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Introduction to Data Mining
03 - Statistical Testing

Benjamin Paaßen

WS 2023/2024, Bielefeld University

Exercise Prep

- First sheet is out TODAY with deadline friday in 11 days (Nov 03, noon, i.e. 12:00)

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  - sometimes, language models get it wrong and you need to be able to check the result
  - you need to take full responsibility for correctness of the result
  - we want to understand how students use the available tools

Example

B

Write python code to automatically identify outliers, which are defined as any students with a pre-test score higher than 3 standard deviations above the mean. Write python code that removes these outliers from the data set.

Example

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B

Write python code to automatically identify outliers, which are defined as any students with a pre-test score higher than 3 standard deviations above the mean. Write python code that removes these outliers from the data set.

You can identify and remove outliers from a dataset using Python with the help of libraries like NumPy and pandas. Here's an example code to do this:

python Copy code

```python
import numpy as np
import pandas as pd

# Generate a sample dataset (you can replace this with your own data)
data = {
    'StudentID': [1, 2, 3, 4, 5, 6, 7, 8, 9, 10],
    'PreTestScore': [85, 92, 88, 110, 60, 105, 95, 102, 80, 120]
}

# Create a DataFrame from the data
df = pd.DataFrame(data)

# Calculate the mean and standard deviation of the PreTestScore
mean_score = df['PreTestScore'].mean()
std_dev = df['PreTestScore'].std()

# Define a threshold for identifying outliers
threshold = mean_score + 3 * std_dev

# Identify outliers
outliers = df[df['PreTestScore'] > threshold]

# Remove outliers from the dataset
df = df[df['PreTestScore'] <= threshold]
```

Python introduction

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Recap: Random variables

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- marginal probabilities for single variables, joint/conditional probabilities for combinations of variables
- most important density functions for continuous variables (in this lecture): Gaussian (!), t-distribution
- means of random variables tend to be Gaussian (central limit theorem)

Preamble: Study Design

Setup

Imagine you have invented a cool pedagogical intervention

Child with VR glasses by Julia M Cameron (Link); Usage according to pexels license.

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... but how do you know if it is effective?

Attempt 1: Single Class Study

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class of students

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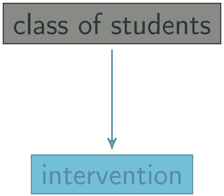

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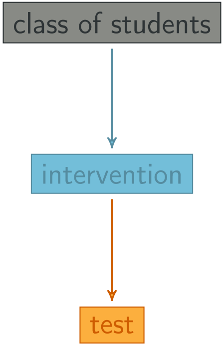

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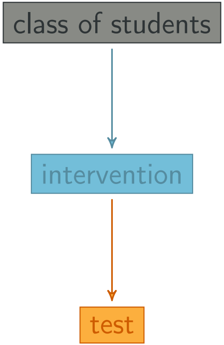

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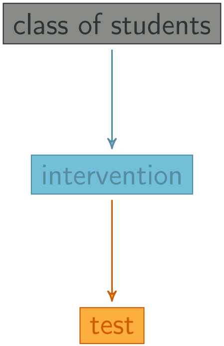

What are the problems here?

- Maybe the students already knew everything before the intervention

Attempt 2: Pre- and post-test

class of students

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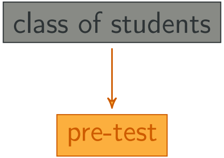

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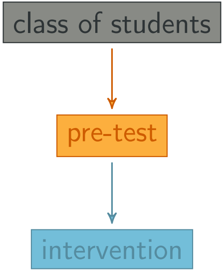

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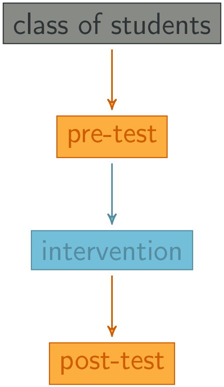

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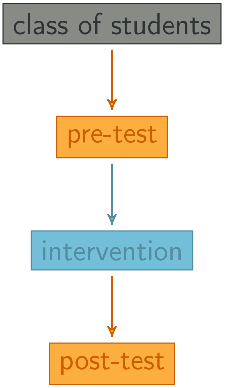

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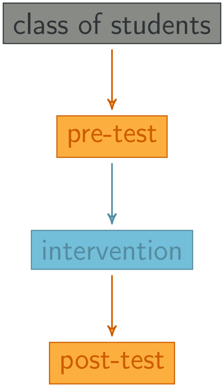

What are the problems here?

- Maybe the learning is not due to the intervention but something else

Attempt 3: Controlled study

intervention group

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intervention group

↓

intervention

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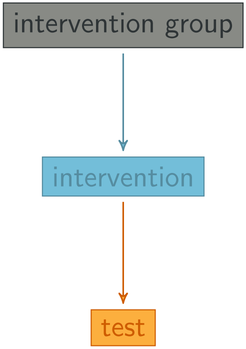

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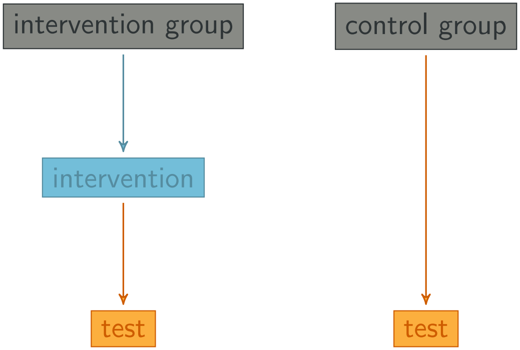

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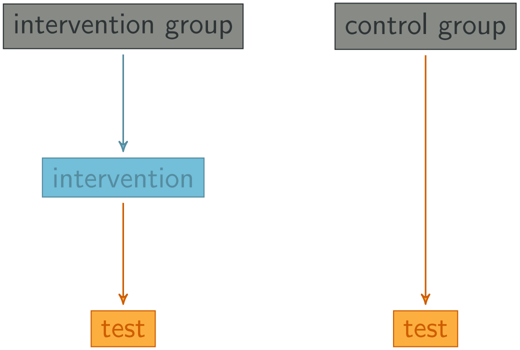

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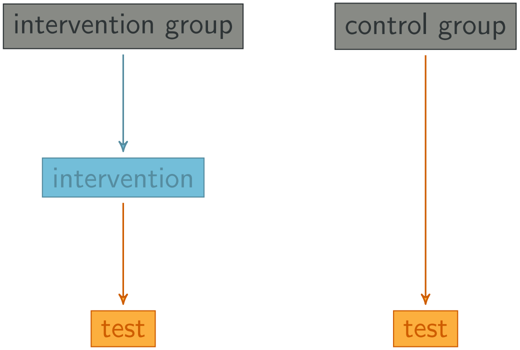

What are the problems here?

- Maybe students in the intervention condition are just better

Attempt 4: Controlled pre-/post-test study

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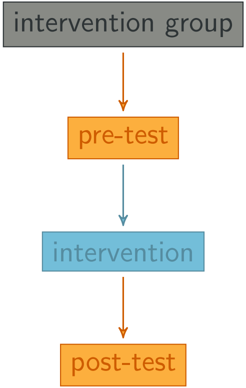

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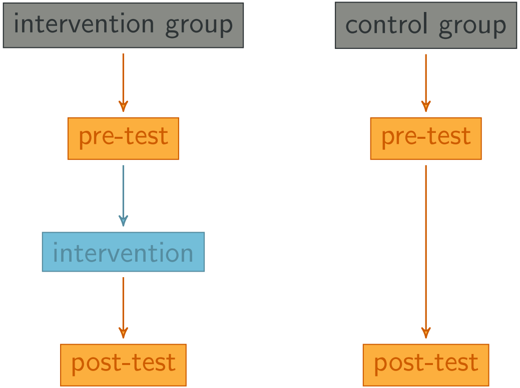

Further aspects

- Regression to the mean if intervention group are the (seemingly) weaker students

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- ... many more (Kulik and Fletcher 2016)

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	control condition				
student	1	2	3	4	5
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attention! This fictional data set is too small for actual statistics! This is only for illustration purposes!

