743- Regression and Time Series

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The Multiple Regression Model

Elements of Matrix Algebra

Definition:

A matrix is a rectangular array of real numbers:

$$A = A_{mn} = \left\| a_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is $(m \times n)$- **rectangular matrix**, where $\boldsymbol{m}$ is the number of **rows**, and $\boldsymbol{n}$ is the number of **columns**.

If $\boldsymbol{m} = \boldsymbol{n}$, then $A = \left\| a_{ki} \right\|_{k,j=\overline{1,n}}$ is called $(n \times n)$ - **square matrix**.

Elements of Matrix Algebra

- **Examples.**

$$A_{23} = \begin{bmatrix} 6 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}$$ is a (2×3) -matrix, $\boldsymbol{m} = 2$, $\boldsymbol{n} = 3$.

$$A_{32} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$$ is a (3×2) -matrix, $\boldsymbol{m} = 2$, $\boldsymbol{n} = 3$.

$$A_{1n} = X_n = [x_1 \quad x_2 \quad \ldots \quad x_n]$$ is a $(1\times n)$ -matrix.

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- **Addition of Matrices.**

If $A = \left\| a_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$ and $B = \left\| b_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$,

then $A + B = C = \left\| c_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$,

where $c_{kj} = a_{kj} + b_{kj}$.

Similarly, can be defined $A + B + C + \cdots$

Elements of Matrix Algebra

- **Example.**

Let $A_{23} = \begin{bmatrix} 6 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}$, and $B_{23} = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & -5 \end{bmatrix}$,

then

$$C_{23} = A + B = \begin{bmatrix} 7 & -1 & 4 \\ 5 & 0 & -4 \end{bmatrix}.$$

Elements of Matrix Algebra

- **Multiplication of a Matrix by a Real Number.**

Let $A_{mn} = \left\| a_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$, and $l \in R$,

then $l\, A_{mn} = \left\| l\, a_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$.

Elements of Matrix Algebra

- **Example.**

$$\text{If} \quad A_{23} = \begin{bmatrix} 6 & 0 & 1 \\ 3 & -2 & 7 \end{bmatrix} \quad \text{and} \quad l = -2,$$

then

$$l\,A_{23} = \begin{bmatrix} 6(-2) & 0(-2) & 1(-2) \\ 3(-2) & -2(-2) & 7(-2) \end{bmatrix} = \begin{bmatrix} -12 & 0 & -2 \\ -6 & 4 & -14 \end{bmatrix}.$$

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- **Matrix Multiplication.**

Let $A_{mp} = \left\| a_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,p}}$ and $B_{pn} = \left\| a_{kj} \right\|_{k=\overline{1,p},\, j=\overline{1,n}}$,

then

$$C_{mn} = A_{mp} B_{pn} = \left\| c_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$$

is a $(m \times n)$ -rectangular matrix with elements:

$$c_{kj} = \sum_{i=1}^{p} a_{ki} b_{ij}.$$

Elements of Matrix Algebra

Remark 1.

$AB \neq BA$ (in general), moreover $\boldsymbol{BA}$ may be **undefined.**

- **Example 1.**

Let $A_{32} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$ and $B_{23} = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix}$.

Then

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1) $C_{33} = A_{32}B_{23} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -2 & 0 \\ 2 & -1 & -3 \\ 8 & 0 & 8 \end{bmatrix}$

is a (3×3) -matrix, while

2) $C_{22} = B_{23}A_{32} = \begin{bmatrix} 4 & -1 & -1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 10 \end{bmatrix}$

is a (2×2) -matrix.

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- **Example 2.**

(a) $$A_{13}B_{32} = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \end{bmatrix} = C_{12},$$

(b) $B_{32}A_{13}$ is **undefined** because of the dimensions of $\boldsymbol{A}$ and $\boldsymbol{B}$,

(c) $$A_{14}B_{41} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = [30] = C_{11},$$

(d) $B_{41}A_{14} = C_{44}$ is (4×4)-square matrix.

Elements of Matrix Algebra

- **The Identify Matrix**

Definition 1.

The matrix $I_n = \|d_{kj}\|_{kj=\overline{1,n}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$, where

$d_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$ is called $(n \times n)$ -**identity matrix.**

Definition 2.

The matrix $O = \|a_{kj}\|$, for which $a_{kj} = 0$, for all $k = \overline{1,m}$ and

$j = \overline{1,n}$ is called **O-matrix or zero-matrix.**

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Definition 3.

Let $A_{mn} = \left\| a_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$ and $B_{mn} = \left\| b_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$, then

$A_{mn} = B_{mn} \Leftrightarrow a_{kj} = b_{kj}$ for all $k = \overline{1,m}$ and $j = \overline{1,n}$.

- **Properties of O and I matrices.**

1) $\boldsymbol{A} + \boldsymbol{O} = \boldsymbol{O} + \boldsymbol{A} = \boldsymbol{A}$;

2) $\boldsymbol{IA} = \boldsymbol{AI} = \boldsymbol{A}$.

Elements of Matrix Algebra

- **The Inverse Matrix**

Definition 4.

Let $A_{mn} = \left\| a_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$ be $(n \times n)$ -square matrix.

If a matrix, denoted by A_n^{-1}, can be found such that

$$A_n A_n^{-1} = A_n^{-1} A_n = I_n,$$

then A_n^{-1} is called the **inverse** of A_n.

