Incompleteness and Intensionality via Universal Property of Arithmetic

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Introduction

"Violations" of (GII)

Identify a theory as a set of axioms. To formalise PA in itself, we need a formula ax(x) to enumerate all PA axioms.

Example ([Feferman, 1960])

There is a formula ax(x) representing the PA axioms,

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$$\begin{cases} \varphi \in \mathsf{PA} & \mathsf{PA} \vdash \mathsf{ax}(\lceil \varphi \rceil) \\ \varphi \not\in \mathsf{PA} & \mathsf{PA} \vdash \neg \mathsf{ax}(\lceil \varphi \rceil) \end{cases}$$

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A Mathematical Perspective

Externally equivalent notions might be internally different in PA, because weaker meta-theories identify less notions.

Central Question for Intensional Aspects of (GII)
How to identify the correct copy of PA in weaker meta-theories?

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Universal Property of Arithmetic

Very briefly, categorical logic is the following equivalence,

$$\mathcal{C}[-]$$
: {F-Theories} /equiv. \simeq {F-Categories}

Examples of F: Coherent, Heyting, Boolean, ...

- C[T] is the category of T-classes.
- Formalising mathematics in $T\simeq$ working internally in $\mathcal{C}[T]$.

Theorem

 $\mathcal{C}[\mathsf{PA}]$ is the initial Boolean cat. with parametrised NNO

Similar results for CA, the Σ_1 -fragment of $I\Sigma_1$ w.r.t. coherent cat., and for HA w.r.t. Heyting cat.; cf. [Ye, 2023].

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Abstract Categorical Framework

Let \mathcal{L} be some family of left exact cat. (specified algebraically):

- Suppose C is the initial object in L;
- And \mathbb{C} is the initial object in $\mathcal{L}_{\mathcal{C}}$.

The coding functor is induced by initiality of C,

$$\lceil - \rceil : \mathcal{C} \to \operatorname{Ext}(\mathbb{C}) := \mathcal{C}(1, \mathbb{C}) \in \mathcal{L}.$$

Provability is formalised via internal global section,

$$\Gamma : \operatorname{Ext}(\mathbb{C}) \to \mathcal{C}, \quad \alpha \mapsto \begin{array}{c} \Gamma(\alpha) & \longrightarrow & \mathbb{C}_1 \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\langle \Gamma \Gamma, \alpha \rangle} \mathbb{C}_0 \times \mathbb{C}_0 \end{array}$$

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The Constructive Case

We say $\mathcal L$ is constructive if $\mathcal L$ is closed under Artin glueing.

Theorem

If \mathcal{L} *is constructive, then* \square *satisfies the following conditions:*

- (P) \Box *is a pointed functor* $\eta: 1_{\mathcal{C}} \Rightarrow \Box$;
- (L) It has the Löb's property: $\Box 1 = 1$ is the initial \Box -algebra,



About the Proof

The Constructive Case

- The above structure only requires $\mathbb{C} \in \mathcal{L}_{\mathcal{C}}$.
- Joyal studied list pretopoi (cf. [van Dijk and Oldenziel, 2020]). The proof uses initiality of C, e.g. 1 is projective.
- However, there are much more general proofs which works with any $(\mathcal{C}, \mathbb{C}, 1 \Rightarrow \square)$; see [Ramesh, 2023].

Theorem (Non-Constructive Case [Feferman, 1960])
Let $C[PA]_{\Sigma}$ be the smallest coherent full subcategory containing the PNNO. If $\mathbb{C} \in C[PA]_{\Sigma}$, then (GII) holds.

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Outlooks

For HA

The existence of $1 \Rightarrow \Box$ for HA is surprising:

- The usual formalisation of HA does not seem to satisfy (P).
- There are stronger axiomatisations that does [Visser, 1982].
- What is the "correct" provability logic of HA?

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Löb's theorem for PA seems to suggest ($\mathcal{C}[\mathsf{PA}],\square$) also satisfies (L). Is there is nice categorical proof of this?

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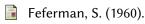
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