Dynamics on the Character Variety of the Fricke Spaces of Orientable Surfaces on Two Generators

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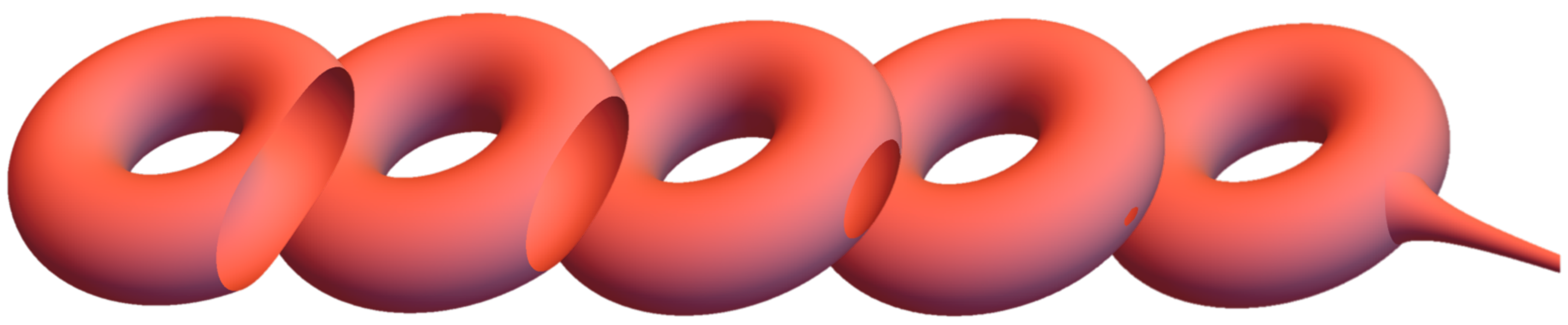
Research at Summer@ICERM 2018 with Jonghyun Lee and Dr. William Goldman

Abstract

The Fricke Spaces of the orientable topological surfaces on two generators lie in a family of cubic surfaces composed of points which represent the hyperbolic metrics on the surface by using SL(2,ℂ) character varieties. The mapping class group of the torus acts ergodicly on its Fricke space, a result proved by William Goldman in his paper *The modular group action on real SL(2)-characters of a one-holed torus* – we visualize every step of this argument using Mathematica.

The Once-Punctured Torus and One-Holed Torus

We will be focusing on tori with one boundary component. If the length of that geodesic boundary is non-trivial, then we get a torus with a hole (left). If the boundary length is 0, then we have a torus with a puncture (right).



Fundamental Group and Mapping Class Group

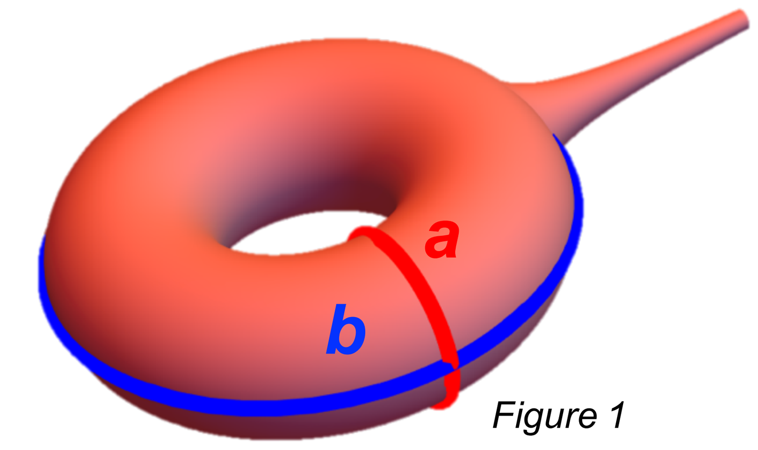


Figure 1

The once-punctured torus is a topological surface on two generators, that is, the set of homotopy (continuous deformation) classes of curves on its surface is the fundamental group F_2, the free group on two generators, specifically the red and blue curves on the torus to the left.

