743- Regression and Time Series

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Stationary Models
&
The Autocorrelation Function

Basic Definitions

Definition 1.

Let $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$ be a time series with $\boldsymbol{E}[\boldsymbol{X_t}^2] < \infty$.

The **mean function** of $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots,\}$ is

$$m_X(t) = E(X_t), \quad t \in Z.$$

The **covariance function** of $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$ is

$$g_X(r,s) = Cov(X_r, X_s) = E[(X_r - m_X(r))(X_S - m_X(s))]$$

for all $r, s \in Z$.

Basic Definitions

Definition 2.

A time series $\{X_t, t = 0, \pm 1, \ldots\}$ is called **(weakly) stationary** if

(i) $\mu_X(\mathbf{t})$ is independent of t, that is, $\mu_X(\mathbf{t}) = m =$ **constant**, and

(ii) $\gamma_X(t+h, t)$ is independent of t, for each $h \in Z$.

Remark 1.

In view of condition (ii), whenever we use the term covariance function with reference to a stationary time series $\{X_t, t = 0, \pm 1, \ldots\}$ we shall mean the function γ_X of **one** variable, defined by

$$\gamma_X(h) := \gamma_X(h, 0) = \gamma_X(t+h, t).$$

Basic Definitions

The function $\gamma_X(.)$ will be referred to as the **autocovariance function** and $\gamma_X(h)$ as its value at lag $\boldsymbol{h}$.

Definition 3.

Let $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$ be a stationary time series.

The **autocovariance function** (**ACVF**) of $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$ at lag h is

$$g_X(h) = Cov(X_{t+h}, X_t) = Cov(X_h, X_0).$$

The **autocorrelation function** (**ACF**) of $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$ at lag h is

$$r_X(h) \equiv \frac{g_X(h)}{g_X(0)} = Cor(X_{t+h}, X_t).$$

Simple Examples: IID Noise

Example 1. (IID Noise).

If $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$ is ***iid*** noise with $\boldsymbol{E}[\boldsymbol{X_t^2}] = \sigma^2 < \infty$, then the first requirement of Definition 2 is obviously satisfied, since $\mathrm{E}(\boldsymbol{X_t}) = 0$ for all $\boldsymbol{t}$.

$$g_X(t+h,t) = \begin{cases} s^2, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0, \end{cases}$$

We use the notation $\{\boldsymbol{X_t}\} \sim \boldsymbol{IID}\,(0, \sigma^2)$, to indicate that the random variables $\boldsymbol{X_t}$ are independent and identically distributed random variables, each with mean 0 and variance σ^2.

Simple Examples: White Noise

Example 2. (White noise).

Definition.

A sequence of random variables $\{ \boldsymbol{\varepsilon_t} = \boldsymbol{Z_t}, \boldsymbol{t} = 0, \pm 1, \ldots \}$ is called **white noise** (with mean **0** and variance $\boldsymbol{\sigma^2}$):

$\boldsymbol{\varepsilon_t} \sim \boldsymbol{WN}(\mathbf{0}, \boldsymbol{\sigma^2})$, if $\mathrm{E}[\boldsymbol{\varepsilon_t}] = 0$, and

$$E[e_t e_s] = s^2 d_{ts} = \begin{cases} s^2, & \text{if } t = s \\ 0, & \text{if } t \neq s. \end{cases}$$

Clearly $\{ \boldsymbol{\varepsilon_t}, \boldsymbol{t} = 0, \pm 1, \ldots \}$ is **stationary** with the same covariance function as IID-noise:

$$g_e(t+h, t) = s^2 d_{h0} = \begin{cases} s^2, & \text{if } h = 0 \\ 0, & \text{if } h \neq 0. \end{cases}$$

Properties
of
Autocorrelation Function (ACF)
and
Autocovariance Function (ACVF)

Basic Properties of ACVF and ACF

- **Basic (Elementary) Properties of ACVF** $\gamma(\,.\,)$:

1) $g(0) \geq 0$

2) $|g(h)| \leq g(0)$ for all h ,

3) $g(h) = g(-h)$ for all h , that is , $g(\cdot)$ is even function.

Proof.

The first property is simply the statement that $Var(X_t) \geq 0$, since

$$g(0) = g_X(0) = Cov(X_t, X_t) = Var(X_t) \geq 0.$$

Basic (Elementary) Properties of ACVF γ(.)

The second is an immediate consequence of the fact that correlations are less than or equal to 1 in absolute value (or the Cauchy-Schwarz inequality),

$$|g_X(h)| = |Cov(X_{t+h}, X_t)| \leq \sqrt{Var(X_{t+h}, X_{t+h})}\sqrt{Var(X_t, X_t)}$$
$$= g_X(0) = 1.$$

The third is established by observing that

$$g(h) = Cov(X_{t+h}, X_t) = Cov(X_t, X_{t+h}) = g(-h).$$

Basic (Elementary) Properties of ACF $\rho(.)$

Remark 1.

