Dynamics on the Character Variety of the Fricke Spaces of Orientable Surfaces on Two Generators

Research at Summer@ICERM 2018 by Ajeet Gary, Jonghyun Lee and Dr. William Goldman

Abstract The Fricke Spaces of the orientable topological surfaces on two generators lie in a family of cubic surfaces, where by using SL(2,ℂ) character varieties we may represent each hyperbolic metric on these structures as a unique point on the cubic surfaces. The mapping class group action on the torus acts ergodicly on the Fricke space, a result proved by Dr. William Goldman – we visualize every step of this argument using Mathematica, creating interactive tools.

The Once-Punctured Torus, Fundamental Group and Mapping Class Group

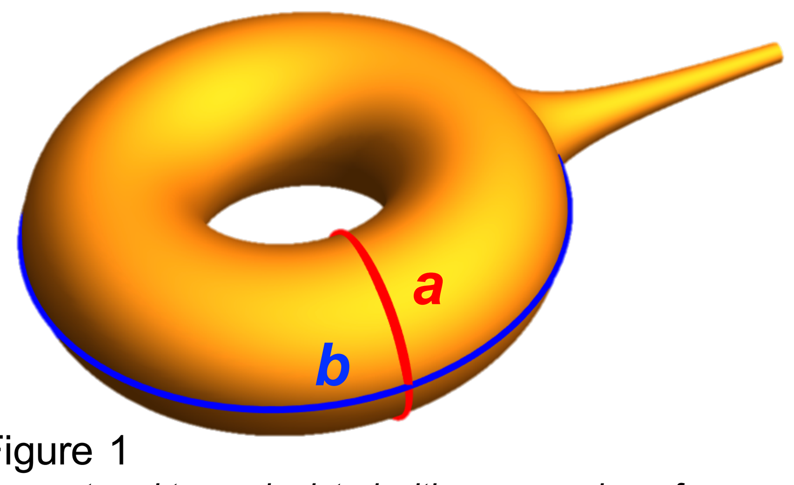


Figure 1
A punctured torus depicted with one member of each of the two unique curves up to homotopy

The once-punctured torus is a torus with one point removed, a topological surface on two generators, that is, the set of homotopy (continuous deformation) classes of curves on its surface is the fundamental group F_2, the free group on two generators, specifically the red and blue curves in *Figure 1*. If you name these curves *a* and *b*, then every homotopy class of curves can be represented with a word on *a* and *b*, such as $aaba^{-1}b$. Geometrically, to get from one curve to another we apply Dehn Twists, which is a construction that wraps a curve around one extra time in the meridian direction (like *a*) or in the longitude direction (like *b*). Algebraically, this is called the Mapping Class Group (MCG) action on the torus, and corresponds to altering a word by applying *a*, *b*, a^{-1}, or b^{-1}.

