Xfields
physics manual

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1 FFT method

We illustrate the method in a single dimension then extend to multiple dimensions.

2 Space charge

We assume that the bunch travels rigidly along s with velocity $\beta_0 c$:

$$\rho\left(x, y, s, t\right) = \rho_0\left(x, y, s - \beta_0 ct\right) \tag{1}$$

$$\mathbf{J}\left(x, y, s, t\right) = \beta_0 c\, \rho_0\left(x, y, s - \beta_0 ct\right) \hat{\mathbf{i}}_s \tag{2}$$

We define an auxiliary variable ζ as the position along the bunch:

$$\zeta = s - \beta_0 ct\,. \tag{3}$$

We call K the lab reference frame in which we have defined all equations above, and we introduce a boosted frame K' moving rigidly with the reference particle. The coordinates in the two systems are related by a Lorentz transformation [3]:

$$ct' = \gamma_0\left(ct - \beta_0 s\right) \tag{4}$$

$$x' = x \tag{5}$$

$$y' = y \tag{6}$$

$$s' = \gamma_0\left(s - \beta_0 ct\right) = \gamma_0 \zeta \tag{7}$$

The corresponding inverse transformation is:

$$ct = \gamma_0\left(ct' + \beta_0 s'\right) \tag{8}$$

$$x = x' \tag{9}$$

$$y = y' \tag{10}$$

$$s = \gamma_0\left(s' + \beta_0 ct'\right) \tag{11}$$

The quantities $\left(c\rho, J_x, J_y, J_s\right)$ form a Lorentz 4-vector and therefore they are transformed between K and K' by relationships similar to the Eqs. 4-6 [3]:

$$c\rho'\left(\mathbf{r}', t'\right) = \gamma_0\left[c\rho\left(\mathbf{r}\left(\mathbf{r}', t'\right), t\left(\mathbf{r}', t'\right)\right) - \beta_0 J_s\left(\mathbf{r}\left(\mathbf{r}', t'\right), t\left(\mathbf{r}', t'\right)\right)\right] \tag{12}$$

$$J_s'\left(\mathbf{r}', t'\right) = \gamma_0\left[J_s\left(\mathbf{r}\left(\mathbf{r}', t'\right), t\left(\mathbf{r}', t'\right)\right) - \beta_0 c\rho\left(\mathbf{r}\left(\mathbf{r}', t'\right), t\left(\mathbf{r}', t'\right)\right)\right] \tag{13}$$

where the transformations $\mathbf{r}\left(\mathbf{r}', t'\right)$ and $t\left(\mathbf{r}', t'\right)$ are defined by Eqs. 8 and 11 respectively. The transverse components J_x and J_y of the current vector are invariant for our transformation, and are anyhow zero in our case.
Using Eq. 2 these become:

$$\rho'\left(\mathbf{r}', t'\right) = \frac{1}{\gamma_0}\rho\left(\mathbf{r}\left(\mathbf{r}', t'\right), t\left(\mathbf{r}', t'\right)\right) \tag{14}$$

$$J_s'\left(\mathbf{r}', t'\right) = 0 \tag{15}$$

Using Eqs. 1 and 8-10, we obtain:

$$\rho\left(x', y', s(s', t'), t(s', t')\right) = \rho_0\left(x', y', s(s', t') - \beta_0 c\, t(s', t')\right) \tag{16}$$

From Eq. 7 we get:

$$s(s', t') - \beta_0 c\, t(s', t') = \frac{s'}{\gamma_0} \tag{17}$$

where the coordinate t' has disappeared.
We can therefore write:

$$\rho'\left(x', y', s', t'\right) = \frac{1}{\gamma_0}\rho_0\left(x', y', \frac{s'}{\gamma_0}\right) \tag{18}$$

The electric potential in the bunch frame is solution of Poisson's equation:

$$\frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial y'^2} + \frac{\partial^2 \phi'}{\partial s'^2} = -\frac{\rho'(x', y', s')}{\varepsilon_0} \tag{19}$$

