

# The Flexibility of Models and the Bias-Variance Trade-Off

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Introduction à l'apprentissage automatique

GYMINF 2021

# Table of Contents

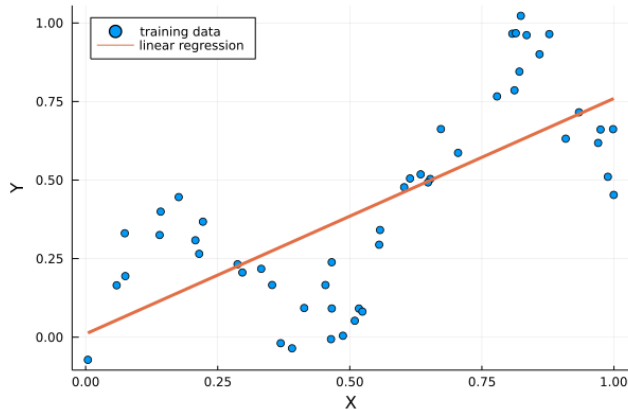
## 1. Polynomial Regression

## 2. K Nearest-Neighbors

## 3. Underfitting and Overfitting

## 4. Bias-Variance Decomposition

# When Linear Methods are Not Flexible Enough

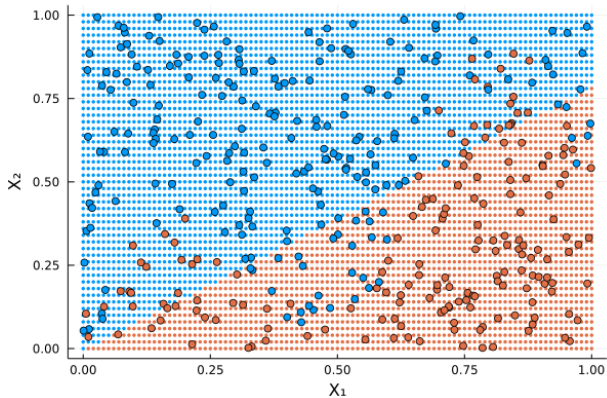


Data Generating Process  
 $Y = 0.3 \sin(10X) + 0.7X + \epsilon$

There is structure in the data that linear regression fails to capture.

Linear regression has a high reducible error.

# When Linear Methods are Not Flexible Enough



Data Generating Process

$Y = \text{red if}$

$$\sigma(20(0.3 \sin(10X_1) + 0.7X_1 - X_2)) > \epsilon$$

There is structure in the data that logistic regression fails to capture.

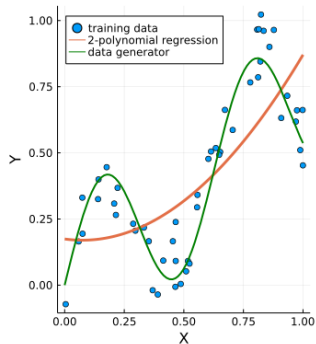
Logistic regression has a high reducible error.

large dots: training data

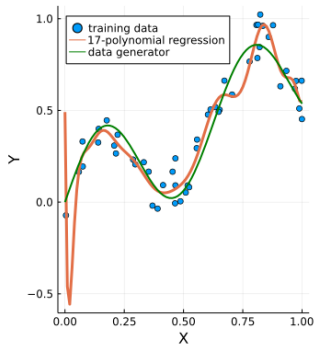
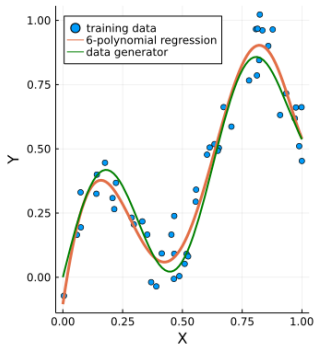
small dots: classification with logistic regression  
at decision threshold 0.5

