Monotone Inequalities on Ising and Potts Model

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Introduction of Ising and Potts Model

Let $G = (V, E)$ be a finite graph, a configuration of the graph is a function $\sigma : V \to Q$ which assigns each vertex $x \in V$ a spin value $\sigma(x) \in Q$, where Q is the state space.
In the **Ising model**, the state space $Q = \{+1, -1\}$. Each spin $x \in V$ can take either of two spin values, $+1$ ("spin up") and -1 ("spin down").
In the **Potts model**, the state space $Q = \{1, 2, \ldots, q\}$. Each spin may take $q \geq 3$ (rather than only two) different values.

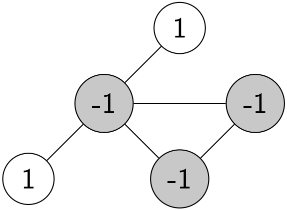


Figure: Ising configuration

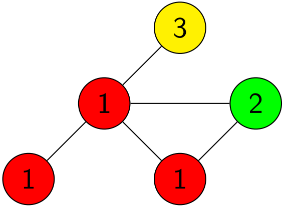


Figure: Potts configuration

Hamiltonian and Gibbs Measure

For a spin system, given a configuration σ, define its **Hamiltonian** as

$$H(\sigma) = \sum_{x \sim y} U(\sigma(x), \sigma(y)) + \sum_{x} V(\sigma(x)),$$

where $U : S \times S \to \mathbb{R}$ represents the neighbor-interaction of the spin system, and $V : S \to \mathbb{R} \cup \{\infty\}$ represents the influence of the external field.

For each configuration, define the **Gibbs measure** via

$$\mu_g(\sigma) = \frac{1}{Z_g} \exp\{-\beta H(\sigma)\},$$

where Z_g is the normalizing constant known as the partition function and β is the inverse temperature.

Correlation Factor

We call a model is **ferromagnetic** if two neighbor spins are more likely to be the same, and a model is **antiferromagnetic** if two neighbor spins are more likely to be different. This property is reflected in its Hamiltonian.

In the ferromagnetic Ising model, the Hamiltonian is

$$U(\sigma(x), \sigma(y)) = -\sigma(x)\sigma(y), \quad V(\sigma(x)) = -h(x)\sigma(x).$$

In the antiferromagnetic Ising model,

$$U(\sigma(x), \sigma(y)) = +\sigma(x)\sigma(y), \quad V(\sigma(x)) = -h(x)\sigma(x).$$

In the ferromagnetic Potts model,

$$U(\sigma(x), \sigma(y)) = -\mathbf{1}_{\sigma(x)=\sigma(y)}, \quad V(\sigma(x)) = -h(x)\mathbf{1}_{\sigma(x)=1}.$$

In the antiferromagnetic Potts model,

$$U(\sigma(x), \sigma(y)) = \mathbf{1}_{\sigma(x)=\sigma(y)}, \quad V(\sigma(x)) = -h(x)\mathbf{1}_{\sigma(x)=1}.$$

Monotone Inequalities

In the ferromagnetic Ising model, many monotone inequalities are established to find out the correlation between different states. Denote expectation with respect to Gibbs measure μ by $\langle\cdot\rangle$. If there exists an external field g, denote the expectation as $\langle\cdot\rangle_g$.

Theorem (C. M. Fortuin-P. W. Kasteleyn-J. Ginibre 1971)

For arbitrary external field g, the Gibbs measure satisfies the following FKG inequality.

$$\langle\sigma_x\sigma_y\rangle_g \geq \langle\sigma_x\rangle_g\langle\sigma_y\rangle_g.$$

Theorem (Griffiths–Kelly-Sherman,1967)

Denote $\sigma_A = \prod_{i\in A}\sigma_i$, if the external field g is positive, i.e. $g(u) > 0$ for all $u \in V$ then

$$\langle\sigma_A\rangle_g \geq 0,$$

$$\langle\sigma_A\sigma_B\rangle_g \geq \langle\sigma_A\rangle_g\langle\sigma_B\rangle_g.$$

