Unsteady Stream Depletion when Pumping from Semiconfined Aquifer

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Abstract: A solution is obtained for flow depleted from a stream when water is pumped from an adjacent well in a semiconfined aquifer. The streambed partially penetrates the aquitard, which forms the top boundary of the pumped aquifer, and the distance between the well and stream is assumed large enough to allow the stream to be modeled with a zero width. The governing partial differential equations for this problem are shown to be equivalent to the equation postulated and solved by Boulton for flow to a well in a delayed-yield aquifer. Consequently, drawdown curves and plots of stream depletion versus time are all found to have two inflection points. However, unlike the solution behavior for Boulton's problem, aquifer recharge furnished by the stream causes all drawdown and stream depletion curves to approach horizontal asymptotes as time becomes infinite. The solution calculated herein is general enough to reduce to a solution calculated earlier by Hunt when the aquitard becomes impermeable.

DOI: 10.1061/(ASCE)1084-0699(2003)8:1(12)

CE Database keywords: Groundwater depletion; Aquifers; Streams; Differential equations.

Introduction

Water abstracted from a well beside a stream also removes flow from the stream in a process known as stream depletion. The effects of stream depletion are particularly important in small streams, where engineers and groundwater hydrologists may be required to assess and control the environmental consequences of stream depletion. If a constant pumping rate continues for a sufficiently long period of time to reach steady flow, then stream depletion and well abstraction become identical. However, well abstraction always exceeds stream depletion before steady-flow conditions are reached. Thus, unsteady solutions for this problem allow water resource managers to devise well pumping schedules that can reduce the harmful effects of stream depletion to acceptable levels.

Theis (1941) obtained the first unsteady solution for this problem by considering abstraction from a well beside a fully penetrating stream. The stream edge was modeled as an infinitely long, straight boundary of zero drawdown, and the stream depletion was obtained in the form of a definite integral that Theis evaluated with an infinite series. Glover and Balmer (1954) rewrote the Theis solution in terms of the complimentary error function, and, to this day, the Theis solution is known by groundwater hydrologists as the Glover–Balmer solution. Hantush (1965) obtained a solution for the problem considered by Theis when the fully penetrating streambed is lined with a less permeable aquitard. Jenkins (1968) and Wallace et al. (1990) showed how superposition and time translation are used with the Theis solution to obtain solutions for more general pumping schedules. Spalding and Khaleel (1991) and Sophocleous et al. (1995) used numerical models to assess the result of simplifying assumptions that were made to obtain the Theis and Hantush solutions. Zlotnik and Huang (1999) used a solution obtained by Grigoryev (1957) to model stream depletion from a partially penetrating, finite-width stream with a semipermeable bottom.

Hunt (1999) obtained a solution for flow to a well beside a stream in an aquifer that extended to infinity in all directions. The streambed slightly penetrated the aquifer, and the distance between the well and stream was a sufficient number of stream widths to allow the stream to be modeled with a zero width. A similar problem is solved herein except that the aquifer is capped with a semipermeable aquitard that contains a relatively thin layer of standing water. The streambed partially penetrates the aquitard, and the distance between the well and stream is assumed large enough to allow the stream to be modeled with a zero width.

Governing Equations

A definition sketch for the problem is shown in Fig. 1. The pumped aquifer has a drawdown φ_1, a transmissivity T, and an elastic storage coefficient or storativity S. The overlying aquitard has a free-surface drawdown φ_2, a permeability K', a porosity σ, a specific storage S_s and a saturated thickness B'. The river has a width b, and the aquitard thickness beneath the river is B''. A well at a distance L from the stream edge abstracts a flow Q from the aquifer, and the vertical coordinate z is measured upward from the horizontal plane of the initial free surface location in the aquitard. Ultimately, the stream width b will be allowed to approach zero.

Flow in the aquifer is assumed to be largely horizontal and, therefore, is described by the Dupuit equition

$$T\left(\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}\right)=S\frac{\partial\varphi_1}{\partial t}-w \tag{1}$$

where w = specific discharge in the positive z direction at the boundary between the aquifer and aquitard. The aquitard has a

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Note. Discussion open until June 1, 2003. Separate discussions must be submitted for individual papers. To extend the closing date by one month, a written request must be filed with the ASCE Managing Editor. The manuscript for this paper was submitted for review and possible publication on January 18, 2002; approved on June 3, 2002. This paper is part of the ***Journal of Hydrologic Engineering***, Vol. 8, No. 1, January 1, 2003. ©ASCE, ISSN 1084-0699/2003/1-12–19/$18.00.

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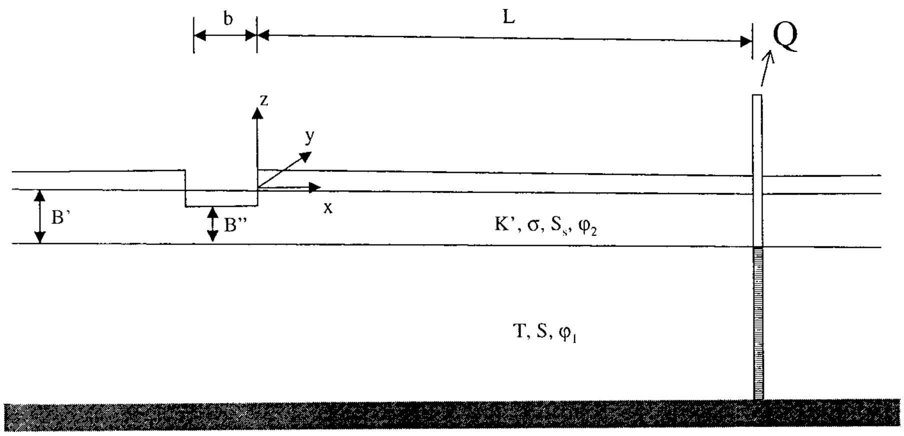