# Polynomial Regression

$$\hat{Y} = \theta_0 + \theta_1 X + \theta_2 X^2 + \dots \theta_d X^d$$

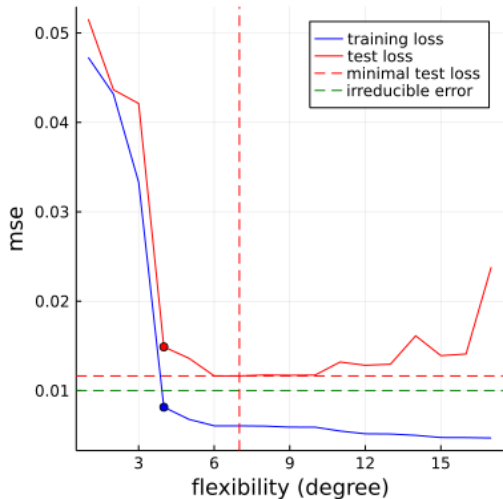


underfitting



overfitting

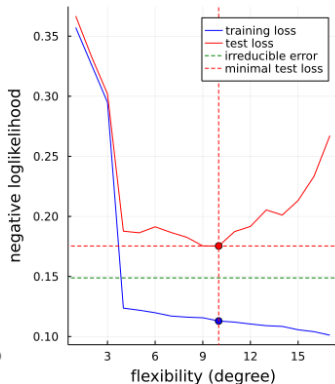
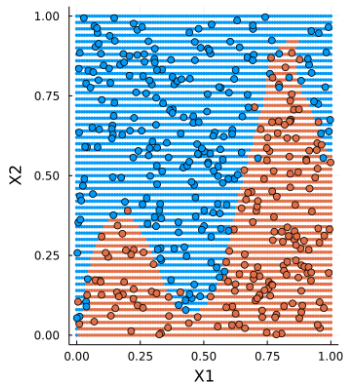
# Polynomial Regression



- ▶ Polynomial regression with degree 1 is linear regression.
- ▶ For small degrees ( $<6$ ) training loss and test loss are high.
- ▶ Minimal test loss is reached for intermediate degrees; it is only slightly higher than the irreducible error.
- ▶ For high degrees ( $>10$ ) the method is too flexible; it can fit peculiarities of the given training set. Therefore the training loss is lowest there, but the test loss is higher.

# Multiple Polynomial Classification

$$\hat{Y} = \theta_0 + \theta_1 X_1 + \theta_2 X_2 + \theta_3 X_1 X_2 + \theta_4 X_1^2 + \theta_5 X_2^2 + \cdots + \theta_{(d+1)(d+2)/2} X_2^d$$



# Summary

- ▶ Polynomial regression is like multiple regression with the additional features being higher orders of the original feature(s).
- ▶ The degree is a **hyper-parameter** that controls the flexibility of the method.
- ▶ The training loss decreases with increasing flexibility (degree).
- ▶ The test loss is U-shaped as a function of the flexibility.
- ▶ The lowest test loss is not lower than the irreducible error.