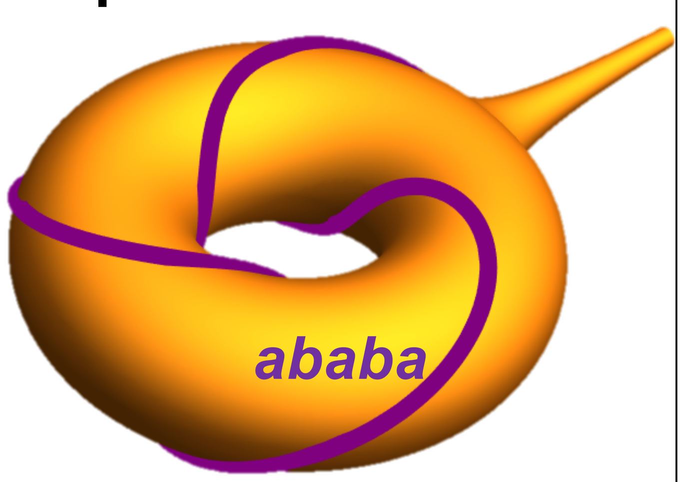


Figure 2
A punctured torus with the curve ababa

Hyperbolic Metrics on the Once-Punctured Torus

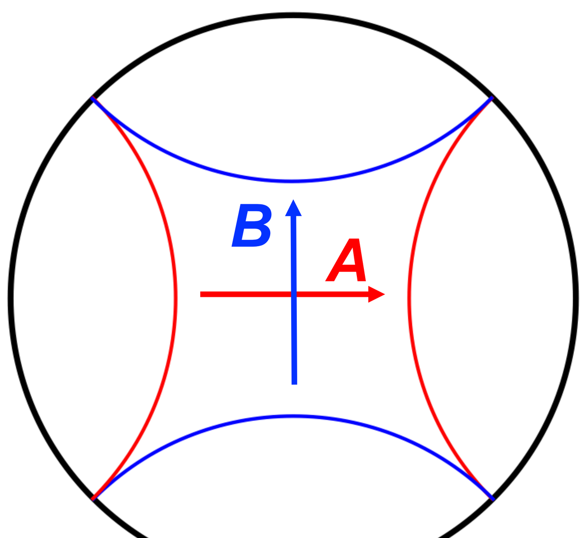


Figure 3
Hyperbolic metric on the once-punctured torus raised to the universal cover.

The once-punctured torus emits only hyperbolic metrics, meaning that the fundamental domain should be an ideal quadrilateral on, for example, the Poincaré disc, as opposed to a normal quadrilateral for the torus with no boundary component. We now move our discussion of the hyperbolic metrics on a torus to the universal cover – in *Figure 3* we can see two red sides that are identified, corresponding to *a* in *Figure 1*, and two identified blue sides corresponding to curve *b*. The hyperbolic metric is entirely described by two hyperbolic isometries *A* and *B* that take the sides to each other. We can tessellate the entire covering space with this ideal quadrilateral, allowing us then to see the geodesic between any two points. In *Figure 4* we see an example of this – the middle red-blue-green-purple quadrilateral is the fundamental domain, and each surrounding quadrilateral is the translation under the hyperbolic isometry *A*, *B*, A^{-1}, or B^{-1}. In the main quadrilateral we see two points connected by a black geodesic, as well as black geodesics connecting the first point to the image of the second in each of the four translated quadrilaterals. In pink we have highlighted the geodesic in the metric we are trying to represent – the shortest distance between two points in the universal cover sometimes passes over to the other copies of the fundamental domain, and this information is completely encapsulated in our pictures like *Figure 3*. We draw pictures of hyperbolic geodesics on the Poincaré disc like *Figure 3* using the University of Maryland's Experimental Geometry Lab's HyperbolicToolkit_v1.0.m Mathematica package.

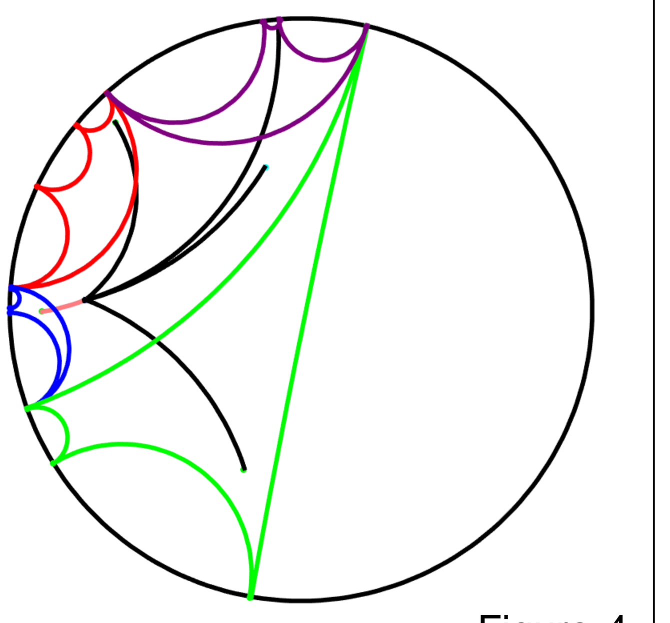


Figure 4
Ideal quadrilateral tessellated one step with the geodesic between two points highlighted.

Representations of Metrics using SL(2,ℂ) Character Varieties

$$\pi_1 = F_2 \longrightarrow SL(2,\mathbb{C})$$

$$\{a,b\} \longrightarrow \{A,B\}$$

We've reduced the task of parameterizing the hyperbolic metrics on the punctured torus to assigning a hyperbolic metric, that is, a matrix in SL(2,ℂ), to each of the two elements of the fundamental group. This group homomorphism, after taking the categorical quotient, is called a character variety.