Fig. 1. Definition sketch for flow to well in semipermeable aquifer when well is beside stream

relatively small permeability, so that flow in the aquitard has a significant vertical component and is described by the following partial differential equation:

$$K'\left(\frac{\partial^2\varphi_2}{\partial x^2}+\frac{\partial^2\varphi_2}{\partial y^2}+\frac{\partial^2\varphi_2}{\partial z^2}\right)=S_s\frac{\partial\varphi_2}{\partial t} \tag{2}$$

Relatively small drawdowns are assumed in both the aquifer and aquitard, which allows the following linearized boundary condition to be imposed on the initial location of the free surface:

$$K'\frac{\partial\varphi_2(x,y,0,t)}{\partial z}+\sigma\frac{\partial\varphi_2(x,y,0,t)}{\partial t}=0 \tag{3}$$

Not all terms in Eq. (2) are of equal importance. Scales in the horizontal and vertical directions may be chosen as L and B', respectively. Since both terms in Eq. (3) must be of equal importance, the time scale is $\sigma B'/K'$. Therefore, magnitudes of terms in Eq. (2) relative to the third term are

$$\left(\frac{B'}{L}\right)^2 \quad \left(\frac{B'}{L}\right)^2 \quad 1 \quad \frac{B'S_s}{\sigma} \tag{4}$$

Since $B'/L \ll 1$ and $B'S_s/\sigma$ has an order of about 10^{-3}, Eq. (2) reduces to

$$K'\frac{\partial^2\varphi_2}{\partial z^2}=0 \tag{5}$$

which states that the vertical specific discharge w does not change with the z coordinate. Therefore, w is given by the second term of Eq. (3), and since φ_2 is a linear function of z, the following relationship is obtained for the aquitard:

$$w=-\sigma\frac{\partial\varphi_2(x,y,0,t)}{\partial t}=K'\frac{\varphi_2(x,y,0,t)-\varphi_1(x,y,t)}{B'} \tag{6}$$

where φ_1 and φ_2 have been assumed to be continuous at the boundary between the aquitard and aquifer.

Thus, Eq. (6) shows that Eqs. (1) and (2) can be replaced with the following two equations:

$$T\left(\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}\right)=S\frac{\partial\varphi_1}{\partial t}+\left(\frac{K'}{B'}\right)(\varphi_1-\varphi_2) \tag{7}$$

$$\sigma\frac{\partial\varphi_2}{\partial t}+\left(\frac{K'}{B'}\right)(\varphi_2-\varphi_1)=0 \tag{8}$$

where $\varphi_1\equiv\varphi_1(x,y,t)=$aquifer drawdown and $\varphi_2\equiv\varphi_2(x,y,0,t)$ $=$free surface drawdown in the aquitard.

It will be useful in what follows to show that Eqs. (7) and (8) are completely equivalent to the differential equation that was postulated empirically by Boulton (1963) for drawdown in a delayed-yield aquifer. Integration of Eq. (8) gives

$$\varphi_2=\left(\frac{K'/B'}{\sigma}\right)\int_0^t\varphi_1(x,y,\tau)\exp\left[-\left(\frac{K'/B'}{\sigma}\right)(t-\tau)\right]d\tau \tag{9}$$

Since $\varphi_1(x,y,0)=0$, integration of Eq. (9) by parts gives

$$\varphi_2-\varphi_1=\int_0^t\frac{\partial\varphi_1(x,y,\tau)}{\partial\tau}\exp\left[-\left(\frac{K'/B'}{\sigma}\right)(t-\tau)\right]d\tau \tag{10}$$

Inserting Eq. (10) into Eq. (7) gives the Boulton equation

$$T\left(\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}\right)=S\frac{\partial\varphi_1}{\partial t}+(K'/B')\times\int_0^t\frac{\partial\varphi_1(x,y,\tau)}{\partial\tau}\exp\left[-\left(\frac{K'/B'}{\sigma}\right)(t-\tau)\right]d\tau \tag{11}$$

in which Boulton's empirically postulated delay index has its reciprocal α given by

$$\alpha=\left(\frac{K'/B'}{\sigma}\right) \tag{12}$$

Thus, Eq. (11) is identical to the partial differential equation given by Boulton for pumping from a well in a delayed-yield aquifer, and every term in this equation is seen to have a clearly defined physical meaning. Eq. (12) has been obtained previously by Cooley and Case (1973). Eqs. (7)–(8) apparently have not been obtained previously. It is also worth mentioning that Neuman (1972) has obtained a solution for an entirely different aquifer geology that gives a solution behavior that is very similar to the Boulton solution behavior.