From Eq. 18 we can write:

$$\frac{\partial^2 \phi'}{\partial x'^2} + \frac{\partial^2 \phi'}{\partial y'^2} + \frac{\partial^2 \phi'}{\partial s'^2} = -\frac{1}{\gamma_0 \varepsilon_0}\rho_0\left(x', y', \frac{s'}{\gamma_0}\right) \tag{20}$$

We now make the substitution:

$$\zeta = \frac{s'}{\gamma_0} \tag{21}$$

obtained from Eq. 7, which allows to rewrite Eq. 20 as:

$$\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} + \frac{1}{\gamma_0^2}\frac{\partial^2 \phi'}{\partial \zeta^2} = -\frac{1}{\gamma_0 \varepsilon_0}\rho_0\left(x, y, \zeta\right) \tag{22}$$

Here we have dropped the "$'$" sign from x and y as these coordinates are unaffected by the Lorentz boost.
The quantities $\left(\frac{\phi}{c}, A_x, A_y, A_s\right)$ form a Lorentz 4-vector, we can show that the s component of the vector potential in the lab frame vanishes:

$$\phi = \gamma_0\left(\phi' + \beta_0 c A'_s\right) \tag{23}$$

$$A_s = A'_s + \beta_0 \frac{\phi'}{c} \tag{24}$$

In the bunch frame the charges are at rest therefore $A'_x = A'_y = A'_z = 0$ therefore:

$$\phi = \gamma_0 \phi' \tag{25}$$

$$A_s = \beta_0 \frac{\phi'}{c} = \frac{\beta_0}{\gamma_0 c}\phi \tag{26}$$

Combining Eq. 25 with Eq. 22 we obtain the equation in ϕ:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{\gamma_0^2}\frac{\partial^2 \phi}{\partial \zeta^2} = -\frac{1}{\varepsilon_0}\rho_0\left(x, y, \zeta\right) \tag{27}$$

3 Lorentz force

We stay in the thin lens approximation so we approximate the velocity vector of the particle as:

$$\mathbf{v} = \beta c\,\hat{\mathbf{i}}_s \tag{28}$$

We want to compute the Lorentz force acting on the particle:

$$\begin{aligned}\mathbf{F} &= q\left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \beta c\,\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A})\right)\\ &= q\left(-\nabla\phi - \frac{\beta_0}{\gamma_0 c}\frac{\partial \phi}{\partial t}\hat{\mathbf{i}}_s + \beta c\,\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A})\right)\end{aligned} \tag{29}$$

We compute the vector product:

$$\begin{aligned}\hat{\mathbf{i}}_s \times (\nabla \times \mathbf{A}) &= \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right)\hat{\mathbf{i}}_x + \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{\mathbf{i}}_y\\ &= \left(\frac{\partial A_s}{\partial x} - \frac{\partial A_x}{\partial s}\right)\hat{\mathbf{i}}_x + \left(\frac{\partial A_s}{\partial y} - \frac{\partial A_y}{\partial s}\right)\hat{\mathbf{i}}_y + \underbrace{\left(\frac{\partial A_s}{\partial s} - \frac{\partial A_s}{\partial s}\right)}_{=0}\hat{\mathbf{i}}_s\\ &= \nabla A_s - \frac{\partial \mathbf{A}}{\partial s}\end{aligned} \tag{30}$$

We replace:

$$\mathbf{F} = q\left(-\nabla\phi - \frac{\beta_0}{\gamma_0 c}\frac{\partial \phi}{\partial t}\hat{\mathbf{i}}_s + \beta\beta_0\nabla\phi - \frac{\beta\beta_0}{\gamma_0}\frac{\partial \phi}{\partial s}\hat{\mathbf{i}}_s\right) \tag{31}$$

