PyPIC
physics manual

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1 Space charge

We assume that the bunch travels rigidly along s with velocity $\beta_0 c$:

$$\rho\,(x, y, s, t) = \rho_0\,(x, y, s - \beta_0 ct) \tag{1}$$

$$\mathbf{J}\,(x, y, s, t) = \beta_0 c\,\rho_0\,(x, y, s - \beta_0 ct)\,\hat{\mathbf{i}}_s \tag{2}$$

We define an auxiliary variable ζ as the position along the bunch:

$$\zeta = s - \beta_0 ct\,. \tag{3}$$

We call K the lab reference frame in which we have defined all equations above, and we introduce a boosted frame K' moving rigidly with the reference particle. The coordinates in the two systems are related by a Lorentz transformation [3]:

$$ct' = \gamma_0\,(ct - \beta_0 s) \tag{4}$$

$$x' = x \tag{5}$$

$$y' = y \tag{6}$$

$$s' = \gamma_0\,(s - \beta_0 ct) = \gamma_0 \zeta \tag{7}$$

The corresponding inverse transformation is:

$$ct = \gamma_0\left(ct' + \beta_0 s'\right) \tag{8}$$

$$x = x' \tag{9}$$

$$y = y' \tag{10}$$

$$s = \gamma_0\left(s' + \beta_0 ct'\right) \tag{11}$$

The quantities $\left(c\rho, J_x, J_y, J_s\right)$ form a Lorentz 4-vector and therefore they are transformed between K and K' by relationships similar to the Eqs. 4-6 [3]:

$$c\rho'\left(\mathbf{r'}, t'\right) = \gamma_0\left[c\rho\left(\mathbf{r}\left(\mathbf{r'}, t'\right), t\left(\mathbf{r'}, t'\right)\right) - \beta_0 J_s\left(\mathbf{r}\left(\mathbf{r'}, t'\right), t\left(\mathbf{r'}, t'\right)\right)\right] \tag{12}$$

$$J'_s\left(\mathbf{r'}, t'\right) = \gamma_0\left[J_s\left(\mathbf{r}\left(\mathbf{r'}, t'\right), t\left(\mathbf{r'}, t'\right)\right) - \beta_0 c\rho\left(\mathbf{r}\left(\mathbf{r'}, t'\right), t\left(\mathbf{r'}, t'\right)\right)\right] \tag{13}$$

where the transformations $\mathbf{r}\left(\mathbf{r'}, t'\right)$ and $t\left(\mathbf{r'}, t'\right)$ are defined by Eqs. 8 and 11 respectively. The transverse components J_x and J_y of the current vector are invariant for our transformation, and are anyhow zero in our case.
Using Eq. 2 these become:

$$\rho'\left(\mathbf{r'}, t'\right) = \frac{1}{\gamma_0}\rho\left(\mathbf{r}\left(\mathbf{r'}, t'\right), t\left(\mathbf{r'}, t'\right)\right) \tag{14}$$

$$J'_s\left(\mathbf{r'}, t'\right) = 0 \tag{15}$$

Using Eqs. 1 and 8-10, we obtain:

$$\rho\left(x', y', s(s', t'), t(s', t')\right) = \rho_0\left(x', y', s(s', t') - \beta_0 c\, t(s', t')\right) \tag{16}$$

From Eq. 7 we get:

$$s(s', t') - \beta_0 c\, t(s', t') = \frac{s'}{\gamma_0} \tag{17}$$

where the coordinate t' has disappeared.
We can therefore write:

$$\rho'\left(x', y', s', t'\right) = \frac{1}{\gamma_0}\rho_0\left(x', y', \frac{s'}{\gamma_0}\right) \tag{18}$$

The electric potential in the bunch frame is solution of Poisson's equation:

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{\rho'(x', y', s')}{\varepsilon_0} \tag{19}$$

From Eq. 18 we can write:

$$\frac{\partial^2\phi'}{\partial x'^2} + \frac{\partial^2\phi'}{\partial y'^2} + \frac{\partial^2\phi'}{\partial s'^2} = -\frac{1}{\gamma_0\varepsilon_0}\rho_0\left(x', y', \frac{s'}{\gamma_0}\right) \tag{20}$$