Application of Monotone Inequalities

Corollary: Ferromagnetic Ising model has monotonicity of magnetism with regard to temperature and external field.
In the ferromagnetic Ising model with positive external field $h > 0$, and $0 \le \beta_1 \le \beta_2$, then

$$\langle \sigma_o \rangle_{\beta_1,h} \le \langle \sigma_o \rangle_{\beta_2,h}.$$

If $\beta > 0$, and $0 \le h_1 \le h_2$, then

$$\langle \sigma_o \rangle_{\beta,h_1} \le \langle \sigma_o \rangle_{\beta,h_2}.$$

Proof.

$$\frac{d\langle\sigma_o\rangle_{\beta,h}}{d\beta} = \frac{d}{d\beta}\frac{\sum_\sigma \sigma_o e^{-\beta H(\sigma)}}{\sum_\sigma e^{-\beta H(\sigma)}}$$

$$= \frac{\sum_\sigma \sigma_o \left(\sum_{x\sim y}\sigma_x\sigma_y + \sum_x h(x)\sigma_x\right) e^{-\beta H(\sigma)}}{\left(\sum_\sigma e^{-\beta H(\sigma)}\right)^2}$$

$$- \frac{\sum_\sigma \sigma_o e^{H(\beta,\sigma)} \sum_\sigma e^{H(\beta,\sigma)} \left(\sum_{x\sim y}\sigma_x\sigma_y + \sum_x h(x)\sigma_x\right)}{\left(\sum_\sigma e^{-\beta H(\sigma)}\right)^2}$$

$$= \sum_{x\sim y}\langle\sigma_o\sigma_x\sigma_y\rangle_{\beta,h} + \sum_x h(x)\langle\sigma_x\sigma_o\rangle_{\beta,h}$$

$$- \sum_{x\sim y}\langle\sigma_o\rangle_{\beta,h}\langle\sigma_x\sigma_y\rangle_{\beta,h} - \sum_x h(x)\langle\sigma_x\rangle_{\beta,h}\langle\sigma_o\rangle_{\beta,h}$$

≥ 0 (by GKS inequality)

Theorem (Ding-Song-Sun, 2023)

In ferromagnetic Ising model, let $g: V \to [-\infty, \infty]$ *and* $h: V \to [0, \infty]$ *be such that* $\min\{|g_v|, h_v\} < \infty$ *for all* $v \in V$*. Then for any* $o \in V$*,*

$$\langle \sigma_o \rangle_{g+h} - \langle \sigma_o \rangle_{g-h} \leqslant \langle \sigma_o \rangle_h - \langle \sigma_o \rangle_{-h}.$$

Corollary

$$0 \leq \langle \sigma_u \sigma_v \rangle_g - \langle \sigma_u \rangle_g \langle \sigma_v \rangle_g \leqslant \langle \sigma_u \sigma_v \rangle_0.$$

If we take $G = \Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$*, then*

$$\langle \sigma_0 \rangle_h^+ - \langle \sigma_0 \rangle_h^- \leqslant \langle \sigma_0 \rangle^+ - \langle \sigma_0 \rangle^- \leqslant C_1(\beta) e^{-C_2(\beta) N},$$

Generalization

Question: Can this inequality be generalized to Potts model ?

$$\mu_{g+h}(\sigma_o = 1) - \mu_{g-h}(\sigma_o = 1) \leq \mu_h(\sigma_o = 1) - \mu_{-h}(\sigma_o = 1)?$$

The answer is NO.

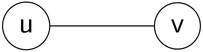


Figure: Counterexample1

Consider a simple graph with two vertices $V = \{u, v\}$, let $Q = \{1, 2, 3\}$, $h(u) = 1, h(v) = 0, g(u) = 0, g(v) = 1$, then

$$\mu_{g+h}(\sigma_u = 1) - \mu_{g-h}(\sigma_u = 1) = 0.7345,$$

$$\mu_h(\sigma_u = 1) - \mu_{-h}(\sigma_u = 1) = 0.7236.$$

Next, we study the Gibbs measure of the antiferromagnetic (AF)-Potts model without external field on d-ary trees. Let $\mathbb{T}_n^d$ be the d-ary tree. We then use $\partial\mathbb{T}_n^d$ for the set of leaves. Denote $\mu_{n,\beta}^{\xi}$ as the Gibbs measure with temperature β, tree depth n and boundary condition $\partial\mathbb{T}_n^d = \xi$.

