separated.
- **Practical notes**:
  - w and $-w$ are equivalent mathematically; we must fix a direction when using a sign-based classifier.
  - Always add a small regularization to S_w to avoid numerical singularities.