The potentials will have the same form as the sources (this can be shown explicitly using the Lorentz transformations):

$$\phi(x, y, s, t) = \phi\left(x, y, t - \frac{s}{\beta_0 c}\right) \tag{32}$$

For a function in this form we can write:

$$\frac{\partial \phi}{\partial s} = \frac{\partial}{\partial \zeta} = -\frac{1}{\beta_0 c}\frac{\partial \phi}{\partial t} \tag{33}$$

obtaining:

$$\mathbf{F} = q\left(-\nabla\phi + \frac{\beta_0^2}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s + \beta\beta_0\nabla\phi - \frac{\beta\beta_0}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s\right) \tag{34}$$

Reorganizing:

$$\mathbf{F} = -q(1 - \beta\beta_0)\nabla\phi - \frac{\beta_0(\beta - \beta_0)}{\gamma_0}\frac{\partial \phi}{\partial \zeta}\hat{\mathbf{i}}_s \tag{35}$$

Explicit dependencies:

$$F_x(x, y, \zeta(t)) = -q(1 - \beta\beta_0)\frac{\partial \phi}{\partial x}(x, y, \zeta(t)) \tag{36}$$

$$F_y(x, y, \zeta(t)) = -q(1 - \beta\beta_0)\frac{\partial \phi}{\partial y}(x, y, \zeta(t)) \tag{37}$$

$$F_z(x, y, \zeta(t)) = -q\left(1 - \beta\beta_0 - \frac{\beta_0(\beta - \beta_0)}{\gamma_0}\right)\frac{\partial \phi}{\partial \zeta}(x, y, \zeta(t)) \tag{38}$$

Over the single interaction we neglect the particle slippage:

$$\beta = \beta_0 \tag{39}$$
$$\zeta(t) = \zeta \tag{40}$$

(in any case one would need to take into account also the dispersion in order to have the right slippage).
gives the following simplification:

$$F_x(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial x}(x,y,\zeta) \tag{41}$$
$$F_y(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial y}(x,y,\zeta) \tag{42}$$
$$F_z(x,y,\zeta) = -q(1-\beta_0^2)\frac{\partial\phi}{\partial \zeta}(x,y,\zeta) \tag{43}$$

In this way the force over the single interaction becomes independent on time and therefore we can compute the kicks simply as:

$$\Delta\mathbf{P} = \frac{L}{\beta_0 c}\mathbf{F} \tag{44}$$

from which we can compute the kicks on the normalized momenta ($P_0 = m_0\beta_0\gamma_0 c$):

$$\Delta p_x = \frac{m_0}{m}\frac{\Delta P_x}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial x}(x,y,\zeta) \tag{45}$$
$$\Delta p_y = \frac{m_0}{m}\frac{\Delta P_y}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial y}(x,y,\zeta) \tag{46}$$
$$\Delta\delta \simeq \Delta p_z = \frac{m_0}{m}\frac{\Delta P_z}{P_0} = -\frac{qL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{\partial\phi}{\partial \zeta}(x,y,\zeta) \tag{47}$$

Of your beam includes particles of different species (tracking of fragments), note that heree q is the charge of the kicked particle while m_0 is the mass of the reference particle.

3.1 2.5D approximation

For large enough values of γ_0, Eq. 22 can be approximated by:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\frac{1}{\varepsilon_0}\rho_0(x,y,\zeta) \tag{48}$$

which means that we can solve a simple 2D problem for each beam slice (identified by its ζ).

3.2 Modulated 2D

Often the beam distribution can be factorized as:

$$\rho_0(x, y, \zeta) = Nq_0\lambda_0(\zeta)\rho_\perp(x, y) \tag{49}$$

where:

$$\int \lambda_0(z)\, dz = 1 \tag{50}$$

$$\int \rho_\perp(x, y)\, dx\, dy = 1 \tag{51}$$

In this case the potential can be factorized as:

$$\phi(x, y, \zeta) = q_0\lambda_0(\zeta)\phi_\perp(x, y) \tag{52}$$

where $\phi_\perp(x, y)$ is the solution of the following 2D Poisson equation:

$$\frac{\partial^2\phi_\perp}{\partial x^2} + \frac{\partial^2\phi_\perp}{\partial y^2} = -\frac{1}{\varepsilon_0}\rho_\perp(x, y) \tag{53}$$

