

Analysis, Topology, Optimization, Machine Learning, and Computational Analysis

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1 Notation, Set Theory, and Logic

Definition 1.1. Sets, Set Builder Notation and Hats

A set is a collection of objects.

Example: $A = \text{The set of all hats.}$

We call the objects in the set elements.

Example: A Cowboy hat is a type of hat and thus belongs in the set of all hats: A .

Set builder notation is a way of describing a set using mathematical, logical symbols, or words. Look at the following example:

$$E = \{x \in \mathbb{N} : x = 2n \text{ where } n \in \mathbb{N}\}$$

This reads: E is the set of all x in \mathbb{N} such that $x = 2n$ where n is in \mathbb{N}

This set is also known as the even numbers.

When talking about functions, another common way of describing a set is:

$$C_X^Y = \{f : X \rightarrow Y | f \text{ is continuous}\}$$

This reads: C_X^Y is the set of all functions f mapping from X to Y such that f is continuous.

Definition 1.2. Common Sets of Numbers

$$\mathbb{N} = \{1, 2, \dots\}$$

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}$$

$$\mathbb{N}_m = \{1, 2, \dots, m\} \text{ where } m \in \mathbb{N}$$

\mathbb{R} is the set of Real Numbers

$$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$$

$$\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$$

$$\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$$

$$\mathbb{R}_0^- = \{x \in \mathbb{R} : x \leq 0\}$$

Definition 1.3. Primitives, Symbols, and Formulas

The logical or and not are both primitives and are written:

logical or: \vee

and

not : \neg

Respectively.

The symbol for statements are written: L, M, N, O, P, Q, \dots etc

A well formed formula is a sequence of symbols which is accepted by the formal grammar .

So then $L \vee M$ is a new formula composed of L, \vee , and M .

$$N := L \vee M$$

Which is read: N is defined as L or M .

So N is true if:

- L is true.
- M is true.
- L and M are both true.

Further, N is false if: L and M are both false.

Similarly, we can define a new statement: $N := \neg M$

In this case, if M is true then N is false.

In the same vain, if M is false then N is true.

Definition 1.4. And

Let A, B be statements.

$$A \wedge B := \neg((\neg A) \vee (\neg B))$$

Definition 1.5. Intersection and Union

Let A, B be sets.

$$A \cap B := \{x : x \in A \wedge x \in B\}$$

$$A \cup B := \{x : x \in A \vee x \in B\}$$

Definition 1.6. Compliment

Let A be a set.

Then: $x \notin A := \neg(x \in A)$ and $A^c := \{x : x \notin A\}$

Definition 1.7. Logical Implication and Equivalence

Let A, B be statements:

$$A \Rightarrow B := (\neg A) \vee B$$

Further:

$$A \Leftrightarrow B := (A \Rightarrow B) \wedge (B \Rightarrow A)$$

Definition 1.8. Theory Crafting

A **mathematical theory** is a collection of well formed statements.

We write $\vdash P$ if and only if a statement P is said to be a part of our theory M

Axiom 1.1. Propositional Logic

Let A, B and C be a statements.

Then we have the following axioms:

- $\vdash A \vee A \Rightarrow A$
- $\vdash A \Rightarrow A \vee B$
- $\vdash A \vee B \Rightarrow B \vee A$
- $\vdash (A \Rightarrow C) \Rightarrow (A \vee B \Rightarrow B \vee C)$

These statements I am taking as the initial part of our theory M .

Rule Of Inference 1.1. Principle of Substitution

If we have a formula P which depends on the statement Q , then we may replace instances of Q with a formula R as long as Q is not dependent on some assumption earlier in the proof.

Rule Of Inference 1.2. Logical Deduction

Let P and Q be statements.

First, we suppose that $\boxed{\vdash} P$.

If we can then show that $\boxed{\vdash} Q$,

then it is proven that $\vdash P \Rightarrow Q$

Note, this does not mean that either $\vdash P$ or $\vdash Q$

To show this, instead of writing: \vdash we will write $\boxed{\vdash}$ meaning that there is a dependence on an assumption.

Rule Of Inference 1.3. Modus Ponens

Let A, B be statements.

If $\vdash A \Rightarrow B$ and $\vdash A$ then $\vdash B$

This can be written as a single theorem:

$[(A \Rightarrow B) \wedge A] \Rightarrow B$

Note that you can't use the Principle of Substitution on A after you use Modus Ponens.

Theorem 1.1. First Theorem

Let A be a statement.

Then $A \Rightarrow A \vee A$

Proof:

We know that:

$\vdash A \Rightarrow A \vee B$

[Axiom 2]

By replacing B with A we then have:

$\vdash A \Rightarrow A \vee A$

[Principle of Substitution.]

Theorem 1.2. Second Theorem

Let A be a statement.

Then $A \Leftrightarrow A \vee A$

Proof:

Since $\vdash A \Rightarrow A \vee A$ and $\vdash A \vee A \Rightarrow A$

[Axiom 1 and the first theorem]

And so we know:

$\vdash A \Leftrightarrow A \vee A$

[Definition of Equivalence]

Theorem 1.3. The Commutative Property of OR

Let A, B be statements.

Then: $A \vee B \Leftrightarrow B \vee A$

Proof:

By Axiom 3 we know:

$\vdash A \vee B \Rightarrow B \vee A$

[Axiom 3]

We can then Substitute B for C :

$\vdash A \vee C \Rightarrow C \vee A$

[Principle of Substitution]

Next we substitute A for B :

$\vdash B \vee C \Rightarrow C \vee B$

[Principle of Substitution]

Finally we substitute C for A :

$\vdash B \vee A \Rightarrow A \vee B$

[Principle of Substitution]

And so by definition of equivalence we have that:

$\vdash A \vee B \Leftrightarrow B \vee A$

[Definition of Equivalence]

Theorem 1.4. Shakespeare's theorem

Let B be a statement.

Then $B \vee \neg B$

Proof:

We have the Axiom:

$\vdash A \Rightarrow A \vee B$

[Axiom 2]

We can then replace A with B and we have:

$\vdash B \Rightarrow B \vee B$

[Principle of Substitution.]

$\vdash \neg B \vee (B \vee B)$

[Def. of Implication]

$\vdash \neg B \vee B$

[Second Theorem]

By the commutative property of OR we can then have that:

$\vdash B \vee \neg B$

[Commutative Property of Or]

Theorem 1.5. The Tautology Theorem

Let A be a statement.

Then: $A \Rightarrow A$ and further $A \Leftrightarrow A$

Proof:

Let B be a statement.

From the proof Shakespeare's theorem we know that:

$\vdash \neg B \vee B$

[Proof of Shakespeare's theorem.]

Replace B with A .

