Solution

DPS-PT-3

Class 12 - Mathematics

Section A

1. Let I = $\int \frac{x^2}{x^6-a^6}dx$, then we have

   $I = \int \frac{x^2}{(x^3)^2-(a^3)^2}dx$

   Let x^3 = t then $3x^2$ dx = dt

   Or $x^2 dx = \frac{dt}{3}$

   So, $I = \frac{1}{3}\int \frac{dt}{t^2-(a^3)^2}$

   $= \frac{1}{3} \times \frac{1}{2a^3}\log\left|\frac{t-a^3}{t+a^3}\right|$ [Since $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$]

   $I = \frac{1}{6a^3}\log\left|\frac{x^3-a^3}{x^3+a^3}\right| + c$

2. Let $I = \int_1^2 \frac{1}{x(1+\log_e x)^2}dx$

   Also let 1 + $\log_e$x = t

   $\Rightarrow \frac{1}{x}dx = dt$

   Also, when x = 1, t = 1 and when x = 2, t = 1 + $\log_e 2$

   Hence,

   $I = \int_1^{1+\log_e 2} \frac{1}{t^2}dt$

   $= -\frac{1}{t}|1 + \frac{log_e 2}{1}$

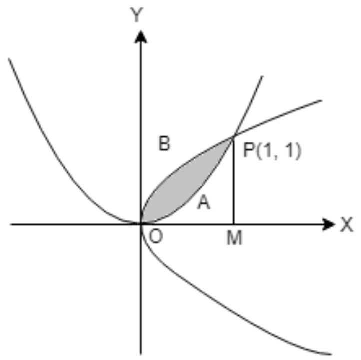
   $= 1 - \frac{1}{1+\log_e 2}$

   $= \frac{\log_e 2}{1+\log_e 2}$

3. The equations of parabolas are

   $y^2 = x$ ....(1)

   and $x^2 = y$ ....(2)

   From (2), $y = x^2$.........(3)

   putting this value of y in (1),we get,

   $x^4 = x \; or \; x(x^3-1) = 0$

   x=0,1

   From (3) ,y=0,1

   Therefore, parabolas (1) and (2) intersect in O(0,0),P(1,1).

   From P, draw PM $\perp$ x-axis.

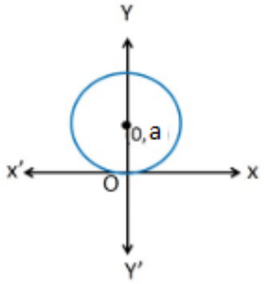
   Required area

   =Area of region OAPB=Area of region OBPM-area of region OAPM

   =$\int_0^1 \sqrt{x}dx - \int_0^1 x^2 dx$

   $= \left[\frac{x^{3/2}}{3/2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$

$= \frac{2}{3}[x^{3/2}]_0^1 - \frac{1}{3}\left[x^3\right]_0^1$
$= \frac{2}{3}[1-0] - \frac{1}{3}[1-0] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$sq.units

4. The given equation may be written as $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \left(y - x\frac{dy}{dx}\right)$
   $\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \left(y - x\frac{dy}{dx}\right)^2$ [on squaring both sides]
   $\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx}$
   $\Rightarrow$ $(1 - x)^2 \left(\frac{dy}{dx}\right)^2$ + 2xy $\frac{dy}{dx}$ + $(1 + y^2)$ = 0
   Clearly, it is a differential equation of order = 1 and degree = 2.
   Order=1, Degree=2
5. Rearranging the terms we get:
   $\frac{dy}{y}$ =tan x dx
   $\Rightarrow \int \frac{dy}{y} = \int \tan x\, dx + c$
   $\Rightarrow$ log|y| - log|sec x| = logc
   $\Rightarrow$ log|y| + log|cos x| = logc
   $\Rightarrow$ y cos x = c
   y = 1 when x = 0
   $\therefore$1 $\times$ cos 0 = c
   $\therefore$c = 1
   $\Rightarrow$ y cos x = 1
   $\Rightarrow$ y = $\frac{1}{\cos c}$
   $\Rightarrow$ y = sec x
   $\Rightarrow$ y = sec x.
   This is the required.
6. 

