Selected Topics in Mathematics of Learning

High-Dimensional Statistics

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Department of Data Science, FAU

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Part VI

Sparse vector autoregressive models

Motivation

- Traditional assumptions:
  - Data are multivariate Gaussian
  - Observations are independent and identically distributed (i.i.d.)

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Key Idea: Vector Autoregressive Models (VARs) are powerful tools for capturing temporal correlations in high-dimensional data.

Outline

- Preliminaries of VAR
- Estimation of Sparse VAR using LASSO
- LASSO: Estimation
- LASSO: Properties

1. Preliminaries of VAR

A time series $\{X_t\}_{t\in\mathbb{Z}} = \{(X_{j,t})_{j=1,\ldots,d}\}_{t\in\mathbb{Z}}$ follows a **VAR(p)** model if:

$$X_t = \Phi_1 X_{t-1} + \ldots + \Phi_p X_{t-p} + \epsilon_t, \quad t \in \mathbb{Z},$$

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where:

- $\Phi_1, \ldots, \Phi_p$ are $d \times d$ matrices.
- $\{\epsilon_t\}_{t\in\mathbb{Z}}$ is a white noise series with the following statistics:

$$\mathbb{E}[\epsilon_t] = 0, \quad \mathbb{E}[\epsilon_t \epsilon_t'] = \Sigma_\epsilon, \quad \mathbb{E}[\epsilon_s \epsilon_t'] = 0 \quad \text{for } s \neq t.$$

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The equations are:

$$X_{1,t} = \Phi_{1,11} X_{1,t-1} + \Phi_{1,12} X_{2,t-1} + \epsilon_{1,t},$$
$$X_{2,t} = \Phi_{1,21} X_{1,t-1} + \Phi_{1,22} X_{2,t-1} + \epsilon_{2,t}.$$

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Please note the following:

- $\mathbb{E}[\epsilon_t] = 0$: This indicates that the white noise has a mean (or expected value) of 0 for all time points t. Essentially, the noise fluctuates around zero on average.
- $\mathbb{E}[\epsilon_t \epsilon_t'] = \Sigma_\epsilon$: Here, Σ_ϵ is the covariance matrix of ϵ_t, which characterizes the variance (for scalar ϵ_t) or the relationships between components (for vector-valued ϵ_t). For scalar white noise, this reduces to $\mathbb{E}[\epsilon_t^2] = \sigma_\epsilon^2$, where σ_ϵ^2 is the variance.
- $\mathbb{E}[\epsilon_s \epsilon_t'] = 0$ for $s \neq t$: This indicates that ϵ_t values at different time points t are uncorrelated. That is, the noise at time t does not depend on or influence the noise at any other time s. For vector-valued ϵ_t, this implies that the cross-covariance between vectors at different times is 0.

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where:

  - Φ_1 is a scalar (instead of a matrix since $d = 1$).
  - ϵ_t is white noise:

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- A VAR(1) model is:
  - **Strictly stable** if:

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  - **Marginally stable** if $|\Phi_1| = 1$,
  - **Unstable** if $|\Phi_1| > 1$.

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General Stability Condition: For a VAR(p) model, stability requires that its characteristic polynomial satisfies:

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Assumption: In this part of the course, we assume all VAR models are strictly stable unless otherwise stated.

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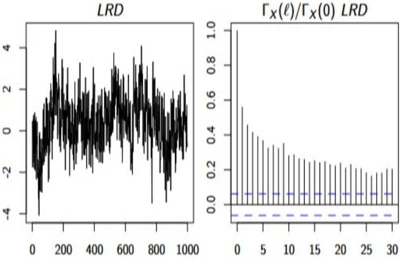
Implications of stationarity:

- The statistical properties of the time series do not change over time.
- Stationarity ensures that the VAR model's behavior is predictable and consistent over time, making it suitable for modeling and forecasting.

1.2 Stationarity: Long-range dependence (LRD)



Left: Time Series:

- The time series shows a pattern with persistent, slowly decaying behavior over time, indicating that observations are strongly correlated even at long time lags.