An autocorrelation function (**ACF**) $\rho(.)$ has all the properties of an autocovariance function (**ACVF**) $\gamma(.)$, and satisfies the **additional condition**:

4) $\rho(0) = 1.$

In particular, we can say that a function $\rho(.)$ is the autocorrelation function (**ACF**) of a stationary process **if and only if** it is an **ACVF** with $\rho(0) = 1.$

Examples of ACVF and ACF: White Noise .

If { $\boldsymbol{\varepsilon_t}$, $\boldsymbol{t}$ = 0, ±1, …} is a **white noise** (with mean **0** and variance $\boldsymbol{\sigma^2}$): $\boldsymbol{\varepsilon_t} \sim \boldsymbol{WN}$ ($\mathbf{0}$, $\boldsymbol{\sigma^2}$), then

$$g_e(h) = s^2 d_{h0} = \begin{cases} s^2, & \text{if } h = 0 \\ 0 & \text{if } |h| \neq 0. \end{cases}$$

and

$$r_e(h) = \frac{g_e(h)}{g_e(0)} = d_{h0} = \begin{cases} 1, & \text{if } h = 0 \\ 0 & \text{if } |h| \neq 0. \end{cases}$$

Examples of ACVF and ACF: MA(1) process

Find the **ACVF** $\gamma(\,.\,)$ and **ACF** $\rho(.)$ for **MA(1) process**, and show that **MA(1) process** is a stationary process.

Definition: First-order moving average or MA(1) process.
A sequence of random variables $\{X_t\, , t = 0, \pm 1, \ \ldots\}$ is called **first-order moving average or MA(1) process** if X_t satisfies the equation

$$X_t = Z_t + q\, Z_{t-1}, \ \ t = 0, \pm 1, \ldots, \qquad (1)$$

where $Z_t \sim WN(0, \sigma^2)$, and θ is a real-valued finite constant.

From (1) we see that $\mathbf{E}[X_t] = 0$; $E[X_t^{\,2}] = \sigma^2(1+ \theta^2) < \infty$, and

Examples of ACVF and ACF: MA(1) process

$$g_X(t+h,t)=\begin{cases} s^2(1+q^2), & \text{if } h=0 \\ s^2 q, & \text{if } h=\pm 1 \\ 0, & \text{if } |h|>1. \end{cases} \qquad (2)$$

Thus, $\{\boldsymbol{X_t},\ \boldsymbol{t}=0, \pm 1, \ldots\}$ is stationary.
The autocorrelation function $\{\boldsymbol{X_t}\}$ is

$$r_X(h)=\frac{g_X(h)}{g_X(0)}=\begin{cases} 1, & \text{if } h=0 \\ q/(1+q^2), & \text{if } h=\pm 1 \\ 0, & \text{if } |h|>1. \end{cases}$$

Characterization of ACVF and ACF

Autocovariance functions have another **fundamental property**, namely that of **nonnegative definiteness**, which gives a **Characterization of Covariance Functions.**

Definition 1.

A real-valued function ***k***(.) defined on the integers is called **nonnegative definite (nnd)** if for any $n \in N$, and $a_i \in R$,

$$\sum_{i,j=1}^{n} k(i-j) a_i a_j \geq 0 \qquad (1)$$

for all positive integers ***n*** and vectors $\boldsymbol{a} = (\boldsymbol{a_1}, \ldots, \boldsymbol{a_n})'$ with real-valued components $\boldsymbol{a_i}$.

Characterization of ACVF and ACF

Theorem 1.

A real-valued function $g(h),\ h \in Z = \{0, \pm 1, \pm 2, \ldots\}$ defined on the integers is the **ACVF** of a stationary time series $\{X_t, t = 0, \pm 1, \ldots\}$ if and only if it is **nonnegative definite.**

Proof.

Let $g(h),\ h \in Z = \{0, \pm 1, \pm 2, \ldots\}$ be the **ACVF** of a stationary time series $\{X_t, t = 0, \pm 1, \ldots\}$, then $\gamma(h)$ is **nnd.**

Indeed, for any real numbers $a_1, \ldots, a_n$, we have

$$\sum_{i,j=1}^{n} a_i g(i-j) a_j = \sum_{i,j=1}^{n} a_i a_j Cov(X_i, X_j) = Var\left(\sum_{i,j=1}^{n} a_i X_i\right) \geq 0,$$

Characterization of ACVF and ACF

To prove the **converse result**, we show that **there exists a stationary time series** $\{X_t\ , t = 0, \pm 1, \ \ldots\}$ with **ACVF** $\gamma(.)$ **satisfying:**

$\gamma(.)$ is **even, real-valued, and nonnegative definite**,

which is difficult to establish.

A slightly stronger statement can be made, namely, that under the specified conditions there exists a **stationary Gaussian** time series $\{X_t\}$ with mean **0** and **ACVF** $\gamma(.)$.