Commutator Relation and Cubic Equation

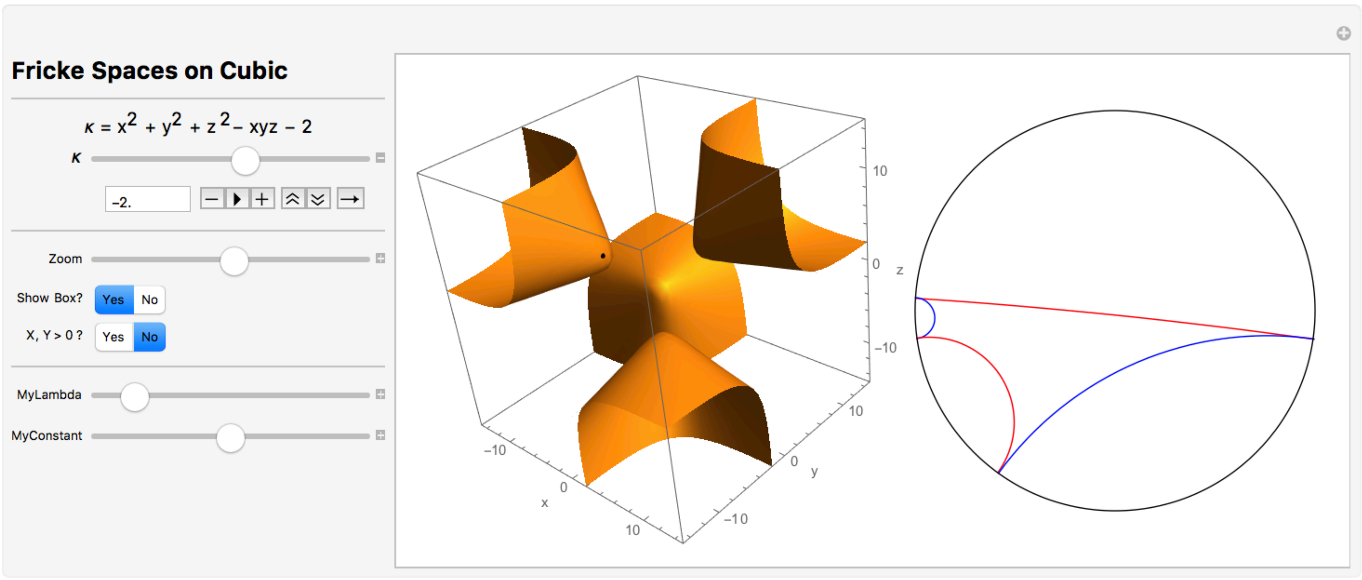
$$x = Tr(A) \qquad k = Tr(ABA^{-1}B^{-1}) = -2 \rightarrow$$