Eqs. (7) and (8) hold everywhere except at the pumped well and along the stream. The pumped well is accounted for by adding to the right side of Eq. (7)

$$-Q\delta(x-L)\delta(y) \tag{13}$$

where $\delta=$Dirac's delta function. Along the stream, continuity in the aquifer requires that the difference between the lateral outflow and inflow on either side of the stream equal the vertical seepage through the aquitard from the stream

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$$T\frac{\partial\varphi_1(0,y,t)}{\partial x}-T\frac{\partial\varphi_1(-b,y,t)}{\partial x}=K'\frac{\varphi_1-0}{B''}b \quad (14)$$

When $b\to 0$, Eq. (14) shows that the stream can be accounted for by adding to the right side of Eq. (7)

$$\delta(x)\lambda\varphi_1 \quad (15)$$

where the resistance coefficient λ is given by

$$\lambda=K'\frac{b}{B''} \quad (16)$$

Furthermore, Eqs. (14) and (16) show that the total stream depletion is given by

$$\Delta Q(t)=\lambda\int_{-\infty}^{\infty}\varphi_1(0,y,t)dy \quad (17)$$

Eq. (14) shows that $\partial\varphi_1/\partial x$ is discontinuous along $x=0$, which implies that φ_1 is continuous along the stream at $x=0$. The value of φ_2 at the stream is zero. However, the steady-flow value of φ_2 on either side of the stream is seen from Eq. (8) to equal φ_1, which means that a discontinuity occurs in φ_2 along $x=0$. This is the price that is paid for dropping horizontal spatial derivatives from Eq. (2) to obtain Eq. (8).

Problem Statement

The equations developed in the preceding section allow the problem shown in Fig. 1 to be described by the solution of the following equations:

$$T\left(\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}\right)=S\frac{\partial\varphi_1}{\partial t}+\left(\frac{K'}{B'}\right)(\varphi_1-\varphi_2)$$
$$-Q\delta(x-L)\delta(y)+\delta(x)\lambda\varphi_1 \quad (18)$$

$$\sigma\frac{\partial\varphi_2}{\partial t}+\left(\frac{K'}{B'}\right)(\varphi_2-\varphi_1)=0 \quad (19)$$

$$\text{Limit}_{r\to\infty}\,\varphi_1=0 \quad (r=\sqrt{x^2+y^2}) \quad (20)$$

$$\varphi_1(x,y,0)=\varphi_2(x,y,0,0)=0 \quad (21)$$

Eqs. (17)–(21) reduce to the same set of equations solved earlier by Hunt (1999) for a single aquifer when $K'\to 0$.

Eqs. (17)–(21) can be simplified considerably by introducing the following dimensionless variables:

$$(\varphi_1^*,\varphi_2^*,x^*,y^*,t^*,K^*,\lambda^*,\varepsilon,\Delta Q^*)$$
$$=\left(\frac{\varphi_1 T}{Q},\frac{\varphi_2 T}{Q},\frac{x}{L},\frac{y}{L},\frac{tT}{SL^2},\frac{(K'/B')L^2}{T},\frac{\lambda L}{T},\frac{S}{\sigma},\frac{\Delta Q}{Q}\right) \quad (22)$$

This allows Eqs. (17)–(21) to be rewritten as follows:

$$\Delta Q=\lambda\int_{-\infty}^{\infty}\varphi_1(0,y,t)dy \quad (23)$$

$$\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_1}{\partial y^2}=\frac{\partial\varphi_1}{\partial t}+K(\varphi_1-\varphi_2)-\delta(x-1)\delta(y)$$
$$+\delta(x)\lambda\varphi_1 \quad (24)$$

$$\frac{\partial\varphi_2}{\partial t}+\varepsilon K(\varphi_2-\varphi_1)=0 \quad (25)$$

$$\text{Limit}_{r\to\infty}\,\varphi_1=0 \quad (r=\sqrt{x^2+y^2}) \quad (26)$$

$$\varphi_1(x,y,0)=\varphi_2(x,y,0,0)=0 \quad (27)$$

where the asterisk superscript has been omitted for notational convenience.

Problem Solution

The Fourier and Laplace transforms of φ_1 and φ_2 will be denoted by an upper case letter and an overbar, respectively. Thus, the double transform of φ_i is defined by

$$\overline{\Phi}_i(x,\alpha,p)=\int_{-\infty}^{\infty}\int_0^{\infty}\varphi_i(x,y,t)e^{-pt+i\alpha y}dt\,dy \quad (28)$$

and the inverse Fourier and Laplace transforms give φ_i

$$\varphi_i(x,y,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\alpha y}$$
$$\times\left[\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\overline{\Phi}(x,\alpha,p)e^{pt}dp\right]d\alpha \quad (29)$$

where $i=1,2$, and $\gamma=$constant chosen so that the integration path of the contour integral lies to the right of all singularities and branch points in the complex p plane.

Application of Eq. (28) to Eqs. (24)–(25) gives the following two equations for $\overline{\Phi}_1$ and $\overline{\Phi}_2$:

$$\frac{d^2\overline{\Phi}_1}{dx^2}-\alpha^2\overline{\Phi}_1=p\overline{\Phi}_1+K(\overline{\Phi}_1-\overline{\Phi}_2) \quad (-\infty<x<0)$$
$$=p\overline{\Phi}_1+K(\overline{\Phi}_1-\overline{\Phi}_2)-\frac{\delta(x-1)}{p} \quad (0<x<\infty)$$
$$(30)$$

$$p\overline{\Phi}_2+\varepsilon K(\overline{\Phi}_2-\overline{\Phi}_1)=0 \quad (-\infty<x<\infty) \quad (31)$$