$\vdash \neg A \vee A$

[Principle of Substitution.]

By the definition of Implication we have that:

$\vdash A \Rightarrow A$

[Def. of Logical Implication]

Finally by the definition of equivalence we have that:

$\vdash A \Leftrightarrow A$

[Def. of Equivalence]

Theorem 1.6. The Double Negative Theorem

Let A be a statement.

Then $\neg(\neg A) \Leftrightarrow A$

Proof:

From Shakespeare's theorem we have:

$\vdash B \vee \neg B$

[Shakespeare's Theorem]

Replace B with $\neg A$

$\vdash \neg A \vee \neg(\neg A)$

[Principle of Substitution]

Then by the definition of implication we have:

$\vdash A \Rightarrow \neg(\neg A)$

[Def. of Logical Implication]

For the next proof we will need the following, substitute A with $\neg A$

$\vdash \neg A \Rightarrow \neg(\neg(\neg A))$

[Principle of Substitution] *1

Now for the other direction, start with axiom 4:

$\vdash (A \Rightarrow C) \Rightarrow (A \vee B \Rightarrow B \vee C)$

[Axiom 4]

Substitute C with $\neg(\neg A)$

$\vdash (A \Rightarrow \neg(\neg A)) \Rightarrow (A \vee B \Rightarrow B \vee \neg(\neg A))$

[Principle of Substitution]

Substitute A with $\neg A$

$\vdash (\neg A \Rightarrow \neg(\neg(\neg A))) \Rightarrow (\neg A \vee B \Rightarrow B \vee \neg(\neg(\neg A)))$

[Principle of Substitution]

Substitute B with A

$\vdash (\neg A \Rightarrow \neg(\neg(\neg A))) \Rightarrow (\neg A \vee A \Rightarrow A \vee \neg(\neg(\neg A)))$	[Principle of Substitution]
We have that $\vdash (\neg A \Rightarrow \neg(\neg(\neg A)))$ from the last part:	
$\vdash \neg A \vee A \Rightarrow A \vee \neg(\neg(\neg A))$	[Modus Ponens knowing: *1]
We have Shakespeare's theorem and so:	
$\vdash A \vee \neg(\neg(\neg A))$	[Modus Ponens knowing: $\vdash \neg A \vee A$]
We know that OR is commutative and so:	
$\vdash \neg(\neg(\neg A)) \vee A$	[The Commutative Property of OR]
By the definition of implication we know:	
$\vdash \neg(\neg A) \Rightarrow A$	[Def. of Logical Implication.]
And finally by the definition of equivalence we know:	
$\vdash \neg(\neg A) \Leftrightarrow A$	[Def. of Logical Equivalence.]

Theorem 1.7. Contrapositive

Let A, B be statements.

$(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$ and $(A \Leftrightarrow B) \Leftrightarrow (\neg A \Leftrightarrow \neg B)$

Theorem 1.8. Captain Morgan's Laws for Logic

Let A, B be statements.

Then: $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$ and $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$

Proof:

Using the Deductive method we start with the assumption that:

$\vdash \neg(A \wedge B)$	[Deductive Method]
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By the definition of and we know the following are equivalent.

$\vdash \neg(\neg(\neg A \vee \neg B))$	[Def. And]
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By the previous theorem we know that:

$\vdash \neg A \vee \neg B$	[Double Negative Theorem]
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Then the Deductive Method tells us that:

$\vdash \neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$	[Deductive Method Conclusion]
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And so we are done with the first part.

Using the Deductive Method again, we start with the assumption that:

$\vdash \neg A \wedge \neg B$	[Deductive Method]
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By definition of and, we know that:

$\vdash \neg(\neg(\neg A) \vee \neg(\neg B))$	[Def. of And]
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By the double negative theorem we know that:

$\vdash \neg(A \vee B)$	[Double Negative Theorem]
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And so by the Deductive Method we know:

$\vdash \neg A \wedge \neg B \Leftrightarrow \neg(A \vee B)$	[Conclusion of the Deductive Method]
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Which gives us both statements.

Theorem 1.9. Shakespeare's Contradiction

Let B be a statement and define: $\perp := B \wedge \neg B$

Then $\neg \perp$

Start with Shakespeare's theorem.

$$\vdash B \vee \neg B$$

$$\vdash \neg\neg(B \vee \neg B)$$

$$\vdash \neg(\neg B \wedge \neg\neg B)$$

$$\vdash \neg(\neg B \wedge B)$$

$$\vdash \neg(B \wedge \neg B)$$

$$\vdash \neg\perp$$

And so it's not the case that contradictions are part of our theory.

Theorem 1.10. Logical Selection

Let A, B be statements.

$$\vdash A \Rightarrow A \vee B$$

$$\vdash \neg A \Rightarrow \neg A \vee B$$

$$\vdash \neg A \Rightarrow \neg A \vee \neg B$$

$$\vdash \neg A \Rightarrow \neg(A \wedge B)$$

$$\vdash \neg\neg A \vee \neg(A \wedge B)$$

$$\vdash A \vee \neg(A \wedge B)$$

$$\vdash \neg(A \wedge B) \vee A$$

$$\vdash (A \wedge B) \Rightarrow A$$

Theorem 1.11. Modus Ponens

Let A, B be statements.

Then: $[(A \Rightarrow B) \wedge A] \Rightarrow B$

$$\boxed{\vdash} (A \Rightarrow B) \wedge A$$

$$\boxed{\vdash} (\neg A \vee B) \wedge A$$

$$\boxed{\vdash} \neg(\neg(\neg A \vee B) \vee \neg A)$$

Theorem 1.12. Logical Reduction

Let A, B be statements.

$$[(A \vee B) \wedge \neg B] \Rightarrow A$$

From Modus Ponens we have:

$$\vdash [(A \Rightarrow B) \wedge A] \Rightarrow B$$

$$\vdash [(\neg A \vee B) \wedge A] \Rightarrow B$$

$$\vdash [(\neg C \vee B) \wedge C] \Rightarrow B$$

$$\vdash [(\neg C \vee A) \wedge C] \Rightarrow A$$

$$\vdash [(\neg B \vee A) \wedge B] \Rightarrow A$$

$$\vdash [(A \vee \neg B) \wedge B] \Rightarrow A$$

$$\vdash [(A \vee \neg\neg B) \wedge \neg B] \Rightarrow A$$

$$\vdash [(A \vee B) \wedge \neg B] \Rightarrow A$$

Rule Of Inference 1.4. *Contradiction*

Let P be a statement.