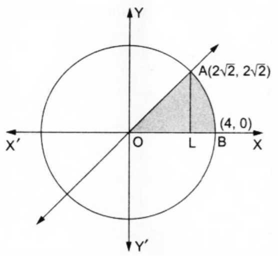
   The equation of the family of circles touching x-axis at origin is,
   $x^2 + (y-a)^2 = a^2$, a being radius of circle.........(1)
   differentiating w.r.t x, we get,
   $2x + 2(y-a)y' = 0$
   $\Rightarrow x + (y-a)y' = 0$
   $\Rightarrow y - a = \frac{-x}{y'}$
   $\Rightarrow y + \frac{x}{y'} = a$
   Put a and y – a in eq (1),we get,
   $x^2 + \left(\frac{-x}{y'}\right)^2 = \left(y + \frac{x}{y'}\right)^2$
   $\Rightarrow x^2 + \frac{x^2}{y'^2} = y^2 + \frac{x^2}{y'^2} + 2.y\frac{x}{y'}$
   $\Rightarrow x^2 - y^2 = \frac{2xy}{y'}$
   $\Rightarrow y' = \frac{2xy}{x^2-y^2}$

Section B

7. Given, $I=\int_0^1 \frac{x^4+1}{x^2+1}dx \Rightarrow I=\int_0^1 \frac{(x^4-1)+2}{x^2+1}dx$
$=\int_0^1 \frac{(x^2-1)(x^2+1)+2}{x^2+1}dx$
$[\because (a^2-b^2)=(a-b)(a+b)$
$=\int_0^1 \left[\frac{(x^2-1)(x^2+1)}{x^2+1}+\frac{2}{x^2+1}\right]dx$
$\Rightarrow \quad I=\int_0^1 \left[x^2-1+\frac{2}{x^2+1}\right]dx$
$\Rightarrow I=\left[\frac{x^3}{3}-x+2\tan^{-1}x\right]_0^1$
$\therefore I=\frac{1}{3}-1+2\tan^{-1}1-0=-\frac{2}{3}+2\times\frac{\pi}{4}=\frac{3\pi-4}{6}$

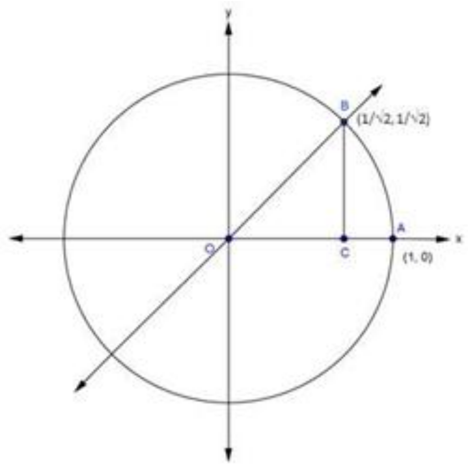
8. According to the question, $I=\int \frac{\sin(x-a)}{\sin(x+a)}dx$
Put $x+a=t \Rightarrow$ dx = dt
$\therefore I=\int \frac{\sin(t-a-a)}{\sin t}dt=\int \frac{\sin(t-2a)}{\sin t}dt$
$=\int \frac{\sin t\cos 2a-\cos t\sin 2a}{\sin t}dt$
$[\because sin(A-B)=sinAcosB-cosAsinB]$
$=\int \cos 2a dt-\int \sin 2a\cdot\cot t dt$
= $cos2a[t]-sin2a[log|sint|]+C_1$
= $(x+a)cos2a-sin2alog|sin(x+a)|+C_1$
[put t = x + a)
= $xcos2a-sin2alog|sin(x+a)|+C1$