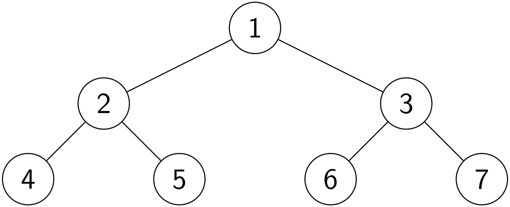


Figure: binary tree of depth-2

We initially let $n = 2$,$d = 3$. We aim to determine the distribution of the root when the colors of all leaves are given, i.e. $\mu_{2,\beta}^{\xi}(\sigma(v_1) = c)$, and its relationship with β and ξ.

AF-Potts Model on Tree

We consider the most-likely color of the spin at root

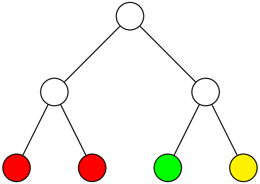
$$S(\beta, \xi) = \max_{c \in Q} \mu^{\xi}_{2,\beta}(\sigma(v_1) = c),$$

and together with the maximal probability among all boundary conditions.

$$S_m(\beta) = \max_{\xi} S(\beta, \xi).$$

Question1: Does $S(\beta, \xi)$ increase along with β ?
Question2: Under what boundary condition ξ does $S(\beta, \xi) = S_m(\beta)$?
Question3: Does $S_m(\beta)$ increase along with β?
...

Question1: Does $S(\beta, \xi)$ increase along with β ?
Answer1: No. Consider the boundary condition $\xi = (1, 1, 2, 3)$,



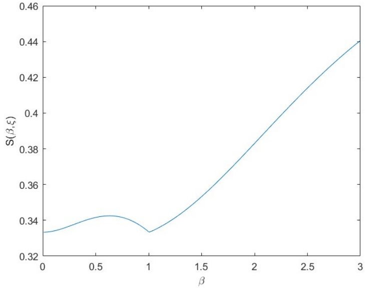
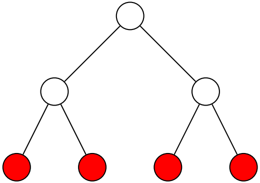
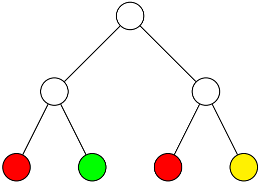


Figure: Relation between β and $S(\beta, \xi)$

Question 2: Under what boundary condition ξ does $S(\beta, \xi) = S_m(\beta)$? Does $S(\beta, (1, 1, 1, 1) = S_m(\beta)$ holds?
Answer 2: No. When$\beta < \log 2(\sqrt{2} + 1)$, the uniform boundary condition is maximum. However, when $\beta \geq \log 2(\sqrt{2} + 1)$, the maximum boundary condition is $\xi = (1, 2, 1, 3)$.





When $\beta = 0$, all the spins are independent.
When $\beta = +\infty$, all the adjacent spins must have different color.
When $q = 3$, $\beta = +\infty$ we call a boundary condition is frozen boundary, if the root is uniquely determined under this boundary condition.

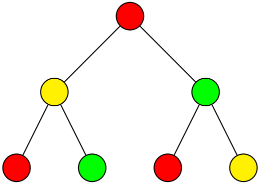


Figure: Frozen Boundary

If the tree has depth n, there are $\frac{2^{2^n-1}}{3^{2^n}}$ frozen boundary conditions.

Question3: Does $S_m(\beta)$ increase along with β?
Answer:$n = 2, 3$ Yes. For ordinary n, we conjecture yes.

Definition

Let v_1 be the root of the tree. The Potts model has **uniqueness** on the infinite d-ary tree if, for all colours $c \in Q$, it holds that

$$\limsup_{n \to \infty} \max_{\xi : \partial \mathbb{T}_n^d \to Q} \left| \mu_{n,\beta}^{\xi} \left[\sigma\left(v_1\right) = c \right] - \frac{1}{q} \right| = 0.$$

It has non-uniqueness otherwise.

