Mathematical Analysis of Neuron Propagation

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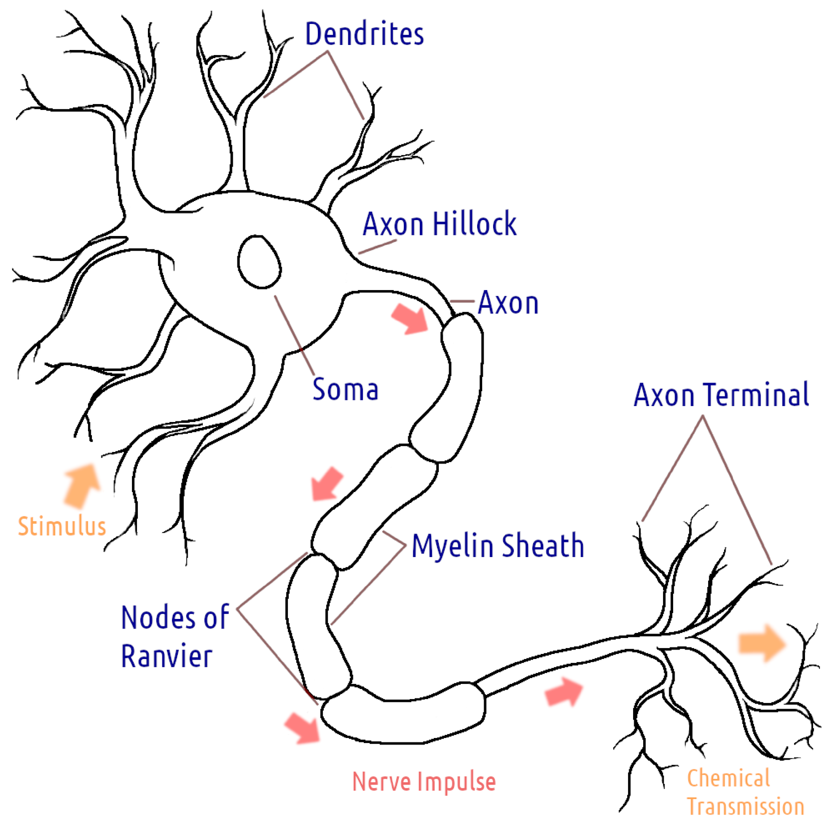

Figure 1: General structure of a neuron. Action potentials (neuron spiking) travel down the axon, which was the main neuron biology studied by Hodgkin and Huxley.

Abstract

Hodgkin and Huxley created a mathematical model for the signals generated by an axon of a giant squid axon. The model is a system of four differential equations that are difficult to solve analytically. Thus, many simpler methods have been developed to study the model. Here we cover a two dimensional reduction of the Hodgkin-Huxley model and analysis it using phase planes.

Figure 2: Alan Hodgkin (right) and Andrew Huxley (left)

Introduction

Beginning in 1938, Alan Hodgkin and Andrew Huxley studied the electro-physiological behavior of a giant squid axon, which is nearly 100 times larger than our own. Hodgkin found a way to measure the ionic flow of the axon by taking a fine glass tube containing a chlorided silver wire and inserting it through the axon to act as an electrode. This enabled them to record the potential difference between the interior and exterior of the axon, where they discovered what we now know as an action potential. They received a Nobel Prize for their work in 1963. [1].

Acknowledgements

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Ionic Flow

1. Voltage gated channels allow sodium ions (Na^+) and potassium ions (K^+) to flow in and out of the neuron.
2. When a stimulus causes the inside of the cell to reach a threshold it will trigger the flow of Na^+ into the cell, the start of the action potential.
3. Na^+ flow is inhibited around +40 mV.
4. K^+ flow out of the cell starts around +40 mV.
5. Leaky gates allow K^+ to leak back into the membrane until the cell is at an equilibrium state