Statistical questions

Assume we have recorded pre- and post-test scores for control and intervention group

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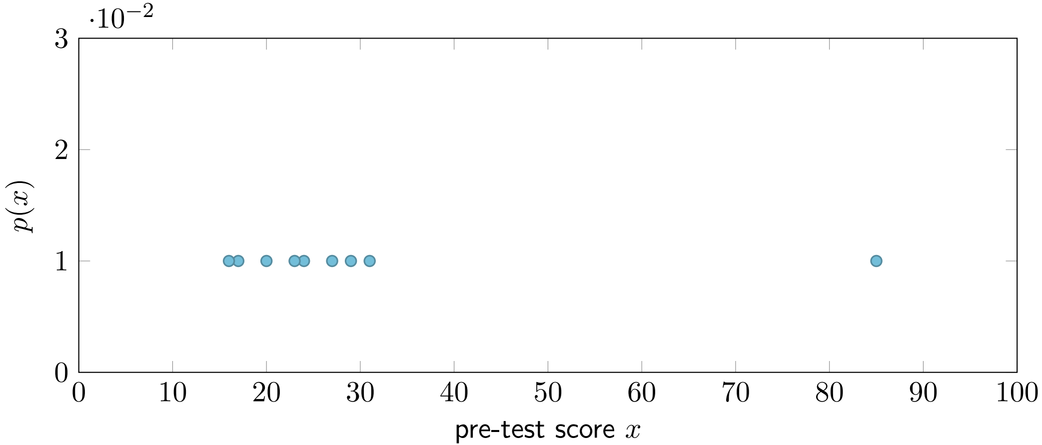
- Do students improve between pre- and post-test? (paired statistical testing)

Outlier detection

Example: Outlier detection



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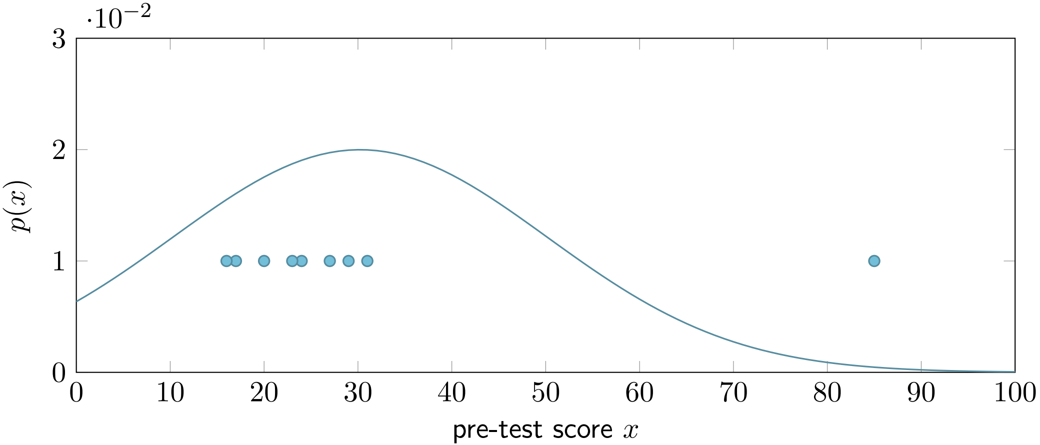

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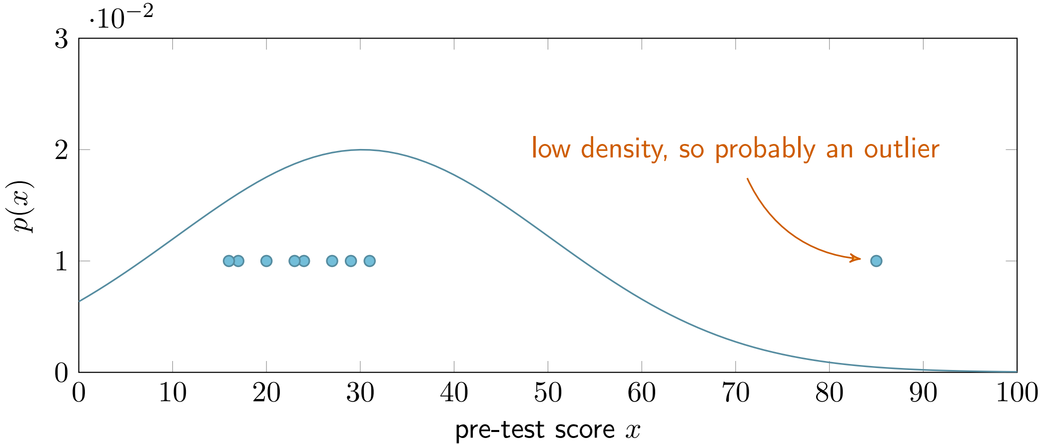

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We call points x_i **outliers** for threshold $\epsilon > 0$ if $p(x_i) < \epsilon$.

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- Approach 1: perform outlier detection separately for each dimension

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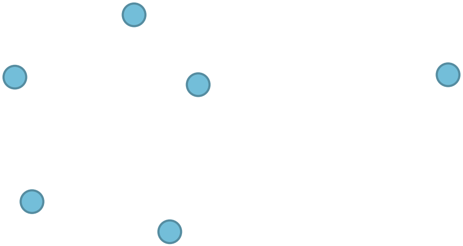

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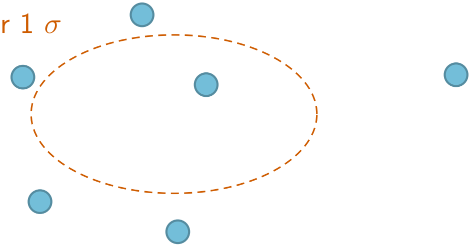
Gaussian density; iso-line for 1 σ



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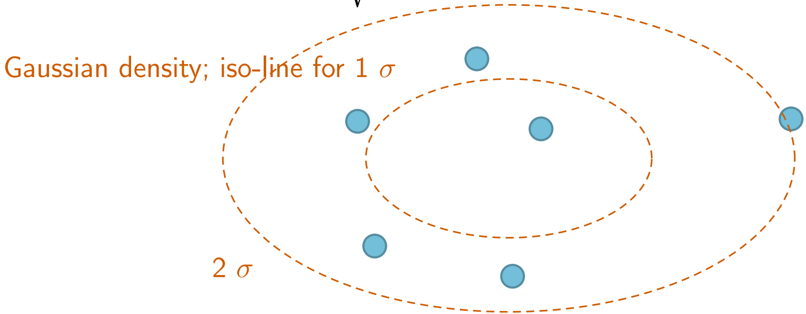

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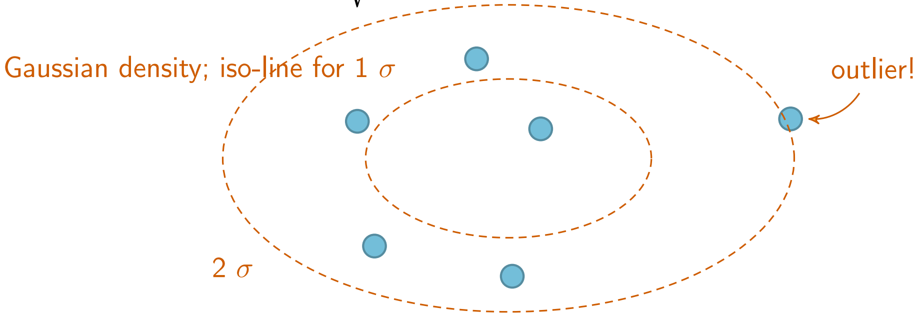

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$\Rightarrow$ iterative models (remove an outlier, fit again, etc.)