ARMA (p, q) Processes:
AutoRegressive- Moving-Average Process

ARMA (p, q) Processes

The MA(q) Process.

Definition.

We say that a stationary time series is ***q*-correlated** if $\gamma(h) = 0$ whenever $|h| > q$.

For example,

- a white noise sequence is then **0-corrlelated**, while
- the **MA(1)** process is **1-correlated**.
- the moving average process of order q, **MA(q),** defined below is ***q* -correlated**, and
- perhaps surprisingly, the **converse is also true** (see **Proposition** below).

The MA(q) Process

Definition.

A time-series $\{X_t, t = 0, \pm 1, \ldots\}$ is called a **moving-average process of order** q (or **MA(**q**)** –process**)** if it satisfies the equation

$$X_t = Z_t + q_1 Z_{t-1} + \ldots + q_q Z_{t-q}, \qquad (1)$$

where $\{Z_t = \varepsilon_t\} \sim WN\,(\mathbf{0}, \sigma^2)$, and $\theta_1, \ldots, \theta_q$ are constants.

Remark 1.

It is a simple matter to check that **(1)** defines a **stationary** time series that is **strictly stationary** if $\{Z_t\}$ is **iid noise**.

The MA(q) Process

Remark 2.

The importance of **MA(*q*)** processes derives from the fact that **every *q*-correlated process is an MA(*q*) process.**
This is the content of the following proposition, whose proof can be found in TSTM, Section 3.2.

Proposition.

If $\{X_t, t = 0, \pm 1, \ldots\}$ is a **stationary *q* -correlated** time series with mean 0, then it can be represented as the **MA(*q*)** process:

$$X_t = Z_t + q_1 Z_{t-1} + \ldots + q_q Z_{t-q}.$$

Remark 3.

The extension of this result to the case $q = \infty$ is essentially **Wold's decomposition**.

The AR(p) Process

Definition.

A stationary time-series $\{X_t, t = 0, \pm 1, \ldots\}$ is called **autoregressive of order** p (or **AR(p) – process**) if X_t satisfies the equation

$$X_t - f_1 X_{t-1} - \ldots - f_p X_{t-p} = Z_t, \quad t = 0, \pm 1, \ldots \qquad (2)$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, and $\phi_1, \ldots, \phi_p$ are some constants.

We will show that under some conditions on constants $\phi_1, \ldots, \phi_p$
there is in fact **exactly one stationary solution** of (2).

ARMA (*p*, *q*) Processes

Definition 1.

A time-series $\{X_t, t = 0, \pm 1, \ldots\}$ is called **autoregressive-moving-average process of order (*p*, *q*)** , or **ARMA (*p*, *q*) –processes,** if

- $\{X_t\}$ is **stationary**, and
- for every $t = 0, \pm 1, \ldots,$ X_t satisfies the equation

$$X_t - f_1 X_{t-1} - \ldots - f_p X_{t-p} = Z_t + q_1 Z_{t-1} + \ldots + q_q Z_{t-q}, \qquad (3)$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and the polynomials

$$f(z) = 1 - f_1 z - \ldots - f_p z^p \text{ and } q(z) = 1 + q_1 z + \ldots + q_q z^q$$

have no common factors.

ARMA (p, q) Processes

Definition 2.

A time-series $\{\boldsymbol{Y_t}, \boldsymbol{t} = 0, \pm 1, \ldots, \}$ is called **ARMA** (p, q) **- process with mean μ** if

$$X_t := Y_t - \mu$$

is an **ARMA** (p, q) **-process** according to Definition1**.**

ARMA (*p*, *q*) Processes

Remark 1.

It is convenient to use the more concise form of (3)

$$f(B)X_t = q(B)Z_t, \qquad (4)$$

where $\Phi(.)$ and $\theta(.)$ are the ***p*-th** and ***q*-th** degree polynomials

$$f(z) = 1 - f_1 z - \ldots - f_p z^p \text{ and } q(z) = 1 + q_1 z + \ldots + q_q z^q,$$

and B is the **backward shift operator:**

$$BX_t = X_{t-1}, \quad B^j X_t = X_{t-j}, \quad B^j Z_t = Z_{t-j}, \quad j = 0, \pm 1, \ldots$$

Remark 2.

To study the **ARMA** (p, q) **–processes,** first we need to introduce and discuss a class of processes, called **Linear Processes.**

Linear Processes

The class of **linear time series models**, which includes the class of autoregressive moving-average (**ARMA**) models, provides a general framework for studying stationary processes.

In fact,

every second-order stationary process is either

- **a linear process or**
- **can be transformed to a linear process**

by subtracting a deterministic component.

This result is known as **Wold's decomposition** and will be discussed later.

Linear Processes

Definition.