If $P \Rightarrow \perp$ then $\neg P$

Proof:

Suppose $\boxed{\vdash} P \Rightarrow \perp$

[Deductive Method]

$\boxed{\vdash} \neg P \vee \perp$

[Def. Implication]

$\boxed{\vdash} \neg P$

[We know $\vdash \neg \perp$]

$\therefore \vdash [(P \Rightarrow \perp) \Rightarrow (\neg P)]$

By this theorem we have an extension of the deductive method and our new rule of Inference:

Method of Contradiction:

If we assume P

And then we can deduce \perp

Then we know $\neg P$

Theorem 1.13. *Principle of Explosion*

Let A be a statement.

Definition 1.9. *Subsets and Set Equality*

Let A, B be sets.

$(A \subset B) := (x \in A \Rightarrow x \in B)$

$(A = B) := (A \subset B) \wedge (B \subset A)$

Theorem 1.14. *De Morgan's laws*

Let A, B be sets.

Then: $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$

Proof:

Let $x \in (A \cap B)^c$

$\Leftrightarrow x \notin A \cap B$

[Def. Compliment]

$\Leftrightarrow \neg(x \in A \cap B)$

[Def. \notin]

$\Leftrightarrow \neg(x \in A \wedge x \in B)$

[Def. Intersection]

$\Leftrightarrow \neg(x \in A) \vee \neg(x \in B)$

[Captain Morgan's laws for logic]

$\Leftrightarrow x \notin A \vee x \notin B$

[Def. \notin]

$\Leftrightarrow x \in A^c \vee x \in B^c$

[Def. Compliment]

$\Leftrightarrow x \in A^c \cup B^c$

[Def. Union]

$\therefore (A \cap B)^c = A^c \cup B^c$

[Def. Subset and Set Equality]

Let $x \in (A \cup B)^c$

$\Leftrightarrow x \notin A \cup B$

$\Leftrightarrow \neg(x \in A \cup B)$

$\Leftrightarrow \neg(x \in A \wedge x \in B)$

$\Leftrightarrow \neg(x \in A) \vee \neg(x \in B)$

$\Leftrightarrow x \notin A \vee x \notin B$

$$\begin{aligned} &\Leftrightarrow x \in A^c \vee x \in B^c \\ &\Leftrightarrow x \in A^c \cup B^c \\ &\therefore (A \cup B)^c = A^c \cap B^c \end{aligned}$$

Theorem 1.15. *The union only makes things larger*

Let A, B be sets.

Then: $A \subset A \cup B$

Proof:

Let $x \in A$ and $x \in B$ be statements.

Then by Axiom 2 we know:

$$x \in A \Rightarrow x \in A \vee x \in B$$

[Axiom 2 where the statements are $x \in A$ and $x \in B$]

Then by definition of union we know:

$$x \in A \Rightarrow x \in A \cup B$$

[Def. Union]

And so by definition of subsets we know:

$$A \subset A \cup B$$

[Def. of Subsets]

Theorem 1.16. *Union and Intersection Distributive Properties*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof:

$$\text{Let } x \in A \cap (B \cup C) \Rightarrow x \in A \wedge x \in B \cup C$$

Suppose that $x \notin C$

$$\Rightarrow x \in B$$

$$\Rightarrow x \in A \wedge x \in B$$

$$\Rightarrow x \in A \cap B$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C)$$

$$[(A \cap B) \subset (A \cap B) \cup (A \cap C)]$$

Suppose that $x \notin B$

$$\Rightarrow x \in C$$

$$\Rightarrow x \in A \text{ and } x \in C$$

$$\Rightarrow x \in A \cap C$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C)$$

$$[(A \cap C) \subset (A \cap B) \cup (A \cap C)]$$

Therefore:

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$$

Axiom 1.2. *Peano's Axioms*

- $1 \in \mathbb{N}$
- $(\forall n \in \mathbb{N})(\exists n' \in \mathbb{N})$

Definition 1.10. Power Set

Let $X \neq \phi$

$$2^X := \{V : V \subseteq X\}$$

Definition 1.11. Injection, Surjection, and Bijection

Let A, B be sets and function $f : A \rightarrow B$. f is said to be an Injection if:

$$(\forall x, y \in A)(f(x) = f(y) \Rightarrow x = y)$$

f is said to be a Surjection if:

$$(\forall y \in B)(\exists x \in A)(f(x) = y)$$

f is said to be a Bijection if it is both an Injection and a Surjection.

Definition 1.12. Carnality

Let A and B be sets.

Then $\text{card}(A) = \text{card}(B) \Leftrightarrow \exists T : A \rightarrow B$ where T is a bijection.

Further, if $\text{card}(A) = n \Leftrightarrow \exists T : A \rightarrow \mathbb{N}_n$ where T is a bijection.

And finally, if $\exists T : A \rightarrow B$ where T is a surjection, then $\text{card}(A) \geq \text{card}(B)$

Theorem 1.17. It's pretty big.

Let $X \neq \phi$ and $M = \text{card}(X) < \text{card}(\mathbb{N})$

Then: $\text{card}(2^X) = 2^M$

Theorem 1.18. It's always bigger

Let $X \neq \phi$ then: $\text{card}(X) \leq \text{card}(2^X)$

Definition 1.13. Aleph null

$$\aleph_0 := \text{card}(\mathbb{N})$$

Definition 1.14. Aleph one

$$\aleph_1 := \text{card}(2^{\mathbb{N}})$$

Definition 1.15. How big are the reals?

$$\text{card}(\mathbb{R}) = \aleph_1$$

Definition 1.16. K - Combinations

Let S be a non empty finite set where $n = \text{card}(S)$

A K -combination of S is a subset: $K \subset S$ where $\text{card}(K) = k$

We then have the collection: $C(S, k) = \{K \subset S : \text{card}(K) = k\}$

The number of K -combinations $= \text{card}(C(S, k)) = \binom{n}{k}$

Definition 1.17. Equivalence Relations

Let $S \neq \phi$

Then \cong is called an Equivalence Relation if:

- $(\forall a \in S)(a \cong a)$ [Reflexive]
- $(\forall a, b \in S)(a \cong b \Leftrightarrow b \cong a)$ [Symmetric]
- $(\forall a, b, c \in S)(a \cong b \wedge b \cong c \Rightarrow a \cong c)$ [Transitive]

Reference

Definition 1.18. Equivalence Class

Let $S \neq \phi$ and $a \in S$ and \cong be an equivalence relation on S .