Theorem (Galanis-Goldberg-Yang,2018)

When $d = 2$,$q = 3$, the 3 -state Potts model on the binary tree has uniqueness for all β.

Spreading Law

Take $d = 2$ and $q = 3$. Consider the sub-tree rooted at two sons of v_1, we aim to find the relationship between the distribution of two layers. Let $p = e^{-\beta}$.

Question: Is there any relationship between $\mu^{\xi}_{n,\beta}$, $\mu^{\xi|_L}_{n-1,\beta}$ and $\mu^{\xi|R}_{n-1,\beta}$?

Here $\xi|_L$ represents $(\xi_{2^n}, \xi_{2^n+1}, \cdots, \xi_{2^n+2^{n-1}-1})$ and $\xi|_R$ represents $(\xi_{2^n+2^{n-1}}, \xi_{2^n+2^{n-1}+1}, \cdots, \xi_{2^n+2^{n-1}-1})$. For simplicity, let $\mu_1(x) = \mu^{\xi}_{n,\beta}(\sigma(v_1) = x), \mu_2(x) = \mu^{\xi|_L}_{n-1,\beta}(\sigma(v_1) = x)$ and $\mu_3(x) = \mu^{\xi|_R}_{n-1,\beta}(\sigma(v_1) = x)$. Then we have

$$\mu_1(x) = \frac{(1-(1-p)\mu_2(x))(1-(1-p)\mu_3(x))}{\sum_{i=1}^{3}(1-(1-p)\mu_2(i))(1-(1-p)\mu_3(i))}.$$

$$\mu_1(x) = \frac{(1-(1-p)\mu_2(x))(1-(1-p)\mu_3(x))}{\sum_{i=1}^{3}(1-(1-p)\mu_2(i))(1-(1-p)\mu_3(i))}.$$

The distribution $\mu_{n,\beta}^{\xi}(\sigma(v_1)=\cdot)$ can be described as a point $A=(x,y,z)$ on the simplex $S=\{(x,y,z)\in\mathbb{R}^3, x,y,z\geq 0, x+y+z=1\}$. For any two distribution A, B on the simplex, define an operation $*: S\times S\to S$ by the law in (1), i,e.

$$A*B=(\frac{(1-(1-p)x_A)(1-(1-p)x_B)}{r},$$

$$\frac{(1-(1-p)y_A)(1-(1-p)y_B)}{r},\frac{(1-(1-p)z_A)(1-(1-p)z_B)}{r}),$$

where r is a normalizing constant.

The spreading of the distribution is a process like:

- Starting from 2^n points $A_0^1, A_0^2, \cdots, A_0^{2^n}$. The coordinate of each point is at $(1,0,0), (0,1,0)$ or $(0,0,1)$.
- At each time $t \geq 1$, we take $A_t^i = A_{t-1}^{2i-1} * A_{t-1}^{2i}$.
- We aim to prove: $\lim_{n\to\infty} A_n^1 = (1/3, 1/3, 1/3)$.

Question: Is there any function $f : S \to \mathbb{R}$ such that for some $\epsilon > 0$ and arbitrary $A, B \in S$

- $(1/3, 1/3, 1/3)$ is the minimum of the function,
- $f(A * B) < \max(f(A), f(B))$?
- $\max_i f(A_k^i) < \max_i f(A_{k-1}^i)$?

Example : $f(A) = \frac{\max\{x,y,z\}}{\min\{x,y,z\}}$,$f(A) = x^2 + y^2 + z^2$ and $f(A) = -(x\log x + y \log y + z \log z)$ is not true.

Two-step Recursion

Theorem (Galanis-Goldberg-Yang, 2018)

If we take $f(A) = \frac{\max\{x_A, y_A, z_A\}}{\min\{x_A, y_A, z_A\}}$, *then*

1. *For sufficient large* n, k, *their is* $\max_i f(A_k^i) \leq \frac{53}{27}$.
2. *Under the condition of (1), the two-step recursion works. i.e. for* $k' > k$,$\max_i f(A_{k'}^i) < \max_i f(A_{k'-2}^i)$.
3. *letting* $n \to \infty$, $f(A_n^1) \to 1$.

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