The general solution of Eqs. (30)–(31) for $\overline{\Phi}_1$ that vanishes as $|x|\to\infty$ is

$$\overline{\Phi}_1(x,\alpha,p)=Ae^{mx} \quad (-\infty<x<0)$$
$$=Be^{mx}+Ce^{-mx} \quad (0<x<1)$$
$$=De^{-mx} \quad (1<x<\infty) \quad (32)$$

where A, B, C and $D=$constants and $m(\alpha,p)$ is given by

$$m(\alpha,p)=\sqrt{\alpha^2+\frac{p(p+K+\varepsilon K)}{p+\varepsilon K}} \quad (33)$$

Integration of Eqs. (24) and (30) across $x=0$ and $x=1$, respectively, shows that the four unknown constants in Eq. (32) can be found by requiring $\overline{\Phi}_1$ to be continuous across $x=0$ and $x=1$ and $\partial\overline{\Phi}_1/\partial x$ to have discontinuities of $\lambda\overline{\Phi}_1(0,\alpha,p)$ and $-1/p$ across $x=0$ and $x=1$, respectively. The final result, when substituted back into Eq. (32), is

$$\overline{\Phi}_1(x,\alpha,p)=\frac{e^{m(x-1)}}{p(\lambda+2m)} \quad (-\infty<x\leqslant 0)$$
$$=\frac{e^{-m|x-1|}}{2pm}-\frac{e^{-m(x+1)}}{2pm}+\frac{e^{-m(x+1)}}{p(\lambda+2m)}$$
$$(0\leqslant x<\infty) \quad (34)$$

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Eqs. (23), (28), and (34) show that the Laplace transform of the stream depletion is

$$\Delta\overline{Q}(p)=\lambda\overline{\Phi}_1(0,0,p)=\frac{\lambda e^{-m_0}}{p(\lambda+2m_0)} \tag{35}$$

where

$$m_0\equiv m(0,p)=\sqrt{\frac{p(p+K+\varepsilon K)}{p+\varepsilon K}} \tag{36}$$

The inversion of Eqs. (35)–(36) will be considered in the next section.

An expression for the drawdown can be obtained by manipulating Eq. (34) into the following form:

$$\overline{\Phi}_1(x,\alpha,p)=\frac{e^{-m|x-1|}}{2pm}-\frac{\lambda e^{-m(|x|+1)}}{2pm(\lambda+2m)}\quad(-\infty<x<\infty) \tag{37}$$

The second term on the right side of Eq. (37) can be rewritten in the form of an integral to obtain

$$\overline{\Phi}_1=\frac{e^{-m|x-1|}}{2pm}-\frac{\lambda}{2}\int_0^\infty e^{-\xi\lambda/2}\frac{e^{-m(\xi+1+|x|)}}{2pm}d\xi \tag{38}$$

The first term on the right of Eq. (38) is the solution when $\lambda=0$ and, therefore, is the double transform of the solution for flow to a well at $(x,y)=(1,0)$ in a delayed-yield aquifer of infinite extent when no stream is present. The second term in the integrand of Eq. (38) is the same function with $|x-1|$ replaced by $(\xi+1+|x|)$. Therefore, if the solution of Boulton's problem for flow to a well in a delayed-yield aquifer of infinite extent is denoted by

$$W(r,t,\varepsilon,K)=\text{Inverse}\left\{\frac{e^{-m|x-1|}}{2pm}\right\}\qquad (r=\sqrt{(x-1)^2+y^2}) \tag{39}$$

then the inverse of Eq. (38) can be written in the following way:

$$\varphi_1(x,y,t)=W(r,t,\varepsilon,K)-\frac{\lambda}{2}\int_0^\infty e^{-\xi\lambda/2}\times W(R,t,\varepsilon,K)d\xi\qquad (R=\sqrt{(\xi+1+|x|)^2+y^2}) \tag{40}$$

The calculation of $W(r,t,\varepsilon,K)$ will be considered in a later section.

Inversion of Eq. (35)

Eq. (35) can be inverted by using the integral

$$\frac{e^{-m_0}}{(\lambda+2m_0)}=\frac{e^{\lambda/2}}{2}\int_1^\infty e^{-\xi\lambda/2}e^{-\xi m_0}d\xi \tag{41}$$

together with a second integral given by Gradshteyn and Ryzhik (1965, Eq. 3.325)

$$e^{-\xi m_0}=\frac{2}{\sqrt{\pi}}\int_0^\infty\left(\exp-x^2-\frac{\xi^2}{4x^2}m_0^2\right)dx \tag{42}$$

where m_0^2 can be manipulated into the form

$$m_0^2=K+p-\frac{\varepsilon K^2}{p+\varepsilon K} \tag{43}$$

This allows Eq. (35) to be rewritten as a double integral.

$$\Delta\overline{Q}(p)=\frac{\lambda e^{\lambda/2}}{\sqrt{\pi}}\int_1^\infty e^{-\xi\lambda/2}\int_0^\infty e^{-x^2}\frac{\exp\left(-\frac{\xi^2}{4x^2}m_0^2\right)}{p}dx\,d\xi \tag{44}$$

The integrand in Eq. (44) is seen from Eq. (43) to have a first-order pole at $p=0$ and an essential singularity at $p=-\varepsilon K$. Since εK is often a small number, these two singularities are usually very close to each other in the p plane. This makes it difficult to obtain accurate numerical values from an inversion of Eq. (44).

