Möbius Transformations Acting on Chains in the Complex Plane represented by Hermitian Matrices

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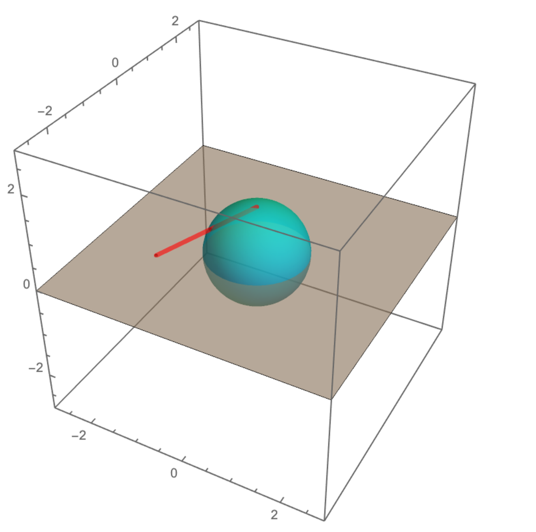
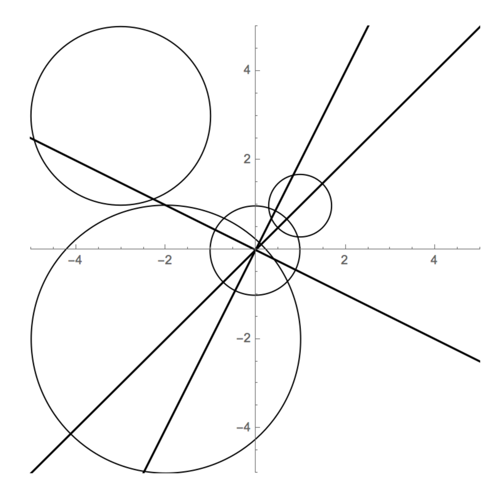


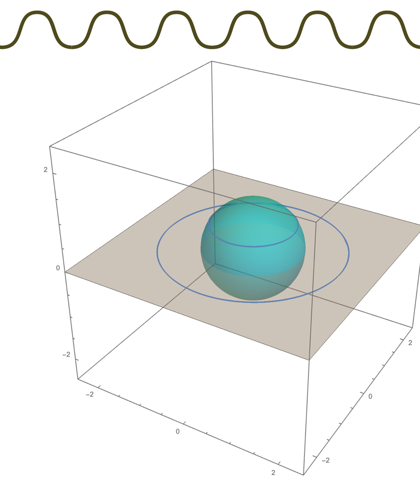
Stereographic Projection

The stereographic projection is a conformal mapping (preserving right angles) from the 2-Sphere (the unti sphere embedded in 3-space) to the complex plane. A ray is drawn from the north pole of the 2-sphere to the complex plane, and the point it hits on the complex plane is identified with the point the ray intersects on the 2-sphere. Complex infinity is identified with the north pole itself.

Chains in the Complex Plane

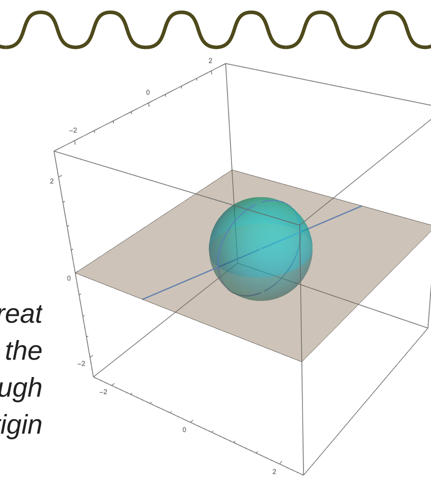
A Chain is the name we give to the set of all circles and lines in the complex plane. Circles can be identified uniquely by 3 points, namely the center, represented by a complex number, and the radius, represented by a nonnegative real scalar, and lines can be identified by a slope and y-intercept. Lines can be thought of as circles with one point at complex infinity. This is corroborated by the inverse stereographic projection back onto the 2-sphere, as all lines correspond to circles on the 2-sphere that pass through the North pole.