The time series $\{X_t, t = 0, \pm 1, \ldots\}$ is called a **linear process** if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \quad \text{for all } t = 0, \pm 1, \ldots, \qquad (1)$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants satisfying

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

In terms of the **backward shift operator B**: $B^j Z_t = Z_{t-j}$,

(1) can be written more compactly as

$$X_t = \psi(B) Z_t, \quad \text{where} \quad \psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j. \qquad (2)$$

Linear Processes

Definition.

A linear process is called a **moving average of infinite order,** or **MA(∞) – process,** if $\psi_j = 0$ for all $j < 0$, that is, if

$$X_t = \psi(B)Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}. \qquad (3)$$

Remark 1.

The operator $\psi(B)$ can be thought of as a **linear filter**, which when applied to the **white noise** **"input"** series $\{Z_t\}$ produces the **"output"** $\{X_t\}$ (see Section 4.3).

Remark 2.

The following proposition shows that a linear filter, when applied to any stationary input series, produces a stationary output series.

Proposition 1.

Let $\{Y_t, t = 0, \pm 1, \ \ldots\}$ be a stationary time series with mean 0 and covariance function $\gamma_Y(h)$. If $\{\psi_j, t = 0, \pm 1, \ \ldots, \}$ is a sequence of real numbers satisfying

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty,$$

then the time-series

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} = \psi(B) Y_t \qquad (4)$$

is stationary with mean 0 and autocovariance function

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h + k - j) \qquad (5)$$

Linear Processes

Remark

In the special case where $\{X_t\}$ is **a linear process**, that is, $Y_t = Z_t \sim WN(0, \sigma^2)$, we have

$$g_X(h) = s^2 \sum_{j=-\infty}^{\infty} y_j y_{j+h}. \qquad (6)$$

AR(1) –process

Definition.

A time-series $\{X_t, t = 0, \pm1, \ldots\}$ is called **centered first-order autoregressive, or AR(1)- process** if it satisfies the equation

$$X_t = j\, X_{t-1} + Z_t, \quad t = 0, \pm 1, \ldots, \qquad (1)$$

where $Z_t \sim WN(0, s^2), \quad |f| < 1,$

and Z_t is uncorrelated with X_t for each $s < t$.

Find the **ACVF and ACF** of X_t, and show that **AR(1) - process** is a stationary process.

We show that there is in fact exactly one such solution of (1).

AR(1) –process

Solution.

By taking expectations on each side of (1), and using the fact that $\mathbf{E}[Z_t] = 0$, we see at once that $\mathbf{E}[X_t] = 0$.

To find the **autocorrelation function (ACF)**, we use the fact that Z_t is uncorrelated with X_s for each $s < t$, that is,

$$\boldsymbol{Cov\,(Z_t\,, X_s) = 0 \quad \text{for} \quad s < t\,.} \tag{2}$$

We have for $h > 0$,

$$\gamma_X(h) = Cov(X_t, X_{t-h}) = Cov(\phi X_{t-1} + Z_t, X_{t-h})$$

$$= Cov(\phi X_{t-1}, X_{t-h}) + Cov(Z_t, X_{t-h}) \quad \text{(since by (2), } Cov(Z_t, X_{t-h}) = 0)$$

$$= \phi\gamma_X(h-1) + 0$$

$$= \phi^2\gamma_X(h-2) = \cdots = \phi^h\gamma_X(0).$$

AR(1) –process

Observing that ***γ*(*h*) = *γ*(-*h*)**, and using definition of correlation function, we find that

$$r_X(h) = \frac{g_X(h)}{g_X(0)} = j^{|h|}, \quad h = 0, \pm 1, \ldots \qquad (3)$$

<u>**To find**</u> the **covariance function (ACF)**, observe that by the linearity of the covariance function in each of its arguments, and the fact that $\boldsymbol{Z_t}$ is uncorrelated with $\boldsymbol{X_{t-1}}$, that is,
$\boldsymbol{Cov\,(Z_t\,,X_{t-1}) = 0}$, we find

$$\begin{aligned} g_X(0) &= Cov(X_t, X_t) = Cov(j\,X_{t-1} + Z_t, j\,X_{t-1} + Z_t) \\ &= j^2 Cov(X_{t-1}, X_{t-1}) + j\,Cov(X_{t-1}, Z_t) \\ &+ j\,Cov(Z_t, X_{t-1}) + Cov(Z_t, Z_t) \\ &= j^2 g_X(0) + s^2. \end{aligned}$$

AR(1) –process

Thus,

$$g_X(0) = j^2 g_X(0) + s^2.$$

Hence

$$g_X(0) = \frac{s^2}{1 - f^2}. \qquad (4)$$

From (3) and (4) we obtain

$$g_X(h) = r_X(h) g_X(0) = f^{|h|} \frac{s^2}{1 - f^2}, \quad h = 0, \pm 1, \ldots$$

AR(1) –process Revisited

In Example 1, an **AR(1)** process $\{X_t\,, t = 0, \pm 1, \ldots\}$ was defined as a **stationary solution** of the equations

$$X_t - f\, X_{t-1} = Z_t, \qquad (5)$$

where $Z_t \sim WN(0, s^2)$, $|f| < 1$, and Z_t is uncorrelated with X_t for each $s < t$.

