

Analysis, Topology, Optimization, Machine Learning, and Computational Analysis

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1 Notation, Set Theory, and Logic

Definition 1.1. Common Sets of Numbers

$$\mathbb{N} = \{1, 2, \dots\}$$

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}$$

$$\mathbb{N}_m = \{1, 2, \dots, m\} \text{ where } m \in \mathbb{N}$$

\mathbb{R} is the set of Real Numbers

$$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$$

$$\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$$

$$\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$$

$$\mathbb{R}_0^- = \{x \in \mathbb{R} : x \leq 0\}$$

Definition 1.2. Sets and Set Builder Notation

Set builder notation is a way of describing a set using mathematical or logical symbols.

Look at the following example:

$$E = \{x \in \mathbb{N} : x = 2n \text{ where } n \in \mathbb{N}\}$$

This reads: E is the set of all x in \mathbb{N} such that $x = 2n$ where n is in \mathbb{N}

This set is also known as the even numbers.

When talking about functions, another common way of describing a set is:

$$C_X^Y = \{f : X \rightarrow Y | f \text{ is continuous}\}$$

This reads: C_X^Y is the set of all functions f mapping from X to Y such that f is continuous.

Definition 1.3. Subsets

Let A, B be sets.

$$A \subset B \Leftrightarrow (x \in A \Rightarrow x \in B)$$

Theorem 1.1. Union and Intersection Distributive Properties

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Definition 1.4. Power Set

Let $X \neq \phi$

$$2^X := \{V : V \subseteq X\}$$

2 Topology**Definition 2.1. Topology**

Let $X \neq \phi$

Further let $\tau \subseteq 2^X$ such that:

$$\begin{aligned} & \phi, X \in \tau \\ & (\forall A \neq \phi) \left(\{U_\alpha\}_{\alpha \in A} \subseteq \tau \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \tau \right) \\ & (\forall m \in \mathbb{N}) \left(\{U_j\}_{j \in \mathbb{N}_m} \Rightarrow \bigcap_{j=1}^m U_j \in \tau \right) \end{aligned}$$

Definition 2.2. Relative Topology

Let $X \neq \phi$ and $Z \subset X$

Then the relative topology on Z is written as follows:

$$\tau_Z = \{Z \cap U : U \in \tau_X\}$$

Theorem 2.1. The Relative Topology is a Topology on Z

Let $E \in \tau_Z$

$$\Rightarrow E = Z \cap U \subset Z$$

$$\Rightarrow \tau_Z \subseteq 2^Z$$

And so we have met the first criteria.

Next:

$$\phi \in \tau \Rightarrow Z \cap \phi \in \tau_Z \Rightarrow Z \cap \phi = \phi \in \tau_Z$$

Next:

$$X \in \tau \Rightarrow Z \cap X \in \tau_Z \Rightarrow Z \cap X = Z \in \tau_Z$$

Next: Let $A \neq \phi$ and $\{U_\alpha\}_{\alpha \in A} \in \tau_Z$

$$\Rightarrow \exists \{V_\alpha\}_{\alpha \in A} \subset \tau \text{ such that: } U_\alpha = Z \cap V_\alpha$$

$$\Rightarrow \bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} Z \cap V_\alpha$$

Theorem 2.2. Topologies on Finite Spaces

Let $E \neq \phi$ and $\text{card}(E) = \phi$ then $\tau_E = 2^E$

3 Change

Definition 3.1. *Metric*

Let X be a non-empty set.

Let $d : X \times X \rightarrow \mathbb{R}_0^+$ such that:

- $(\forall x, y \in X) d(x, y) = 0 \Leftrightarrow x = y$
- $(\forall x, y \in X) d(x, y) = d(y, x)$
- $(\forall x, y, z \in X) d(x, z) \leq d(x, y) + d(y, z)$

Then d is called a metric and (X, d) is called a metric space.

Reference

Definition 3.2. *Limit of a function*

Let $T : X \rightarrow Y$ where (X, d_X) and (Y, d_Y) are metric spaces.

