Betweenness in Enriched Categories

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Main Result

Theorem

There are functors
$$\textbf{BetSp} \underset{R}{\overset{L}{\rightleftarrows}} \textbf{EnCat}$$

Examples of Betweenness

- Pasch (1882):

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- Glivenko (1937): In a lattice $(L, \wedge, \vee, \leq)$,

$$(a \wedge x) \vee (x \wedge b) = x = (a \vee x) \wedge (x \vee b)$$

Formally

Definition

A relation $[\cdot,\cdot,\cdot] \subseteq X^3$ is a betweenness if

(B1) Symmetry: $[a, x, b] \longleftrightarrow [b, x, a]$,

(B2) Reflexivity: $[a, b, b]$ holds for all $a, b \in X$,

(B3) Minimality: $[a, b, a]$ and $[b, a, b]$, then $a = b$

(B4) Transitivity: $[a, x, b]$ and $[a, y, x]$, then $[a, y, b]$.

$$\longleftarrow\!\!-\!\!-\!\!-\; a \;-\!\!-\!\!-\; x \;-\!\!-\!\!-\; b \;-\!\!-\!\!-\!\!\longrightarrow$$

Road Systems

Definition (Bankston, 2013)

A road system on X is a family $\mathcal{R} \subseteq 2^X$ such that:

1. $\{a\} \in \mathcal{R}$ for all $a \in X$,
2. for all $a, b \in X$ there exists $R \in \mathcal{R}$ such that $a, b \in R$.

Definition

x is between a and b whenever

$$x \in \bigcap_{a,b \in R \in \mathcal{R}} R$$

"All roads from a to b go via x"

New Example

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$$d(a, x) + d(x, b) \quad \geq \quad d(a, b)$$

R : **EnCat** $\longrightarrow$ **BetSp**

On a $\mathcal{V}$-category $\mathcal{A}$ define $[-,-,-]_V \subseteq \mathrm{ob}(A)^3$

$$[A,B,C]_V \text{ if and only if } M_{ABC} \text{ and } M_{CBA} \text{ are split epi}$$

Where $M_{ABC} : \mathcal{A}(A,B) \otimes \mathcal{A}(B,C) \longrightarrow \mathcal{A}(A,C)$

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This definition is symmetric (B1)

$$[A,B,C] \longleftrightarrow [C,B,A]$$

Axiom B2

$[A, B, B]$ holds $\forall A, B \in \mathrm{ob}(\mathcal{A})$.

$$\begin{tikzcd}
\mathcal{A}(A,B) \otimes \mathcal{A}(B,B) \arrow[r, "M"] & \mathcal{A}(A,B) & \mathcal{A}(A,A) \otimes \mathcal{A}(A,B) \arrow[l, "M"'] \\
\mathcal{A}(A,B) \otimes I \arrow[u, "1 \otimes j_B"] \arrow[ur, "r"'] & & I \otimes \mathcal{A}(A,B) \arrow[u, "j_A \otimes 1"'] \arrow[ul, "l"]
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\end{tikzcd}$$

$$M_{ABB}(1 \otimes j_B) r^{-1} = \mathrm{Id}$$

Axiom B4

if $[A, X, B]$ and $[A, Y, X]$ then $[A, Y, B]$

$$
\begin{tikzcd}
(\mathcal{A}(A,Y) \otimes \mathcal{A}(Y,X)) \otimes \mathcal{A}(X,B) \arrow[rr, "a"] \arrow[d, bend right, "M_{AYX} \otimes 1"'] & & \mathcal{A}(A,Y) \otimes (\mathcal{A}(Y,X) \otimes \mathcal{A}(X,B)), \arrow[d, "1 \otimes M_{YXB}"] \\
\mathcal{A}(A,X) \otimes \mathcal{A}(X,B) \arrow[u, bend right, "\varphi_{AYX} \otimes 1"'] \arrow[dr, bend right, "M_{AXB}"'] & & \mathcal{A}(A,Y) \otimes \mathcal{A}(Y,B) \arrow[dl, "M_{AYB}"] \\
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$$M_{AYB}(1 \otimes M_{YXB})a(\varphi_{AYX} \otimes 1)\varphi_{AXB} = \mathrm{Id}$$

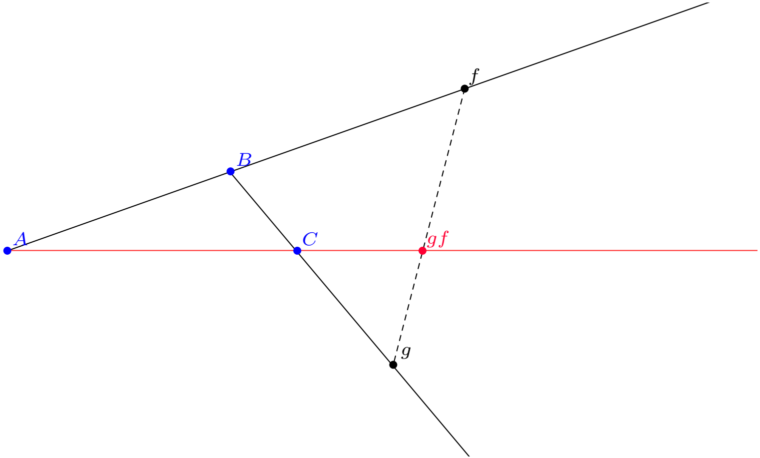
Axiom B3

Since $[A, B, A]_{\mathcal{V}}, [B, A, B]_{\mathcal{V}}$ is an equivalence relation,

$$(\mathcal{A}, \mathcal{V}) \longmapsto (\mathrm{ob}(\mathcal{A})/\sim, [-,-,-]_{\mathcal{V}})$$

defines the object part of our functor L : **EnCat** $\longrightarrow$ **BetSp** .

$\mathbb{R}^2$ is a Category



We can think of points $f \in \overrightarrow{AB}$ as morphisms $f : A \longrightarrow B$

the functor **BetSp** $\longrightarrow$ **EnCat**

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$$([x, b, a] \text{ and } [x, c, b]) \xrightarrow{B4} [x, c, a]$$

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$$([x,b,a] \text{ and } [x,c,b]) \xrightarrow{B4} [x,c,a]$$

- Unit element: $X \subseteq X(a,a)$

$$[x,a,a] \text{ holds for all } x \in X \text{ by B2}$$

Functoriality

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Thus we can view Y as enriched over 2^X via $f^{-1}(Y(f(a), f(b)))$

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X(a,x) \cap X(x,b) \arrow[r] \arrow[d] & X(a,b) \arrow[d] \\
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and

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commute.

Thus $f : X \longrightarrow Y \ \in$ **EnCat**

- F.William Lawvere (1973) *Metric Spaces, Generalized Logic, and Closed Categories*, Rendiconti del Seminario Matematico e Fisico di Milano, 43: 135 – 166
- V. Pambuccian (2011) *The Axiomatics of Ordered Geometry I. Ordered Incidence Spaces*, Expositiones Mathematicae 29: 24 – 66