To show that such a solution exists and is the unique stationary solution of (5), we consider the linear process defined by

$$X_t = \sum_{j=0}^{\infty} f^j Z_{t-j} \qquad (6)$$

Observe that the series (6) with coefficients ϕ^j for $j \geq 0$ is absolutely summable, since $|\phi| < 1$.

AR(1) –process Revisited

It is easy to verify directly that the process (6) is a solution of (5), and by Proposition 1, it is also stationary with mean **0** and **ACVF**

$$g_X(h) = \sum_{j=0}^{\infty} f^j f^{j+h} s^2 = \frac{s^2 f^h}{1-f^2}, \qquad \text{for } h \geq 0.$$

To show that (6) is the **only stationary solution** of (5) let $\{Y_t\}$ be **any** stationary solution.
Then, iterating (5), we obtain

$$Y_t = f Y_{t-1} + Z_t = Z_t + f Z_{t-1} + f^2 Y_{t-2}$$

$$= \ldots$$

$$= Z_t + f Z_{t-1} + \ldots + f^k Z_{t-k} + f^{k+1} Z_{t-k-1}$$

AR(1) –process Revisited

If $\{Y_t\}$ is stationary, then $E[Y_t^2]$ is finite and independent of t, so that

$$E\left(Y_t - \sum_{j=0}^{\infty} f^j Z_{t-j}\right)^2 = f^{2k+2} E(Y_{t-k-1})^2 \to 0 \quad \text{as} \quad k \to \infty.$$

This implies that Y_t is equal to the mean square limit

$$\sum_{j=0}^{\infty} f^j Z_{t-j},$$

and hence that the process defined by (6) is the unique stationary solution of the equations (5).

Analysis of ARMA Processes

Outline.

In order to fit a mathematical model to time series data, one needs to go through the following four stages:

1. **Identification of the model**
2. **Estimation of the parameters in the model**
3. **Diagnostic checking of the fitted model**
4. **Forecasting of the future values.**

Analysis of ARMA Processes

We consider the ARMA(p,q) Models.

Definition 1.

Let $\boldsymbol{Z_t} \sim \boldsymbol{WN(0, \sigma^2)}$ and let $\boldsymbol{\theta_1, \ldots, \theta_q}$ and $\boldsymbol{\phi_1, \ldots, \phi_p}$ be some constants.

A time series $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$ is called a **zero-mean ARMA(p,q) process** if it satisfies the equation

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \cdots - \phi_p X_{t-p} = Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2} - \cdots \theta_q Z_{t-q}, \quad (1)$$

where the polynomials

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p \quad \text{and} \quad \theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$$

have no common factors.

Analysis of ARMA Processes

Remark.

It is convenient to use the more concise form of (1)

$$\Phi(B)\boldsymbol{X_t} = \boldsymbol{\theta}(B)\boldsymbol{Z_t}, \qquad (2)$$

where ***B*** is the backward shift operator

$$\left(B^j X_t = X_{t-j}, \quad B^j Z_t = Z_{t-j}, \quad j = 0, \pm 1, \ldots\right).$$

Analysis of ARMA Processes

A. Identification of the model

consists of selecting tentative values of p and q in equation (1) based on the observed time series data.

B. Estimation of the model

involves statistical estimation of the parameters $\theta_1, \dots, \theta_q$ and $\phi_1, \dots, \phi_p$ in the model tentatively selected in Part A.

C. Diagnostic checking

implies that based on some appropriate criterion we decide whether the model selected in Part A and estimated in Part B adequately fits the given time series.

Identification of the model

Identification of the model for ARMA processes involves:

1. **Existence and Uniqueness of the solution**
2. **Causality**
3. **Invertibility**
4. **Specification** of $\boldsymbol{p}$ and $\boldsymbol{q}$
5. **Specification** of the characteristics of the process:
   ACVF $\gamma(h)$ and **ACF** $\rho(h)$.

Identification of the model

Recall:

1. A time-series $\{X_t, t = 0, \pm 1, \ldots\}$ is called **ARMA**(p, q) – **processes**, if $\{X_t\}$ is stationary and satisfies the equation

$$X_t - f_1 X_{t-1} - \ldots - f_p X_{t-p} = Z_t + q_1 Z_{t-1} + \ldots + q_q Z_{t-q}, \qquad (1)$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and the polynomials

$$f(z) = 1 - f_1 z - \ldots - f_p z^p \text{ and } q(z) = 1 + q_1 z + \ldots + q_q z^q$$

have no common factors.

2. A time-series $\{Y_t, t = 0, \pm 1, \ldots\}$ is called **ARMA** (p, q) **- process with mean μ** if $X_t := Y_t - \mu$ is an **ARMA** (p, q) **- process** according to 1.

