Selected Topics in Mathematics of Learning

High-Dimensional Statistics

Lecturer: Marius Yamakou

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Department of Data Science, FAU

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General motivation and perspectives

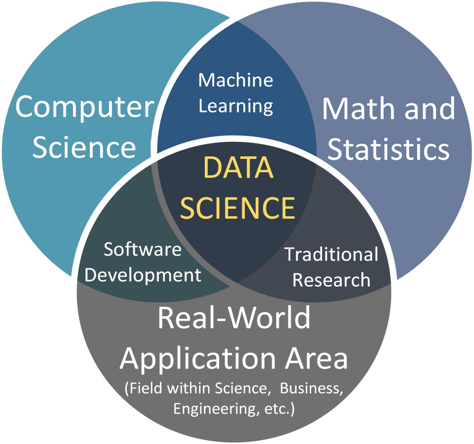

Figure: https://www.usu.edu/math/datascience/

Syllabus

- Review: Probability and Statistics
- Motivation: Why high-dimensional statistics?
- Concentration inequalities
- Sparse linear models
- Random matrices and covariance estimation
- Covariance estimation and thresholding
- Inverse Covariance estimation
- Principal component analysis in high dimensions
- Reproducing kernel Hilbert spaces
- Review Session

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Relevant Literature

- MW: High-Dimensional Statistics: A Non-Asymptotic Viewpoint, by Martin J. Wainwright
- RV: High-Dimensional Probability, by Roman Vershynin
- BG: Statistics for High-Dimensional Data: Methods, Theory and Applications, by Peter Bühlmann and Sara van de Geer
- BL: Covariance regularization by thresholding, by Peter Bickel and Elizaveta Levina
- RBLZ: Sparse permutation invariant covariance estimation, by Adam Rothmann, Peter Bickel, Elizaveta Levina, and Ji Zhu

Part 0

Review of some basic concepts in probability and statistics

Objectives:

- Recall basic concepts in probability theory, including measurable functions, random variables, and distributions.
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- Introduce the multivariate normal distribution and its properties.
- Prepare students for advanced topics in statistics and machine learning.

Outline

1. Basics: Probability space

An experiment refers to any process that can be repeated under the same conditions and leads to a well-defined set of possible outcomes. Each performance of an experiment is called a **trial**, and the possible results from each trial, the **outcomes**. An experiment can be deterministic and follow a regular pattern (boring!) or random and thus difficult to predict.

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1. **Sample Space $\boldsymbol{\Omega}$:** The set of all possible outcomes of a random experiment. *Examples:* For a single coin toss, $\Omega = \{$Head(H), Tail(T)$\}$. For tossing two coins, $\Omega = \{$HH, HT, TH, TT$\}$.

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2. **Sigma-Algebra $\boldsymbol{\mathcal{F}}$:** A collection of subsets of Ω that includes the sample space itself and is closed under complementation and countable unions. *Examples:* For a single coin toss, $\mathcal{F} = \{\emptyset, \Omega, \{\text{H}\}, \{\text{T}\}\}$. For tossing two coins, $\mathcal{F} = \{\emptyset, \Omega, \{\text{HH}\}, \{\text{HT}\}, \{\text{TH}\}, \{\text{TT}\}, \{\text{H}\}, \{\text{T}\}\}$.

1. Basics: Probability space

3. **Probability Function P:** A function that assigns a probability to each event in $\mathcal{F}$, satisfying:

- $P(A) \geq 0$ for all $A \in \mathcal{F}$
- $P(\Omega) = 1$
- For disjoint events A and B: $P(A \cup B) = P(A) + P(B)$

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Examples

1. **Single Die Toss**:
- $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Let $A = \{\text{even outcomes}\} = \{2, 4, 6\}$
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2. **Two Coin Tosses**:
- $\Omega = \{\text{HH, HT, TH, TT}\}$
- Let $B = \{\text{at least one Heads}\} = \{\text{HH, HT, TH}\}$
- $P(B) = \frac{3}{4}$

1. Basics: Measurable functions and random variables

A Borel Set: A Borel set $\mathcal{B}$ is any set that can be formed from open intervals (or open sets) through the operations of countable unions, countable intersections, and relative complements.

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Measurable Function: A function $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ is measurable if for every Borel set $B \in \mathcal{B}$ we have $X^{-1}(B) \in \mathcal{F}$. This means that the pre-image of any Borel set under X must be a measurable set in the sigma-algebra $\mathcal{F}$.