The Bromwich integral, which is introduced in texts such as Hildebrand (1976), can be used to show that the inverse of Eq. (44) is given by

$$\Delta Q(t)=\frac{\lambda e^{\lambda/2}}{\sqrt{\pi}}\left[\int_1^\infty e^{-\xi\lambda/2}\int_{\xi/2\sqrt{t}}^\infty e^{-x^2}dx\,d\xi+\int_1^\infty e^{-\xi\lambda/2}\times\int_{\xi/2\sqrt{t}}^\infty e^{-x^2}\left(\frac{1}{2\pi i}\oint_{p=-\varepsilon K}\frac{e^{-(\xi^2/4x^2)m_0^2+pt}}{p}dp\right)dx\,d\xi\right] \tag{45}$$

where the first term is the contribution from the pole at the origin and the second term is the result of integrating around the essential singularity at $p=-\varepsilon K$. (The contour around $p=-\varepsilon K$ must not include the point $p=0$ in its interior.) The first term on the right of Eq. (45) can be evaluated in terms of complementary error functions, and the second term can be simplified by changing the integration variable from x to α with the substitution $x=\xi/(2\alpha\sqrt{t})$ and then carrying out the integration with respect to ξ. This gives the result

$$\Delta Q(t)=\text{erfc}\left(\frac{1}{2\sqrt{t}}\right)-\exp\left(\frac{\lambda}{2}+\frac{t\lambda^2}{4}\right)\text{erfc}\left(\frac{1}{2\sqrt{t}}+\frac{\lambda\sqrt{t}}{2}\right)-\lambda\int_0^1F(\alpha,t)G(\alpha,t)d\alpha \tag{46}$$

where erfc(x)=complementary error function and the functions $F(\alpha,t)$ and $G(\alpha,t)$ are given by

$$F(\alpha,t)=\exp\left(-\frac{1}{4t\alpha^2}\right)\sqrt{\frac{t}{\pi}}-\frac{\alpha t\lambda}{2}\times\exp\left(\frac{\lambda}{2}+\frac{t\alpha^2\lambda^2}{4}\right)\text{erfc}\left(\frac{\alpha\lambda\sqrt{t}}{2}+\frac{1}{2\alpha\sqrt{t}}\right) \tag{47}$$

$$G(\alpha,t)=-\frac{1}{2\pi i}\oint_{p=-\varepsilon K}\frac{e^{-t\alpha^2m_0^2+pt}}{p}dp \tag{48}$$

The contour integral in Eq. (48) can be evaluated by noting that the exponential function in the integrand has a zero imaginary part and a real part of

$$-Kt[\alpha^2+\varepsilon(1-\alpha^2)-2\alpha\sqrt{\varepsilon(1-\alpha^2)}\cos(\theta)]$$

on the circle

$$|p+\varepsilon K|=\alpha K\sqrt{\varepsilon}/\sqrt{1-\alpha^2}$$

Integrating around this contour, adding 0, $\frac{1}{2}$, or 1 when the origin lies outside, upon, or within this circle and carrying out a number of integral evaluations and manipulations ultimately leads to the following result:

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$$G(\alpha,t)=\frac{1}{2}\left[1-e^{-(a+b)}I_0(2\sqrt{ab})\right.$$

$$\left.+\left(\frac{b-a}{a+b}\right)\int_0^{(a+b)}e^{-\xi}I_0\left(2\xi\frac{\sqrt{ab}}{(a+b)}\right)d\xi\right] \quad (49)$$

where

$$a=\varepsilon Kt(1-\alpha^2) \quad (50)$$

$$b=Kt\alpha^2 \quad (51)$$

Substituting a power series for I_0 in Eq. (49) and integrating termwise finally gives the following series expansion:

$$G(\alpha,t)=\frac{1}{2}\left[1-e^{-(a+b)}I_0(2\sqrt{ab})\right.$$

$$\left.+\left(\frac{b-a}{a+b}\right)\sum_{n=0}^{\infty}\binom{2n}{n}P(2n+1,a+b)\left(\frac{\sqrt{ab}}{a+b}\right)^{2n}\right] \quad (52)$$

where $\binom{2n}{n}$ = binomial coefficient and $P(\nu,x)$ = an incomplete gamma function defined, for example, by Abramowitz and Stegun (1964) and Press et al. (1986). Calculated values of $G(\alpha,t)$ vary between 0 and 1 when $0<a<\infty$ and $0<b<\infty$, which is always true.

Round-off errors became a problem when t became too large in Eq. (47). This was avoided by using the following asymptotic formula:

$$F(\alpha,t)\sim\frac{\exp\left(-\frac{1}{4t\alpha^2}\right)}{2\alpha z\sqrt{\pi}}\left[1+\frac{2}{\lambda\left(1+\frac{1}{\lambda t\alpha^2}\right)^2}\right.$$

$$\left.\times\left(1-\frac{3}{2z^2}+\frac{15}{4z^4}-\frac{105}{8z^6}+\cdots\right)\right]$$

$$\left(z\equiv\frac{\alpha\lambda\sqrt{t}}{2}+\frac{1}{2\alpha\sqrt{t}}\right) \quad (53)$$

Eq. (53) was used when $z\geqslant 3$. The relative error when $z=3$ has an order of 1%.