Existence and Uniqueness

Theorem 1.
A **stationary solution** $\{X_t, t = 0, \pm 1, \ldots\}$ of equation (1) **exists** (and is also **unique**) if and only if

$$f(z) = 1 - f_1 z - \ldots - f_p z^p \neq 0 \quad \text{for all} \quad |z| = 1 \qquad (2)$$

Causality

Definition.

An **ARMA (p , q)** process $\{X_t\ , t = 0, \pm 1, \ldots\}$ is called **causal, or a causal function** of $\{Z_t\}$, or **future – independent function** of $\{Z_t\}$, or more concisely a **causal autoregressive process**, if there exist constants $\{\psi_j\ , j = 0, 1, \ldots\}$ such that

$$\sum_{j=0}^{\infty} |y_j| < \infty \quad \text{and} \quad X_t = \sum_{j=0}^{\infty} y_j Z_{t-j} \quad \text{for all} \quad t = 0, \pm 1, \ldots \qquad (3)$$

Theorem 2.

An **ARMA (p , q)** process $\{X_t\ , t = 0, \pm 1, \ldots\}$ is **causal** if and only if

$$f(z) = 1 - f_1 z - \ldots - f_p z^p \neq 0 \quad \text{for all} \quad |z| \leq 1. \qquad (4)$$

Causality

Remark.

The coefficients $\{\psi_j\}$ can be found from equations

$$y_0 = 1$$

$$y_1 - f_1 y_0 = q_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y_j - \sum_{k=1}^{p} f_k y_{j-k} = q_j, \quad j = 0, 1, \ldots, \qquad (5)$$

where $q_0 := 1$; $q_j := 0$ for $j > q$, and $y_j := 0$ for $j < 0$.

Invertibility

Definition.

An **ARMA (*p* , *q*)** processes $\{X_t , t = 0, \pm 1, \ldots\}$ is called **invertible** if there exist constants $\{\pi_j , j = 0,1, \ldots\}$ such that

$$\sum_{j=0}^{\infty} |p_j| < \infty \quad \text{and} \quad Z_t = \sum_{j=0}^{\infty} p_j X_{t-j} \quad \text{for all} \quad t = 0, \pm 1, \ldots.$$

Theorem 3.

An **ARMA (*p* , *q*)** processes $\{X_t , t = 0, \pm 1, \ldots\}$ is **invertible** if and only if

$$q(z) = 1 + q_1 z + \ldots + q_q z^q \neq 0 \quad \text{for all} \quad |z| \leq 1. \qquad (6)$$

Invertibility

Remark.

The coefficients $\{\boldsymbol{\pi_j}, \boldsymbol{j} = 0, \pm 1, \ldots\}$ can be found from equations

$$p_j + \sum_{k=1}^{q} q_k p_{j-k} = -f_j, \quad j = 0, 1, \ldots, \qquad (7)$$

where $f_0 := 1; \quad f_j := 0$ for $j > p$, and $p_j := 0$ for $j < 0$.

A Special Case: ARMA (1,1) process

The time series $\{X_t\,, t = 0, \pm 1, \ \ldots\}$ is an **ARMA (1,1)** process if it is stationary and satisfies

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1} \quad \text{for every } t = 0, \pm 1, \ \ldots, \qquad (8)$$

where $\{Z_t\} \sim WN\,(\mathbf{0}, \sigma^2)$ and $\phi + \theta \neq \mathbf{0}$.
Using the backward shift operator notation, (8) can be written as

$$\Phi(B)X_t = \theta(B)Z_t\,, \qquad (9)$$

where $\Phi(B) = 1 - \Phi B$ and $\theta(B) = 1 + \theta B$ are linear filters.

A Special Case: ARMA (1,1) process

Theorem 1.

1. A stationary solution of the **ARMA (1,1)** equations exists if and only if $\boldsymbol{\phi} \neq \pm 1$, and the unique stationary solution $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$ of (8) is given by the **MA(∞)**-process:

$$X_t = Z_t + (f + q)\sum_{j=1}^{\infty} f^{j-1} Z_{t-j}, \quad t = 0, \pm 1, \ldots \qquad (10)$$

2. If $|f| < 1$, then the unique stationary solution $\{\boldsymbol{X_t}, \boldsymbol{t} = 0, \pm 1, \ldots\}$, given by (10) is **causal** function of $\{\boldsymbol{Z_t}\}$, since $\boldsymbol{X_t}$ can be expressed in terms of the current and past values $\boldsymbol{Z_s}, \boldsymbol{s} \leq \boldsymbol{t}$.

A Special Case: ARMA (1,1) process

3. If $|\boldsymbol{\phi}| > 1$ then the unique stationary solution of equation (8) is given by

$$X_t = -q f^{-1} Z_t - (f + q) \sum_{j=1}^{\infty} f^{-j-1} Z_{t+j}, \quad t = 0, \pm 1, \ldots \quad (11)$$

The solution is **non-causal**, since $\boldsymbol{X_t}$ is then a function of $\boldsymbol{Z_s}$, $\boldsymbol{s \geq t}$.

