

# Analysis, Topology, Optimization, Machine Learning, and Computational Analysis

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## 1 Notation, Set Theory, and Logic

**Definition 1.1. *Common Sets of Numbers***

$$\mathbb{N} = \{1, 2, \dots\}$$

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}$$

$$\mathbb{N}_m = \{1, 2, \dots, m\} \text{ where } m \in \mathbb{N}$$

$\mathbb{R}$  is the set of Real Numbers

$$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$$

$$\mathbb{R}_0^+ = \{x \in \mathbb{R} : x \geq 0\}$$

$$\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\}$$

$$\mathbb{R}_0^- = \{x \in \mathbb{R} : x \leq 0\}$$

**Definition 1.2. *Sets and Set Builder Notation***

*Set builder notation is a way of describing a set using mathematical, logical symbols, or words. Look at the following example:*

$$E = \{x \in \mathbb{N} : x = 2n \text{ where } n \in \mathbb{N}\}$$

*This reads:  $E$  is the set of all  $x$  in  $\mathbb{N}$  such that  $x = 2n$  where  $n$  is in  $\mathbb{N}$*

*This set is also known as the even numbers.*

*When talking about functions, another common way of describing a set is:*

$$C_X^Y = \{f : X \rightarrow Y | f \text{ is continuous}\}$$

*This reads:  $C_X^Y$  is the set of all functions  $f$  mapping from  $X$  to  $Y$  such that  $f$  is continuous.*

**Definition 1.3. *Primitives***

*The logical or and not are both primitives and are written:*

*logical or :  $\vee$*

*and*

*not :  $\neg$*

Respectively.

Statements are written:  $L, M, N, O, P, Q, \dots$  etc

A statement is a sequence of words or symbols which is either true or false.

So then  $L \vee M$  is a new statement composed of  $L, \vee$ , and  $M$ .

We can then use this as the definition of a new statement:

$$N := L \vee M$$

Which is read:  $N$  is defined as  $L$  or  $M$ .

So  $N$  is true if:

$L$  is true,  $M$  is true, or  $N$  and  $M$  are both true.

#### **Definition 1.4. And**

Let  $A, B$  be sets.

$$A \wedge B := \neg((\neg A) \vee (\neg B))$$

#### **Definition 1.5. Intersection and Union**

Let  $A, B$  be sets.

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

$$A \cup B = \{x : x \in A \vee x \in B\}$$

#### **Definition 1.6. Subsets**

Let  $A, B$  be sets.

$$A \subset B \Leftrightarrow (x \in A \Rightarrow x \in B)$$

#### **Theorem 1.1. The union only makes things larger**

Let  $A, B$  be sets.

Then:  $A \subset A \cup B$

#### **Theorem 1.2. Union and Intersection Distributive Properties**

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

*Proof:*

$$\text{Let } x \in A \cap (B \cup C) \Rightarrow x \in A \wedge x \in B \cup C$$

Suppose that  $x \notin C \Rightarrow x \in B$

$$\Rightarrow x \in A \wedge x \in B$$

$$\Rightarrow x \in A \cap B$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C)$$

$$[(A \cap B) \subset (A \cap B) \cup (A \cap C)]$$

Suppose that  $x \notin B \Rightarrow x \in C$

$$\Rightarrow x \in A \text{ and } x \in C$$

$$\Rightarrow x \in A \cap C$$

$$\Rightarrow x \in (A \cap B) \cup (A \cap C)$$

$$[(A \cap C) \subset (A \cap B) \cup (A \cap C)]$$

Therefore:

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$$

**Definition 1.7. Power Set**

Let  $X \neq \phi$

$$2^X := \{V : V \subseteq X\}$$

## 2 Topology

**Definition 2.1. Topology**

Let  $X \neq \phi$

Further let  $\tau \subseteq 2^X$  such that:

$$\begin{aligned} & \phi, X \in \tau \\ & (\forall A \neq \phi) \left( \{U_\alpha\}_{\alpha \in A} \subseteq \tau \Rightarrow \bigcup_{\alpha \in A} U_\alpha \in \tau \right) \\ & (\forall m \in \mathbb{N}) \left( \{U_j\}_{j \in \mathbb{N}_m} \Rightarrow \bigcap_{j=1}^m U_j \in \tau \right) \end{aligned}$$