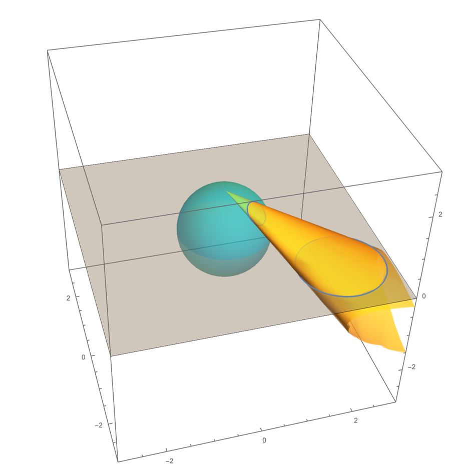
Circles around the axis through the North and South poles map to circles centered at the origin

Prime Meridians, that is, great circles passing through the North pole, map to lines through the origin



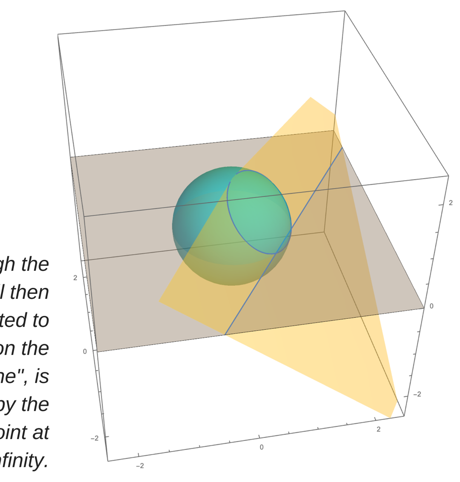
Stereographic Projection sends Chains to Chains

This can be proven by showing that any circle on the 2-sphere is a Möbius transformation of the unit circle on the 2-sphere.



Circles not centered at the origin can also be projected to from circles on the sphere. Specifically, points on the top hemisphere map outside the unit circle, and points on the bottom hemisphere map inside the unit circle.

Any circle that passes through the North pole on the sphere will then have a point being projected to complex infinity. The chain on the complex plane here, the "line", is actually fully represented by the line you see as well as a point at complex infinity.



Homogeneous Coordinates

We identify the complex 2–vector $\begin{pmatrix} z \\ w \end{pmatrix}$ with all complex multiples $x\begin{pmatrix} z \\ w \end{pmatrix}$ where x is a complex scalar.

When w ≠ 0 we generally use $\begin{pmatrix} z/w \\ 1 \end{pmatrix}$ to represent the complex point z/w in the plane, that is, we normalize the second coordinate to 1.

If w = 0, this vector is non–normalizable. The vector $\begin{pmatrix} z \\ 0 \end{pmatrix}$ is then a complex multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so we use this as the standard form.

$\begin{pmatrix} z \\ 1 \end{pmatrix}$ represents the complex point z, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents complex infinity.

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represents 0 in the complex plane, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ does not represent anything.

In mathematics, homogeneous coordinates are a system of coordinates used in projective geometry. They have the advantage that the coordinates of points, including points at infinity, can be represented using finite coordinates. If the homogeneous coordinates of a point are multiplied by a non-zero complex scalar then the resulting coordinates represent the same point. Any point in the projective plane is represented by a pair (X, Y), where X, and Y are not both 0. When Y is not 0 the point represented is the point X/Y in the Complex plane. When Y is 0 the point represented is a point at complex infinity.

Hermitian matrices

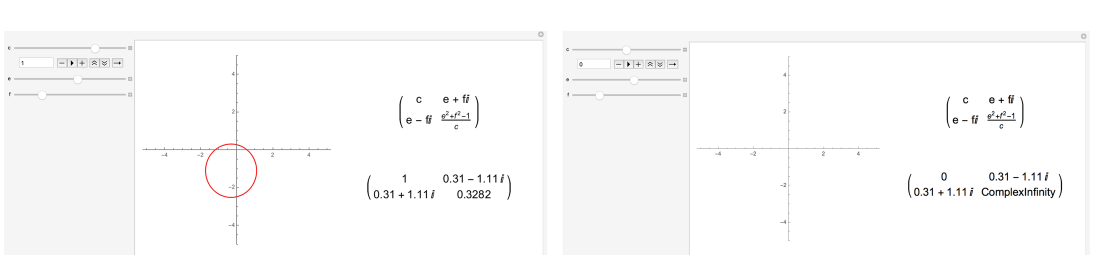
See the following example:

$$\begin{bmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{bmatrix}$$

A Hermitian matrix (or self-adjoint matrix) is a complex square matrix that is equal to its own conjugate transpose. It can be understood as the complex extension of real symmetric matrices. The entries on the main diagonal (top left to bottom right) of any Hermitian matrix are necessarily real because they have to be equal to their complex conjugate. Also, because of conjugation, for complex-valued entries the off-diagonal elements are equal to the conjugate of the element across the main diagonal.

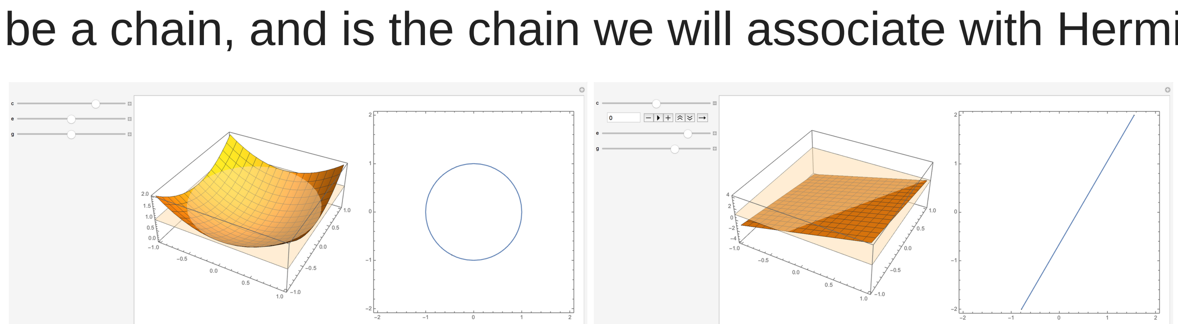
Hermitian Forms

Since we only need 3 real variables to describe all chains in the plane, we can restrict the Hermitian matrices we're looking at to just those with determinate −1, which we'll call $\mathcal{H}_{-1}$. A Hermitian Form is a function associated with a Hermitian Matrix that takes a complex pair to a real number, that is, for $H \in \mathcal{H}_{-1}$, the Hermitian Form is $f_H(v,w) = v^T H \overline{w}$.



Our first attempts to display the chains with Hermitian matrices yielded problems like dividing by zero. We had trouble creating a smooth model that allowed the user to swing through all circles and lines.

The Hermitian form f associated with a hermitian matrix H is a surface over the complex plane. The level curve of this surface f = 1 will always be a chain, and is the chain we will associate with Hermitian matrix H.



We were able to overcome this obstacle and make a smooth model by plotting the chain as a level curve of the Hermitian form, and allowing the user to manipulate the entries of the Hermitian matrix.

Complex Fractional Linear Transformations aka Möbius Transformations

Complex Fractional Linear Transformations, as the name suggests, are transformations of the form

$$f(z) = \frac{az+b}{cz+d}$$

where a,b,c and d are complex constants. This set of transformations is special and arises often. Using our homogeneous coordinates, Möbius transformations take the form of 2x2 complex matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \sim \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}$$

So, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on $z \in \mathbb{C}$ represented in homogenous coordinates behaves the same as the function $f(z) = \dfrac{az+b}{cz+d}$ on z.

Möbius Transformations map Chains to Chains

Now that we have shown that chains can be represented by a unique Hermitian matrix, it is easy to show that Möbius transformations preserve chains. When we say that a Möbius transformation M acts on a Hermitian matrix H we mean that M(H) = MHM* where M* is the complex transpose of M.

(MHM*)* = M**H*M* = MHM*

meaning that the image of the Hermitian matrix under the Mobius transformation is also a Hermitian matrix, that is, chains are mapped to other chains under Möbius transformations.