Remark.

If $A_n^{-1} = \left\| b_{kj} \right\|_{kj=\overline{1,n}}$, then $\sum_{i=1}^{n} a_{ki} b_{ij} = d_{kj}$.

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Note: Let A and B be two matrices whose inverses exist.

Let $C = AB$. Then the inverse of the matrix C exists and

$$C^{-1} = B^{-1}A^{-1}.$$

The Woodbury Theorem:

$$\left(A + BCD\right)^{-1} = A^{-1} - A^{-1}B\left[C^{-1} + DA^{-1}B\right]^{-1} DA^{-1}$$

where the inverses

$$A^{-1}, C^{-1} \quad \text{and} \quad \left[C^{-1} + DA^{-1}B\right]^{-1} \quad \text{exist.}$$

Elements of Matrix Algebra

Note: The **Woodbury Theorem** can be used to find the inverse of some pattern matrices:

For Example:

$$\begin{bmatrix} b & a & \cdots & a \\ a & b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & b \end{bmatrix}^{-1} = \begin{bmatrix} c & d & \cdots & d \\ d & c & \cdots & d \\ \vdots & \vdots & \ddots & \vdots \\ d & d & \cdots & c \end{bmatrix}$$

where:

$$d = -\frac{a}{(b-a)\big(b+a(n-1)\big)} \quad \text{and} \quad c = \frac{1}{b-a}\left[\frac{b+a(n-2)}{b+a(n-1)}\right].$$

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Example- Note 1: For $n = 2$

$$d = -\frac{a}{(b-a)(b+a)} = -\frac{a}{b^2 - a^2}$$

and $$c = \frac{1}{b-a}\left[\frac{b}{b+a}\right] = \frac{b}{b^2 - a^2}$$

Thus $$\begin{bmatrix} b & a \\ a & b \end{bmatrix}^{-1} = \frac{1}{b^2 - a^2}\begin{bmatrix} b & -a \\ -a & b \end{bmatrix}$$

Elements of Matrix Algebra

Example- Note 2: For special case $a = 0$, we have

$$\begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{bmatrix}^{-1} = \begin{bmatrix} 1/b & 0 & \cdots & 0 \\ 0 & 1/b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/b \end{bmatrix}$$

Since in this case

$$d = -\frac{a}{(b-a)\left(b+a(n-1)\right)} = 0$$

$$c = \frac{1}{b-a}\left[\frac{b+a(n-2)}{b+a(n-1)}\right] = \frac{1}{b}.$$

Elements of Matrix Algebra

- **The Transpose of a Matrix.**

Definition 5.

Let $A_{mn} = \left\| a_{kj} \right\|_{k=\overline{1,m},\, j=\overline{1,n}}$.

The **transpose** of $\boldsymbol{A}$, denoted by A', is defined to be a matrix obtained from $\boldsymbol{A}$ by **interchanging corresponding rows and columns** of $\boldsymbol{A}$, that is first with first, second by second, and so on.

$$\text{Thus } A' = A'_{nm} = \left\| a_{jk} \right\|_{j=\overline{1,n},\, k=\overline{1,m}}.$$

Property.

The transpose of product: $(ABC)' = C'B'A'$.

Elements of Matrix Algebra

- **Example 1.**

If $A_{32} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 4 & 3 \end{bmatrix}$, then $A'_{23} = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$.

- **Example 2.**

If $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then $X' = [x_1 \quad x_2 \quad x_3]$,

and $X'X = x_1^2 + x_2^2 + x_3^2$.

Elements of Matrix Algebra

- **A matrix Expression for a system of Linear Equations.**

Consider the systems of $\boldsymbol{n}$ equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \qquad (1)$$

Elements of Matrix Algebra

- **A matrix Expression for a system of Linear Equations.**

Denoting by

$$A_n = \begin{bmatrix} a_{11} a_{12} \cdots a_{1n} \\ a_{21} a_{22} \cdots a_{2n} \\ \cdots \cdots \cdots \cdots \cdots \\ a_{n1} a_{n2} \cdots a_{nn} \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad (2)$$

and using matrix operations we obtain that (1) is equivalent to (**matrix form** of (1)):

$$(1) \quad \Leftrightarrow \quad A_n X = B \qquad (3)$$

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If A_n has an inverse A_n^{-1}, then the solution of (3) (and hence (1)) is given by

$$X = A_n^{-1} B. \qquad (4)$$

Thus, to solve the system (1) follow the **steps**.

Step 1. Specify the matrices $\boldsymbol{A}$, $\boldsymbol{X}$ and $\boldsymbol{B}$ as in (2).

Step 2. Write (1) in equivalent matrix form (3).

Step 3. Find the inverse A^{-1} of $\boldsymbol{A}$.

Step 4. Multiply A^{-1} by $\boldsymbol{B}$, to get $X = A^{-1}B.$

Elements of Matrix Algebra

- **Example.**

Solve the system of equations $\begin{cases} 2x_1 + x_2 = 5 \\ x_1 - x_2 = 1 \end{cases}$

(observe that $x_1 = 2, x_2 = 1$).

- **Matrix - Solution.**

We have

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

Elements of Matrix Algebra

- **Matrix - Solution.**

For inverse A^{-1} we have

$$A^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}.$$

So,

$$X = A^{-1}B = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and hence the solution is $x_1 = 2, x_2 = 1$.