**Definition 2.2. Relative Topology**

Let  $X \neq \phi$  and  $Z \subset X$

Then the relative topology on  $Z$  is written as follows:

$$\tau_Z = \{Z \cap U : U \in \tau_X\}$$

**Theorem 2.1. The Relative Topology is a Topology on  $Z$** 

Let  $E \in \tau_Z$

$$\Rightarrow E = Z \cap U \subset Z$$

$$\Rightarrow \tau_Z \subseteq 2^Z$$

And so we have met the first criteria.

Next:

$$\phi \in \tau \Rightarrow Z \cap \phi \in \tau_Z \Rightarrow Z \cap \phi = \phi \in \tau_Z$$

Next:

$$X \in \tau \Rightarrow Z \cap X \in \tau_Z \Rightarrow Z \cap X = Z \in \tau_Z$$

Next: Let  $A \neq \phi$  and  $\{U_\alpha\}_{\alpha \in A} \in \tau_Z$

$$\Rightarrow \exists \{V_\alpha\}_{\alpha \in A} \subset \tau \text{ such that: } U_\alpha = Z \cap V_\alpha$$

$$\Rightarrow \bigcup_{\alpha \in A} U_\alpha = \bigcup_{\alpha \in A} Z \cap V_\alpha$$

**Theorem 2.2. Topologies on Finite Spaces**

Let  $E \neq \phi$  and  $\text{card}(E) = \phi$  then  $\tau_E = 2^E$

### 3 Change

#### Definition 3.1. *Metric*

Let  $X$  be a non-empty set.

Let  $d : X \times X \rightarrow \mathbb{R}_0^+$  such that:

- $(\forall x, y \in X) d(x, y) = 0 \Leftrightarrow x = y$
- $(\forall x, y \in X) d(x, y) = d(y, x)$
- $(\forall x, y, z \in X) d(x, z) \leq d(x, y) + d(y, z)$

Then  $d$  is called a metric and  $(X, d)$  is called a metric space.

Reference

#### Definition 3.2. *Limit of a function*

Let  $T : X \rightarrow Y$  where  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces.

Then fix  $x_0 \in X$ .

If:

$$(\exists L \in Y)(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in X)(d(x, x_0) < \delta \Rightarrow d(f(x), L) < \epsilon)$$

Then:

$$\lim_{x \rightarrow x_0} f(x) = L$$

Reference

#### Definition 3.3. *Derivative*

Let  $\hat{f} : \mathbb{R} \rightarrow \mathbb{R}$

Further let  $f = \hat{f}|_U$  where  $U \in \tau_{\mathbb{R}}$

Then  $f$  is said to be differentiable at  $x \in U$  if there exists an  $L_x$  such that:

$$L_x = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If  $L_x$  exists for all  $x \in U$  then we write:

$$\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Reference

#### Theorem 3.1. *Fundamental increment lemma*

Let  $f$  be described as above and be differentiable at  $x$ .

Then there exists a function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that:

$$f(x+h) = f(x) + \frac{d}{dx} f(x)h + \phi(x)h$$

and

$$\lim_{h \rightarrow 0} \phi(h) = 0$$

**Proof:**

Define:  $\phi(h) = \frac{f(x+h) - f(x)}{h} - \frac{d}{dx} f(x)$

Then:  $\phi(h)h = f(x+h) - f(x) - \frac{d}{dx} f(x)h$

Then:  $\phi(h)h + f(x) - \frac{d}{dx} f(x)h = f(x+h)$

And so we have property 1.