Fixed Points of Möbius Transformations

Our Möbius transformations of interest are 2x2 complex matrices with determinate 1, meaning they're invertible and thus have two eigenvalues, each corresponding to (not necessarily distinct) families of eigenvectors equal up to a complex scalar. For each of these sets, the Möbius transformation in question will map all of the points represented by the complex 2-vectors in that set to complex scalar multiples of those vectors, which are also in that set, and since we identify all complex multiples as the same vector anyway, there are really only two (not necessarily distinct) eigenvectors, which, furthermore, are fixed points:

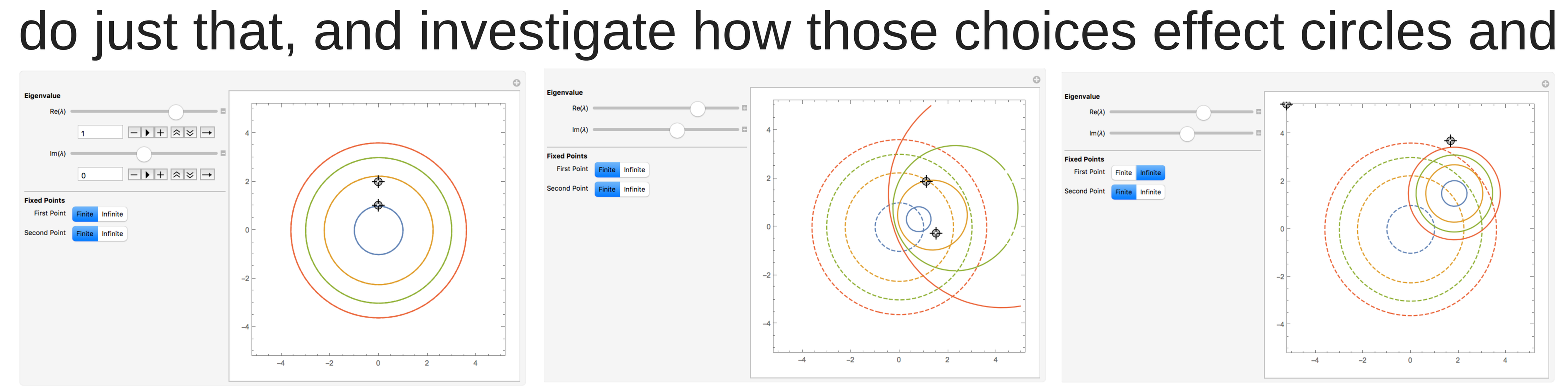
Let M be a Möbius Transformation, that is, $M \in M_2(\mathbb{C})$ with detM = 1.

Let $S_1 = \left\{ x\begin{pmatrix} z_1 \\ w_1 \end{pmatrix} \middle| x \in \mathbb{C}^* \right\}$ and $S_2 = \left\{ x\begin{pmatrix} z_2 \\ w_2 \end{pmatrix} \middle| x \in \mathbb{C}^* \right\}$ be the sets of eigenvectors of M, where $v_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} z_2 \\ w_2 \end{pmatrix}$. Then, $\forall v \in S_1$, $v_1 \sim v$, and $\forall v \in S_2$, $v_2 \sim v$.

So, $Mv_1 = \lambda v_1 \sim v_1$ and $Mv_2 = \lambda v_2 \sim v_2$, meaning that v_1 and v_2 are fixed points of M. In short, M has two spaces of eigenvectors of complex dimension 1, and since we identify all 2–vectors that are complex multiples anyway, these eigenvectors represent just one or two points, and those points map to themselves under M.

Observing the action of Möbius Transformations on Chains by choice of Fixed Points and a Complex Scalar

By the choice of two fixed points, one of which may be at complex infinity, along with a complex scalar, we can specify the Möbius Transformation we want. Below is a model that allows you to do just that, and investigate how those choices effect circles and lines.



(1) When the complex scalar is 1, we have the identity transformation.
(2) Placing a fixed point on a circle forces the circle's image to pass through that point
(3) Placing a fixed point at infinity reduces the transformation to simply a dilation and a translation (or slide).

(1) Lines are special cases, in general, they're mapped to circles. (2) Placing two fixed points on the same line forces that line's image to pass through both. (3) A chain is a line if and only if it has a point at complex infinity. By choosing the point at infinity as one of our fixed points, we ensure that the lines are mapped to lines.