Then the equivalence class $[a]$ is defined as follows:

$$[a] := \{x \in S : x \cong a\}$$

2 Topology

Definition 2.1. Topology

Let $X \neq \phi$

Further let $\tau \subseteq 2^X$ such that:

$$\phi, X \in \tau$$

$$(\forall A \neq \phi) \left(\left\{ U_\alpha \right\}_{\alpha \in A} \subseteq \tau \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \tau \right)$$

$$(\forall m \in \mathbb{N}) \left(\left\{ U_j \right\}_{j \in \mathbb{N}_m} \Rightarrow \bigcap_{j=1^m} U_j \in \tau \right)$$

Definition 2.2. Relative Topology

Let $X \neq \phi$ and $Z \subset X$

Then the relative topology on Z is written as follows:

$$\tau_Z = \{Z \cap U : U \in \tau_X\}$$

Theorem 2.1. The Relative Topology is a Topology on Z

Let $E \in \tau_Z$

$$\Rightarrow E = Z \cap U \subset Z$$

$$\Rightarrow \tau_Z \subseteq 2^Z$$

And so we have met the first criteria.

Next:

$$\phi \in \tau \Rightarrow Z \cap \phi \in \tau_Z \Rightarrow Z \cap \phi = \phi \in \tau_Z$$

Next:

$$X \in \tau \Rightarrow Z \cap X \in \tau_Z \Rightarrow Z \cap X = Z \in \tau_Z$$

Next: Let $A \neq \phi$ and $\{U_\alpha\}_{\alpha \in A} \in \tau_Z$

$$\Rightarrow \exists \{V_\alpha\}_{\alpha \in A} \subset \tau \text{ such that: } U_\alpha = Z \cap V_\alpha$$

$$\Rightarrow \bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} Z \cap V_\alpha$$

3 Change

Definition 3.1. Metric

Let X be a non-empty set..

Let $d : X \times X \rightarrow \mathbb{R}_0^+$ such that:

- $(\forall x, y \in X) d(x, y) = 0 \Leftrightarrow x = y$
- $(\forall x, y \in X) d(x, y) = d(y, x)$
- $(\forall x, y, z \in X) d(x, z) \leq d(x, y) + d(y, z)$

Then d is called a metric and (X, d) is called a metric space.

Reference

Definition 3.2. *Limit of a function*

Let $T : X \rightarrow Y$ where (X, d_X) and (Y, d_Y) are metric spaces.

Then fix $x_0 \in X$.

If:

$$(\exists L \in Y)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in X)(d(x, x_0) < \delta \Rightarrow d(f(x), L) < \epsilon)$$

Then:

$$\lim_{x \rightarrow x_0} f(x) = L$$

Reference

Definition 3.3. *Derivative*

Let $\hat{f} : \mathbb{R} \rightarrow \mathbb{R}$

Further let $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}}$

Then f is said to be differentiable at $x \in U$ if there exists an L_x such that:

$$L_x = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If L_x exists for all $x \in U$ then we write:

$$\frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Reference

Theorem 3.1. *Fundamental increment lemma*

Let f be described as above and be differentiable at x .

Then there exists a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that:

$$f(x+h) = f(x) + \frac{d}{dx}f(x)h + \phi(x)h$$

and

$$\lim_{h \rightarrow 0} \phi(h) = 0$$

Proof:

Define: $\phi(h) = \frac{f(x+h)-f(x)}{h} - \frac{d}{dx}f(x)$

Then: $\phi(h)h = f(x+h) - f(x) - \frac{d}{dx}f(x)h$

Then: $\phi(h)h + f(x) - \frac{d}{dx}f(x)h = f(x+h)$

And so we have property 1.

Next:

$$\begin{aligned} \lim_{h \rightarrow 0} \phi(h) &= \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x) - \frac{d}{dx}f(x)h}{h} \right] = \lim_{h \rightarrow 0} \left[\frac{f(x+h) - f(x)}{h} - \frac{d}{dx}f(x) \right] \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{d}{dx}f(x) = \frac{d}{dx}f(x) - \frac{d}{dx}f(x) = 0 \end{aligned}$$

Definition 3.4. Partial Derivative

Let $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$

Further let $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}^n}$

Then f is said to be differentiable at $x \in U$ with respect to the i 'th component of x if there exists an L_{x_i} such that:

$$L_{x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

If L_{x_i} exists for all $x \in U$ then we write:

$$\frac{\partial}{\partial x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

Reference

Theorem 3.2. Equivalent characterization

Let $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$

Further let $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}^n}$

And let f be differentiable at $x \in U$ with respect to the i 'th component of x , then:

$$\begin{aligned} L_{x_i} &= \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[\frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} - L_{x_i} \right] \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[\frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} - \frac{L_{x_i} \cdot h}{h} \right] \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[\frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n) - \langle L_{x_i}, h \rangle}{h} \right] \end{aligned}$$

Definition 3.5. Gradient

Let $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ and let $f : U \rightarrow \mathbb{R}$ such that $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}^n}$

f is said to be differentiable at $x \in U$ if $\exists \nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that:

$$\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - \langle \nabla f(x), h \rangle|}{\|h\|} = 0$$

Theorem 3.3. Form of the Gradient

Let f be defined as above.

Then $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ where:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix} \quad \forall x \in \mathbb{R}^n$$

is the form of ∇f which satisfies the above statement if f is differentiable.

Reference

Proof:

Suppose ∇f is defined as above and all the partial derivatives exist.

Then:

$$\frac{1}{||h||} |f(x+h) - f(x) - \langle \nabla f(x), h \rangle| = \frac{1}{||h||} \left| f(x+h) - f(x) - \sum_{j=1}^n \frac{\partial}{\partial x_j} f(x) \cdot h_j \right|$$

Definition 3.6. Matrix Functional Differentiability

Let $\hat{T} : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$ and let $T : U \rightarrow \mathbb{R}$ such that $T = \hat{T}|_U$ where $U \in \tau_{\mathbb{R}^{n \times m}}$

T is said to be differentiable at $x \in U$ if $\exists D : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n \times m}$ such that:

$$\lim_{h \rightarrow 0} \frac{|T(x+h) - T(x) - \langle DT(x), h \rangle|}{||h||} = 0$$

where $\langle \cdot, \cdot \rangle$ is an inner product defined on $\mathbb{R}^{n \times m}$

Definition 3.7. Frobenius inner product

The Frobenius inner product is defined as:

$$\langle \cdot, \cdot \rangle_{FB} : \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \rightarrow \mathbb{R} \text{ such that: } \langle A, B \rangle_{FB} = \sum_{i=1}^n \sum_{j=1}^m a_{i,j} b_{i,j} \text{ for all } A, B \in \mathbb{R}^{n \times m}$$

Theorem 3.4. Form of Matrix Functional Derivative

$$DT(x) = \begin{bmatrix} \frac{\partial}{\partial x_{1,1}} & \cdots & \frac{\partial}{\partial x_{1,m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_{n,1}} & \cdots & \frac{\partial}{\partial x_{n,m}} \end{bmatrix}$$

Definition 3.8. Differentiability of a multi-variable function.