$\Rightarrow$ density estimation & outlier exclusion in one go, e.g. one-class SVM

Imputation

Example: Imputation

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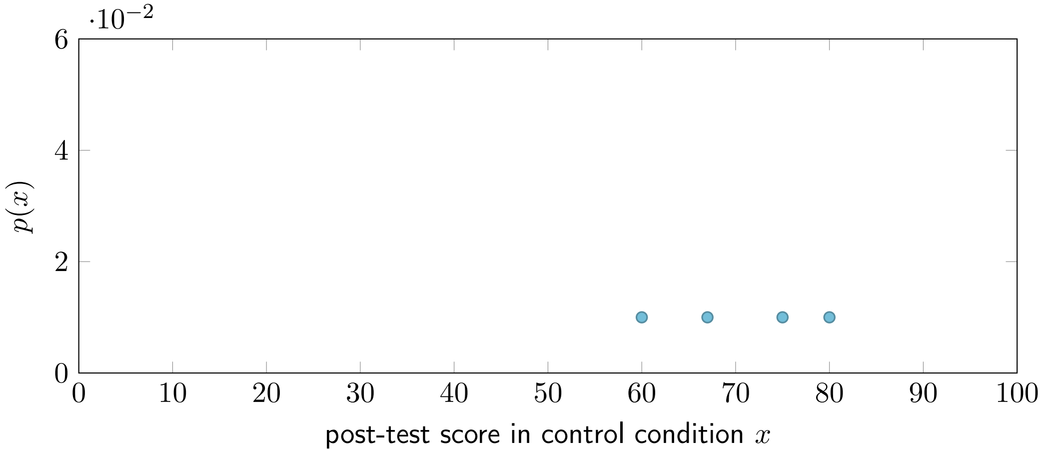

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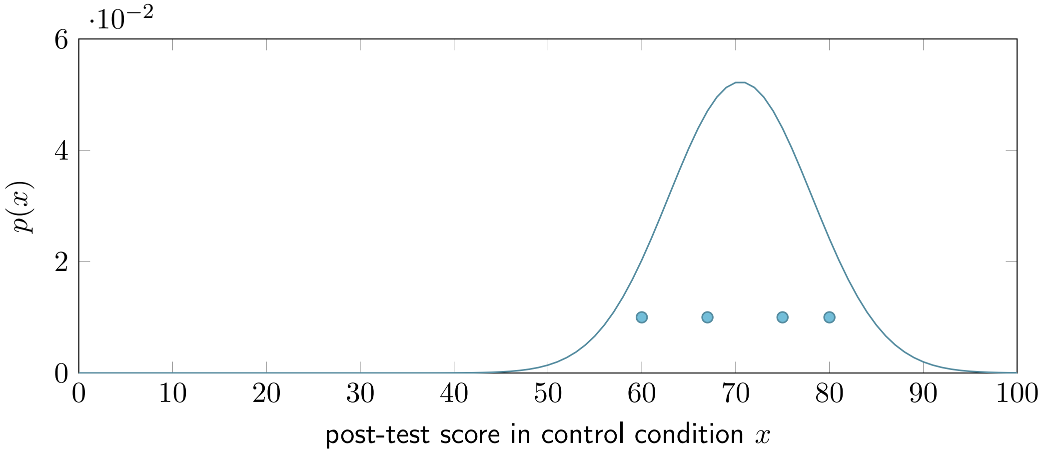

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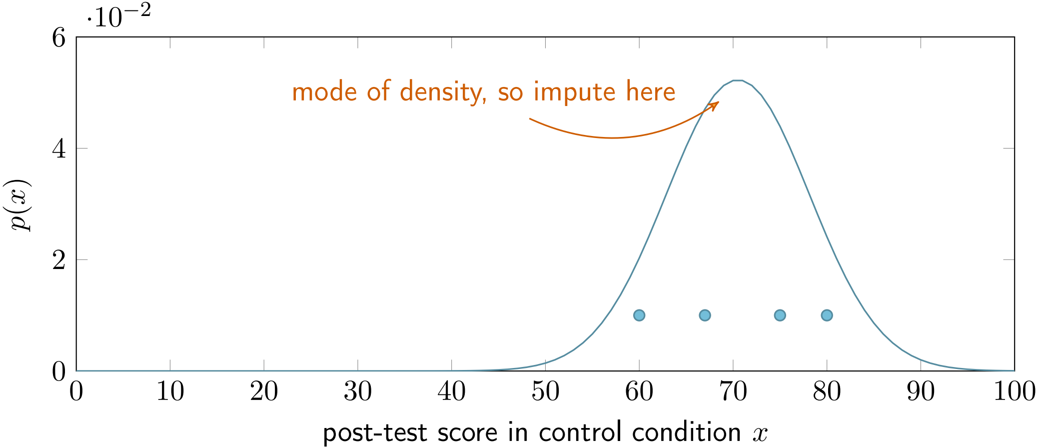

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We call a value $x_{i,j} = \text{NaN}$ a **missing value**.
Assume a probability density p fitted to the data.
Imputation means to replace a missing $x_{i,j}$ with one that maximizes p.

Gaussian imputation

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- Approach 2: maximize conditional $p(x_{i,j}|x_{i,1}, \ldots, x_{i,j-1}, x_{i,j+1}, \ldots, x_{i,n})$

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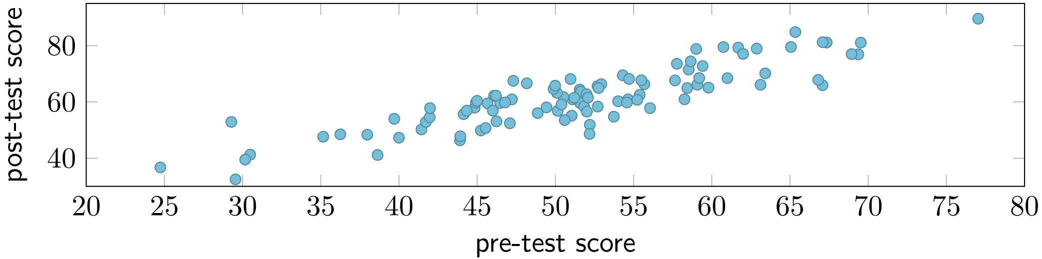

Gaussian imputation

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- Fit Gaussian, i.e. compute empiric mean μ and standard deviation σ of data, excluding missing values
- Replace NaNs with $\mu \Rightarrow$ maximizes $p(x_{i,j})$
- Approach 1 for multi-dimensional data: treat each dimension separately
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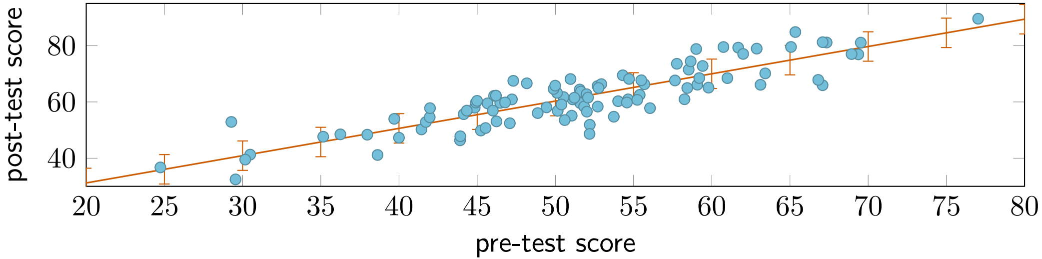

Challenges in imputation

- Constant value imputation artificially decreases variance of data

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- Constant value imputation artificially decreases variance of data

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- better models of conditionals?