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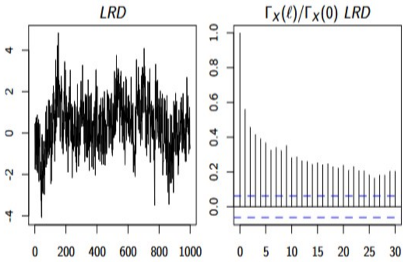

Right: Autocorrelation Function (ACF):

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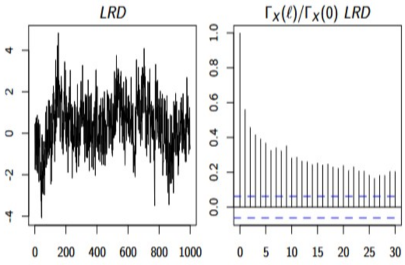

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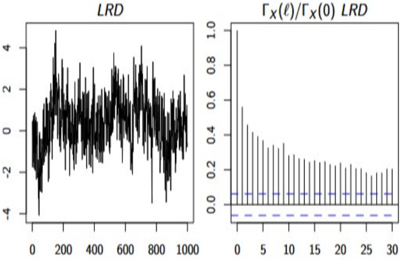

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Stationarity Implications:

- The slow decay in the ACF suggests that the time series is **not stationary**, as stationarity requires the ACF to decay rapidly (e.g., exponentially or geometrically–decreases by a constant ratio in each time step.

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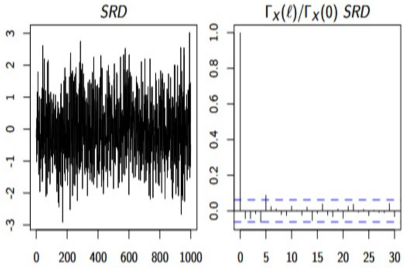

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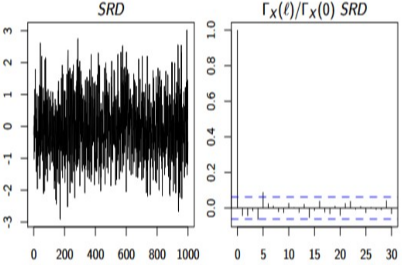

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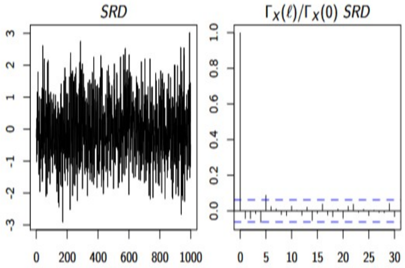

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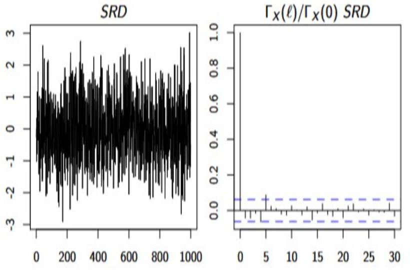

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Stationarity Implications:

- The rapid decay in the ACF suggests that the time series is likely **stationary**.
- Stationary time series exhibit consistent statistical properties over time.

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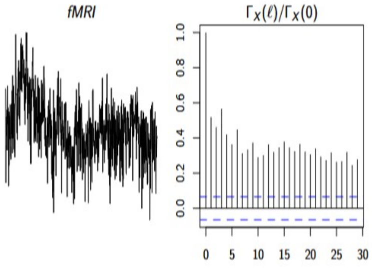

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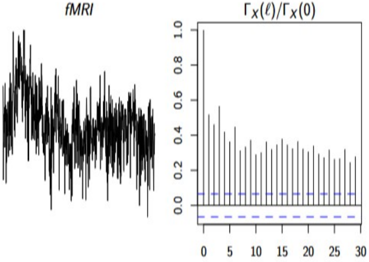

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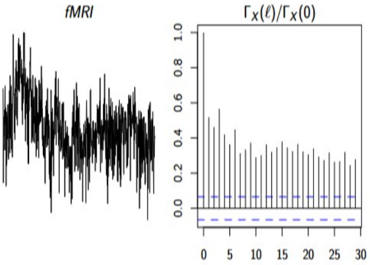

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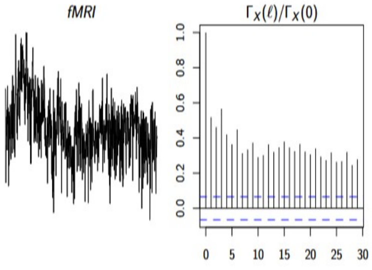

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- However, correlations persist over moderate lags, requiring further investigation to confirm stationarity.