Let $\hat{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that:

$$\hat{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \text{ and } (\forall j \in \mathbb{N}_n)(f_j : \mathbb{R}^m \rightarrow \mathbb{R})$$

Further let $f = \hat{f}|_U$ where $U \in \tau_{\mathbb{R}^m}$

Then f is said to be differentiable at $x \in U$ if there exists a linear operator $J_{f(x)} : \mathbb{R}^m \rightarrow \mathbb{R}^n$ such that:

$$\lim_{h \rightarrow \vec{0}} \frac{\|f(x+h) - f(x) + J_{f(x)}(h)\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^m}} = 0$$

Reference

Theorem 3.5. The Jacobian matrix linear operator for the above system.

So our guess is that:

$$J_{f(x)} = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x) & \cdots & \frac{\partial}{\partial x_n} f_1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_m(x) & \cdots & \frac{\partial}{\partial x_n} f_m(x) \end{bmatrix}$$

since this form is a linear operator mapping from the appropriate space to the appropriate space. It should be noted that the transpose of this matrix can not satisfy the definition of differentiability of a multi-variable function and so it is not the correct linear operator.

Definition 3.9. Matrix operator differentiability

Let $T : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$ such that:

$$T(A) = \begin{bmatrix} T_1(A) \\ \vdots \\ T_n(A) \end{bmatrix} \quad \forall A \in \mathbb{R}^{n \times m} \text{ and } (\forall j \in \mathbb{N}_n)(T_j : \mathbb{R}^{n \times m} \rightarrow \mathbb{R})$$

Then T is said to be differentiable at $A \in \mathbb{R}^{n \times m}$ if there exists a linear operator $D : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$ where:

$$\lim_{h \rightarrow 0} \frac{\|T(A+h) - T(A) + D(h)\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^{n \times m}}} = 0$$

If D exists then it is called the Matrix operator derivative and is written: $D_{\mathbb{R}^{n \times m}} T(A)$

Theorem 3.6. The form of the Matrix operator derivative.

Let T be described as above and differentiable at $A \in \mathbb{R}^{n \times m}$

$$\frac{T(A+h) - T(A)}{\|h\|} = \begin{bmatrix} \frac{T_1(A+h) - T_1(A)}{\|h\|} \\ \vdots \\ \frac{T_n(A+h) - T_n(A)}{\|h\|} \end{bmatrix}$$

and so:

$$\lim_{h \rightarrow 0} \frac{T(A+h) - T(A)}{\|h\|} = \begin{bmatrix} \lim_{h \rightarrow 0} \frac{T_1(A+h) - T_1(A)}{\|h\|} \\ \vdots \\ \lim_{h \rightarrow 0} \frac{T_n(A+h) - T_n(A)}{\|h\|} \end{bmatrix}$$

Definition 3.10. Subspace Differentiability

Let $X = \{X_j\}_{j=1}^n$ be a sequence of finite dimensional vector spaces where $\dim(X_j) = k_j = m_j \times n_j$

Let $T : \prod_{j=1}^n X_j \rightarrow Y$ where Y is a finite dimensional vector space with $\dim(Y) = k_y$

Let $x_j \in X_j$ for some $j \in \mathbb{N}_n$

Where

$$x_j = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n_j} \\ \vdots & \ddots & \vdots \\ x_{m_j,1} & \cdots & x_{m_j,n_j} \end{bmatrix}$$

T is said to be differentiable at $x \in X$ where $x = (x_0, \dots, x_j, \dots, x_{n-1})$ with respect to X_j if there exists a linear operator $D : X_j \rightarrow Y$:

Given $h \in X_j \setminus \{\vec{0}\}$ define $\hat{h} = (0, \dots, h, \dots, 0) \in X$ where h is in the j 'th place of \hat{h} :

$$\lim_{h \rightarrow 0} \frac{\|T(x + \hat{h}) - T(x) + D(h)\|_Y}{\|h\|_{X_j}} = 0$$

Then D is called the subspace derivative of T at x with respect to X_j and is written: $D_{x_j}T(x)$

Definition 3.11. Product space Derivative

Let $X = \{X_j\}_{j=0}^{n-1}$ be a sequence of finite dimensional vector spaces where $\dim(X_j) = k_j$

Let $T : \prod_{j=0}^{n-1} X_j \rightarrow Y$ where Y is a finite dimensional vector space with $\dim(Y) = k_y$

Let $\{x_j\}_{j=0}^{n-1}$ be a sequence of vectors such that: $(\forall j \in \{0, \dots, n-1\})(x_j \in X_j)$

The product space derivative at the point $z \in X$ is:

$$D_X T(z) = \begin{bmatrix} D_{x_0} T(z) \\ \vdots \\ D_{x_{n-1}} T(z) \end{bmatrix}$$

Definition 3.12. Fréchet derivative

Let V, W be normed vector spaces and $U \subset V$ be an open set.

An operator $f : U \rightarrow W$ is said to be Fréchet differentiable if there exists a bounded linear operator $A : V \rightarrow W$ such that:

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0$$

Reference

Theorem 3.7. Fréchet derivative of a bounded linear operator

Let V, W be normed vector spaces and $U \subset V$ be an open set.

Let $\hat{f} : V \rightarrow W$ be a bounded linear operator.

Then lets look at $f = \hat{f}|_U$

My guess is that $A = \hat{f}$

Let $x \in U$ and $h \in U \cap \{h \mid \|h\| \neq 0 \text{ and } x+h \in U\}$, Then:

$$\frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = \frac{\|f(x) + f(h) - f(x) + \hat{f}(h)\|_W}{\|h\|_V} = \frac{\|f(x) + f(h) - f(x) + f(h)\|_W}{\|h\|_V} = 0$$

Thus let $\epsilon > 0$ and $\delta > 0$

Then if $0 < \|h\| < \delta$ we know that $\frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0 < \epsilon$

Therefore:

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0$$

Thus $A = \hat{f}$ is the Fréchet derivative of f .

3.1 Finite Composition Operator

Definition 3.13. *Finite Composition Operator*

Let the collection $X = \{X_j\}_{j=0}^n$ be a finite sequence of sets.

Further let $\{T_j\}_{j=0}^{n-1}$ be a finite sequence of operators such that $(\forall j \in \mathbb{N}_{n-1})(T_j : X_j \rightarrow X_{j+1})$

Then $T^n : X_0 \rightarrow X_n$ defined by:

$$T^n := \bigcirc_{j=0}^{n-1} T_j$$

is called the **Finite Composition Operator defined on X** .

Definition 3.14. *Multi-variable Finite Composition Iteration*

Let the collection $X = \{X_j\}_{j=0}^n$ and $Y = \{Y_j\}_{j=0}^{n-1}$ be finite sequences of sets.