Inversion of Eq. (39)

Boulton (1963) gave an exact solution for $W(r,t,\varepsilon,K)$, when $K=1$ and, consequently, when $L=\sqrt{T/(K'/B')}$, in the form of a definite integral with an infinite upper limit. However, the integrand has an infinite number of oscillations and also decays like $x^{-3/2}$ as x becomes infinite, a decay rate which is too slow to be efficient for numerical computations. Consequently, an expression will be obtained for W in a form that is more useful for computations.

Taking the inverse Fourier transform of the first term on the right side of Eq. (38) gives the Laplace transform of W

$$\overline{W}(r,p,\varepsilon,K)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-m|x-1|-i\alpha y}}{2pm}d\alpha \quad (m=\sqrt{\alpha^2+m_0^2}) \quad (54)$$

Since m is an even function of α, Eq. (54) can be rewritten in the form

$$\overline{W}(r,p,\varepsilon,K)=\frac{1}{2\pi p}\int_0^{\infty}\frac{e^{-|x-1|\sqrt{\alpha^2+m_0^2}}}{\sqrt{\alpha^2+m_0^2}}\cos(\alpha y)d\alpha \quad (55)$$

Two integrals given by Gradshteyn and Ryzhik [1965, Eqs. (3.471(9)) and (3.961(2))] allow Eq. (55) to be evaluated and then rewritten as follows:

$$\overline{W}(r,p,\varepsilon,K)=\frac{K_0(rm_0)}{2\pi p}=\frac{1}{4\pi}\int_0^{\infty}\frac{e^{-(r^2/4x)}}{x}\frac{e^{-xm_0^2}}{p}dx \quad (56)$$

where $K_0(x)$ = zero-order modified Bessel function of the second kind. The Bromwich integral gives the inverse of Eq. (56) as

$$W(r,t,\varepsilon,K)=\frac{1}{4\pi}\int_0^{t}\frac{e^{-(r^2/4x)}}{x}dx$$

$$+\frac{1}{4\pi}\int_0^{t}\frac{e^{-(r^2/4x)}}{x}\left(\frac{1}{2\pi i}\oint_{p=-\varepsilon K}\frac{e^{-xm_0^2+pt}}{p}dp\right)dx \quad (57)$$

where the first term on the right results from the simple pole at the origin and the second term is the contribution from the essential singularity at $p=-\varepsilon K$. Changing integration variables from x to u with $u=r^2/(4x)$ in the first integral and from x to α with $x=t\alpha^2$ in the second integral gives the result

$$W(r,t,\varepsilon,K)=\frac{1}{4\pi}E_1\left(\frac{r^2}{4t}\right)+\frac{1}{2\pi}$$

$$\times\int_0^1\frac{e^{-(r^2/4t\alpha^2)}}{\alpha}\left(\frac{1}{2\pi i}\oint_{p=-\varepsilon K}\frac{e^{-t\alpha^2m_0^2+pt}}{p}dp\right)d\alpha \quad (58)$$

where $E_1(x)$ = exponential integral. Finally, the definition of $G(\alpha,t)$ in Eq. (48) allows Eq. (58) to be written in its final form

$$W(r,t,\varepsilon,K)=\frac{1}{4\pi}E_1\left(\frac{r^2}{4t}\right)-\frac{1}{2\pi}\int_0^1e^{-(r^2/4t\alpha^2)}G(\alpha,t)\frac{d\alpha}{\alpha} \quad (59)$$

Because $G(\alpha,t)$ has an almost discontinuous change on the interval $0\leqslant\alpha\leqslant 1$ for very large values of t, integrals containing G in their integrand are evaluated most accurately with the trapezoidal rule.

It is worth looking briefly at the behavior of Eq. (59) for large and small values of t. For small values of t, Eq. (52) shows that $G(\alpha,t)\to 0$ as $t\to 0$ and Eq. (59) gives

$$W(r,t,\varepsilon,K)\sim\frac{1}{4\pi}E_1\left(\frac{r^2}{4t}\right) \quad (t\to 0) \quad (60)$$

Thus, W becomes independent of the aquitard porosity σ for small values of t, and W behaves as the solution for a confined aquifer with a storage coefficient S as $t\to 0$.

When $t\to\infty$, Eq. (49) can be used to show that

$$G(\alpha,t)=0 \quad \left(\alpha<\sqrt{\frac{\varepsilon}{1+\varepsilon}}\right)$$

$$=1 \quad \left(\sqrt{\frac{\varepsilon}{1+\varepsilon}}<\alpha<1\right) \quad (61)$$

Thus, for large values of t

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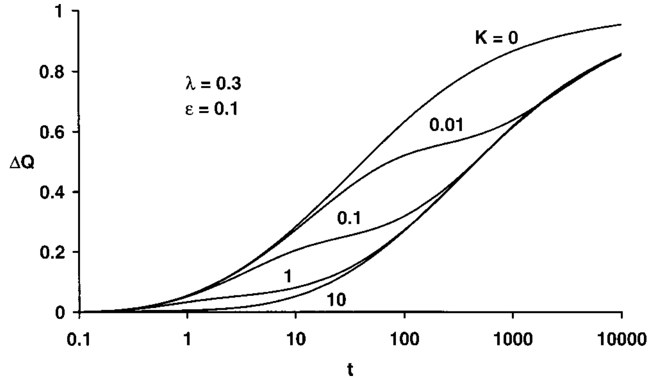