Next:

$$\begin{aligned}\lim_{h \rightarrow 0} \phi(h) &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x) - \frac{d}{dx}f(x)h}{h} \right] = \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} - \frac{d}{dx}f(x) \right] \\ &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} - \lim_{h \rightarrow 0} \frac{d}{dx}f(x) = \frac{d}{dx}f(x) - \frac{d}{dx}f(x) = 0\end{aligned}$$

**Definition 3.4. Partial Derivative**

Let  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$

Further let  $f = \hat{f}|_U$  where  $U \in \tau_{\mathbb{R}^n}$

Then  $f$  is said to be differentiable at  $x \in U$  with respect to the  $i$ 'th component of  $x$  if there exists an  $L_{x_i}$  such that:

$$L_{x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

If  $L_{x_i}$  exists for all  $x \in U$  then we write:

$$\frac{\partial}{\partial x_i} f(x) = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

Reference

**Theorem 3.2. Equivalent characterization**

Let  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$

Further let  $f = \hat{f}|_U$  where  $U \in \tau_{\mathbb{R}^n}$

And let  $f$  be differentiable at  $x \in U$  with respect to the  $i$ 'th component of  $x$ , then:

$$\begin{aligned}L_{x_i} &= \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[ \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} - L_{x_i} \right] \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[ \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} - \frac{L_{x_i} \cdot h}{h} \right] \\ \Leftrightarrow 0 &= \lim_{h \rightarrow 0} \left[ \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n) - \langle L_{x_i}, h \rangle}{h} \right]\end{aligned}$$

**Definition 3.5. Gradient**

Let  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}$  and let  $f : U \rightarrow \mathbb{R}$  such that  $f = \hat{f}|_U$  where  $U \in \tau_{\mathbb{R}^n}$

$f$  is said to be differentiable at  $x \in U$  if  $\exists \nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that:

$$\lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - \langle \nabla f(x), h \rangle|}{\|h\|} = 0$$

**Theorem 3.3. Form of the Gradient**

Let  $f$  be defined as above.

Then  $\nabla f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  where:

$$\nabla f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix} \quad \forall x \in \mathbb{R}^n$$

is the form of  $\nabla f$  which satisfies the above statement if  $f$  is differentiable.

*Reference*

**Proof:**

Suppose  $\nabla f$  is defined as above and all the partial derivatives exist.

Then:

$$\frac{1}{||h||} |f(x+h) - f(x) - \langle \nabla f(x), h \rangle| = \frac{1}{||h||} \left| f(x+h) - f(x) - \sum_{j=1}^n \frac{\partial}{\partial x_j} f(x) \cdot h_j \right|$$

**Definition 3.6. Matrix Functional Differentiability**

Let  $\hat{T} : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}$  and let  $T : U \rightarrow \mathbb{R}$  such that  $T = \hat{T}|_U$  where  $U \in \tau_{\mathbb{R}^{n \times m}}$ .  
 $T$  is said to be differentiable at  $x \in U$  if  $\exists D : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n \times m}$  such that:

$$\lim_{h \rightarrow 0} \frac{|T(x+h) - T(x) - \langle DT(x), h \rangle|}{\|h\|} = 0$$

where  $\langle \cdot, \cdot \rangle$  is an inner product defined on  $\mathbb{R}^{n \times m}$

**Definition 3.7. Frobenius inner product**

The Frobenius inner product is defined as:

$$\langle \cdot, \cdot \rangle_{FB} : \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m} \rightarrow \mathbb{R} \text{ such that: } \langle A, B \rangle_{FB} = \sum_{i=1}^n \sum_{j=1}^m a_{i,j} b_{i,j} \text{ for all } A, B \in \mathbb{R}^{n \times m}$$

**Theorem 3.4. Form of Matrix Functional Derivative**

$$DT(x) = \begin{bmatrix} \frac{\partial}{\partial x_{1,1}} & \cdots & \frac{\partial}{\partial x_{1,m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_{n,1}} & \cdots & \frac{\partial}{\partial x_{n,m}} \end{bmatrix}$$

**Definition 3.8. Differentiability of a multi-variable function.**

Let  $\hat{f} : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that:

$$\hat{f}(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \quad \text{and } (\forall j \in \mathbb{N}_n)(f_j : \mathbb{R}^m \rightarrow \mathbb{R})$$

