Notes On Rudin Functional Analysis

Daniel Drake

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1 Chapter 1, Topological Vector Spaces

Definition 1.1. ***Norms and Normed Spaces***
Let X be a non-empty vector space and $||\cdot|| : X \to \mathbb{R}_0^+$ such that:

- $(\forall x, y \in X)(||x+y|| \leq ||x|| + ||y||)$
- $(\forall x \in X)(\forall \alpha \in \mathbb{R})(||\alpha x|| = |\alpha|||x||)$
- $x \neq \vec{0} \Rightarrow ||x|| > 0$

We then call the ordered pair, $(X, ||\cdot||)$ a Normed space.

Theorem 1.1. ***Every Norm space is also a metric space.***
Let $(X, ||\cdot||)$ be a normed spaced.
Define: $(d(x,y) = ||x-y||)(\forall x, y \in X)$
Then: (X, d) is a metric space.

Proof:
Proving this just comes down to showing that d is a metric.

First, the identity of indiscernibles.
Let $(x, y \in X)(x = y)(d(x,y) = ||x-y|| = ||\vec{0}|| = ||0\vec{0}|| = |0|||\vec{0}|| = 0)$
Next let $(x, y \in X)(d(x,y) = 0 \Rightarrow 0 = ||x-y|| \Rightarrow x - y = \vec{0} \Rightarrow x = y)$

Next, the symmetry property.
Let $x, y \in X$
$d(x,y) = ||x-y|| = |-1|||x-y|| = ||-1(x-y)|| = ||-x+y|| = ||y-x|| = d(y,x)$

Finally, the triangle inequality.
Let $x, y, z \in X$
Then $d(x,y) = ||x-y|| = ||x+z-z-y|| = ||(x-z)+(z-y)|| \leq ||x-z|| + ||z-y|| = d(x,z) + d(z,y)$

Definition 1.2. ***Balls***
Let (X, d) be a metric space, and $B : X \times \mathbb{R}^+ \to 2^X$ such that:

$$B(x,r) = \{z \in X : d(x,z) < r\}$$

Then $B(x,r)$ is called an open ball centered at x with radius r.

Next let $\bar{B} : X \times \mathbb{R}^+ \to 2^X$ such that:

$$\bar{B}(x,r) = \{z \in X : d(x,z) \leq r\}$$

Then $\bar{B}(x, r)$ is called the closed ball centered at x with radius r.

Finally: $B(\vec{0}, 1)$ and $\bar{B}(\vec{0}, 1)$ are called the open and closed unit balls.

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