$$y = Tr(B) \qquad x^2 + y^2 + z^2 - xyz - 2 = k = -2$$

$$z = Tr(AB)$$

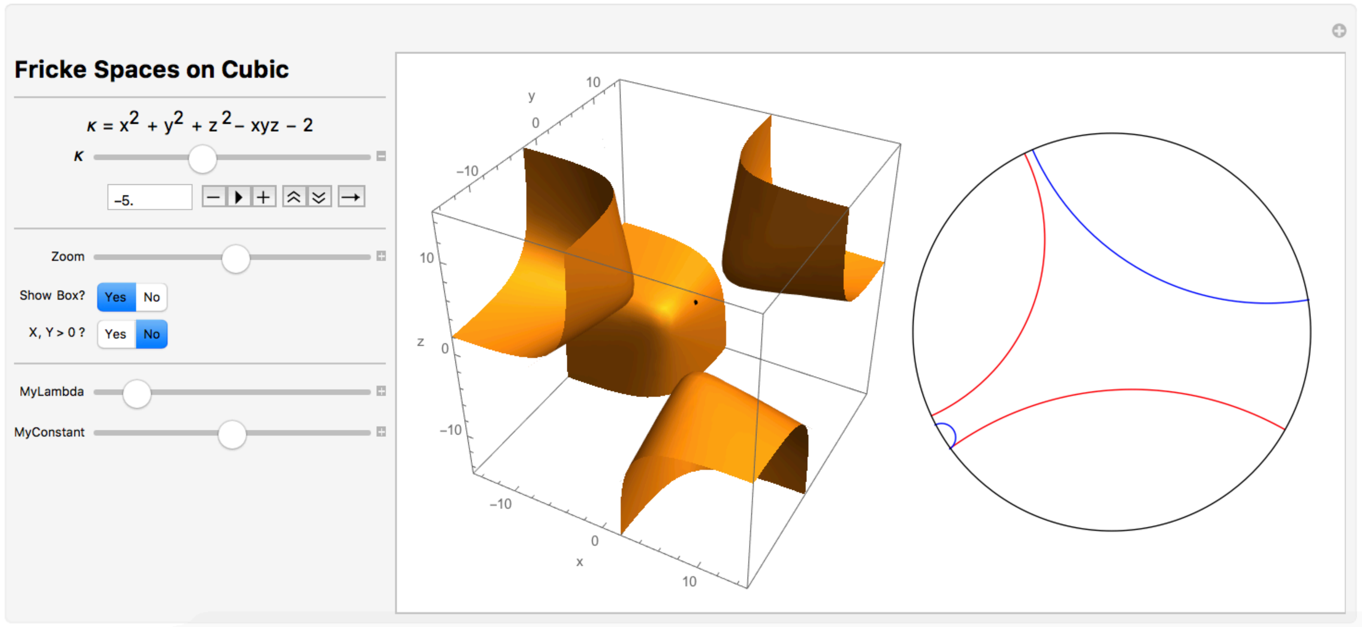
The condition that the geodesic boundary around the puncture has length zero is equivalent to requiring that the trace of the commutator of the representations (under the character variety) of the generators of the fundamental group be -2. If we assign values x, y and z to the traces of *A*, *B* and *AB*, respectively, then this commutator relation is equivalent to the cubic equation above. Different geodesic boundary lengths, such as for the one-holed torus, will force a different k value. Using certain trace relations we are able to derive this cubic equation.

The Fricke Space of the Once-Punctured Torus



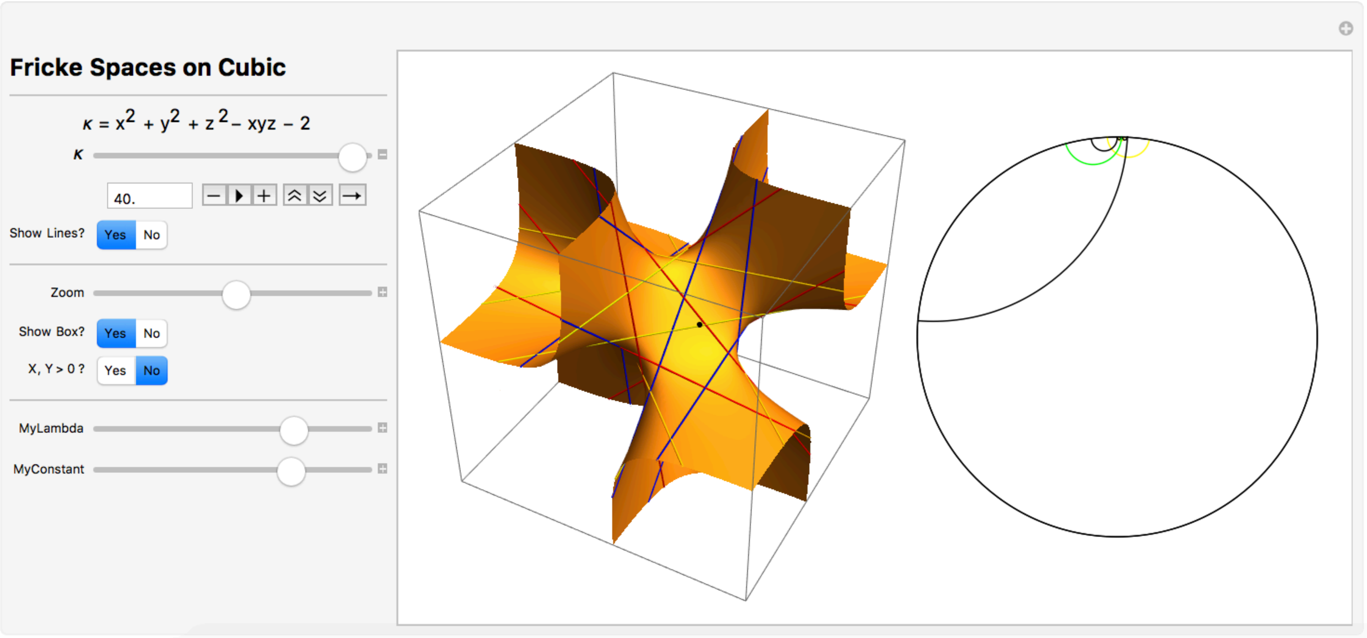
We created an interactive model where the user clicks the cubic surface to see the associated hyperbolic metric on the Poincaré disc side-by-side.