$\Rightarrow$ Regression approaches

Correlations

Correlation: Example

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- Intuition: How strongly are two random variables associated?

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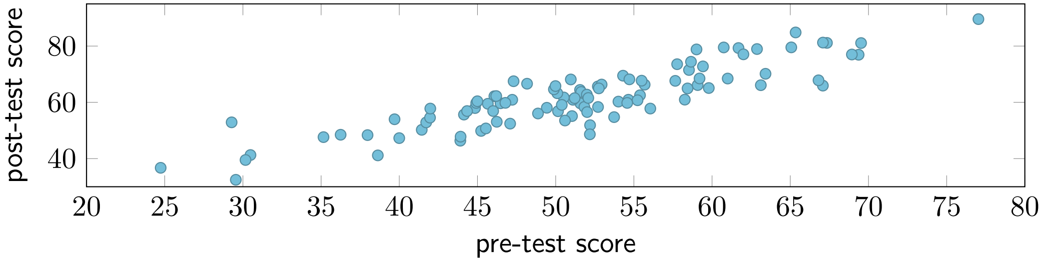

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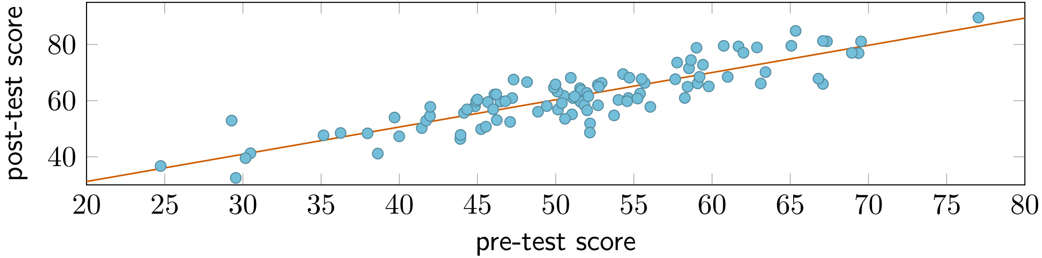

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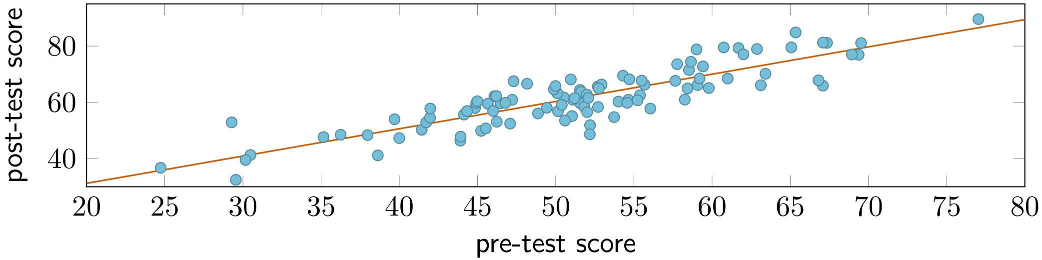

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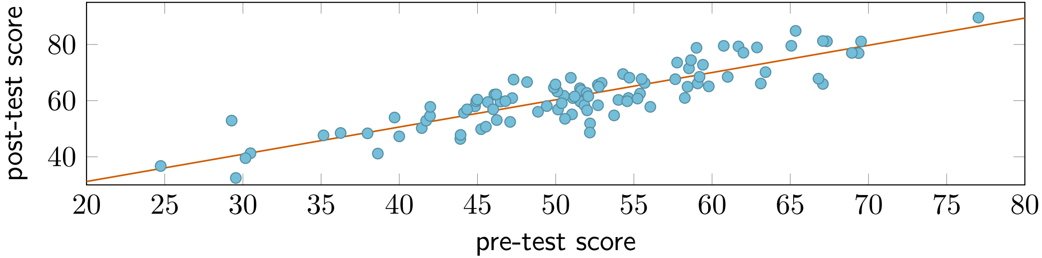

- Problem: What about the **scaling** of the data?

$\Rightarrow$ normalize data beforehand: $\tilde{x} = (x - \mu_x)/\sigma_x$ and $\tilde{y} = (y - \mu_y)/\sigma_y$.

Linear/Pearson correlation

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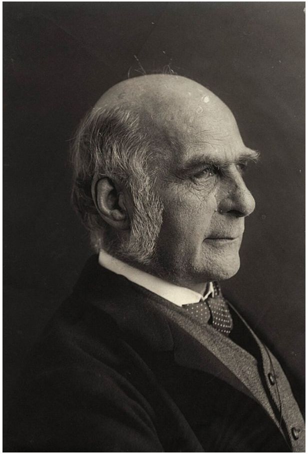
Sir Francis Galton FRS FRAI (/ˈɡɔːltən/; 16 February 1822 – 17 January 1911) was a British polymath and the originator of the eugenics movement during the Victorian era.[1][2]

Galton produced over 340 papers and books. He also developed the statistical concept of correlation and widely promoted regression toward the mean.

In recent years, he has received significant criticism for being a proponent of social Darwinism, eugenics, and scientific racism; he was a pioneer of eugenics, coining the term itself in 1883.



Linear/Pearson correlation

Karl Pearson FRS FRSE[1] (/ˈpɪərsən/; born **Carl Pearson**; 27 March 1857 – 27 April 1936[2]) was an English mathematician and biostatistician. He has been credited with establishing the discipline of mathematical statistics.[3][4] He founded the world's first university statistics department at University College London in 1911, and contributed significantly to the field of biometrics and meteorology. Pearson was also a proponent of social Darwinism and eugenics, and his thought is an example of what is today described as scientific racism.

Linear/Pearson correlation (Galton 1889)

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Linear correlation coefficient

Let $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^2$.

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Linear correlation coefficient

Let $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^2$.
the linear correlation coefficient is defined as:

$$r = \frac{1}{m} \sum_{i=1}^{m} \frac{x_i - \mu_x}{\sigma_x} \cdot \frac{y_i - \mu_y}{\sigma_y},$$

where μ_x and μ_y are the means and σ_x and σ_y are the standard deviations of the data.

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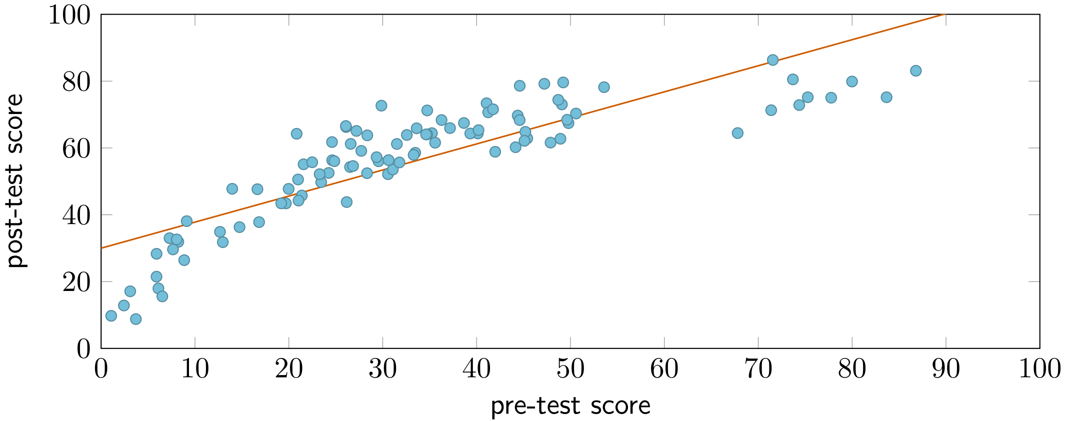
- interpretation: slope of the linear regression line **after** normalization of x and y
- Rules of thumb: $r \leq 0.3$ is very small, $r \in (0.3, 0.5]$ is small, $r \in (0.5, 0.7]$ is moderate, $r \in (0.7, 0.9]$ is high, $r \in (0.9, 1.0]$ is very high (Mukaka 2012)

Problems: Nonlinear correlations



Rank/Spearman correlation

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Charles Edward Spearman, FRS[1][3] (10 September 1863 – 17 September 1945) was an English psychologist known for work in statistics, as a pioneer of factor analysis, and for Spearman's rank correlation coefficient.