We now make the substitution:

$$\zeta = \frac{s'}{\gamma_0} \tag{21}$$

obtained from Eq. 7, which allows to rewrite Eq. 20 as:

$$\frac{\partial^2\phi'}{\partial x^2} + \frac{\partial^2\phi'}{\partial y^2} + \frac{1}{\gamma_0^2}\frac{\partial^2\phi'}{\partial \zeta^2} = -\frac{1}{\gamma_0\varepsilon_0}\rho_0\left(x, y, \zeta\right) \tag{22}$$

Here we have dropped the "$'$" sign from x and y as these coordinates are unaffected by the Lorentz boost.
The quantities $\left(\frac{\phi}{c}, A_x, A_y, A_s\right)$ form a Lorentz 4-vector, we can show that the s component of the vector potential in the lab frame vanishes:

$$\phi = \gamma_0\left(\phi' + \beta_0 c A_s'\right) \tag{23}$$

$$A_s = A_s' + \beta_0\frac{\phi'}{c} \tag{24}$$

In the bunch frame the charges are at rest therefore $A_x' = A_y' = A_z' = 0$ therefore:

$$\phi = \gamma_0\phi' \tag{25}$$

$$A_s = \beta_0\frac{\phi'}{c} = \frac{\beta_0}{\gamma_0 c}\phi \tag{26}$$

Combining Eq. 25 with Eq. 22 we obtain the equation in ϕ:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{1}{\gamma_0^2}\frac{\partial^2\phi}{\partial \zeta^2} = -\frac{1}{\varepsilon_0}\rho_0\left(x, y, \zeta\right) \tag{27}$$

2 Lorentz force

We stay in the thin lens approximation so we approximate the velocity vector of the particle as:

$$\mathbf{v} = \beta c\,\hat{\mathbf{i}}_s \tag{28}$$

We want to compute the Lorentz force acting on the particle:

$$\begin{aligned}\mathbf{F} &= q\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \beta c\,\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A})\right)\\ &= q\left(-\nabla\phi - \frac{\beta_0}{\gamma_0 c}\frac{\partial \phi}{\partial t}\hat{\mathbf{i}}_s + \beta c\,\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A})\right)\end{aligned} \tag{29}$$

We compute the vector product:

$$\begin{aligned}\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) &= \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right)\hat{\mathbf{i}}_x + \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{\mathbf{i}}_y\\ &= \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right)\hat{\mathbf{i}}_x + \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{\mathbf{i}}_y + \underbrace{\left(\frac{\partial A_s}{\partial s} - \frac{\partial A_s}{\partial s}\right)}_{=0}\hat{\mathbf{i}}_s\\ &= \nabla A_s - \frac{\partial \mathbf{A}}{\partial s}\end{aligned} \tag{30}$$

We replace:

$$\mathbf{F} = q\left(-\nabla\phi - \frac{\beta_0}{\gamma_0 c}\frac{\partial \phi}{\partial t}\hat{\mathbf{i}}_s + \beta\beta_0\nabla\phi - \frac{\beta\beta_0}{\gamma_0}\frac{\partial \phi}{\partial s}\hat{\mathbf{i}}_s\right) \tag{31}$$

The potentials will have the same form as the sources (this can be shown explicitly using the Lorentz transformations):

$$\phi(x, y, s, t) = \phi\left(x, y, t - \frac{s}{\beta_0 c}\right) \tag{32}$$

For a function in this form we can write:

$$\frac{\partial \phi}{\partial s} = \frac{\partial}{\partial \zeta} = -\frac{1}{\beta_0 c}\frac{\partial \phi}{\partial t} \tag{33}$$

obtaining:

$$\mathbf{F} = q\left(-\nabla\phi + \frac{\beta_0^2}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s + \beta\beta_0\nabla\phi - \frac{\beta\beta_0}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s\right) \tag{34}$$

Reorganizing:

$$\mathbf{F} = -q(1-\beta\beta_0)\nabla\phi - \frac{\beta_0(\beta-\beta_0)}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s \tag{35}$$