Then fix $x_0 \in X$.

If:

$$(\exists L \in Y)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in X)(d(x, x_0) < \delta \Rightarrow d(f(x), L) < \epsilon)$$

Then:

$$\lim_{x \rightarrow x_0} f(x) = L$$

Reference

Definition 3.3. *Derivative*

Let $\hat{f} : \mathbb{R} \rightarrow \mathbb{R}$

Further let $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}}$

Then f is said to be differentiable at $x \in U$ if there exists an L_x such that:

$$L_x = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If L_x exists for all $x \in U$ then we write:

$$\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Reference

Theorem 3.1. *Fundamental increment lemma*

Let f be described as above and be differentiable at x .

Then there exists a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$f(x+h) = f(x) + \frac{d}{dx} f(x)h + \phi(x)h$$

and

$$\lim_{h \rightarrow 0} \phi(h) = 0$$

Proof:

Define: $\phi(h) = \frac{f(x+h) - f(x)}{h} - \frac{d}{dx} f(x)$

Then: $\phi(h)h = f(x+h) - f(x) - \frac{d}{dx} f(x)h$

Then: $\phi(h)h + f(x) - \frac{d}{dx} f(x)h = f(x+h)$

And so we have property 1.

Next:

$$\begin{aligned}\lim_{h \rightarrow 0} \phi(h) &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x) - \frac{d}{dx}f(x)h}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} - \frac{d}{dx}f(x) \right] \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{d}{dx}f(x) = \frac{d}{dx}f(x) - \frac{d}{dx}f(x) = 0\end{aligned}$$

Definition 3.4. Partial Derivative

Let $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$

Further let $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}^n}$

Then f is said to be differentiable at $x \in U$ with respect to the i 'th component of x if there exists an L_{x_i} such that:

$$L_{x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

If L_{x_i} exists for all $x \in U$ then we write:

$$\frac{\partial}{\partial x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

Reference

Theorem 3.2. Equivalent characterization

Let $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$

Further let $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}^n}$

And let f be differentiable at $x \in U$ with respect to the i 'th component of x , then:

$$\begin{aligned}L_{x_i} &= \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[\frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} - L_{x_i} \right] \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[\frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} - \frac{L_{x_i} \cdot h}{h} \right] \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[\frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n) - \langle L_{x_i}, h \rangle}{h} \right]\end{aligned}$$

Definition 3.5. Gradient

Let $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ and let $f : U \rightarrow \mathbb{R}$ such that $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}^n}$

f is said to be differentiable at $x \in U$ if $\exists \nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that:

$$\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - \langle \nabla f(x), h \rangle|}{\|h\|} = 0$$

Theorem 3.3. Form of the Gradient

Let f be defined as above.

Then $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ where:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix} \quad \forall x \in \mathbb{R}^n$$

is the form of ∇f which satisfies the above statement if f is differentiable.

Reference

Proof:

Suppose ∇f is defined as above and all the partial derivatives exist.

Then:

$$\frac{1}{||h||} |f(x+h) - f(x) - \langle \nabla f(x), h \rangle| = \frac{1}{||h||} \left| f(x+h) - f(x) - \sum_{j=1}^n \frac{\partial}{\partial x_j} f(x) \cdot h_j \right|$$

Definition 3.6. Matrix Functional Differentiability

Let $\hat{T} : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ and let $T : U \rightarrow \mathbb{R}$ such that $T = \hat{T}|_U$ where $U \in \tau_{\mathbb{R}^{n \times m}}$. T is said to be differentiable at $x \in U$ if $\exists D : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n \times m}$ such that:

$$\lim_{h \rightarrow 0} \frac{|T(x+h) - T(x) - \langle DT(x), h \rangle|}{\|h\|} = 0$$

where $\langle \cdot, \cdot \rangle$ is an inner product defined on $\mathbb{R}^{n \times m}$