# Table of Contents

1. Polynomial Regression

**2. K Nearest-Neighbors**

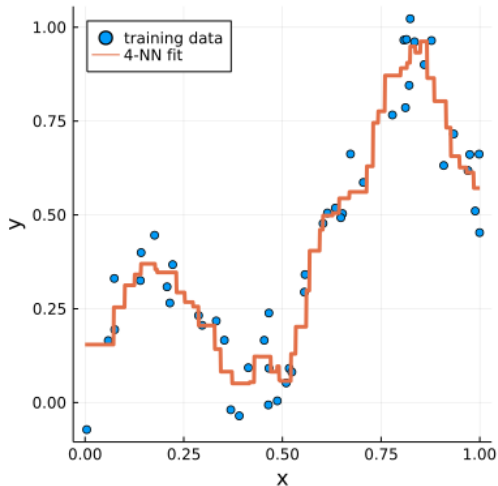
3. Underfitting and Overfitting

4. Bias-Variance Decomposition

# kNN Regression

$$\hat{y} = \frac{1}{K} \sum_{i \in \mathcal{N}_0} y_i$$

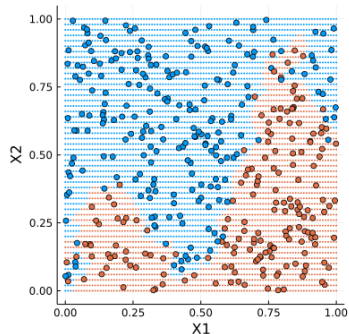
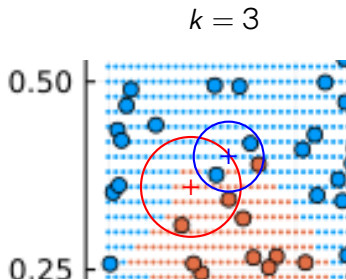
where  $\mathcal{N}_0$  is the set of  $k$  nearest neighbours of  $x_0$ .



# kNN Classification

$$\hat{\Pr}(Y = j|X = x_0) = \frac{1}{K} \sum_{i \in \mathcal{N}_0} I(y_i = j)$$

where  $\mathcal{N}_0$  is the set of  $k$  nearest neighbours of  $x_0$ .



# Properties of kNN

- ▶ kNN is an example of a **non-parametric method**, where the raw training data is used to make predictions.  
This is in contrast to **parametric methods** (like linear regression), where all information about the training data is stored in the parameters  $\theta$ .
- ▶ The number of neighbors  $k$  is a **hyper-parameter**; it controls the assumed smoothness of the function family and does not depend on the training data.
- ▶ If the hyper-parameter  $k$  is small, the function family is **flexible**, which allows to follow the training data accurately. If  $k$  is large the function family is **inflexible** and follows only the main trend of the data.
- ▶ kNN is very fast to fit (no parameters are learnt).
- ▶ kNN is slow to make predictions, because the neighbors of the test point need to be searched in the training set.

# kNN Classification on MNIST



First nearest neighbors classification on MNIST reaches a misclassification rate of 0.075.

This is better than linear regression.

# Table of Contents

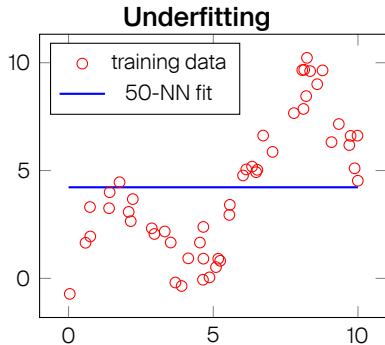
1. Polynomial Regression

2. K Nearest-Neighbors

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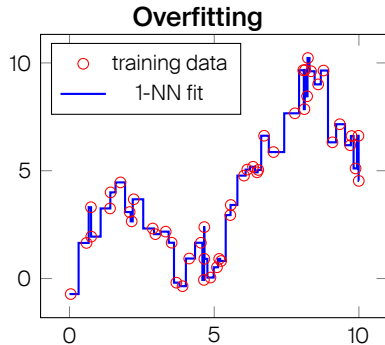
4. Bias-Variance Decomposition

# Underfitting and Overfitting



the function family is not flexible enough.  
training and test error are high.

Hyper-parameters control the flexibility (k for kNN, degree for polynomial regression).



the function family is too flexible.  
training error is low, test error is high.

# Table of Contents

1. Polynomial Regression

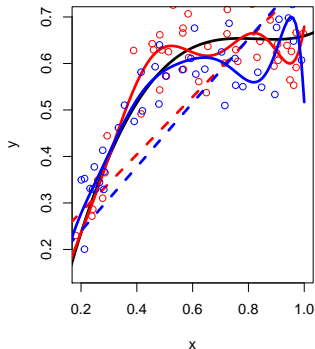
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# Fitting Multiple Training Sets



- ▶ Red and blue data points generated with  $Y = f(X) + \epsilon$  with  $\text{Var}(\epsilon) = 0.06^2$  and  $f(X) = \sin(2X) + 2(X - 0.5)^3 - 0.5X$  (black line)
- ▶ Red/blue lines: fits on red/blue training set
- ▶ The linear fits (dashed lines) are close to each other (small variance) but far away from  $f$  (large bias).
- ▶ The polynomial fits (with  $d = 10$ ) are far from each other (large variance) but close to  $f$  (small bias).