A Special Case: ARMA (1,1) process

Theorem 2.

1. If $|\boldsymbol{\theta}| < \mathbf{1}$ then the **ARMA(1,1)** process $\boldsymbol{X_t}$ is **invertible**, and $\boldsymbol{Z_t}$ is expressed in terms of $\boldsymbol{X_s}$ **,** $\boldsymbol{s \le t}$, by

$$Z_t = X_t - (f + q)\sum_{j=1}^{\infty}(-q)^{j-1}X_{t-j}, \quad t = 0, \pm 1, \ldots$$

2. If $|\boldsymbol{\theta}| > \mathbf{1}$ then the **ARMA(1,1)** process $\boldsymbol{X_t}$ is **non-invertible,** and $\boldsymbol{Z_t}$ is expressed in terms of $\boldsymbol{X_s}$ **,** $\boldsymbol{s \ge t}$, by

$$Z_t = -fq^{-1}X_t + (f + q)\sum_{j=1}^{\infty}(-q)^{-j-1}X_{t+j}, \; t = 0, \pm 1, \ldots$$

A Special Case: ARMA (1,1) process

3. If $|\theta| = 1$ then the **ARMA(1,1)** process X_t is **invertible** in the more general sense that Z_t is a mean square limit of finite linear combinations of X_s , $s \le t$.

4. If the **ARMA(1,1)** process is **non-causal** or **non-invertible** with $|\theta| > 1$ then it is possible to find a new **White Noise** process W_t such that X_t is a **causal** and **invertible** **ARMA(1,1)** process relative to W_t (**Problem 4.10, BD**).

   Therefore, from second-order point of view, nothing is lost by restricting attention to **causal** and **invertible** **ARMA(1,1)** models.
   This assertion is also valid for **higher-order** **ARMA(*p*, *q*)** models.

A Special Case: ARMA (1,1) process

Example 1: Numerical - ARMA (1,1) processes.

Consider the **ARMA(** ***p, q*** **)** processes $\{X_t\}$ satisfying the equation

$$X_t - .5\,X_{t-1} = Z_t + .4\,Z_{t-1}\,,$$

$$\{Z_t\} \sim WN(0, \sigma^2)\ , \Phi = .5\ ; \theta = .4. \qquad (1)$$

Since the autoregressive polynomial $\boldsymbol{\phi(z) = 1 - .5z}$ has a zero at **z = 2 > 1**, which is located outside the unit circle, we conclude that there **exists a unique** **ARMA processes** satisfying (1), and that is also **causal.**

Example 1: Numerical - ARMA (1,1) processes.

The coefficients $\{\psi_j\}$ in the **MA - representation** of $\{X_t\}$ are found directly from by formulas:

$$y_0 = 1,$$

$$y_1 = q_1 + f_1 y_0 = .4 + .5,$$

$$y_2 = .5(.4 + .5),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_j = .5^{j-1}(.4 + .5), \qquad j = 1, 2, \ldots$$

Example 1: Numerical - ARMA (1,1) processes.

The MA polynomial $\boldsymbol{\theta}\mathbf{(z) = 1 + .4z}$ has a zero at $\mathbf{z = -1/.4 = -2.5}$, which is also located outside the unit circle. This implies that $\{X_t\}$ is **invertible** with coefficients $\{\boldsymbol{\pi_j}\}$ given by

$$p_0 = 1,$$

$$p_1 = -(.4 + .5),$$

$$p_2 = -(.4 + .5)(-.4),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_j = -(.4 + .5)(-.4)^{j-1}. \qquad j = 1, 2, \ldots$$

Example 2: An AR (2) processes

Let $\{X_t\}$ be an **AR(2)** process, a solution of equation

$$\boldsymbol{X_t} \ \mathbf{=}\ .7\,\boldsymbol{X_{t-1}} - .1\,\boldsymbol{X_{t-2}} + \boldsymbol{Z_t}, \qquad \{\boldsymbol{Z_t}\} \sim \boldsymbol{WN}(\mathbf{0}, \boldsymbol{\sigma}^{\mathbf{2}}).$$

The autoregressive polynomial for this process has the factorization

$$f(z) = 1 - .7z + .1z^2 = (1 - .5z)(1 - .2z),$$

and is therefore zero at **z = 2** and **z = 5**.
Since these zeros lie outside the unit circle, we conclude that $\{X_t\}$ is a **causal AR(2) process** with coefficients $\{\psi_j\}$ given by

Example 2: An AR (2) processes

$$y_0 = 1,$$
$$y_1 = .7,$$
$$y_2 = .7^2 - .1,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$y_j = .7y_{j-1} - (.1)y_{j-2}, \qquad j = 2, 3, \dots$$

Remark.