Further let  $f = \hat{f}|_U$  where  $U \in \tau_{\mathbb{R}^m}$

Then  $f$  is said to be differentiable at  $x \in U$  if there exists a linear operator  $J_{f(x)} : \mathbb{R}^m \rightarrow \mathbb{R}^n$  such that:

$$\lim_{h \rightarrow \vec{0}} \frac{\|f(x+h) - f(x) + J_{f(x)}(h)\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^m}} = 0$$

Reference

**Theorem 3.5. If a multi-variable function,  $f$ , is differentiable at  $x$  then the linear operator  $J$  is the Jacobian matrix.**

So our guess is that:

$$J_{f(x)} = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x) & \cdots & \frac{\partial}{\partial x_n} f_1(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1} f_m(x) & \cdots & \frac{\partial}{\partial x_n} f_m(x) \end{bmatrix}$$

since this form is a linear operator mapping from the appropriate space to the appropriate space. It should be noted that the transpose of this matrix can not satisfy the definition of differentiability of a multi-variable function and so it is not the correct linear operator.

**Definition 3.9. Matrix operator differentiability**

Let  $T : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$  such that:

$$T(A) = \begin{bmatrix} T_1(A) \\ \vdots \\ T_n(A) \end{bmatrix} \quad \forall A \in \mathbb{R}^{n \times m} \quad \text{and } (\forall j \in \mathbb{N}_n)(T_j : \mathbb{R}^{n \times m} \rightarrow \mathbb{R})$$

Then  $T$  is said to be differentiable at  $A \in \mathbb{R}^{n \times m}$  if there exists a linear operator  $D : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^n$  where:

$$\lim_{h \rightarrow 0} \frac{\|T(A+h) - T(A) + D(h)\|_{\mathbb{R}^n}}{\|h\|_{\mathbb{R}^{n \times m}}} = 0$$

If  $D$  exists then it is called the Matrix operator derivative and is written:  $D_{\mathbb{R}^{n \times m}} T(A)$

**Theorem 3.6. The form of the Matrix operator derivative.**

Let  $T$  be described as above and differentiable at  $A \in \mathbb{R}^{n \times m}$

$$\frac{T(A+h) - T(A)}{\|h\|} = \begin{bmatrix} \frac{T_1(A+h) - T_1(A)}{\|h\|} \\ \vdots \\ \frac{T_n(A+h) - T_n(A)}{\|h\|} \end{bmatrix}$$

and so:

$$\lim_{h \rightarrow 0} \frac{T(A+h) - T(A)}{\|h\|} = \begin{bmatrix} \lim_{h \rightarrow 0} \frac{T_1(A+h) - T_1(A)}{\|h\|} \\ \vdots \\ \lim_{h \rightarrow 0} \frac{T_n(A+h) - T_n(A)}{\|h\|} \end{bmatrix}$$



**Definition 3.10. Subspace Differentiability**

Let  $X = \{X_j\}_{j=1}^n$  be a sequence of finite dimensional vector spaces where  $\dim(X_j) = k_j = m_j \times n_j$

Let  $T : \prod_{j=1}^n X_j \rightarrow Y$  where  $Y$  is a finite dimensional vector space with  $\dim(Y) = k_y$

Let  $x_j \in X_j$  for some  $j \in \mathbb{N}_n$

Where

$$x_j = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n_j} \\ \vdots & \ddots & \vdots \\ x_{m_j,1} & \cdots & x_{m_j,n_j} \end{bmatrix}$$

$T$  is said to be differentiable at  $x \in X$  where  $x = (x_0, \dots, x_j, \dots, x_{n-1})$  with respect to  $X_j$  if there exists a linear operator  $D : X_j \rightarrow Y$ :

Given  $h \in X_j \setminus \{\vec{0}\}$  define  $\hat{h} = (0, \dots, h, \dots, 0) \in X$  where  $h$  is in the  $j$ 'th place of  $\hat{h}$ :

$$\lim_{h \rightarrow 0} \frac{\|T(x + \hat{h}) - T(x) + D(h)\|_Y}{\|h\|_{X_j}} = 0$$