If you name these curves *a* and *b*, then every homotopy class of curves can be represented with a word on *a* and *b*, such as $aaba^{-1}b$. *Figure 2* is an example. Geometrically, to get from one curve to another we apply Dehn Twists, which is a construction that wraps a curve around one extra time in the meridian direction (like *a*) or in the longitude direction (like *b*). Algebraically, this is called the Mapping Class Group (MCG) action on the torus, and corresponds to altering a word by applying *a*, *b*, a^{-1}, or b^{-1}.

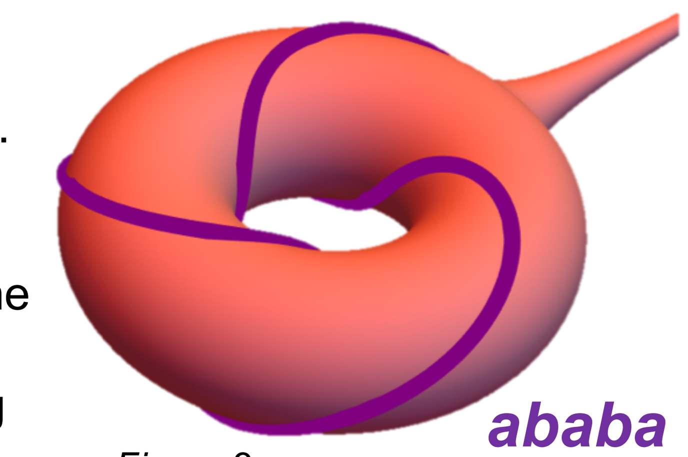
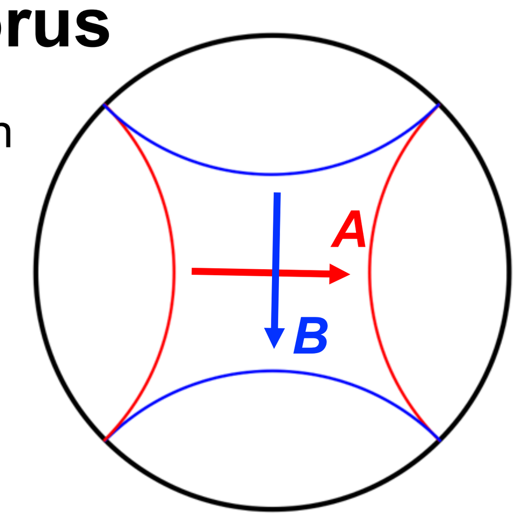


Figure 2

Hyperbolic Metrics on the Once-Punctured Torus



The once-punctured torus emits only hyperbolic metrics, meaning that the fundamental domain should be an ideal quadrilateral, which we depict on the Poincaré disc. We now move our discussion of the hyperbolic metrics on a torus to the universal cover – to the right we can see two red sides that are identified, corresponding to *a* in *Figure 1*, and two identified blue sides corresponding to curve *b*. The hyperbolic metric is entirely described by two hyperbolic isometries *A* and *B* that take the sides to each other.

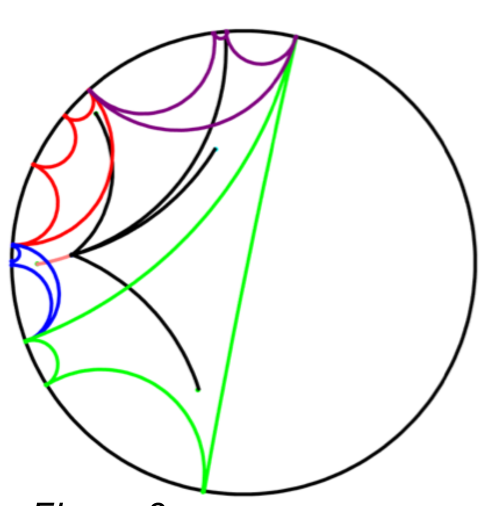


Figure 3

We can tessellate the entire covering space with this ideal quadrilateral, allowing us then to see the geodesic between any two points. In *Figure 3* we see an example of this – the middle red-blue-green-purple quadrilateral is the fundamental domain, and each surrounding quadrilateral is the translation under the hyperbolic isometry *A*, *B*, A^{-1}, or B^{-1}. In the main quadrilateral we see two points connected by a black geodesic, as well as black geodesics connecting the first point to the image of the second in each of the four translated quadrilaterals. In pink we have highlighted the geodesic in the metric we are trying to represent.