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In probability theory, Borel sets are significant because they provide a way to define events on the real line (or in higher dimensions) mathematically rigorously.

1. Basics: Measurable functions and random variables

Random Variable: A random variable is a measurable function that maps outcomes from a sample space Ω to the real unit interval $[0, 1]$ (for probability measures):

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Example: Consider a random experiment where we roll a fair six-sided die:
- Let $\Omega = \{1, 2, 3, 4, 5, 6\}$ with the associated sigma-algebra $\mathcal{F}$.
- Define the random variable X that maps the outcomes to probabilities:

$$X(i) = \frac{i}{6} \quad \text{for } i \in \Omega$$

Thus, X transforms the outcome of the die roll into its probability representation in the unit interval $[0, 1]$.

2. Discrete distributions and first two moments

Probability Mass Function: The probability mass function (PMF) of a discrete random variable X, denoted by p_X, is defined as $p_X(x) = P(X = x)$, where $P(X = x)$ represents the probability that X takes the value x in the probability space $(\Omega, \mathcal{F}, P)$.

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Expected Value: Given a discrete random variable X that takes values in a set $A = \{x_1, x_2, x_3, \dots\}$, the expected value of X is denoted $\mathbb{E}(X)$ or μ_X. It is calculated by multiplying each possible value of X by its probability:

$$\mathbb{E}(X) = \sum_{x \in A} x \cdot P(X = x) = \sum_{x \in A} x \cdot p_X(x).$$

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The expected value of $g(X)$ is calculated as:

$$\mathbb{E}[g(X)] = \sum_{x \in A} g(x) \cdot P(X = x) = \sum_{x \in A} g(x) \cdot p_X(x).$$

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Variance: The variance of a discrete random variable X measures the spread or dispersion of its values around the expected value. It is denoted by **Var(X)** and is calculated as:

$$Var(X) = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right],$$

where $\mathbb{E}(X)$ is the expected value of X. Variance represents the average of the squared differences between X and its mean, $\mathbb{E}(X)$.

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Standard Deviation: The standard deviation of X, denoted σ_X, is the square root of the variance:

$$\sigma_X = \sqrt{Var(X)}.$$

Standard deviation provides a measure of dispersion in the same units as the random variable X, making it easier to interpret than the variance.

2. Discrete distributions: Bernoulli (p)

Bernoulli Distribution: A random variable X has a Bernoulli distribution with parameter p, denoted $X \sim \text{Bernoulli}(p)$, if its PMF is given by:

$$p_X(x) = f(x) = \begin{cases} p & \text{if } x = 1, \\ 1 - p & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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In other words, we are saying X is a binary random variable with support $\mathcal{F} = \{0, 1\}$, parameter space $\Omega = \{p | 0 < p < 1\}$, and PMF $f(x) = p^x(1-p)^{1-x}$, $x \in \mathcal{F}$.

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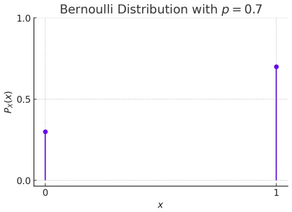
Properties:

- $\mathbb{E}(X) = \sum_{x \in \mathcal{F} = \{0,1\}} x \cdot P(X = x)$
  $= 0 \cdot (1-p) + 1 \cdot (p) = p$
- $[\mathbb{E}(X)]^2 = p^2$
- $\mathbb{E}(X^2) = 0^2 \cdot (1-p) + 1^2 \cdot (p) = p$
- $Var(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$
  $= p(1-p)$



2. Discrete distributions: Binomial (n, p)

Binomial Distribution: A random variable X has a binomial distribution with parameters n and p, denoted $X \sim \text{Binomial}(n, p)$, if its PMF is given by:

$$P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & \text{if } x = 0, 1, 2, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ is the binomial coefficient and the factorial operator ! is defined as $n! := n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$ with $0! = 1$.

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In other words, the random variable X represents the number of successes in n independent Bernoulli trials with success probability p. In this case, the support is $\mathcal{F} = \{0, 1, \ldots, n\}$, and the parameter space is $\Omega = \{(n, p) | n \in \mathbb{Z}^+, 0 < p < 1\}$.