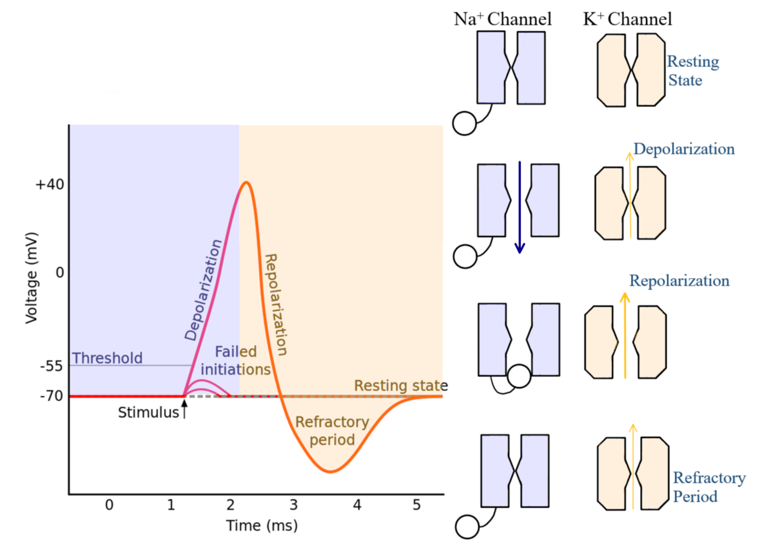


Figure 3: Ionic flow during an action potential (red).

Assumptions

1. Na^+ inactivation (h) can be modeled as the reverse of K^+, such that $h = 0.8 - n$ [2].
2. Na^+ activation (m) can be modeled as a steady state because it changes on a much faster time scale than K^+ (n) [2].

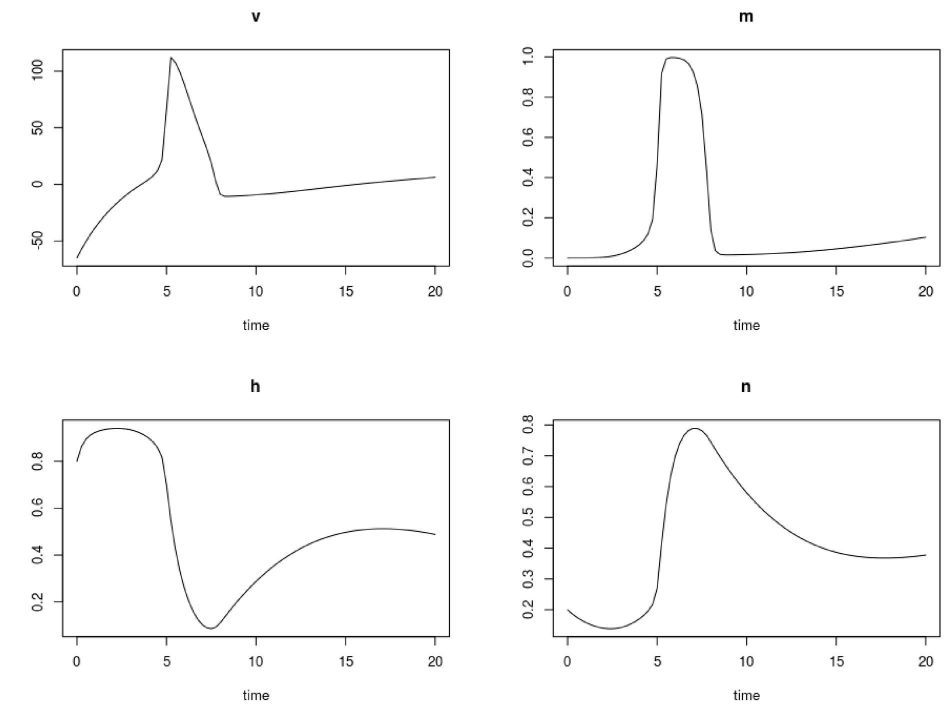
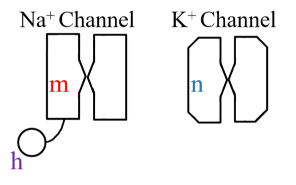


Figure 4: The ion activation and voltage during an action potential, all with time (ms). Top left is voltage, top right is Na^+ activation. The bottom left is Na^+ inactivation and bottom right is K^+ activation. Note how the bottom two graphs are inverse of each other.