Further let $\{T_j\}_{j=0}^{n-1}$ be a finite sequence of operators such that: $(\forall j \in \mathbb{N}_{n-1})(T_j : X_j \times Y_j \rightarrow X_{j+1})$

Let $T^n : X_0 \times \prod_{j=0}^{n-1} Y_j \rightarrow X_n$ where:

$$T^n(x, y) = z_n \text{ where } z_{j+1} = T_j(z_j, \pi_j(y)) \text{ or } z_{j+1} = T_j(z_j) \text{ and } z_0 = x \in X_0$$

Definition 3.15. *Gradient Descent*

Let $E : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable operator.

The method of Gradient Descent says that a local minimum of E can be found using the following iteration:

$$a_{n+1} = a_n - \gamma \nabla E(a_n)$$

Where $\gamma > 0$

Example 3.1. Objective Operator for Data Set Defined Operator Approximation

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ such that $X \times Y$ defines an operator T .

$$E(a) = \sum_{x \in X} ||T(x) - T^n(x, a)||$$

4 Calculus

Definition 4.1. Let $A \subset B$

Then: $X_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \in A^c \end{cases}$

Theorem 4.1. Topological Manipulation

$f_a(x) := \frac{x-a}{x-a}$ where $\forall x \in \mathbb{R} \setminus \{a\}$ and $a \in \mathbb{R}$

Then: $f_a : \mathbb{R} \setminus \{a\} \rightarrow \mathbb{R}$

Question: Is f_a continuous?

Let $g(x) = \prod_{a \in \mathbb{R} \setminus \mathbb{Q}} f_a(x)$

Is g continuous?

Is $g : \mathbb{Q} \rightarrow \mathbb{R}$?

What is: $\int_0^1 g(x) \cdot X_{[0,1]}(x) dx$?

Definition 4.2. The Diamond Problem

To start this problem, we first need to start with a definition.

Let $B_d(0, n) = \{x \in \mathbb{Z}^d : d_1(0, x) \leq n\}$ where $d_1(x, y) = \sum_{j=1}^d |x_j - y_j|$

So since 0 is fixed, it's clear that B_d can be written as:

$$B_d(n) = \{x \in \mathbb{Z}^d : \sum_{j=1}^d |x_j| \leq n\}$$

The proof of this is left to the reader.

So an interesting question that arises from this is that of its cardinality and the set $B_d(n)$ specifically, we would like to be able to calculate $\mu(B_d(n)) = \text{card}(B_d(n))$ without a recursive equation.

A recursive equation is a place we can start though.

First we can start with $\text{card}(B_1(n))$

This turns into looking at the following:

$$B_1(n) = \{x \in \mathbb{Z}^1 : |x| \leq n\}$$

Since we know that $\text{card}(\mathbb{N}_n) = n$ and we can show that $B_1(n) = \mathbb{N}_n \cup -1 \cdot \mathbb{N}_n \cup \{0\}$ and we know that:

$$\mathbb{N}_n \cap -1 \cdot \mathbb{N}_n = \phi, \mathbb{N}_n \cap \{0\} = \phi, -1 \cdot \mathbb{N}_n \cap \{0\} = \phi$$

we know that:

$$\mu(B_1(n)) = \mu(\mathbb{N}_n \cup -1 \cdot \mathbb{N}_n \cup \{0\}) = \mu(\mathbb{N}_n) + \mu(-1 \cdot \mathbb{N}_n) + \mu(\{0\})$$

some more stuff that should be proven is that:

$$\mu(\mathbb{N}_n) = \mu(-1 \cdot \mathbb{N}_n) = n \text{ and } \mu(\{0\}) = 1$$

This gives us our starting point which is that:

$$\mu(B_1(n)) = 2n + 1 = p_1(n)$$

Alright this is great, we have worked out the first dimensional case, but what about the two dimension case? To answer this, I will start with the d dimensional case and make some observations.

We will need the following equation.

$$T_{n+1} : \mathbb{Z}^n \times \mathbb{Z} \rightarrow \mathbb{Z}^{n+1} \text{ where } T(x, a) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ a \end{bmatrix} \text{ for every } x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{Z}^n, a \in \mathbb{Z}$$

With this, we get a recursive characterization of $B_{d+1}(n)$ with $B_d(n)$

$$B_{d+1}(n) = \bigcup_{j=-n}^n T(B_d(n - |j|), j)$$

This needs to be proven.

With this we can actually start working on the 2 dimensional case.

$$B_d(n) = \bigcup_{j=-n}^n T(B_1(n - |j|), j) = \bigcup_{j=-n}^{-1} T(B_1(n - |j|), j) \cup \bigcup_{j=1}^n T(B_1(n - |j|), j) \cup T(B_1(n), 0)$$

Since $T(B_d(n - |j|), j) \cap T(B_d(n - |k|), k) = \phi$ for every $j \neq k$ where $j, k \in \mathbb{Z}$

and we know that: $\mu(T(B_d(n - |j|), j)) = \mu(T(B_d(n - |-j|), -j))$

and that: $\mu(T(B_d(n - |j|), j)) = \mu(B_d(n - |j|))$

We can show that:

$$\begin{aligned} \mu(B_{d+1}(n)) &= \mu\left(\bigcup_{j=-n}^{-1} T(B_d(n - |j|), j) \cup \bigcup_{j=1}^n T(B_d(n - |j|), j) \cup T(B_d(n), 0)\right) \\ &= \mu\left(\bigcup_{j=-n}^{-1} T(B_d(n - |j|), j)\right) + \mu\left(\bigcup_{j=1}^n T(B_d(n - |j|), j)\right) + \mu(T(B_d(n), 0)) \\ &= \mu\left(\bigcup_{j=1}^n T(B_d(n - |j|), j)\right) + \mu\left(\bigcup_{j=1}^n T(B_d(n - |j|), j)\right) + \mu(T(B_d(n), 0)) \\ &= \mu(B_d(n)) + 2 \sum_{j=1}^n \mu(B_d(n - |j|)) \end{aligned}$$

Notice that since we know that $\mu(B_1(n))$ is a polynomial, we then know that $\mu(B_{d+1}(n))$ is also a polynomial because the set of polynomials is a vector space.

so $p_1(n) = 2n + 1$ Knowing this observe:

$$\begin{aligned} \mu(B_2(n)) &= \mu(B_1(n)) + 2 \sum_{j=1}^n \mu(B_1(n - |j|)) < \mu(B_1(n)) + 2 \sum_{j=1}^n \mu(B_1(n)) \\ &= (2n + 1) + 2n(2n + 1) = (2n + 1)(2n + 1) = 4n^2 + 4n + 1 \in P(2) \end{aligned}$$

And so this shows us that $\mu(B_2(n))$ is either still in $P(1)$ or in $P(2)$ but can't be in any other polynomial space.