9. The given circle is $x^2 + y^2 =16$ ...(i)
The given line is y = x ...(ii)



Putting y = x from (ii) into (i), we get
$2x^2 = 16 \Leftrightarrow x^2 = 8 \Leftrightarrow x = 2\sqrt{2}$ [$\because$ x is +ve in the first quad.}
Thus, the point of intersection of (i) and (ii) in the first quadrant is A$(2\sqrt{2},2\sqrt{2})$
Draw AL perpendicular on the x-axis
Therefore required area of region = (area of region OLA) + area of region(LBAL).
$=\int\limits_0^{2\sqrt{2}} xdx+\int\limits_{2\sqrt{2}}^{4}\sqrt{16-x^2}dx$
$=\left[\frac{x^2}{2}\right]_0^{2\sqrt{2}}+\left[\frac{x\sqrt{16-x^2}}{2}+\frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{2\sqrt{2}}^4$
$=\frac{1}{2}\left[(2\sqrt{2})^2-0\right]+\left[\left(0+8\sin^{-1}1\right)-\left(4+8\sin^{-1}\frac{1}{\sqrt{2}}\right]\right.$
$=\left[4+\left(8\times\frac{\pi}{2}\right)-4-\left(8\times\frac{\pi}{4}\right)\right]$ = (2π) sq units.

10. To find area bounded by positive x-axis and curve
$y=\sqrt{1-x^2}$
$x^2 + y^2 = 1$ ...(i)
x = y ...(ii)
Equation (i) represents a circle with centre (0, 0) and meets axes at $(\pm1,0),(0,\pm1)$
Equation (ii) represents a line passing through $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ and they are also points of intersection.
A rough sketch of the curve is as under:-



Thus Required area of Region = Area of bounded Region OABO

A = Region OCBO + Region CABC

$= \int_0^{\frac{1}{\sqrt{2}}} y_1 dx + \int_{\frac{1}{\sqrt{2}}}^{1} y_2 dx$

$= \int_0^{\frac{1}{\sqrt{2}}} x dx + \int_{\frac{1}{\sqrt{2}}}^{1} \sqrt{1-x^2} dx$

$= \left[\frac{x^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} + \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_{\frac{1}{\sqrt{2}}}^{1}$

$= \left[\frac{1}{4} - 0\right] + \left[\left(0 + \frac{1}{2} \cdot \frac{\pi}{2}\right) - \left(\frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{\pi}{4}\right)\right]$

$= \frac{1}{4} + \frac{\pi}{4} - \frac{1}{4} - \frac{\pi}{8}$

$\therefore A = \frac{\pi}{8}$ sq. units.

11. According to question, Given differential equation is,

$\left(x^2-1\right) \frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$

Dividing both sides with (x^2 - 1), we get

$\frac{dy}{dx} + \frac{2x}{x^2-1} y = \frac{2}{\left(x^2-1\right)^2}$

It is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{\left(x^2-1\right)^2}$

We know that ,

$\mathrm{IF} = e^{\int \mathrm{P} x} = e^{\int \frac{2\mathrm{x}}{x^2-1} dx}$

$= e^{\log\left|x^2-1\right|} = x^2 - 1 \left[\begin{array}{l} \text{put } x^2 - 1 = t \Rightarrow 2x dx = dt, \text{ then} \\ \int \frac{2x}{x^2-1} dx = \int \frac{1}{t} dt = \log|t| = \log\left|x^2-1\right| \end{array}\right]$

So, the required general solution is $y \times \mathrm{IF} = \int (Q \times \mathrm{IF}) dx + C$

$\Rightarrow \quad y\left(x^2-1\right) = \int \frac{2}{\left(x^2-1\right)^2} \times \left(x^2-1\right) dx + C$

$\Rightarrow \quad y\left(x^2-1\right) = \int \frac{2}{x^2-1} dx + C$

$\Rightarrow \quad y\left(x^2-1\right) = \frac{2}{2 \times 1} \log\left|\frac{x-1}{x+1}\right| + C \left[\because \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right|\right]$

$\therefore \quad y\left(x^2-1\right) = \log\left|\frac{x-1}{x+1}\right| + C$

which is the required differential equation.