Representations of Metrics using SL(2,ℂ) Character Varieties

$$\pi_1 = F_2 \longrightarrow SL(2,\mathbb{C})$$

$$\{a,b\} \longrightarrow \{A,B\}$$

We've reduced the task of parameterizing the hyperbolic metrics on the punctured torus to assigning a hyperbolic metric, that is, a matrix in SL(2,ℂ), to each of the two elements of the fundamental group. This group homomorphism, after taking the categorical quotient, is called a character variety.

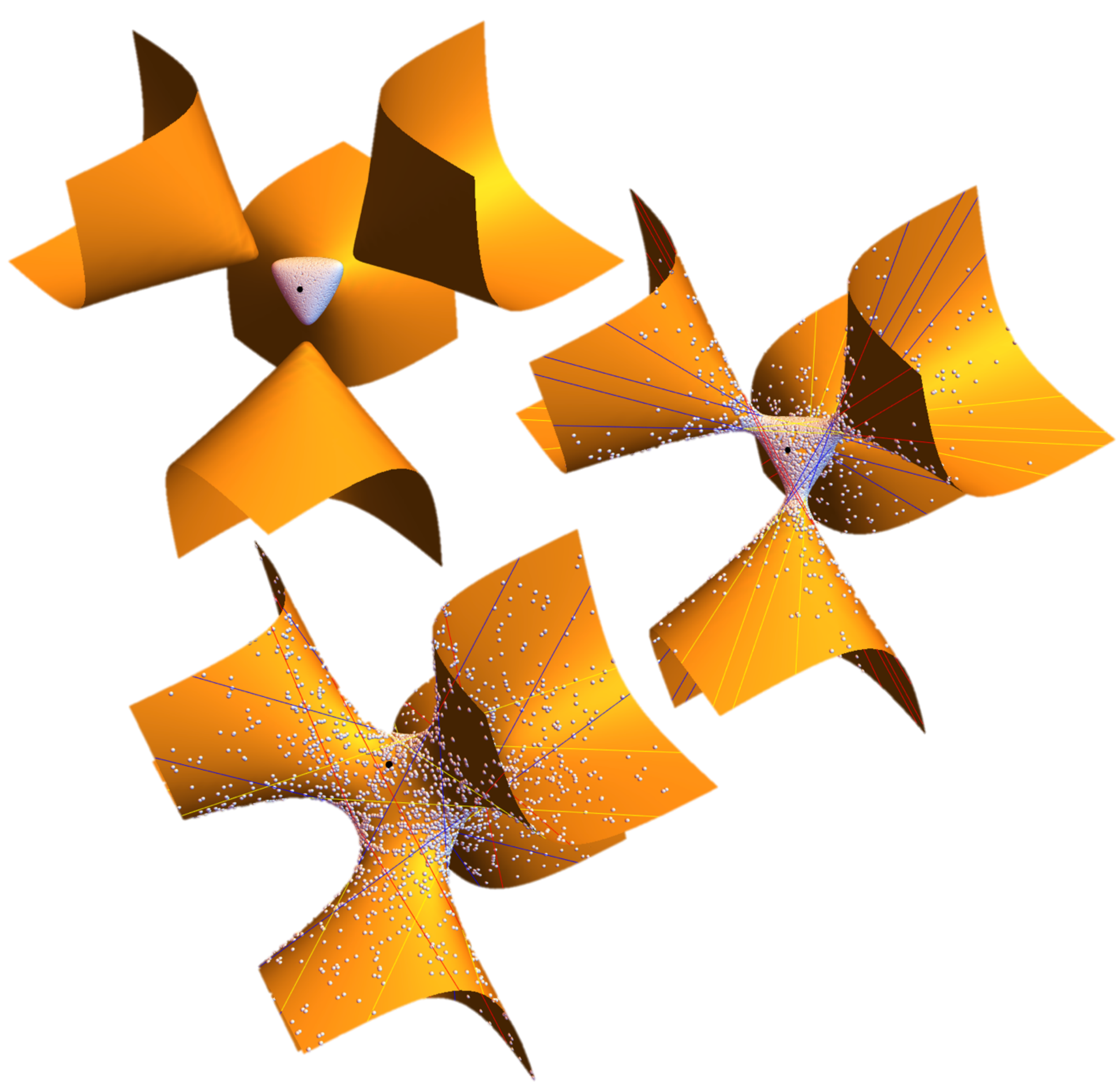
Commutator Relation and Cubic Equation

$$x = \mathrm{Tr}(A)$$
$$y = \mathrm{Tr}(B)$$
$$z = \mathrm{Tr}(AB)$$

If we assign values x, y, and z to the traces of *A*, *B*, and *AB*, then this commutator relation is equivalent to the cubic equation below. Different geodesic