2D Reduction of the Hodgkin-Huxley Equations[2]

$$C_m \frac{dV_m}{dt} = -\bar{g}_{Na} m_\infty^3 (0.8 - n)(V_m - E_{Na}) - \bar{g}_K n^4 (V_m - E_K) - g_L (V_m - E_L) + I_{ext}$$

$$\frac{dn}{dt} = \alpha_n (1 - n) - \beta_n n$$



Parameters	Values Used to Analyze
V_m - membrane potential (mV)	Variable
n - potassium activation	Variable
I_{ext} - externally applied current ($\mu A/cm^2$)	Parameter
C_m - membrane conductance ($\mu F/cm^2$)	$C_m = 1$
$\bar{g}_{Na}$ and $\bar{g}_K$ - maximal ion conductance ($\mu A/cm^2$)	$\bar{g}_{Na} = 120$, $\bar{g}_K = 36$
g_L - leak conductance ($\mu A/cm^2$)	$g_L = 0.3$
E_{Na} and E_K - ion resting potentials ($\mu A/cm^2$)	$E_{Na} = 115$, $E_K = -12$
E_L - neuron resting state ($\mu A/cm^2$)	$E_L = 10.6$

$\alpha_n = 0.01(\frac{10 - V_m}{exp(10 - V_m 10) - 1})$ $\beta_n = 0.125 exp(-V_m 80)$

$\alpha_m = 0.1(\frac{25 - V_m}{exp(25 - V_m 10) - 1})$, $\beta_m = 4 exp(-V_m 18)$ $m_\infty = \frac{\alpha_m}{\alpha_m + \beta_m}$

Nullclines

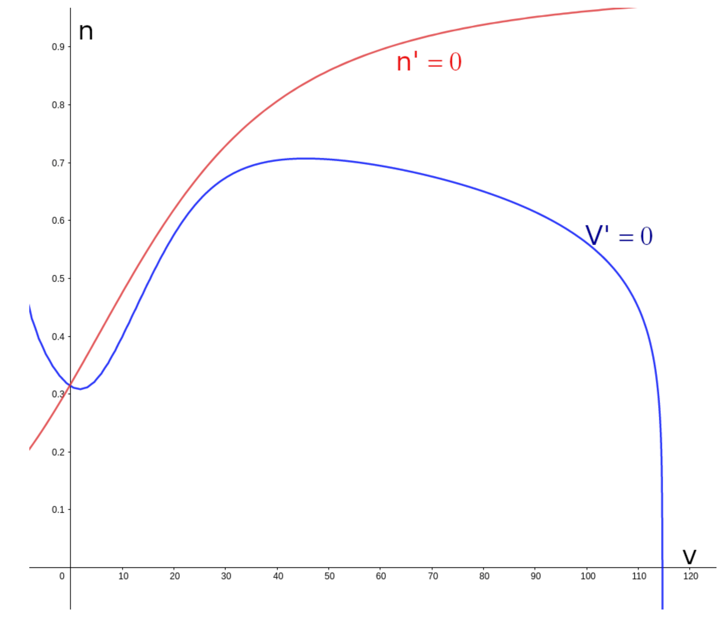


Figure 5: By setting the equations equal to zero, we get the critical point set called nullclines. The intersection of these tell us where our equilibrium solutions are [1].

Equilibrium Solutions

A Jacobian matrix J can be used to evaluate equilibrium points using the trace and determinant of J such that

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}, \quad tr(J) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}, \ and \ \det(J) = \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \cdot \frac{\partial f}{\partial y}.$$