The Eugenics Review

Eugen Rev. 1914 Oct; 6(3): 219–237.

PMCID: PMC2987066
PMID: 21259592

The heredity of abilities

C. Spearman

Rank/Spearman correlation (Spearman 1904)

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Replace x_i and y_i by their **rank**.

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Rank correlation coefficient

Let $(x_1, y_1), \ldots, (x_m, y_m) \in \mathbb{R}^2$.
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- similar interpretation to linear correlation coefficient r – but in rank space
- no direct relation to regression line on the original data; no “drawing” of fit $\Rightarrow$ “non-parametric statistic”
- alternatives: Cohen’s κ, Krippendorff’s α, ...

Non-monotonic relations

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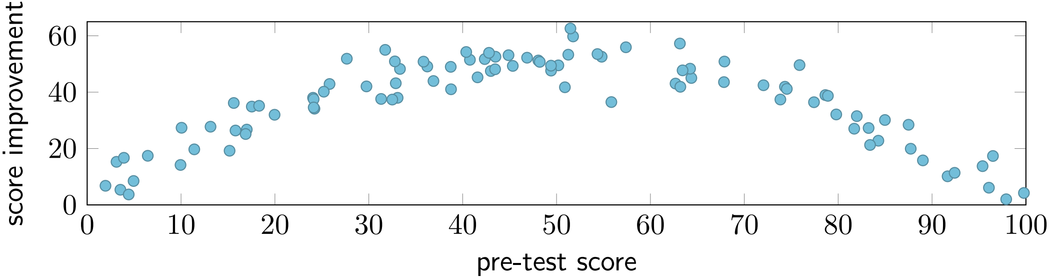

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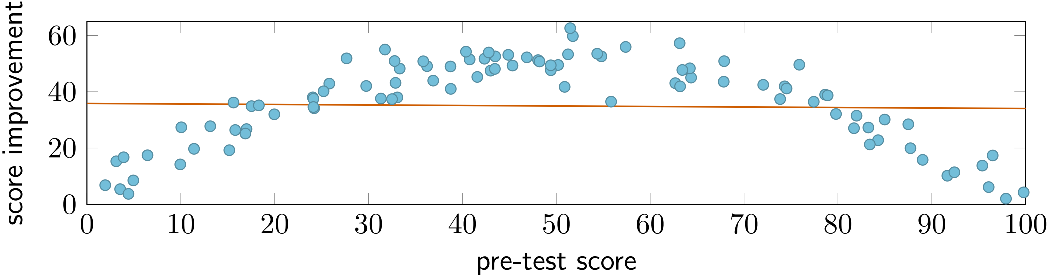

Non-monotonic relations

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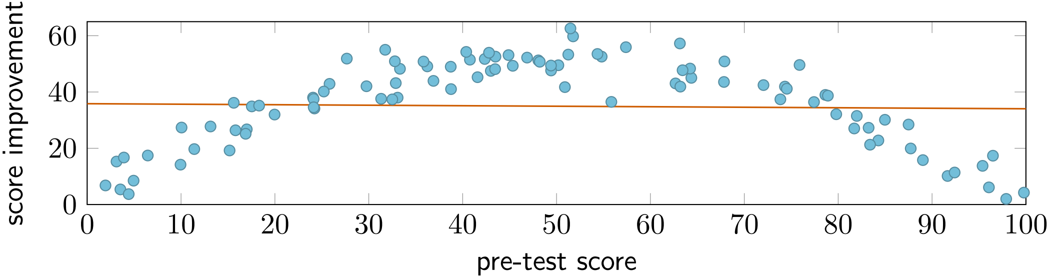

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Non-monotonic relations

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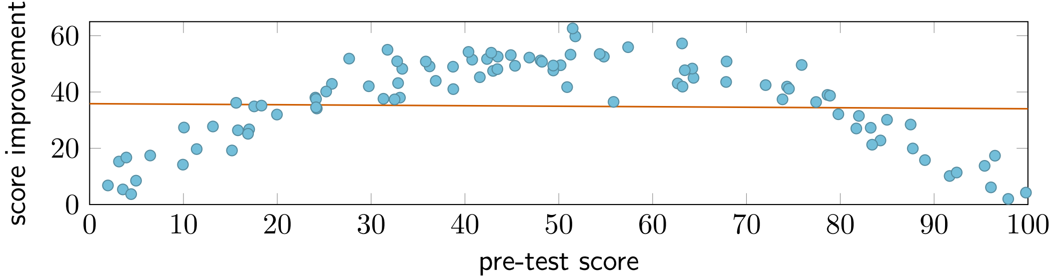

- correlation measures capture only **monotonic** relationships

$\Rightarrow$ independence $\Rightarrow$ no correlation, but not vice versa

Statistical Tests

Example: Bar plot

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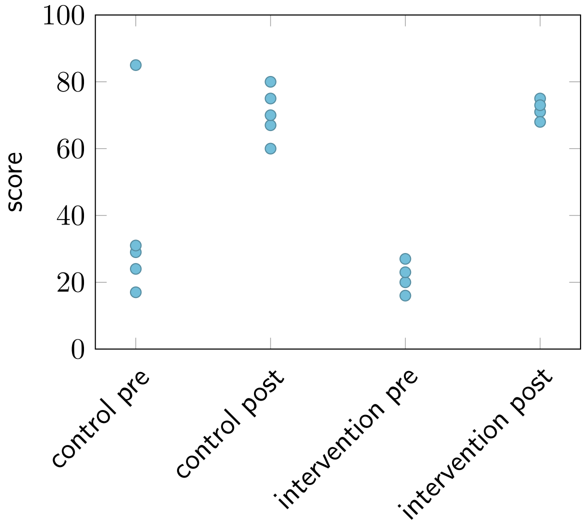

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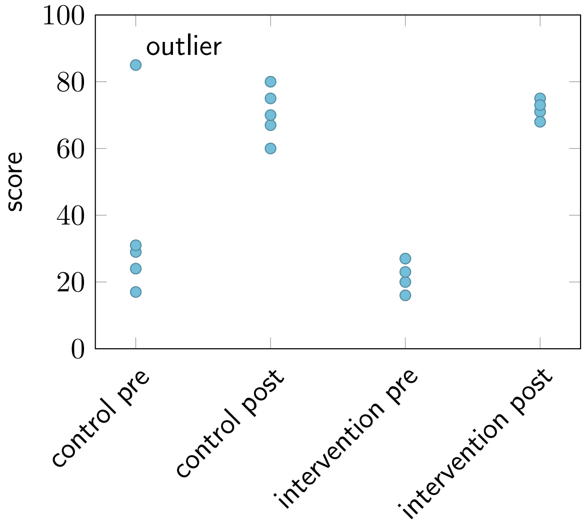