boundary lengths will force a different k value. Using trace relations we are able to derive this cubic equation. In the case that the geodesic boundary is trivial (a puncture), k = -2.

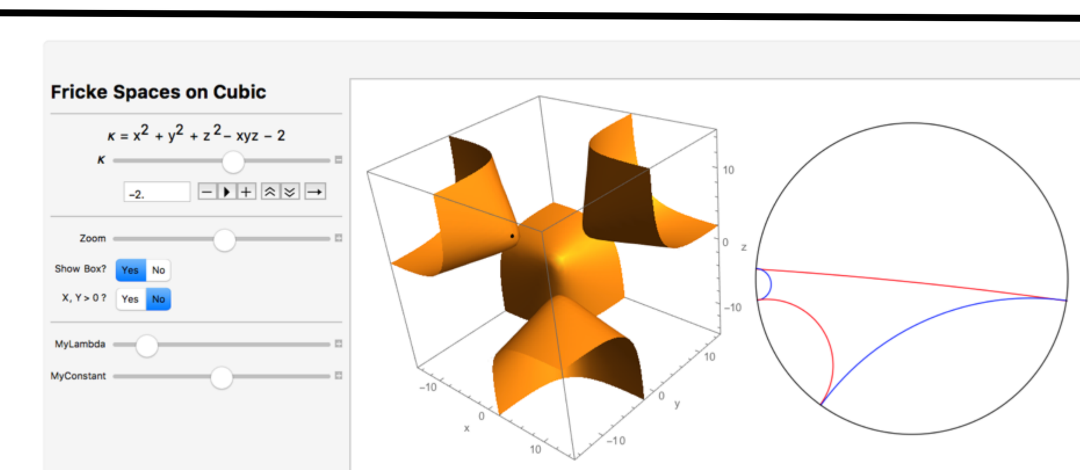
$$k = \mathrm{Tr}(ABA^{-1}B^{-1}) = -2 \rightarrow x^2 + y^2 + z^2 - xyz - 2 = k = -2$$

Ergodicity of the Mapping Class Group Action in the Complement of the Wandering Domain at k > 2 and on the Compact Component -2 < k < 2

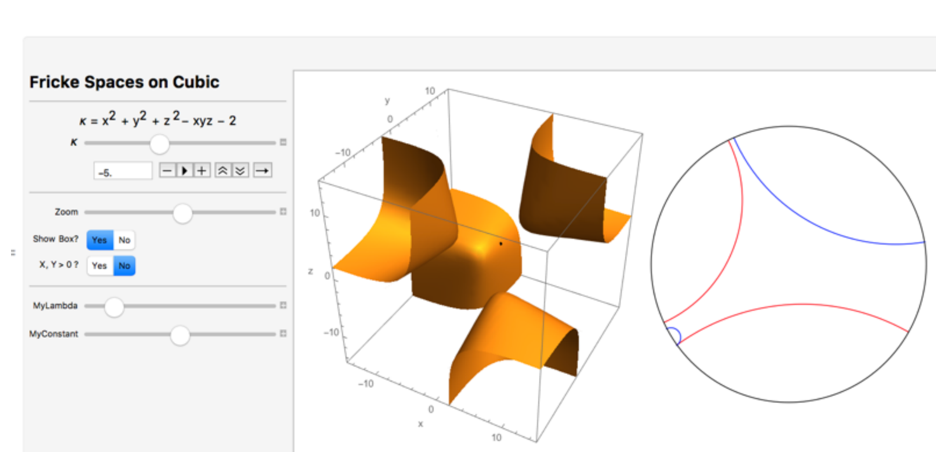


The Fricke Space of the Once-Punctured Torus

We created an interactive model where the user clicks the cubic surface to see the associated hyperbolic metric on the Poincaré disc side-by-side. To the right is the Fricke space for the one-punctured torus.