2. Discrete distributions: Binomial (n, p)

Properties:

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  Changing the index of summation to $k = x - 1$, we get

$$\mathbb{E}(X) = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} p^{k+1} (1-p)^{(n-1)-k}$$

  This can be expressed as: $\mathbb{E}(X) = n \cdot p \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k}$

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  Using the binomial theorem, we have:

  $$\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = (p + (1-p))^{n-1} = 1^{n-1} = 1$$

  Thus, we conclude: $\mathbb{E}(X) = n \cdot p \cdot 1 = n \cdot p$

2. Discrete distributions: Binomial (n, p)

Properties:

- $\mathbb{E}(X) = n \cdot p$
- $\mathbb{E}(X^2) \stackrel{?}{=} n \cdot p \cdot (1-p) + (n \cdot p)^2$
- $Var(X) \stackrel{?}{=} n \cdot p \cdot (1-p)$

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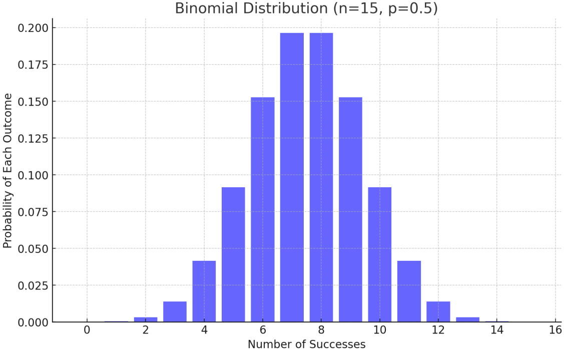

2. Discrete distributions: Binomial (n, p)

Example: Suppose a coin is flipped 3 times. Let X represent the number of heads (successes) in these 3 flips. Assuming the coin is fair, the probability of getting a head on any flip is $p = 0.5$. Since each flip is independent, X follows a binomial distribution with parameters $n = 3$ (number of flips) and $p = 0.5$ (probability of heads). This is denoted as:

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The probability of getting exactly x heads in 3 flips is given by the binomial PMF:

$$P(X = x) = \binom{3}{x}(0.5)^x(0.5)^{3-x}, \quad x = 0, 1, 2, 3$$

Let's calculate the probabilities for different values of x:

- $P(X = 0) = \binom{3}{0}(0.5)^0(0.5)^3 = 1 \times 0.125 = 0.125$
- $P(X = 1) = \binom{3}{1}(0.5)^1(0.5)^2 = 3 \times 0.125 = 0.375$
- $P(X = 2) = \binom{3}{2}(0.5)^2(0.5)^1 = 3 \times 0.125 = 0.375$
- $P(X = 3) = \binom{3}{3}(0.5)^3(0.5)^0 = 1 \times 0.125 = 0.125$

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- $P(X = 3) = \binom{3}{3}(0.5)^3(0.5)^0 = 1 \times 0.125 = 0.125$

In this case, the expected value $\mathbb{E}(X)$ and variance $Var(X)$ can be computed as:

$$\mathbb{E}(X) = n \cdot p = 3 \cdot 0.5 = 1.5$$

$$Var(X) = n \cdot p \cdot (1 - p) = 3 \cdot 0.5 \cdot 0.5 = 0.75$$

2. Discrete distributions: Poisson (λ)

Poisson Distribution: A random variable X has a Poisson distribution with parameter λ, denoted $X \sim \text{Poisson}(\lambda)$, if its PMF is given by:

$$p_X(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & \text{if } x \in \{0, 1, 2, \dots\}, \\ 0 & \text{otherwise.} \end{cases}$$

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$$p_X(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & \text{if } x \in \{0, 1, 2, \dots\}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the Poisson distribution models the probability of a given number of events occurring in a fixed interval of time or space, under the assumption that events occur independently and at a constant rate λ.

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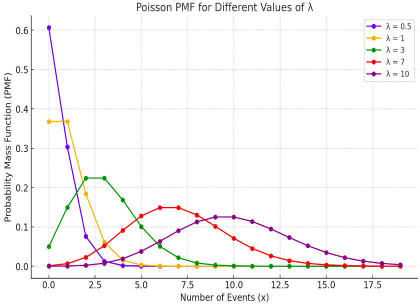

Precisely, λ represents the **average** number of events that occur in an interval, and x represents the **actual** number of events observed in that interval.

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  Hence, the expected value becomes: $\mathbb{E}(X) = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$

- $\mathbb{E}(X^2) \stackrel{?}{=} \lambda + \lambda^2$
- $Var(X) \stackrel{?}{=} \mathbb{E}[(X - \mathbb{E}(X))^2] = \lambda$