The Fricke Space of the Once-Holed Torus



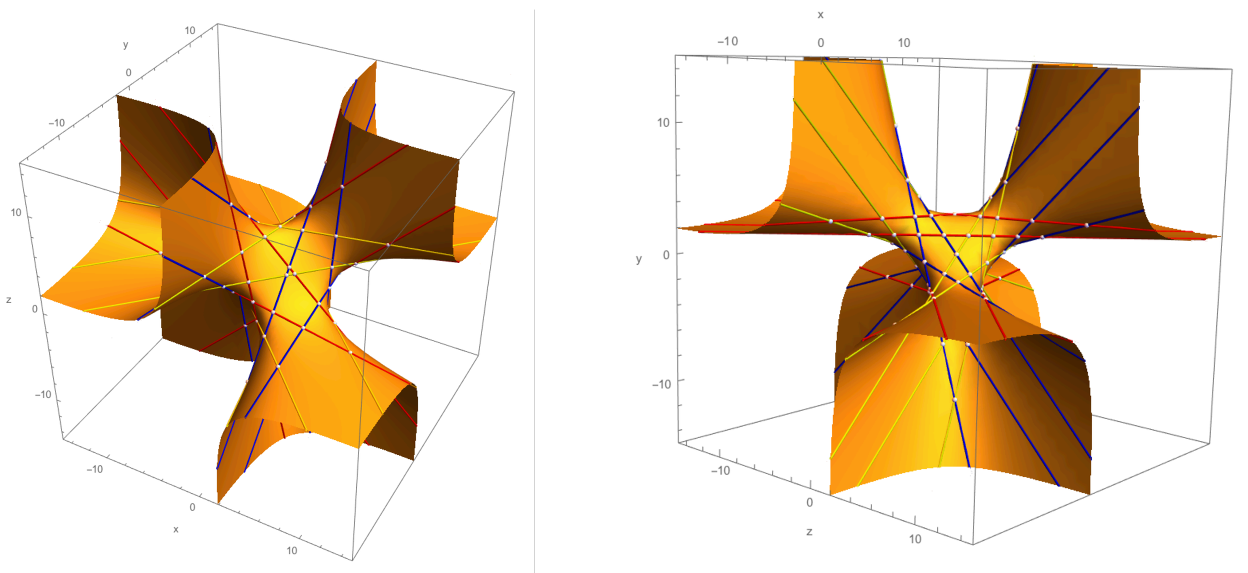
If we lower the level set k below -2 we get the Fricke Space of a torus with a hole with non-trivial geodesic boundary. We see that the fundamental domain in this case is still on the Poincaré disc because these metrics too are hyperbolic, however it is not a closed quadrilateral.

The Fricke Space of the Hyperbolic Pair of Pants



When we raise the level set past 18, an Eckerd point appears and disappears carved out by the 27 lines, creating a new triangle that is of particular interest. This triangle is a wandering domain on the surface under the mapping class group action, and the points in it correspond to metrics on the hyperbolic pair of pants, an example pictures to the cubic's right (as you can see, it's quite hard to resolve).

The Classical 27 Lines on the Projective Cubic



It is a classical result that the general projective cubic surface has 27 lines – in the case of this family of cubics there are three ideals lines at infinity that we can't see here. These 24 non-projective lines are highly symmetrical. We've classified them into "P Lines" and "C Lines" as there are two pairs of parallel lines and two sets of crossing lines, respectively, for each axis x, y and z.

Ergodicity of the Mapping Class Group Action in the Complement of the Wandering Domain at k > 2 and on the Compact Component -2 < k < 2