Definition 3.7. Frobenius inner product

The Frobenius inner product is defined as:

$$\langle \cdot, \cdot \rangle_{FB} : \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \rightarrow \mathbb{R} \text{ such that: } \langle A, B \rangle_{FB} = \sum_{i=1}^n \sum_{j=1}^m a_{i,j} b_{i,j} \text{ for all } A, B \in \mathbb{R}^{n \times m}$$

Theorem 3.4. Form of Matrix Functional Derivative

$$DT(x) = \begin{bmatrix} \frac{\partial}{\partial x_{1,1}} & \cdots & \frac{\partial}{\partial x_{1,m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_{n,1}} & \cdots & \frac{\partial}{\partial x_{n,m}} \end{bmatrix}$$

Definition 3.8. Differentiability of a multi-variable function.

Let $\hat{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that:

$$\hat{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \quad \text{and } (\forall j \in \mathbb{N}_n)(f_j : \mathbb{R}^m \rightarrow \mathbb{R})$$

Further let $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}^m}$

Then f is said to be differentiable at $x \in U$ if there exists a linear operator $J_{f(x)} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that:

$$\lim_{h \rightarrow \vec{0}} \frac{\|f(x+h) - f(x) + J_{f(x)}(h)\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^m}} = 0$$

Reference

Theorem 3.5. If a multi-variable function, f , is differentiable at x then the linear operator J is the Jacobian matrix.

So our guess is that:

$$J_{f(x)} = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x) & \cdots & \frac{\partial}{\partial x_n} f_1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_n(x) & \cdots & \frac{\partial}{\partial x_n} f_n(x) \end{bmatrix}$$

since this form is a linear operator mapping from the appropriate space to the appropriate space. It should be noted that the transpose of this matrix can not satisfy the definition of differentiability of a multi-variable function and so it is not the correct linear operator.

Definition 3.9. Matrix operator differentiability

Let $T : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$ such that:

$$T(A) = \begin{bmatrix} T_1(A) \\ \vdots \\ T_n(A) \end{bmatrix} \quad \forall A \in \mathbb{R}^{n \times m} \quad \text{and } (\forall j \in \mathbb{N}_n)(T_j : \mathbb{R}^{n \times m} \rightarrow \mathbb{R})$$

Then T is said to be differentiable at $A \in \mathbb{R}^{n \times m}$ if there exists a linear operator $D : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$ where:

$$\lim_{h \rightarrow 0} \frac{\|T(A+h) - T(A) + D(h)\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^{n \times m}}} = 0$$

If D exists then it is called the Matrix operator derivative and is written: $D_{\mathbb{R}^{n \times m}} T(A)$

Theorem 3.6. The form of the Matrix operator derivative.

Let T be described as above and differentiable at $A \in \mathbb{R}^{n \times m}$

$$\frac{T(A+h) - T(A)}{\|h\|} = \begin{bmatrix} \frac{T_1(A+h) - T_1(A)}{\|h\|} \\ \vdots \\ \frac{T_n(A+h) - T_n(A)}{\|h\|} \end{bmatrix}$$

and so:

$$\lim_{h \rightarrow 0} \frac{T(A+h) - T(A)}{\|h\|} = \begin{bmatrix} \lim_{h \rightarrow 0} \frac{T_1(A+h) - T_1(A)}{\|h\|} \\ \vdots \\ \lim_{h \rightarrow 0} \frac{T_n(A+h) - T_n(A)}{\|h\|} \end{bmatrix}$$

Definition 3.10. Subspace Differentiability

Let $X = \{X_j\}_{j=1}^n$ be a sequence of finite dimensional vector spaces where $\dim(X_j) = k_j = m_j \times n_j$

Let $T : \prod_{j=1}^n X_j \rightarrow Y$ where Y is a finite dimensional vector space with $\dim(Y) = k_y$

Let $x_j \in X_j$ for some $j \in \mathbb{N}_n$

Where

$$x_j = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n_j} \\ \vdots & \ddots & \vdots \\ x_{m_j,1} & \cdots & x_{m_j,n_j} \end{bmatrix}$$