Section C

12. According to the question, $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16(1 + \sin 2x - 1)} dx$

$\Rightarrow \quad I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16[1 - (1 - \sin 2x)]} dx$

$\Rightarrow I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\left[1 - \left(\cos^2 x + \sin^2 x - 2 \sin x \cos x\right)\right]} dx [\because 1 = \cos^2 x + \sin^2 x]$ and $[\because \sin 2x = 2 \sin x \cos x]$

$\Rightarrow \quad I=\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\left[1-(\cos x-\sin x)^2\right]} dx$

put , $cosx - sinx = t$

$\Rightarrow (-sinx - cosx)dx = dt$

$\Rightarrow (sinx + cosx)dx = -dt$

Lower limit , when x = 0, then t = cos 0 - sin 0 = 1

Upper limit , when x = $\frac{\pi}{4}$, then $t=\cos\frac{\pi}{4}-\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0.$

$\therefore \quad I=\int_1^0 \frac{-dt}{9+16(1-t^2)}$

$\Rightarrow \quad I=\int_0^1 \frac{dt}{9+16(1-t^2)}$

$=\int_0^1 \frac{dt}{25-16t^2}$

$=\frac{1}{16}\int_0^1 \frac{dt}{\left(\frac{5}{4}\right)^2-t^2}$

$=\frac{1}{2\times\frac{5}{4}\times 16}\left[\log\left|\frac{5+4t}{5-4t}\right|\right]_0^1 \left[\because \int \frac{1}{a^2-x^2}dx=\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C\right]$

$=\frac{1}{40}\left[\log\left|\frac{5+4}{5-4}\right|-\log\left|\frac{5}{5}\right|\right]$

$=\frac{1}{40}\left[\log\left(\frac{9}{1}\right)-\log\left(\frac{5}{5}\right)\right]$

$=\frac{1}{40}(\log 9-\log 1)$

$=\frac{1}{40}(\log 9)$ [$\because$ log 1 = 0]

$\Rightarrow I=\frac{1}{40}\log(3)^2$

$=\frac{2}{40}\log 3$ [$\because$log a^n = nlog a]

$\therefore \quad I=\frac{1}{20}\log 3$

13. According to the question ,

Given differential equation is

$\frac{dy}{dx}=\frac{y^2}{xy-x^2}$ ...(i)

Let $F(x,y)=\frac{y^2}{xy-x^2}$

Now, on replacing x by λx and y by λy , we get

$F(\lambda x,\lambda y)=\frac{\lambda^2 y^2}{\lambda^2(xy-x^2)}=\lambda^0\frac{y^2}{xy-x^2}=\lambda^0 F(x,y)$

Thus, the given differential equation is a homogeneous differential equation.

Now, to solve it, put y = vx

$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

From Eq. (i), we get

$v+x\frac{dv}{dx}=\frac{v^2x^2}{vx^2-x^2}=\frac{v^2}{v-1}$

$\Rightarrow \quad x\frac{dv}{dx}=\frac{v^2}{v-1}-v=\frac{v^2-v^2+v}{v-1}$

$\Rightarrow \quad x\frac{dv}{dx}=\frac{v}{v-1} \Rightarrow \frac{v-1}{v}dv=\frac{dx}{x}$

On integrating both sides, we get

$\int\left(1-\frac{1}{v}\right)dv=\int\frac{dx}{x}$

$\Rightarrow \quad v-\log|v|=\log|x|+C$

$\Rightarrow \quad \frac{y}{x}-\log\left|\frac{y}{x}\right|=\log|x|+C$ [put $v=\frac{y}{x}$]

$\Rightarrow \quad \frac{y}{x}-\log|y|+\log|x|=\log|x|+C$ $\left[\because \log\left(\frac{m}{n}\right)=\log m-\log n\right]$

$\therefore \quad \frac{y}{x}-\log|y|=C$

which is the required solution.