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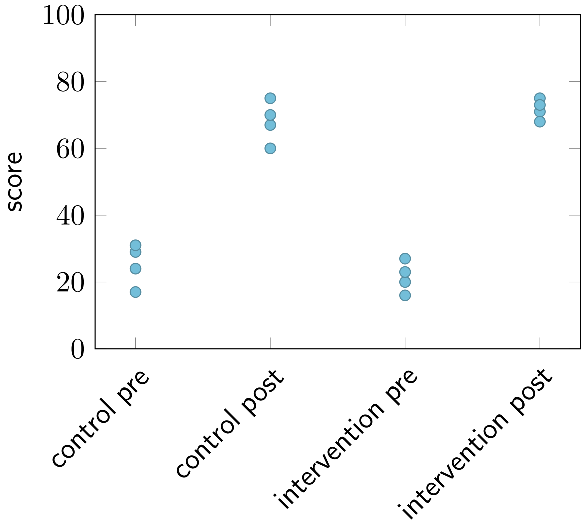

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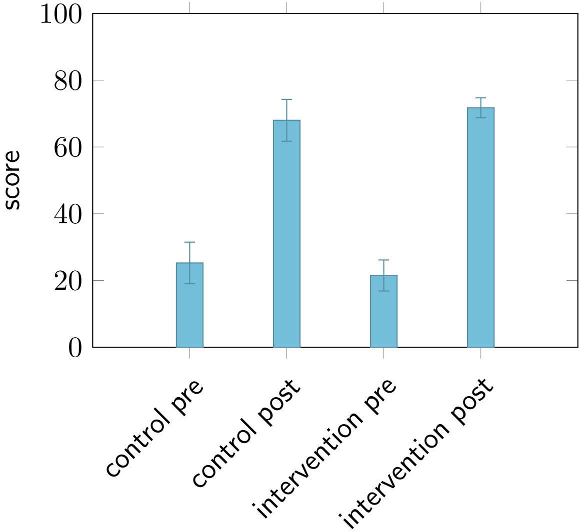

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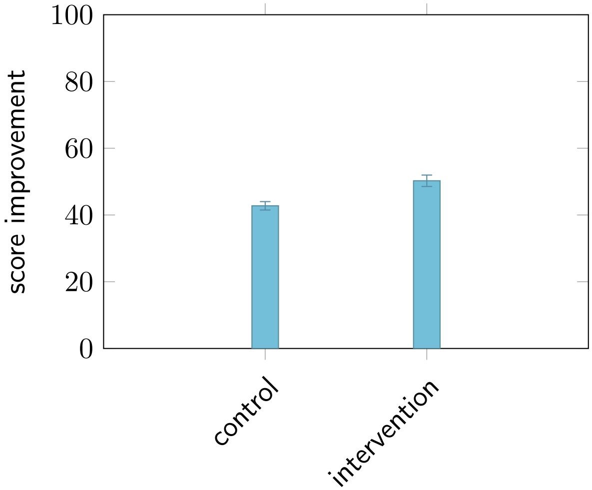
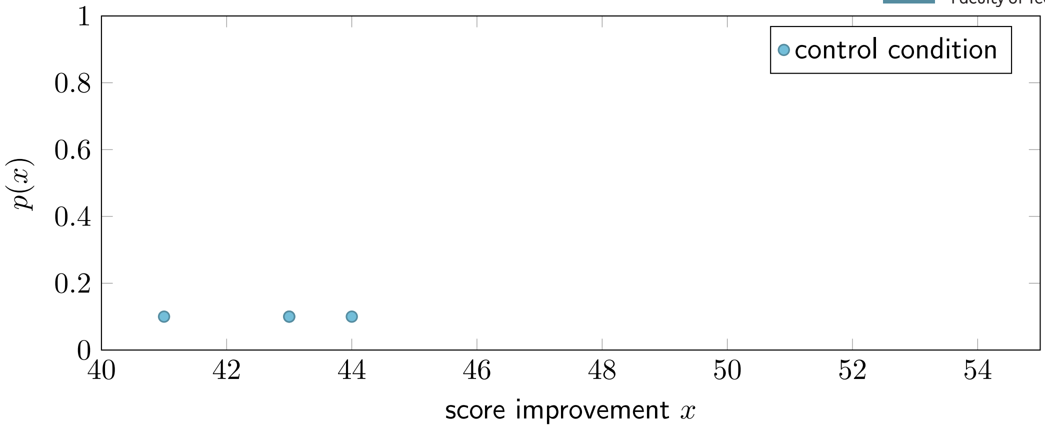
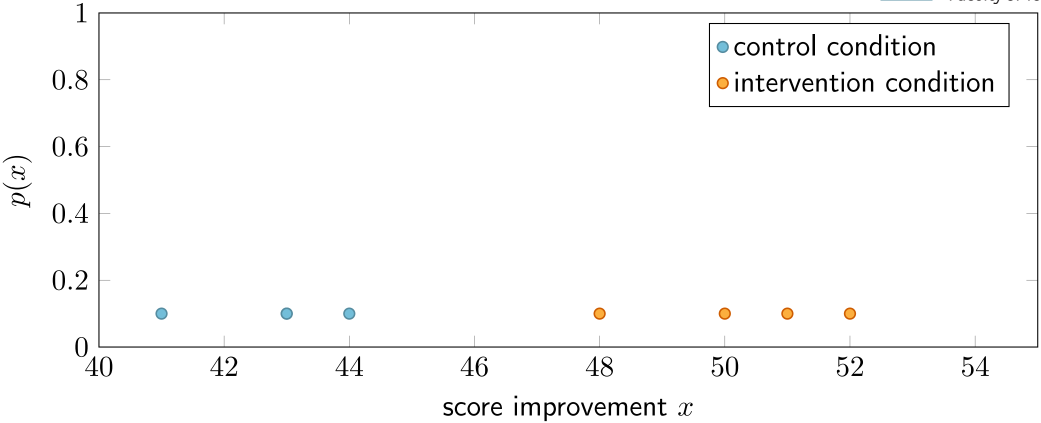


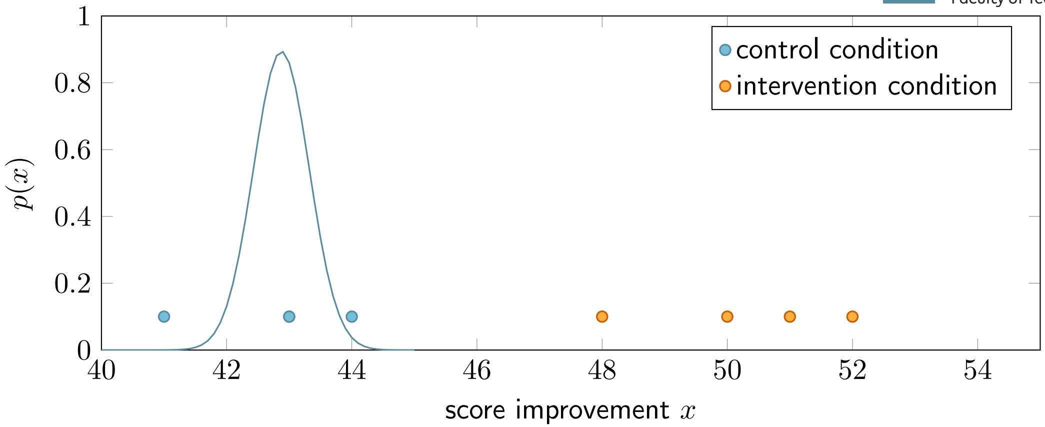
Example (continued)



Example (continued)



Example (continued)



Example (continued)