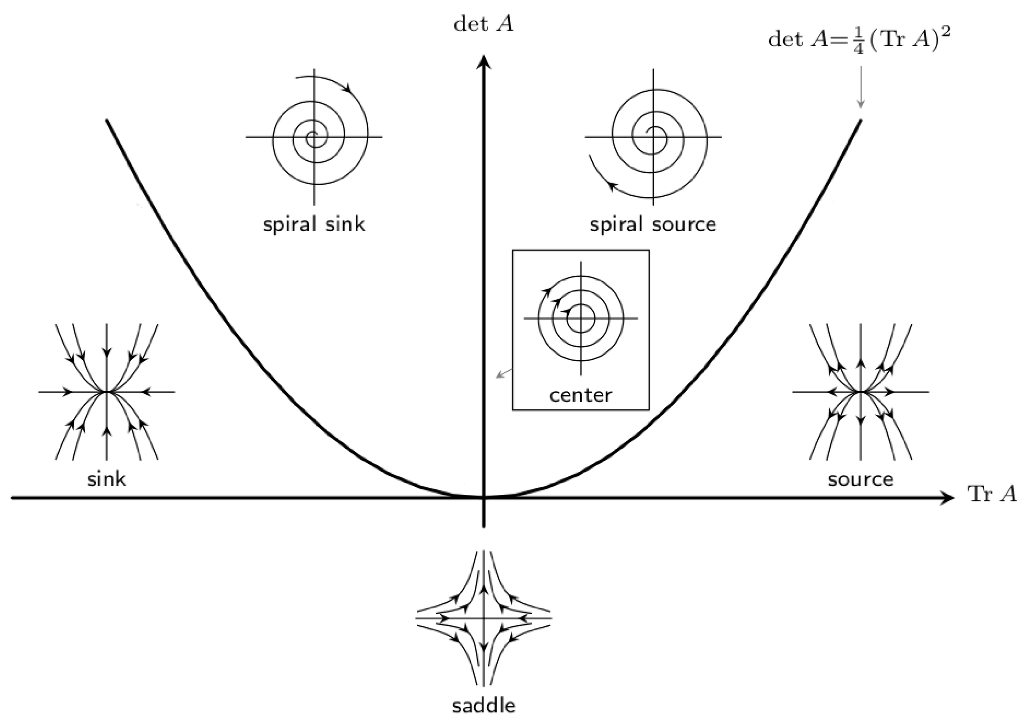


Figure 6: Equilibrium solution behavior given results from the Jacobian matrix analysis.

Bifurcations

These occur when a parameter of a system causes the equilibrium point to change behavior. A **Hopf bifurcation** is when a system has the sudden onset of a limit cycle[2].