# The Bias-Variance Trade-Off

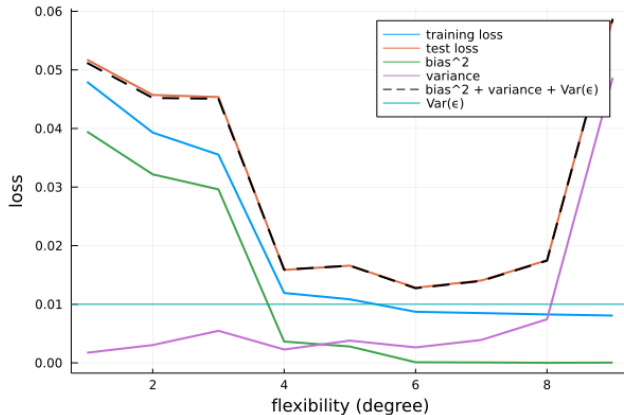
Expected test MSE

$$E_{Y|X, \hat{f}}(Y - \hat{f}(X))^2 = \text{Var}(\hat{f}(X)) + [\text{Bias}(\hat{f}(X))]^2 + \text{Var}(\epsilon)$$

- ▶ Here,  $\text{Var}(\hat{f}(X))$  refers to the variance of the estimator, if it would be fitted multiple times to each time another training set.
- ▶ Generally: flexible methods have higher variance but lower bias than less flexible methods.

# Blackboard: Bias-Variance Decomposition

# The Bias-Variance Trade-Off



For 1000 different training sets of size  $n = 50$ , polynomial fits of degrees 1 to 9 were performed. Each fit was evaluated on a large test set.

The plot shows the average training and test loss (over all 1000 training sets), and the average squared bias and variance (over all test points).

Note that the squared bias, the variance and the irreducible error sum up to the test loss.

# Quiz

Which of the following statements is correct?

1. If we know the variance  $\text{Var}(\epsilon)$  of the irreducible error, we can estimate the reducible error of an estimate  $\hat{f}$  at input  $x$  by computing  $\frac{1}{n} \sum_{i=1}^n (y_i - \hat{f}(x))^2 - \text{Var}(\epsilon)$  with  $y_i = f(x) + \epsilon_i$ .
2. If we know the variance  $\text{Var}(\epsilon)$  of the irreducible error, we can estimate the squared bias at input  $x$  of a machine learning method by computing estimates  $\hat{f}_k$  for different training sets  $k$  and computing  $\frac{1}{n} \sum_{k=1}^n (y_k - \hat{f}_k(x))^2 - \text{Var}(\epsilon) - \text{Var}(f_k(x))$  with  $y_k = f(x) + \epsilon_k$ .
3. If we have access to different training sets, we can estimate the variance of a machine learning method without having access to the variance of the irreducible error or the true data generating function  $f$ .
4. If we have access to different training sets, we can estimate the squared bias of a machine learning method without having access to the variance of the irreducible error or the true data generating function  $f$ .

# Summary

- ▶ Rigid (inflexible) methods
  - ▶ may underfit the data.
  - ▶ may have a large bias.
  - ▶ tend to have little variance when fitted on different training sets.
- ▶ Flexible methods
  - ▶ may overfit the data.
  - ▶ can achieve a low bias.
  - ▶ tend to have high variance when fitted on different training sets.

In expectation over training sets and responses, the test MSE of a method is given by the sum of the method's variance, its squared bias and the irreducible error.