Explicit dependencies:

$$F_x(x, y, \zeta(t)) = -q(1-\beta\beta_0)\frac{\partial \phi}{\partial x}(x, y, \zeta(t)) \tag{36}$$

$$F_y(x, y, \zeta(t)) = -q(1-\beta\beta_0)\frac{\partial \phi}{\partial y}(x, y, \zeta(t)) \tag{37}$$

$$F_z(x, y, \zeta(t)) = -q\left(1 - \beta\beta_0 - \frac{\beta_0(\beta-\beta_0)}{\gamma_0}\right)\frac{\partial \phi}{\partial \zeta}(x, y, \zeta(t)) \tag{38}$$

Over the single interaction we neglect the particle slippage:

$$\beta = \beta_0 \tag{39}$$

$$\zeta(t) = \zeta \tag{40}$$

(in any case one would need to take into account also the dispersion in order to have the right slippage).
gives the following simplification:

$$F_x(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial x}(x,y,\zeta) \tag{41}$$

$$F_y(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial y}(x,y,\zeta) \tag{42}$$

$$F_z(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial \zeta}(x,y,\zeta) \tag{43}$$

In this way the force over the single interaction becomes independent on time and therefore we can compute the kicks simply as:

$$\Delta\mathbf{P} = \frac{L}{\beta_0 c}\mathbf{F} \tag{44}$$

from which we can compute the kicks on the normalized momenta ($P_0 = m_0\beta_0\gamma_0 c$):

$$\Delta p_x = \frac{m_0}{m}\frac{\Delta P_x}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial x}(x,y,\zeta) \tag{45}$$

$$\Delta p_y = \frac{m_0}{m}\frac{\Delta P_y}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial y}(x,y,\zeta) \tag{46}$$

$$\Delta\delta \simeq \Delta p_z = \frac{m_0}{m}\frac{\Delta P_z}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial \zeta}(x,y,\zeta) \tag{47}$$

Of your beam includes particles of different species (tracking of fragments), note that heree q is the charge of the kicked particle while m_0 is the mass of the reference particle.

2.1 2.5D approximation

For large enough values of γ_0, Eq. 22 can be approximated by:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{1}{\varepsilon_0}\rho_0(x,y,\zeta) \tag{48}$$

which means that we can solve a simple 2D problem for each beam slice (identified by its ζ).

2.2 Modulated 2D

Often the beam distribution can be factorized as:

$$\rho_0(x,y,\zeta) = Nq_0\lambda_0(\zeta)\rho_\perp(x,y) \tag{49}$$

where:

$$\int \lambda_0(z)\,dz = 1 \tag{50}$$

$$\int \rho_\perp(x,y)\,dx\,dy = 1 \tag{51}$$

In this case the potential can be factorized as:

$$\phi(x,y,\zeta) = q_0\lambda_0(\zeta)\phi_\perp(x,y) \tag{52}$$

where $\phi_\perp(x,y)$ is the solution of the following 2D Poisson equation:

$$\frac{\partial^2\phi_\perp}{\partial x^2} + \frac{\partial^2\phi_\perp}{\partial y^2} = -\frac{1}{\varepsilon_0}\rho_\perp\,(x,y) \tag{53}$$

The kick can be expressed as:

$$\Delta p_x = \frac{m_0}{m}\frac{\Delta P_x}{P_0} = -\frac{qq_0NL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\,\lambda_0(\zeta)\,\frac{\partial\phi}{\partial x}\,(x,y) \tag{54}$$

$$\Delta p_y = \frac{m_0}{m}\frac{\Delta P_y}{P_0} = -\frac{qq_0NL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\,\lambda_0(\zeta)\,\frac{\partial\phi}{\partial y}\,(x,y) \tag{55}$$

$$\Delta\delta \simeq \Delta p_z = \frac{m_0}{m}\frac{\Delta P_z}{P_0} = -\frac{qq_0NL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\,\frac{d\lambda_0}{d\zeta}(\zeta)\,\phi\,(x,y) \tag{56}$$

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