Phase Plane Analysis

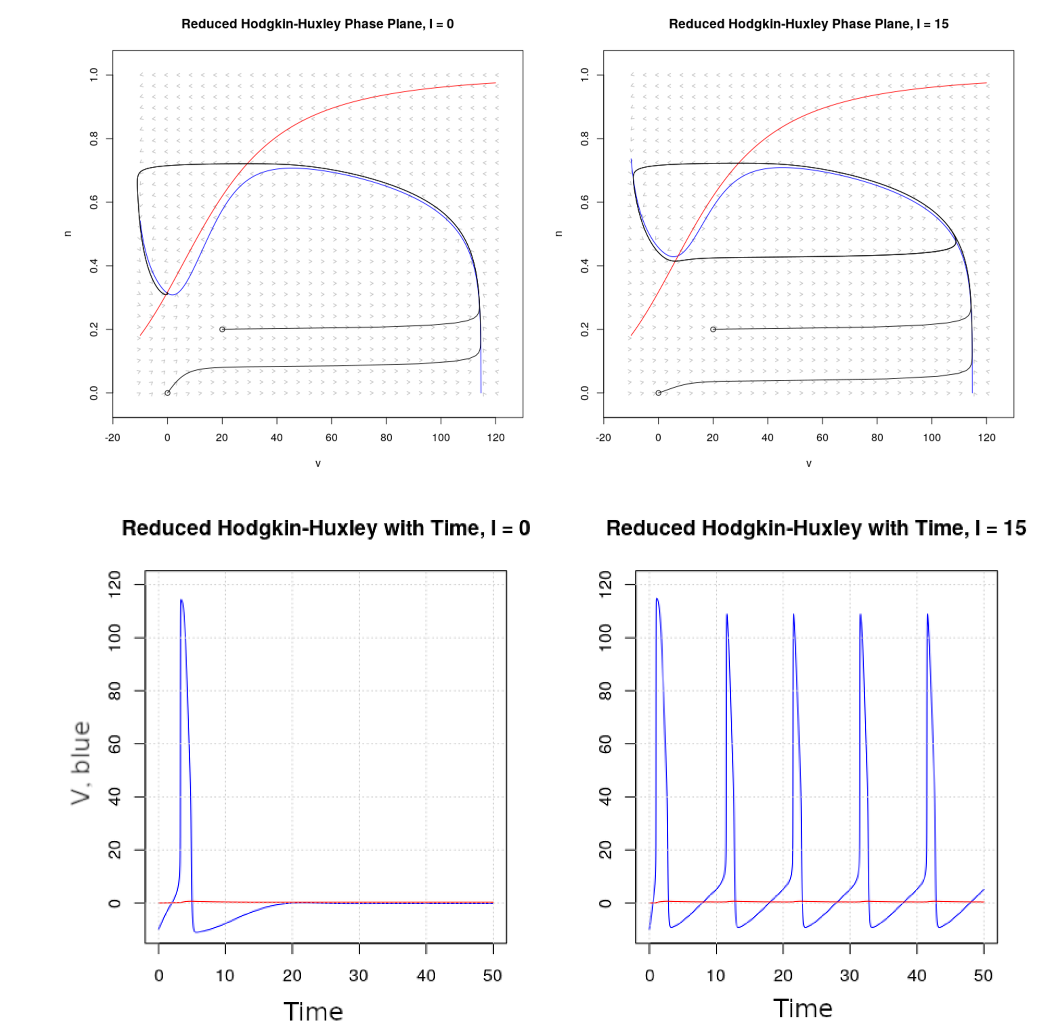


Figure 7: Phase plane of the reduced Hodgkin-Huxley equations with $I_{ext} = 0$ (top left) and $I_{ext} = 15$ (top right). When $I_{ext} = 15$, the system experiences a Hopf bifurcation with the onset of a limit cycle. The bottom two images are showing the change in voltage over time for both $I_{ext} = 0$ (left) and $I_{ext} = 15$ (right).

Fun Application

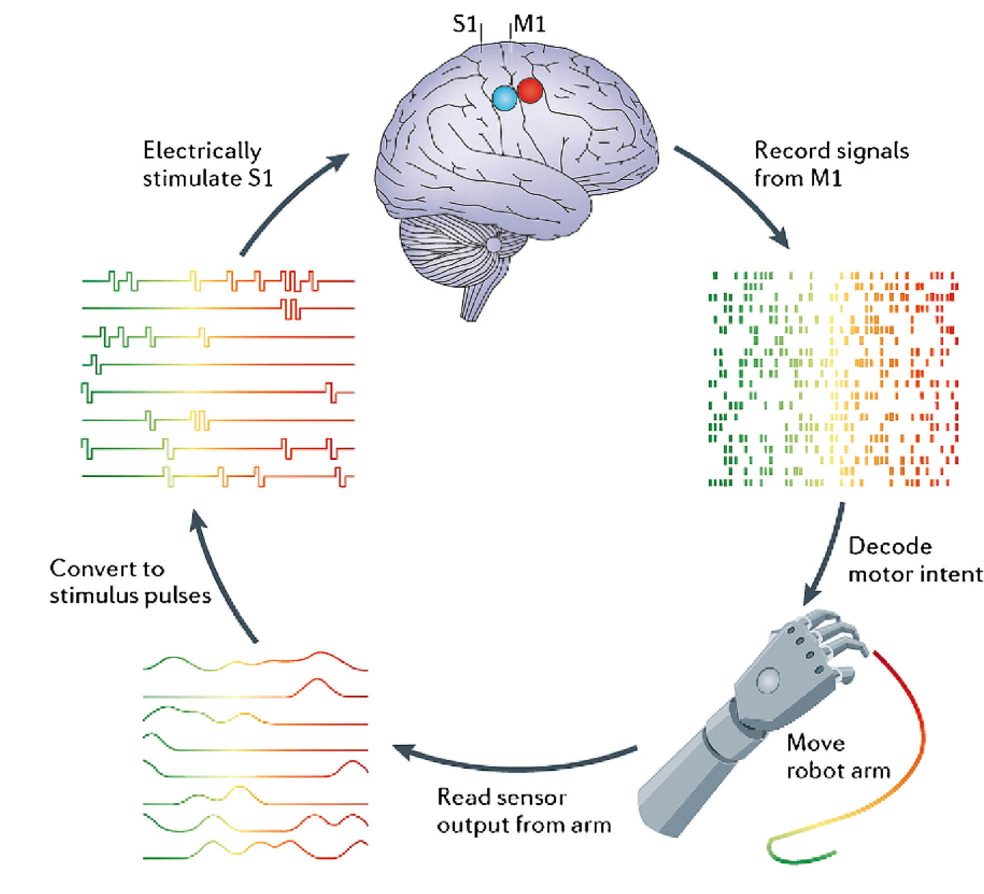


Figure 8: Brain computer interfaces are chips that can be planted in the brain and used to send and receive signals for use with robotic limbs.

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