Fig. 2. Typical stream-depletion behavior

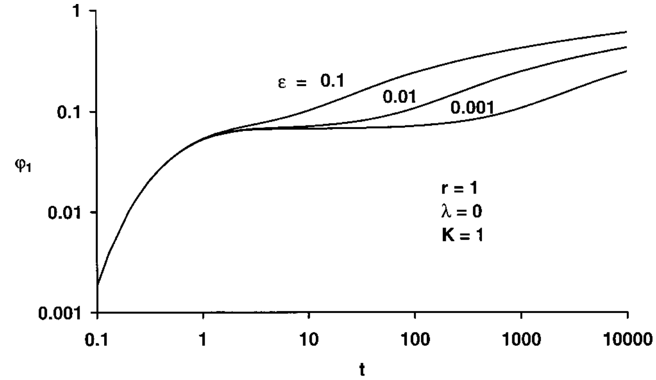

Fig. 3. Drawdowns for Boulton solution when no stream is present ($\lambda=0$)

$$W(r,t,\varepsilon,K)\sim\frac{1}{4\pi}E_1\left(\frac{r^2}{4t}\right)-\frac{1}{2\pi}\int_{\sqrt{\varepsilon/1+\varepsilon}}^{1}e^{-(r^2/4t\alpha^2)}\frac{d\alpha}{\alpha}\quad(62)$$

Setting $u=r^2/(4t\alpha^2)$ in the integral then gives the behavior as $t\to\infty$

$$W(r,t,\varepsilon,K)\sim\frac{1}{4\pi}E_1\left[\frac{r^2(1+\varepsilon)}{4t\varepsilon}\right]\quad(t\to\infty)\qquad(63)$$

Since $\varepsilon\ll1$ in most applications, and since $t\varepsilon$ is seen from Eq. (22) to be independent of the aquifer elastic storage coefficient S, Eq. (63) shows that drawdowns become nearly independent of S as $t\to\infty$. Thus, the aquifer behaves as an unconfined aquifer with porosity σ at large values of time.

Eq. (59) can be inserted into Eq. (40) and the integration with respect to ξ can be carried out to obtain the following expression for the drawdown:

$$\varphi_1(x,y,t)=W(r,t,\varepsilon,K)-\frac{1}{4\pi}E_1\left(\frac{R_0^2}{4t}\right)+\int_0^1 H(\alpha,x,y,t)\,d\alpha\qquad(64)$$

where r is defined in Eq. (39) and R_0 is defined by

$$R_0=\sqrt{(1+|x|)^2+y^2}\qquad(65)$$

The function $H(\alpha,x,y,t)$ is given by

$$H(\alpha,x,y,t)=\frac{1}{2\pi\alpha}\exp\left(-\frac{R_0^2}{4t\alpha^2}-\frac{\lambda}{2}\right)\left[\sqrt{\left(\frac{R_0}{\alpha}\right)^2-y^2}-\sqrt{R_0^2-y^2}\right]+\frac{\lambda}{4}\sqrt{\frac{t}{\pi}}e^{-(R_0^2/4t\alpha^2)}G(\alpha,t)e^{\theta^2}\operatorname{erfc}(\theta)\qquad(66)$$

where θ is used to represent the following combination of terms:

$$\theta=\frac{\alpha\sqrt{t}}{2}\left(\lambda+\frac{\sqrt{R_0^2-y^2}}{t\alpha^2}\right)\qquad(67)$$

It is much more efficient to use Eq. (64) to calculate φ_1 than to insert numerical values for W from Eq. (59) in Eq. (40).

Results and Discussion

A typical dimensionless plot of stream depletion versus time is shown in Fig. 2. The curve for $K=0$ is given by the first two terms on the right side of Eq. (46) and is identical to the result given earlier by Hunt (1999). In other words, since Eq. (49) gives the limit $G(\alpha,t)=0$ when $K=0$, Eq. (46) shows that the solution obtained herein for a semiconfined aquifer contains the earlier solution that was obtained for pumping from either an unconfined or completely confined aquifer.

Fig. 2 shows that increasing K causes the stream depletion to be less at smaller values of time than the depletion calculated for $K=0$. This is because increasing K allows a larger portion of water pumped from the well at early values of time to come from water stored in the overlying aquitard. However, at larger values of time drawdowns in the aquitard and aquifer approach each other and all curves become asymptotic to one. In other words, because the volume of water stored in the aquitard is finite, any well pumped for a sufficiently long period of time will ultimately take all of its flow from the adjacent stream.

Fig. 3 shows a typical plot of the Boulton solution for drawdown in an observation well calculated from Eq. (59). As shown by, Eq. (40) this result is identical to the result obtained herein when $\lambda=0$, and setting $K=1$ in this plot is shown by Eq. (22) to be equivalent to choosing $L=\sqrt{T/(K'/B')}$. However, it is both interesting and useful to point out that L does not have to be chosen in this way. For example, since L has no obvious physical meaning when $\lambda=0$, one can choose L to be radial distance to the observation well, in which case Eqs. (60) and (63) show that drawdowns become independent of K at both small and large values of time. The implication is that K can be determined by matching the mathematical and experimental solutions only at intermediate values of time, that S can be determined from Eq. (60) only at small values of time, and that σ can be determined from Eq. (63) only at large values of time. In fact, the easiest way to determine both T and σ in a pumping test analysis is to use Eq. (63) to justify an application of Jacob's straight-line approximation for values of time that are sufficiently large to allow an experimental semilog plot of drawdown versus time to become asymptotic to a straight line.