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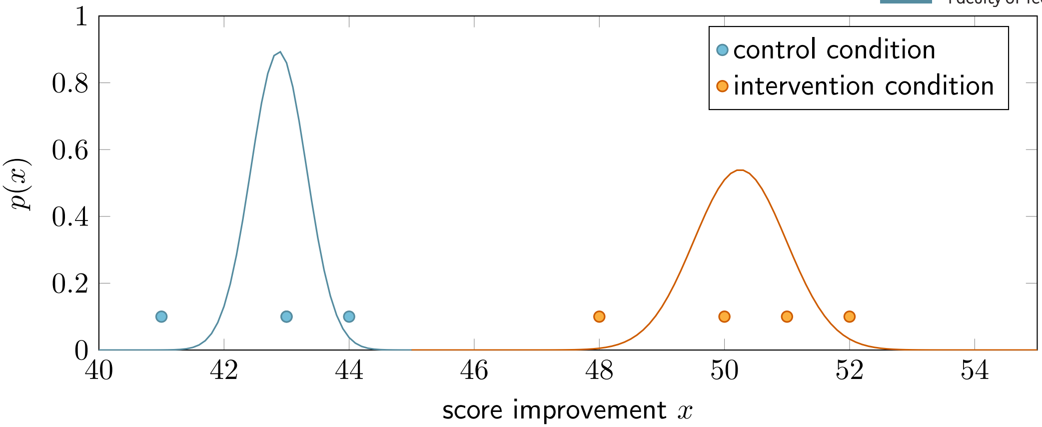

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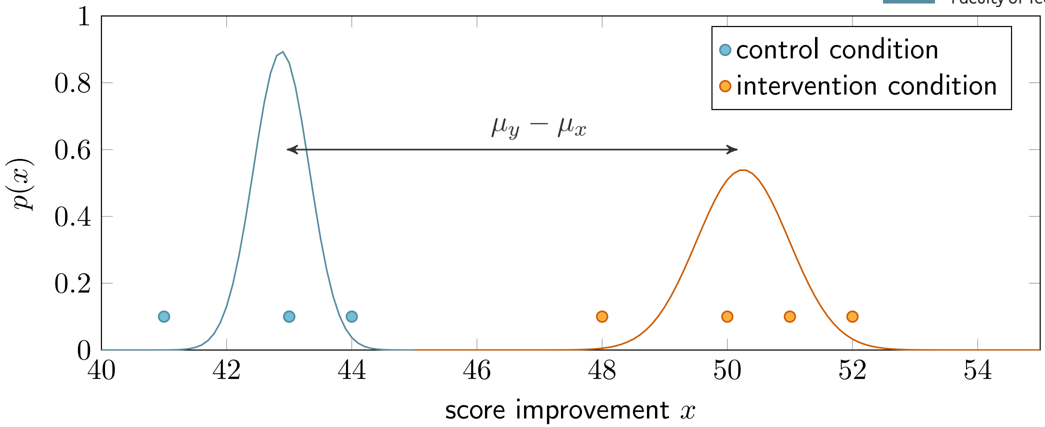

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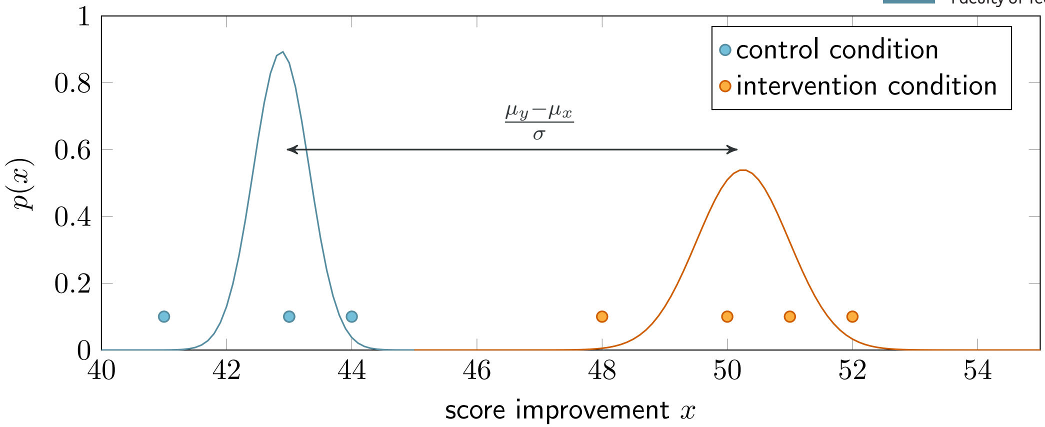

Example (continued)

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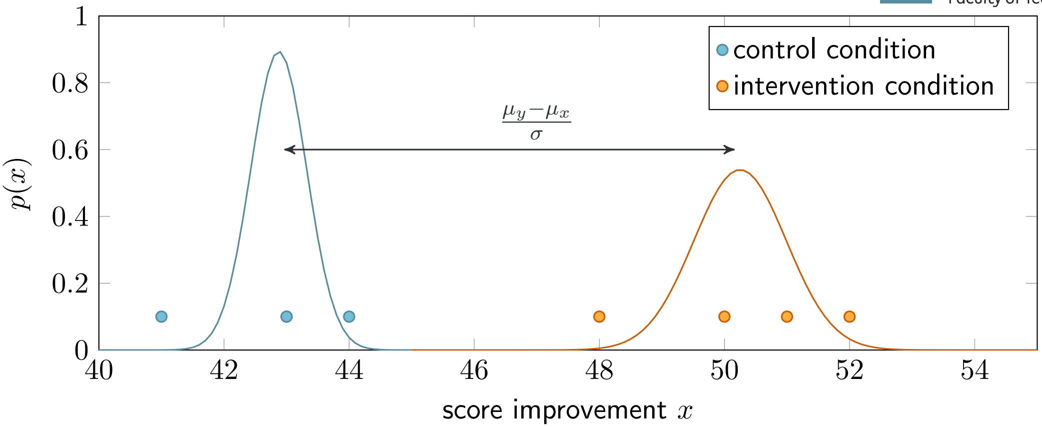

- What is the probability of $\frac{\mu_y - \mu_x}{\sigma}$ being that large just by random chance?

Null hypothesis testing: Motivation

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- Type II error: Failing to reject the null hypothesis even though it did not generate the observed effect (statistical power)

Null hypothesis testing framework

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- Tests differ in statistic, assumptions, and model

t-Test (Welch version)

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William Sealy Gosset (13 June 1876 – 16 October 1937) was an English statistician, chemist and brewer who served as Head Brewer of Guinness and Head Experimental Brewer of Guinness and was a pioneer of modern statistics. He pioneered small sample experimental design and analysis with an economic approach to the logic of uncertainty. Gosset published under the pen name **Student** and developed most famously Student's t-distribution – originally called Student's "z" – and "Student's test of statistical significance".[1]

t-Test (Welch version)

(Student 1908; Welch 1947)

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- Under the null hypothesis, μ_x and μ_y stem from the same Gaussian with $\tilde{\sigma}$ and t is t-distributed with parameter

$$\nu = \frac{\tilde{\sigma}^4}{\frac{\sigma_x^4}{n_x^2 \cdot (n_x - 1)} + \frac{\sigma_y^4}{n_y^2 \cdot (n_y - 1)}}$$

t-distribution illustration

- For our toy data set: $\tilde{\sigma} \approx 1$, $t = 7.39$, $\nu = 4.93$, $p = 0.0007$