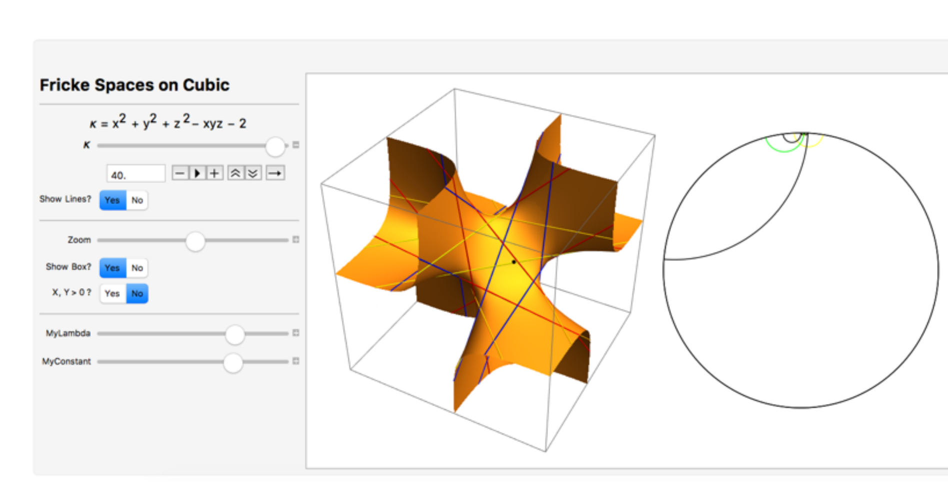
The Fricke Space of the One-Holed Torus



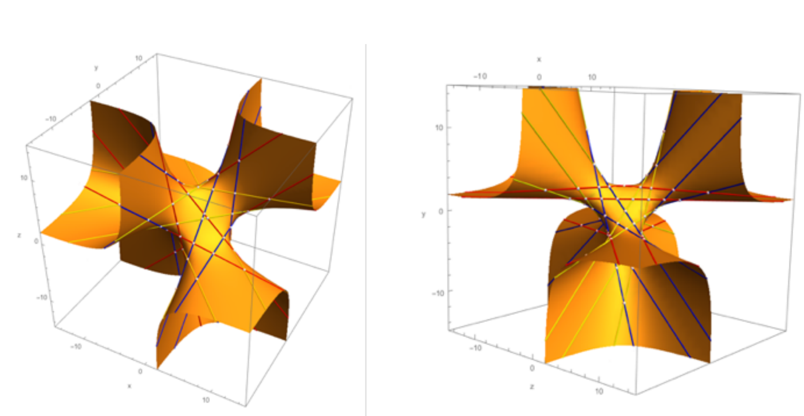
If we lower the level set k below -2 we get the Fricke Space of a torus with a hole with non-trivial geodesic boundary. We see that the fundamental domain in this case is still on the Poincaré disc because these metrics too are hyperbolic, however it is not a closed quadrilateral.

The Fricke Space of the Hyperbolic Pair of Pants

When we raise the level set past 18, an Eckerd point appears and disappears carved out by the 27 lines, creating a new triangle that is of particular interest. This triangle is a wandering domain on the surface under the mapping class group action, and the points in it correspond to metrics on the hyperbolic pair of pants.



The Classical 27 Lines on the Projective Cubic



It is a classical result that the general projective cubic surface has 27 lines. The 24 non-projective lines are highly symmetrical. We have classified them into "P Lines" and "C Lines" as there are two pairs of parallel lines and two sets of crossing lines, respectively, for each axis x, y, and z.