Fig. 4 shows a typical plot of drawdown versus time for an observation well when λ and K are finite. As might have been anticipated from the plot shown in Fig. 3, drawdown curves shown in Fig. 4 all have two inflection points. However, these curves also differ from curves plotted in Fig. 3 by having horizontal asymptotes at large values of time. In other words, the source of recharge provided by the stream at a finite distance from the pumped well allows steady flow to be approached as time becomes infinite. More specifically, Eq. (37) and the final-value theorem of Laplace-transform theory can be used to obtain the following expression for steady-flow drawdowns:

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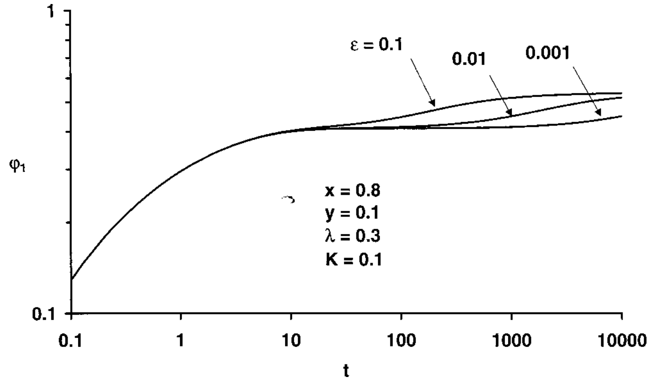

Fig. 4. Drawdowns when stream is present ($\lambda = 0.3$)

$$\varphi_1(x,y,\infty) = \frac{1}{4\pi}\ln\left[\frac{(|x|+1)^2+y^2}{|x-1|^2+y^2}\right] + \frac{1}{2\pi}\int_0^\infty \frac{(\xi+|x|+1)e^{-\xi\lambda/2}}{(\xi+|x|+1)^2+y^2}d\xi \qquad (68)$$

Additional reasons for the stream depletion behavior shown in Fig. 2 also become apparent after seeing the behavior of the drawdown curves shown in Fig. 4. Stream depletion is proportional to values of drawdown beneath the stream, as shown by Eq. (17), and this suggests that flow depletion and drawdown curves must have similar behaviors. Thus, all flow depletion curves in Fig. 2 have two inflection points and become asymptotic to the horizontal line $\Delta Q = 1$ as time becomes infinite.

Conclusions

A solution has been obtained for flow to a well in a semiconfined aquifer when the well is beside a stream. Distance between the well and stream is sufficiently large to allow the stream width to be approximated as zero, and the stream partially penetrates the aquitard, which forms the top boundary of the pumped aquifer. The governing partial differential equations for this problem have been shown to be equivalent to the equation postulated and solved by Boulton (1963) for flow to a well in a delayed-yield aquifer. Consequently, drawdown curves have the two inflection points that are characteristic of drawdown plots obtained for the Boulton solution. Unlike the Boulton solution, though, these drawdown curves have horizontal asymptotes that are approached as time becomes infinite, when stream depletion and well abstraction values become identical. Plots of stream depletion versus time have been seen to have two inflection points and to approach the horizontal asymptote $\Delta Q = 1$ as time becomes infinite. Two reasons have been given for this behavior: first, water pumped at smaller times comes mainly from water stored in the aquitard while water pumped at very large times comes from the stream, and, second, stream depletion and drawdown curves must have similar behaviors since stream depletion is proportional to drawdown beneath the stream.

Acknowledgments

The writer is indebted to David Scott of Environment Canterbury and Julian Weir of Lincoln Ventures for pointing out that this problem was worth solving.

Notation

The following symbols are used in this paper:

A,B,C,D = constants;
a,b = variables defined by Eqs. (50) and (51);
B' = aquitard saturated thickness;
B'' = aquitard thickness beneath stream;
b = stream width;
E_1 = exponential integral;
Erfc = complementary error function;
F = function defined by Eq. (47);
G = function defined by Eq. (48);
H = function defined by Eq. (66);
K_0 = zero-order modified Bessel function of second kind;
K' = aquitard permeability;
L = shortest distance between well and stream edge;
m = variable defined by Eq. (33);
m_0 = variable defined by Eq. (36);
P = incomplete gamma function;
p = Laplace transform parameter;
Q = well abstraction flow;
R,R_0,r = radial coordinates;
S = aquifer elastic storage coefficient (storativity);
S_s = aquitard specific storage;
T = aquifer transmissivity;
t = time;
u = integration variable;
W = Boulton's solution for flow to well in delayed-yield aquifer;
$\overline{W}$ = Laplace transform of W;
w = vertical component of specific discharge;
x,y = horizontal Cartesian coordinates;
z = vertical Cartesian coordinate, variable defined by Eq. (53);
α = Boulton's delay index, integration variable;
ΔQ = stream depletion flow;
δ = Dirac's delta function;
ε = S/σ;
θ = function defined by Eq. (67);
λ = stream-bed resistance coefficient;
ξ = integration variable;
σ = aquitard porosity;
τ = integration variable;
Φ_i = Fourier transform of φ_i;
$\overline{\Phi}_i$ = double transform of φ_i;
$\overline{\varphi}_i$ = Laplace transform of φ_i;
φ_1 = aquifer drawdown; and
φ_2 = aquitard free-surface drawdown.

Superscript

$*$ = dimensionless variable.

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