That claim needs proof.

This leads me to a general extension of this.

$$\mu(B_d(n)) < (2n + 1)^d$$

Showing this would tell us that:

$$\mu(B_d(n)) \in \bigcup_{k=1}^d P(k)$$

We need to get a lower bound to show that $\mu(B_d(n)) \in P(d)$

One candidate inequality we could start with is:

$$n^d < \mu(B_d(n))$$

Once we have proven that $\mu(B_d(n)) \in P(d)$ we can then know the following equation always has a solution.

$$A_d^n x = b_d^n$$

Where:

$$A_d^n = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 2^d & 2^{d-1} & 2^{d-2} & \dots & 2^2 & 2 & 1 \\ 3^d & 3^{d-1} & 3^{d-2} & \dots & 3^2 & 3 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ (d-2)^d & (d-2)^{d-1} & (d-2)^{d-2} & \dots & (d-2)^2 & d-2 & 1 \\ (d-1)^d & (d-1)^{d-1} & (d-1)^{d-2} & \dots & (d-1)^2 & d-1 & 1 \\ d^d & d^{d-1} & d^{d-2} & \dots & d^2 & d & 1 \end{bmatrix} \text{ and } b_d^n = \begin{bmatrix} \mu(B_d(0)) \\ \mu(B_d(1)) \\ \mu(B_d(2)) \\ \vdots \\ \mu(B_d(d-2)) \\ \mu(B_d(d-1)) \\ \mu(B_d(d)) \end{bmatrix}$$

And is $\mu(B_d(n)) \in P(d)$ then we have $d + 1$ unknowns and $d + 1$ equations.

5 Surjective Continuous Non-decreasing Bounded Functionals

Let $B = \{f : \mathbb{R} \rightarrow [0, 1] \mid f \text{ is surjective, continuous, and non-decreasing.}\}$

Theorem 5.1. B is convex.

Let $f, g \in B$ and $h(x) := \lambda f(x) + (1 - \lambda)g(x)$ where $\lambda \in [0, 1]$

Then h is still continuous since the linear combination of continuous functions is continuous.

Proof:

Since both f and g are surjective and non-decreasing, then there exists x_0, y_0, x_1, y_1 in \mathbb{R} such that:

$$f(x_0) = 0 = g(y_0) \text{ and } f(x_1) = 1 = g(y_1)$$

Suppose WLOG that $x_0 \leq y_0$ and $x_1 \leq y_1$

Then we know that:

$$h(x_0) = \lambda f(x_0) + (1 - \lambda)g(x_0) = \lambda 0 + (1 - \lambda)0 = 0$$

and

$$h(y_1) = \lambda f(y_1) + (1 - \lambda)g(y_1) = \lambda 1 + (1 - \lambda)1 = 1$$

Now if we pick $\alpha \in [0, 1]$ by the intermediate value theorem, we know that there exists an $x_\alpha \in [x_0, y_1]$ such that:

$$h(x_\alpha) = \alpha$$

Since α was arbitrary element, I have shown that h is surjective.

Finally, let $x_0 < x_1$ be elements in \mathbb{R}

Then we know that $f(x_0) \leq f(x_1)$ and $g(x_0) \leq g(x_1)$

$\Rightarrow \lambda f(x_0) \leq \lambda f(x_1)$ and $(1 - \lambda)g(x_0) \leq (1 - \lambda)g(x_1)$

$\Rightarrow \lambda f(x_0) + (1 - \lambda)g(x_0) \leq \lambda f(x_1) + (1 - \lambda)g(x_1)$

$\Rightarrow h(x_0) \leq h(x_1)$

Thus h is non-decreasing.

Since h is surjective, continuous, and non-decreasing, then $h \in B$

Thus B is convex.

Theorem 5.2. B is translation invariant.

Let $f \in B$ and $g(x) := f(x + c)$ where $c \in \mathbb{R}$

f is continuous and so is the addition operator so g is continuous.

Proof:

Let $\alpha \in [0, 1]$ since f is surjective then $\exists x \in \mathbb{R} \cap f(x) = \alpha$

Then $g(x - c) = f(x + c - c) = f(x) = \alpha$ and so g is surjective.

Let $x < y$ be elements in \mathbb{R}

Then $f(x) \leq f(y) \Rightarrow f(x + c) \leq f(y + c)$

$\Rightarrow g(x) \leq g(y)$ and so g is non-decreasing.

Thus $g \in B$ and B is therefore translation invariant.

Theorem 5.3. B is not complete.

Theorem 5.4. Every element in B can be decomposed as a finite non-trivial convex combination

6 A topological description of finite metric spaces.

Finite spaces are of interest because they describe the world of computers. This being the case, we would still like to do analysis on these spaces and analysis starts with topological descriptions.

Theorem 6.1. The First Rule of Induced Topologies on Finite Metric Subspaces is:

Let $V \subset X$ where $0 < \text{card}(V) =: N < \text{card}(\mathbb{N})$ and X is a metric space.

Then the subspace topology on V is the discrete topology.

Proof:

The associated topological space on V is:

$$\tau_V = \{V \cap U : U \in \tau_X\}$$

Since V is of finite cardinality, we can uniquely number each element.

Thus:

$$V = \bigcup_{i=1}^N \{v_i\}$$

Further:

$$V_{\min} := \min\{d(x, y) : x, y \in V\}$$

Let $v \in V$ and $\epsilon_V = \frac{V_{\min}}{2}$

Then $B(v, \epsilon_V) \in \tau_X \Rightarrow V \cap B(v, \epsilon_V) \in \tau_V$

However $V \cap B(v, \epsilon_V) = \{v\}$ and since v was arbitrary, we thus know that: $(\forall v \in V)(\{v\} \in \tau_V)$

We can now prove that $\tau_V = 2^V$

By definition we know that $\tau_V \subset 2^V$

Let $E \in 2^V$ then $E = \bigcup_{j=1}^M \{v_j\}$ where $M \leq N$

Since we know every $\{v_j\}$ is open we know that E is open and thus: $E \in \tau_V$

Thus the induced topology on a finite subset of a Metric space is the discrete topology.

Lemma 6.1. Everything is Continuous when your domain is finite.

Let $V \subset X$ and $Y \neq \emptyset$ where X, Y are metric spaces and $0 < \text{card}(V) = N < \text{card}(\mathbb{N})$

Then every $f : V \rightarrow Y$ is continuous.

Proof:

Let $U \in \tau_Y$

Then: $f^{-1}(U) \subset V$ where $f : V \rightarrow Y$ is arbitrary.