Then  $D$  is called the subspace derivative of  $T$  at  $x$  with respect to  $X_j$  and is written:  $D_{x_j}T(x)$

**Definition 3.11. *Product space Derivative***

Let  $X = \{X_j\}_{j=0}^{n-1}$  be a sequence of finite dimensional vector spaces where  $\dim(X_j) = k_j$

Let  $T : \prod_{j=0}^{n-1} X_j \rightarrow Y$  where  $Y$  is a finite dimensional vector space with  $\dim(Y) = k_y$

Let  $\{x_j\}_{j=0}^{n-1}$  be a sequence of vectors such that:  $(\forall j \in \{0, \dots, n-1\})(x_j \in X_j)$

The product space derivative at the point  $z \in X$  is:

$$D_X T(z) = \begin{bmatrix} D_{x_0} T(z) \\ \vdots \\ D_{x_{n-1}} T(z) \end{bmatrix}$$

**Definition 3.12. Fréchet derivative**

Let  $V, W$  be normed vector spaces and  $U \subset V$  be an open set.

An operator  $f : U \rightarrow W$  is said to be Fréchet differentiable if there exists a bounded linear operator  $A : V \rightarrow W$  such that:

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0$$

Reference

**Theorem 3.7. Fréchet derivative of a bounded linear operator**

Let  $V, W$  be normed vector spaces and  $U \subset V$  be an open set.

Let  $\hat{f} : V \rightarrow W$  be a bounded linear operator.

Then let's look at  $f = \hat{f}|_U$

My guess is that  $A = \hat{f}$

Let  $x \in U$  and  $h \in U$  with  $\|h\| \neq 0$  and  $x+h \in U$ , Then:

$$\frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = \frac{\|f(x) + f(h) - f(x) + \hat{f}(h)\|_W}{\|h\|_V} = \frac{\|f(x) + f(h) - f(x) + f(h)\|_W}{\|h\|_V} = 0$$

Thus let  $\epsilon > 0$  and  $\delta > 0$

Then if  $0 < \|h\| < \delta$  we know that  $\frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0 < \epsilon$

Therefore:

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x+h) - f(x) + Ah\|_W}{\|h\|_V} = 0$$

Thus  $A = \hat{f}$  is the Fréchet derivative of  $f$ .

**3.1 Finite Composition Operator****Definition 3.13. Finite Composition Operator**

Let the collection  $X = \{X_j\}_{j=0}^n$  be a finite sequence of sets.

Further let  $\{T_j\}_{j=0}^{n-1}$  be a finite sequence of operators such that  $(\forall j \in \mathbb{N}_{n-1})(T_j : X_j \rightarrow X_{j+1})$

Then  $T^n : X_0 \rightarrow X_n$  defined by:

$$T^n := \bigcirc_{j=0}^{n-1} T_j$$

is called the **Finite Composition Operator defined on  $X$** .

**Definition 3.14. Multi-variable Finite Composition Iteration**

Let the collection  $X = \{X_j\}_{j=0}^n$  and  $Y = \{Y_j\}_{j=0}^{n-1}$  be finite sequences of sets.

Further let  $\{T_j\}_{j=0}^{n-1}$  be a finite sequence of operators such that:  $(\forall j \in \mathbb{N}_{n-1})(T_j : X_j \times Y_j \rightarrow X_{j+1})$

Let  $T^n : X_0 \times \prod_{j=0}^{n-1} Y_j \rightarrow X_n$  where:

$$T^n(x, y) = z_n \text{ where } z_{j+1} = T_j(z_j, \pi_j(y)) \text{ or } z_{j+1} = T_j(z_j) \text{ and } z_0 = x \in X_0$$

**Definition 3.15. Gradient Descent**

Let  $E : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable operator.