T is said to be differentiable at $x \in X$ where $x = (x_0, \dots, x_j, \dots, x_{n-1})$ with respect to X_j if there exists a linear operator $D : X_j \rightarrow Y$:

Given $h \in X_j \setminus \{\vec{0}\}$ define $\hat{h} = (0, \dots, h, \dots, 0) \in X$ where h is in the j 'th place of \hat{h} :

$$\lim_{h \rightarrow 0} \frac{\|T(x + \hat{h}) - T(x) + D(h)\|_Y}{\|h\|_{X_j}} = 0$$

Then D is called the subspace derivative of T at x with respect to X_j and is written: $D_{x_j}T(x)$

Definition 3.11. *Product space Derivative*

Let $X = \{X_j\}_{j=0}^{n-1}$ be a sequence of finite dimensional vector spaces where $\dim(X_j) = k_j$

Let $T : \prod_{j=0}^{n-1} X_j \rightarrow Y$ where Y is a finite dimensional vector space with $\dim(Y) = k_y$

Let $\{x_j\}_{j=0}^{n-1}$ be a sequence of vectors such that: $(\forall j \in \{0, \dots, n-1\})(x_j \in X_j)$

The product space derivative at the point $z \in X$ is:

$$D_X T(z) = \begin{bmatrix} D_{x_0} T(z) \\ \vdots \\ D_{x_{n-1}} T(z) \end{bmatrix}$$

Definition 3.12. Fréchet derivative

Let V, W be normed vector spaces and $U \subset V$ be an open set.

An operator $f : U \rightarrow W$ is said to be Fréchet differentiable if there exists a bounded linear operator $A : V \rightarrow W$ such that:

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0$$

Reference

Theorem 3.7. Fréchet derivative of a bounded linear operator

Let V, W be normed vector spaces and $U \subset V$ be an open set.

Let $\hat{f} : V \rightarrow W$ be a bounded linear operator.

Then let's look at $f = \hat{f}|_U$

My guess is that $A = \hat{f}$

Let $x \in U$ and $h \in U$ with $\|h\| \neq 0$ and $x+h \in U$, Then:

$$\frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = \frac{\|f(x) + f(h) - f(x) + \hat{f}(h)\|_W}{\|h\|_V} = \frac{\|f(x) + f(h) - f(x) + f(h)\|_W}{\|h\|_V} = 0$$

Thus let $\epsilon > 0$ and $\delta > 0$

Then if $0 < \|h\| < \delta$ we know that $\frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0 < \epsilon$

Therefore:

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0$$

Thus $A = \hat{f}$ is the Fréchet derivative of f .

3.1 Finite Composition Operator**Definition 3.13. Finite Composition Operator**

Let the collection $X = \{X_j\}_{j=0}^n$ be a finite sequence of sets.

Further let $\{T_j\}_{j=0}^{n-1}$ be a finite sequence of operators such that $(\forall j \in \mathbb{N}_{n-1})(T_j : X_j \rightarrow X_{j+1})$

Then $T^n : X_0 \rightarrow X_n$ defined by:

$$T^n := \bigcirc_{j=0}^{n-1} T_j$$

is called the **Finite Composition Operator defined on X** .

Definition 3.14. Multi-variable Finite Composition Iteration

Let the collection $X = \{X_j\}_{j=0}^n$ and $Y = \{Y_j\}_{j=0}^{n-1}$ be finite sequences of sets.

Further let $\{T_j\}_{j=0}^{n-1}$ be a finite sequence of operators such that: $(\forall j \in \mathbb{N}_{n-1})(T_j : X_j \times Y_j \rightarrow X_{j+1})$

Let $T^n : X_0 \times \prod_{j=0}^{n-1} Y_j \rightarrow X_n$ where:

$$T^n(x, y) = z_n \text{ where } z_{j+1} = T_j(z_j, \pi_j(y)) \text{ or } z_{j+1} = T_j(z_j) \text{ and } z_0 = x \in X_0$$

Definition 3.15. Gradient Descent

Let $E : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable operator.