Thus: $f^{-1}(U) \in \tau_V$ by the previous theorem.

And so f is continuous.

And so every $f : V \rightarrow Y$ is continuous.

Theorem 6.2. Everything is Lipschitz continuous when your domain is finite.

So if we want to have some form of meaningful topological description of continuity for these spaces, we are going to need a "stronger" form of continuity.

Let $V \subset X$ and $Y \neq \emptyset$ where X, Y are metric spaces and $0 < \text{card}(V) = N < \text{card}(\mathbb{N})$

Further let $f : V \rightarrow Y$, then f is **Lipschitz continuous**.

Proof:

We can simply look at:

$$K := \max \left(\left\{ \frac{d_Y(f(x), f(y))}{d_X(x, y)} : x, y \in V \right\} \right)$$

Then we know that:

$$(\forall x, y \in V) \left(\frac{d_Y(f(x), f(y))}{d_X(x, y)} \leq K \right)$$

And therefore:

$$(\forall x, y \in V) (d_Y(f(x), f(y)) \leq K d_X(x, y))$$

And thus f is Lipschitz continuous.

Definition 6.1. K - Families

Let $V \subset X$ and $Y \neq \emptyset$ where X, Y are Normed spaces and $0 < \text{card}(V) = N < \text{card}(\mathbb{N})$

First we have the set of all operators.

$$\Lambda := \{f | f : V \rightarrow Y\}$$

Next we have the K - Families

$$\Lambda(K) := \{f \in \Lambda | K \text{ is the smallest Lipschitz constant for } f.\}$$

The set: $\Lambda(K)$ is called a K - Family. Next we have the K - Nests

$$B(K) := \{f : V \rightarrow Y | (\forall x, y \in V) (\|f(x) - f(y)\| \leq K \|x - y\|)\}$$

Theorem 6.3. K - Properties

- $B(K) = \bigcup_{L \in [0, K]} \Lambda(L)$

Proof:

Let $f \in B(K) \Rightarrow f$ is lipschitz.

Which means that there exists a unique smallest constant K' where: $0 \leq K' \leq K$

Thus $f \in \Lambda(K') \subset \bigcup_{L \in [0, K]} \Lambda(L)$

Therefore: $B(K) \subseteq \bigcup_{L \in [0, K]} \Lambda(L)$

Let $f \in \bigcup_{L \in [0, K]} \Lambda(L) \Rightarrow \exists L \in [0, K] \cap (\forall x, y \in V) (\|f(x) - f(y)\| \leq L \|x - y\| \leq K \|x - y\|)$

$\Rightarrow f \in B(K)$

$\Rightarrow \bigcup_{L \in [0, K]} \Lambda(L) \subseteq B(K)$

$\therefore B(K) = \bigcup_{L \in [0, K]} \Lambda(L)$

- $L \leq K \Rightarrow B(L) \subseteq B(K)$

Proof:

Let $f \in B(L)$ and $x, y \in V$

$\Rightarrow \|f(x) - f(y)\| \leq L \|x - y\| \leq K \|x - y\|$

$\Rightarrow f \in B(K)$

- $B(K)$ is convex.

Proof:

Let $f, g \in B(K)$ and $h_\lambda(x) = \lambda f(x) + (1 - \lambda)g(x)$ for some $\lambda \in [0, 1]$

Further let $x, y \in V$:

$$\begin{aligned} \|h_\lambda(x) - h_\lambda(y)\| &= \|\lambda f(x) + (1 - \lambda)g(x) - (\lambda f(y) + (1 - \lambda)g(y))\| = \|\lambda f(x) + (1 - \lambda)g(x) - \lambda f(y) - (1 - \lambda)g(y)\| \\ &= \|\lambda(f(x) - f(y)) + (1 - \lambda)(g(x) - g(y))\| \leq \lambda\|f(x) - f(y)\| + (1 - \lambda)\|g(x) - g(y)\| \\ &\leq \lambda K \|x - y\| + (1 - \lambda)K \|x - y\| = K \|x - y\| \end{aligned}$$

$\Rightarrow h_\lambda \in B(K)$

- $\Lambda(K)$ is not convex.

Proof:

Let $K > 0$

If $f \in \Lambda(K) \Rightarrow -f \in \Lambda(K)$

However $\frac{f+(-f)}{2} = 0 \in \Lambda(0) \neq \Lambda(K)$

Lemma 6.2. *Paths through Function spaces*

Let $f \in \Lambda(K)$ and $g \in \Lambda(L)$ then $h_\lambda(x) := \lambda f(x) + (1 - \lambda)g(x)$ at most belongs to the family: $\Lambda(\lambda K + (1 - \lambda)L)$ Further: $H : [0, 1] \rightarrow \Gamma$ where $H(\lambda) = h_\lambda$ is a path through the function space:

$$\Gamma = \bigcup_{\nu \in [0,1]} \Lambda(\nu \cdot \max(L, K))$$

Proof:

$$\begin{aligned} \|h(x) - h(y)\| &= \|\lambda f(x) + (1 - \lambda)g(x) - (\lambda f(y) + (1 - \lambda)g(y))\| = \|\lambda f(x) + (1 - \lambda)g(x) - \lambda f(y) - (1 - \lambda)g(y)\| \\ &= \|\lambda(f(x) - f(y)) + (1 - \lambda)(g(x) - g(y))\| \leq \lambda\|f(x) - f(y)\| + (1 - \lambda)\|g(x) - g(y)\| \\ &\leq \lambda K\|x - y\| + (1 - \lambda)L\|x - y\| = (\lambda K + (1 - \lambda)L)\|x - y\| \end{aligned}$$

Theorem 6.4. *Ordering*

For this theorem, we will need both the domain and co-domain to be finite.

First, Let $V \subset X$ and $W \subset Y$ where X, Y are normed spaces.

Next let $\Lambda := \{f | f : V \rightarrow W\}$ where $0 < \text{card}(V) =: N < \text{card}(\mathbb{N})$ and $0 < \text{card}(W) =: M < \text{card}(\mathbb{N})$

And: $\Lambda(K) := \{f \in \Lambda | K \text{ is the Lipschitz constant for } f.\}$ as before.

Then:

$$K < L \Rightarrow \text{card}(\Lambda(K)) < \text{card}(\Lambda(L))$$

Theorem 6.5. *A vector space over a finite field would only be able to have the discrete topology*

7 Convex Operators

Theorem 7.1. *Invariance under Convex Monotonic Operator Composition*

Reference

Theorem 7.2. *Invariance under Affine Composition*

Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a convex operator, then:

$$g(x) = f(Ax + b)$$

Is also a convex operator.

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