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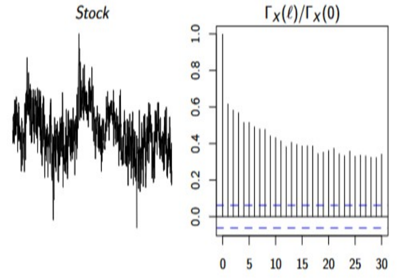

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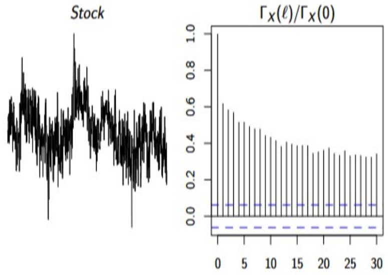

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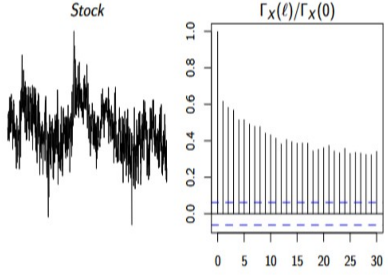

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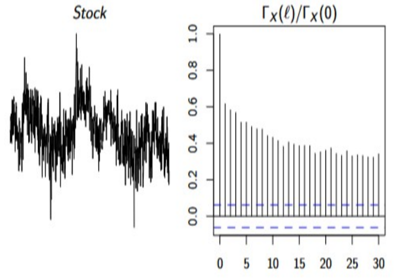

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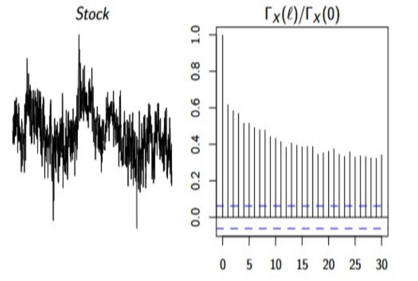

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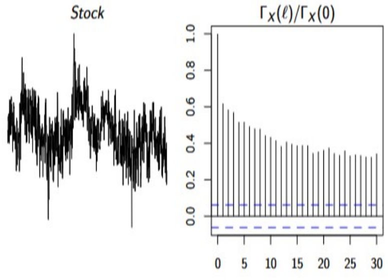

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Stationarity Implications:

- The slow decay in the ACF indicates that the time series may be **weakly stationary**.
- Financial data can exhibit behaviors such as long-range dependence or structural breaks that violate stationarity assumptions.

1.2. Key Points on Stationarity and ACF

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- A time series is stationary if its mean and variance are constant over time, and the autocovariance $\Gamma_X(\ell)$ depends only on the lag ℓ, not on time t.

1.2. Key Points on Stationarity and ACF

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- A time series is stationary if its mean and variance are constant over time, and the autocovariance $\Gamma_X(\ell)$ depends only on the lag ℓ, not on time t.
- In the figures:
  - SRD, fMRI, and stock data are examples of stationary (or weakly stationary) time series.

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   - Stationary time series typically exhibit ACFs that decay rapidly (e.g., exponentially or geometrically) as ℓ increases.
   - Non-stationary time series may show very slow or no decay in the ACF.

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   - **Units:** The autocovariance is measured in the square of the units of X_t, so it depends on the scale of the time series.

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- **Units:** The autocorrelation is dimensionless and does not depend on the scale of the time series.

1.2 Note 2: Covariance and Variance Relationship at lag ℓ

The covariance, $\text{Cov}(X_t, X_{t+\ell}) = \mathbb{E}[(X_t - \mu)(X_{t+\ell} - \mu)]$ will be equal to the variance of the time series, $\text{Var}(X_t) = \mathbb{E}[(X_t - \mu)^2]$, when the lag $\ell = 0$.

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Explanation:

- **Variance:** The variance of a random variable X_t is a special case of the covariance where the two random variables are identical, i.e., the lag is $\ell = 0$:

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- For $\ell \neq 0$, the covariance involves the dependence between X_t and its lagged value $X_{t+\ell}$. Unless the time series is perfectly correlated for that lag ($X_{t+\ell} = X_t$), the covariance at lag ℓ will **not** equal the variance.

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Conclusion: $\text{Cov}(X_t, X_{t+\ell}) = \text{Var}(X_t) \quad$ if and only if $\ell = 0$.