The method of Gradient Descent says that a local minimum of  $E$  can be found using the following iteration:

$$a_{n+1} = a_n - \gamma \nabla E(a_n)$$

Where  $\gamma > 0$

**Example 3.1. Objective Operator for Data Set Defined Operator Approximation**

Let  $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$  such that  $X \times Y$  defines an operator  $T$ .

$$E(a) = \sum_{x \in X} ||T(x) - T^n(x, a)||$$

## 4 Surjective Continuous Non-decreasing Bounded Functionals

Let  $B = \{f : \mathbb{R} \rightarrow [0, 1] \mid f \text{ is surjective, continuous, and non-decreasing.}\}$

**Theorem 4.1.  $B$  is convex.**

Let  $f, g \in B$  and  $h(x) := \lambda f(x) + (1 - \lambda)g(x)$  where  $\lambda \in [0, 1]$

Then  $h$  is still continuous since the linear combination of continuous functions is continuous.

Since both  $f$  and  $g$  are surjective and non-decreasing, then there exists  $x_0, y_0, x_1, y_1$  in  $\mathbb{R}$  such that:

$f(x_0) = 0 = g(y_0)$  and  $f(x_1) = 1 = g(y_1)$

Suppose WLOG that  $x_0 \leq y_0$  and  $x_1 \leq y_1$

Then we know that:

$$h(x_0) = \lambda f(x_0) + (1 - \lambda)g(x_0) = \lambda 0 + (1 - \lambda)0 = 0$$

and

$$h(y_1) = \lambda f(y_1) + (1 - \lambda)g(y_1) = \lambda 1 + (1 - \lambda)1 = 1$$

Now if we pick  $\alpha \in [0, 1]$  by the intermediate value theorem, we know that there exists an  $x_\alpha \in [x_0, y_1]$  such that:

$$h(x_\alpha) = \alpha$$

Since  $\alpha$  was arbitrary element, I have shown that  $h$  is surjective.

Finally, let  $x_0 < x_1$  be elements in  $\mathbb{R}$

Then we know that  $f(x_0) \leq f(x_1)$  and  $g(x_0) \leq g(x_1)$

$\Rightarrow \lambda f(x_0) \leq \lambda f(x_1)$  and  $(1 - \lambda)g(x_0) \leq (1 - \lambda)g(x_1)$

$\Rightarrow \lambda f(x_0) + (1 - \lambda)g(x_0) \leq \lambda f(x_1) + (1 - \lambda)g(x_1)$

$\Rightarrow h(x_0) \leq h(x_1)$

Thus  $h$  is non-decreasing.

Since  $h$  is surjective, continuous, and non-decreasing, then  $h \in B$

Thus  $B$  is convex.

**Theorem 4.2.  $B$  is translation invariant.**

Let  $f \in B$  and  $g(x) := f(x + c)$  where  $c \in \mathbb{R}$

$f$  is continuous and so is the addition operator so  $g$  is continuous.

Let  $\alpha \in [0, 1]$  since  $f$  is surjective then  $\exists x \in \mathbb{R} \cap f(x) = \alpha$

Then  $g(x - c) = f(x + c - c) = f(x) = \alpha$  and so  $g$  is surjective.

Let  $x < y$  be elements in  $\mathbb{R}$

Then  $f(x) \leq f(y) \Rightarrow f(x + c) \leq f(y + c)$

$\Rightarrow g(x) \leq g(y)$  and so  $g$  is non-decreasing.

Thus  $g \in B$  and  $B$  is therefore translation invariant.

**Theorem 4.3.  $B$  is not complete.**

**Theorem 4.4. Every element in  $B$  can be decomposed as a finite non-trivial convex combination from  $B$**

## 5 K-Operators

Let  $\hat{f} : X \rightarrow Y$  where  $X, Y$  are topological spaces.

Let  $V \subset X$  and  $W \subset Y$  where  $\text{card}(V) < \text{card}(\mathbb{N})$  and  $\text{card}(W) < \text{card}(\mathbb{N})$  Then define  $f : V \rightarrow W$  where the associated topological spaces are:

$$\tau_V = \{V \cap U : U \in \tau_X\} \text{ and } \tau_W = \{W \cap U : U \in \tau_Y\}$$