While it is a simple matter to calculate ψ_j numerically for any j, it is possible also to give an explicit solution of these difference equations using the theory of linear difference equations (see TSTM, Section 3.6).

Special Cases of the ARMA(p,q) Model

1. AR(1) Model (First-Order Autoregressive Model).

Taking $\boldsymbol{p} = \mathbf{1}$ and $\boldsymbol{q} = \mathbf{0}$ we get

$$\boldsymbol{X_t - \phi_1 X_{t-1} = Z_t}\,. \tag{1}$$

We have

$$\sigma_X^2 = \frac{\sigma^2}{1-\phi_1^2};$$

$$\rho_0 = 1;\ \ \rho_k = \phi_1 \rho_{k-1},\ \ \rho_k = \phi_1^{|k|} \quad \text{for} \ \ k = \pm 1, \pm 2, \ldots$$

$$\gamma(k) = \sigma^2 \frac{\phi_1^{|k|}}{1-\phi_1^2}. \quad \text{for} \ \ k = 0, \pm 1, \pm 2, \ldots.$$

Special Cases of the ARMA (p , q) Model

2. AR(2) Model (Second-Order Autoregressive Model).

Taking $\boldsymbol{p}$ **= 2** and $\boldsymbol{q}$ **= 0** we get

$$\boldsymbol{X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t}\,, \qquad (2)$$

It can be shown that equation (2) will have stationary solution if $\boldsymbol{\phi_1}$ and $\boldsymbol{\phi_2}$ satisfy the following inequalities.

$$-1 < \boldsymbol{\phi_2} < 1 \;,\; \boldsymbol{\phi_1} + \boldsymbol{\phi_2} < 1 \,,\; \boldsymbol{\phi_2} - \boldsymbol{\phi_1} < 1.$$

We have

$$s_X^2 = \frac{s^2}{1 - f_1^2 - f_2^2 - 2 r_1 f_1 f_2},$$

provided that $1 - f_1^2 - f_2^2 - 2 r_1 f_1 f_2 > 0;$

$$r_1 = \frac{f_1}{1 - f_2}; \quad r_2 = f_2 + \frac{f_1^2}{1 - f_2};$$

$$r_k = f_1 r_{k-1} + f_2 r_{k-2}, \ \text{for } k = \pm 3, \pm 4, \ldots$$

Special Cases of the ARMA (p , q) Model

3. MA(1) Model (First-Order Moving Average Model).

Taking $\boldsymbol{p} = \mathbf{0}$ and $\boldsymbol{q} = \mathbf{1}$ we get

$$\boldsymbol{X_t = Z_t - \theta_1 Z_{t-1}} \qquad (3)$$

We have

$$s_X^2 = s^2(1 + q_1^2);$$

$$r_{\pm 1} = \frac{-q_1}{1 + q_1^2};$$

$$r_k = 0 \ \text{for} \ k = \pm 2, \pm 3, \ldots.$$

Special Cases of the ARMA (p , q) Model

4. MA(2) Model (Second-Order Moving Avarage Model).

Taking $\boldsymbol{p = 0}$ and $\boldsymbol{q = 2}$ we get

$$X_t = Z_t - \theta_1 Z_{t-1} - \theta_2 Z_{t-2}\,, \qquad (4)$$

It can be shown that the **invertibility conditions** for **MA(2)** model are

$$-1 < \theta_2 < 1 \;, \theta_1 + \theta_2 < 1 \,, \theta_2 - \theta_1 < 1.$$

We have

$$s_X^2 = (1 + q_1^2 + q_2^2) s^2;$$

$$r_1 = \frac{q_1(q_2 - 1)}{1 + q_1^2 + q_2^2}; \quad r_2 = \frac{-q_2}{1 + q_1^2 + q_2^2}, \quad \text{and}$$

$$r_k = 0 \quad \text{for} \quad k = \pm 3, \pm 4, \ldots.$$

Special Cases of the ARMA (p , q) Model

5. ARMA(1,1) Model.

Taking $\boldsymbol{p = 1}$ and $\boldsymbol{q = 1}$ we get

$$\boldsymbol{X_t - \phi_1 X_{t-1} = Z_t - \theta_1 Z_{t-1}} \qquad (5)$$

It can be shown that under the conditions

$$-1 < \boldsymbol{\theta_1} < 1, -1 < \boldsymbol{\phi_1} < 1 ,$$

the model is **invertible** and **causal.**

We have

$$s_X^2 = \frac{(1 + q_1^2)s^2}{1 + f_1^2 - 2r_1 f_1},$$

$$r_1 = \frac{(f_1 - q_1)(1 - f_1 q_1)}{1 + q_1^2 - 2f_1 q_1}, \quad \text{and}$$

$$r_k = f_1 r_{k-1} \quad \text{for} \quad k = \pm 2, \pm 3, \pm 4, \ldots.$$