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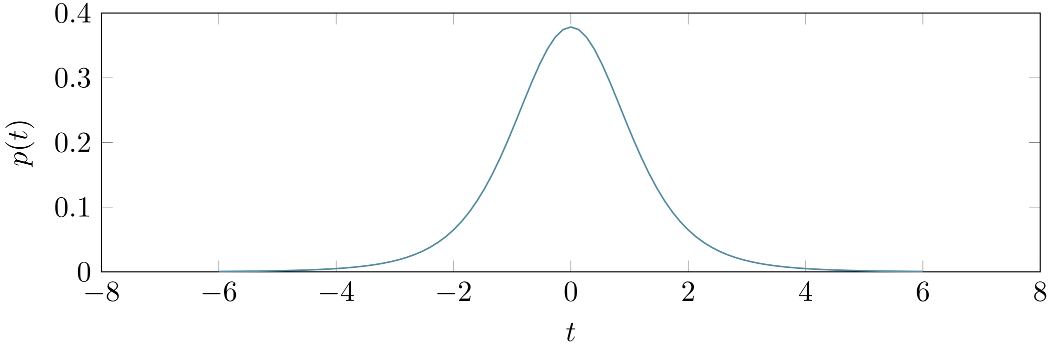

t-distribution illustration

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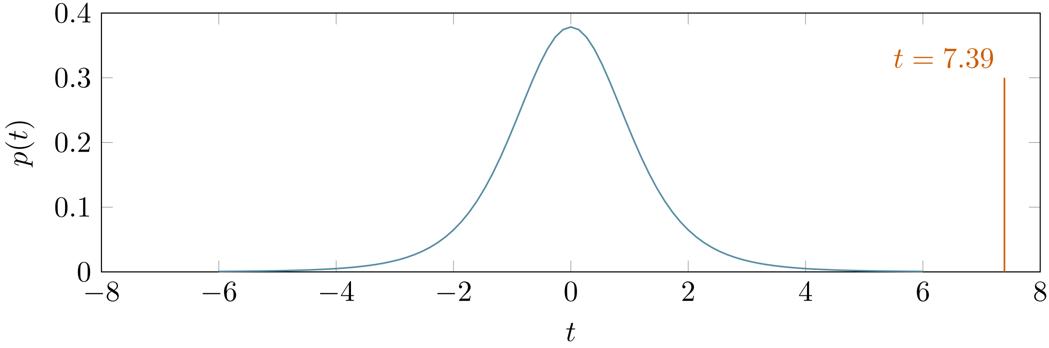

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$\Rightarrow$ one way of verification: check normality of original data distribution via Shapiro-Wilk-test (or other tests) $\Rightarrow$ mean of Gaussian variables is Gaussian

Effect size

(Cohen 1988)

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- Again: σ needs to be a compromise between σ_1 and σ_2, e.g.

$$\sigma = \sqrt{\frac{(n_1 - 1) \cdot \sigma_1^2 + (n_2 - 1) \cdot \sigma_2^2}{n_1 + n_2 - 2}}$$

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$$\sigma = \sqrt{\frac{(n_1 - 1) \cdot \sigma_1^2 + (n_2 - 1) \cdot \sigma_2^2}{n_1 + n_2 - 2}}$$

- Rules of thumb: $d < 0.2$ is very small, $d \in [0.2, 0.5)$ is small, $d \in [0.5, 0.8)$ is moderate, $d \in [0.8, 1.2)$ is large, $d \in [1.2, 2.0)$ is very large, $d \geq 2.0$ is huge (Sawilowsky 2009)

Effect size

(Cohen 1988)

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- Problem: significance testing only tells us whether an effect could be generated by the null hypothesis – not how strong it is
- Effect size: Cohen's d, defined as $d = \frac{|\mu_1 - \mu_2|}{\sigma}$
- Again: σ needs to be a compromise between σ_1 and σ_2, e.g.

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- Side note: largest measured effect sizes in psychology for educational studies

Example: signed rank test

- Question: Are post-test scores significantly higher than pre-test scores?

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$\Rightarrow$ **paired** data

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⇒ **paired** data

student	1	2	3	4	5	6	7	8	9
pre-test score	24	17	29	85	31	20	27	23	16
post-test score	67	60	70	80	75	71	75	73	68

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- Idea 1: If there are more positive than negative numbers, probably yes
- Idea 2: Put higher weights on bigger numbers

Wilcoxon signed rank test (Wilcoxon 1945)

- Sort improvements acc. to absolute value

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$$\Rightarrow T = 43$$

- Under null hypothesis, p_T can be computed exactly (for small n) – or via Gaussian approximation (for large n)

Exact Wilcoxon distribution $n = 3$

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- Under the null hypothesis, any combination of signs amongst the ranks is equally likely

rank	sign combination							
1	-	+	-	-	+	+	-	+
2	-	-	+	-	+	-	+	+
3	-	-	-	+	-	+	+	+
T	-6	-4	-2	0	0	+2	+4	+6

Exact Wilcoxon distribution $n = 3$

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rank	sign combination							
1	-	+	-	-	+	+	-	+
2	-	-	+	-	+	-	+	+
3	-	-	-	+	-	+	+	+
T	-6	-4	-2	0	0	+2	+4	+6

- $p(-6) = p(-4) = p(-2) = p(+2) = p(+4) = p(+6) = \frac{1}{8}$, $p(0) = \frac{1}{4}$

Assumptions of Wilcoxon signed rank test

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- Samples are independent

Assumptions of Wilcoxon signed rank test

- Samples are independent
- Differences are symmetrically distributed

Summary

- This visualization summarizes all key concepts from null hypothesis testing neatly:

  `https://rpsychologist.com/d3/nhst/`

Literature I

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Cohen, Jacob (1988). **Statistical Power Analysis for the Behavioral Sciences**. New York, NY, USA: Routledge. DOI: 10.4324/9780203771587.

Galton, Francis (1889). "I. Co-relations and their measurement, chiefly from anthropometric data". In: **Proceedings of the Royal Society of London** 45.273-279, pp. 135–145. DOI: 10.1098/rspl.1888.0082.

Kulik, James A and JD Fletcher (2016). "Effectiveness of intelligent tutoring systems: a meta-analytic review". In: **Review of educational research** 86.1, pp. 42–78. DOI: 10.3102/0034654315581420. URL: https://www.researchgate.net/profile/J-D-Fletcher/publication/277636218_Effectiveness_of_Intelligent_Tutoring_Systems_A_Meta-Analytic_Review/links/5707a6fe08ae8883a1f7e55f/Effectiveness-of-Intelligent-Tutoring-Systems-A-Meta-Analytic-Review.pdf.

Literature II

UNIVERSITÄT BIELEFELD
Faculty of Technology

Mukaka, Mavuto M (2012). "A guide to appropriate use of correlation coefficient in medical research". In: **Malawi medical journal** 24.3, pp. 69–71. URL: https://www.ajol.info/index.php/mmj/article/view/81576.

Sawilowsky, Shlomo S. (2009). "New effect size rules of thumb". In: **Journal of modern applied statistical methods** 8.2, p. 26. DOI: 10.22237/jmasm/1257035100.

Spearman, Charles (1904). "The Proof and Measurement of Association between Two Things". In: **The American Journal of Psychology** 15.1, pp. 72–101. DOI: 10.2307/1412159.

Student (1908). "The Probable Error of a Mean". In: **Biometrika** 6.1, pp. 1–25. DOI: 10.2307/2331554.

Welch, Bernard Lewis (1947). "The Genralization of Student's Problem when several different population variances are involved". In: **Biometrika** 34.1-2, pp. 28–35. DOI: 10.1093/biomet/34.1-2.28.

Literature III

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Wilcoxon, Frank (1945). “Individual Comparisons by Ranking Methods”. In: **Biometrics Bulletin** 1.6, pp. 80–83. DOI: 10.2307/3001968. URL: http://www.jstor.org/stable/3001968.