The method of Gradient Descent says that a local minimum of E can be found using the following iteration:

$$a_{n+1} = a_n - \gamma \nabla E(a_n)$$

Where $\gamma > 0$

Example 3.1. Objective Operator for Data Set Defined Operator Approximation
Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ such that $X \times Y$ defines an operator T .

$$E(a) = \sum_{x \in X} ||T(x) - T^n(x, a)||$$

4 Surjective Continuous Non-decreasing Bounded Functionals

Let $B = \{f : \mathbb{R} \rightarrow [0, 1] \mid f \text{ is surjective, continuous, and non-decreasing.}\}$

Theorem 4.1. B is convex.

Let $f, g \in B$ and $h(x) := \lambda f(x) + (1 - \lambda)g(x)$ where $\lambda \in [0, 1]$

Then h is still continuous since the linear combination of continuous functions is continuous.

Since both f and g are surjective and non-decreasing, then there exists x_0, y_0, x_1, y_1 in \mathbb{R} such that:

$f(x_0) = 0 = g(y_0)$ and $f(x_1) = 1 = g(y_1)$

Suppose WLOG that $x_0 \leq y_0$ and $x_1 \leq y_1$

Then we know that:

$$h(x_0) = \lambda f(x_0) + (1 - \lambda)g(x_0) = \lambda 0 + (1 - \lambda)0 = 0$$

and

$$h(y_1) = \lambda f(y_1) + (1 - \lambda)g(y_1) = \lambda 1 + (1 - \lambda)1 = 1$$

Now if we pick $\alpha \in [0, 1]$ by the intermediate value theorem, we know that there exists an $x_\alpha \in [x_0, y_1]$ such that:

$$h(x_\alpha) = \alpha$$

Since α was arbitrary element, I have shown that h is surjective.

Finally, let $x_0 < x_1$ be elements in \mathbb{R}

Then we know that $f(x_0) \leq f(x_1)$ and $g(x_0) \leq g(x_1)$

$\Rightarrow \lambda f(x_0) \leq \lambda f(x_1)$ and $(1 - \lambda)g(x_0) \leq (1 - \lambda)g(x_1)$

$\Rightarrow \lambda f(x_0) + (1 - \lambda)g(x_0) \leq \lambda f(x_1) + (1 - \lambda)g(x_1)$

$\Rightarrow h(x_0) \leq h(x_1)$

Thus h is non-decreasing.

Since h is surjective, continuous, and non-decreasing, then $h \in B$

Thus B is convex.

Theorem 4.2. B is translation invariant.

Let $f \in B$ and $g(x) := f(x + c)$ where $c \in \mathbb{R}$

f is continuous and so is the addition operator so g is continuous.

Let $\alpha \in [0, 1]$ since f is surjective then $\exists x \in \mathbb{R} \cap f(x) = \alpha$

Then $g(x - c) = f(x + c - c) = f(x) = \alpha$ and so g is surjective.

Let $x < y$ be elements in \mathbb{R}

Then $f(x) \leq f(y) \Rightarrow f(x + c) \leq f(y + c)$

$\Rightarrow g(x) \leq g(y)$ and so g is non-decreasing.

Thus $g \in B$ and B is therefore translation invariant.

Theorem 4.3. B is not complete.

Theorem 4.4. Every element in B can be decomposed as a finite non-trivial convex combination from B

5 K-Operators

Let $\hat{f} : X \rightarrow Y$ where X, Y are topological spaces.

Let $V \subset X$ and $W \subset Y$ where $\text{card}(V) < \text{card}(\mathbb{N})$ and $\text{card}(W) < \text{card}(\mathbb{N})$ Then define $f : V \rightarrow W$ where the associated topological spaces are:

$$\tau_V = \{V \cap U : U \in \tau_X\} \text{ and } \tau_W = \{W \cap U : U \in \tau_Y\}$$