The kick can be expressed as:

$$\Delta p_x = \frac{m_0}{m}\frac{\Delta P_x}{P_0} = -\frac{qq_0NL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\lambda_0(\zeta)\frac{\partial\phi}{\partial x}(x, y) \tag{54}$$

$$\Delta p_y = \frac{m_0}{m}\frac{\Delta P_y}{P_0} = -\frac{qq_0NL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\lambda_0(\zeta)\frac{\partial\phi}{\partial y}(x, y) \tag{55}$$

$$\Delta\delta \simeq \Delta p_z = \frac{m_0}{m}\frac{\Delta P_z}{P_0} = -\frac{qq_0NL(1-\beta_0^2)}{m\gamma_0\beta_0^2c^2}\frac{d\lambda_0}{d\zeta}(\zeta)\,\phi(x, y) \tag{56}$$

4 FFT solver

We start from a 1D case for illustration and then we generalize.
We assume free space. The potential can be written as the convolution of a Green function with the charge distribution:

$$\phi(x) = \int_{-\infty}^{+\infty}\rho(x')\,G(x-x')dx' \tag{57}$$

We assume that the source is limited to a region of measure L:

$$\rho(x) = \rho(x)\,\Pi\left(\frac{x}{L}\right) \tag{58}$$

where $\Pi(x)$ is a rectangular window defined as:

$$\Pi(x) = \begin{cases} 1 & \text{for } -\frac{1}{2} < x < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{59}$$

We are interested in the electric potential only the region occupied by the sources, so we can compute:

$$\phi_L(x) = \phi(x)\Pi\left(\frac{x}{L}\right) \tag{60}$$

We replace Eq. (58) and Eq. (60) into Eq.(57), obtaining:

$$\phi_L(x) = \Pi\left(\frac{x}{L}\right)\int_{-\infty}^{+\infty}\Pi\left(\frac{x'}{L}\right)\rho(x')\,G(x-x')dx' \tag{61}$$

We change variable into $x'' = x - x'$:

$$\phi_L(x) = \int_{-\infty}^{+\infty}\Pi\left(\frac{x}{L}\right)\Pi\left(\frac{x-x''}{L}\right)\rho(x-x'')\,G(x'')dx'' \tag{62}$$

The integrand vanishes outside the set of the (x, x'') defined by:

$$\begin{cases} -\frac{L}{2} < x < \frac{L}{2} \\ -\frac{L}{2} < (x'' - x) < \frac{L}{2} \end{cases} \tag{63}$$

Combining the two equations we obtain:

$$-L < -\frac{L}{2} + x < x'' < \frac{L}{2} + x < L \tag{64}$$

i.e. the integrand is zero for $|x''| < L$. Therefore in equation (62) we can replace the $G(x'')$ with its truncated version:

$$G_{2L}(x'') = G(x'')\,\Pi\left(\frac{x''}{2L}\right) \tag{65}$$

obtaining:

$$\phi_L(x) = \int_{-\infty}^{+\infty}\Pi\left(\frac{x}{L}\right)\Pi\left(\frac{x-x''}{L}\right)\rho(x-x'')\,G_{2L}(x'')dx'' \tag{66}$$

Since the two window function force the integrand to zero outside the region $|x''| < L$ we can replace $G_{2L}(x'')$ with its replicated version:

$$G_{2LR}(x'') = \sum_{n=-\infty}^{+\infty} G_{2L}(x' - 2nL) = \sum_{n=-\infty}^{+\infty} G(x'' - 2nL)\,\Pi\left(\frac{x'' - 2nL}{2L}\right) \tag{67}$$

obtaining:

$$\phi_L(x) = \int_{-\infty}^{+\infty}\Pi\left(\frac{x}{L}\right)\Pi\left(\frac{x-x''}{L}\right)\rho(x-x'')\,G_{2LR}(x'')dx'' \tag{68}$$

We can go back to the initial coordinate by substituting $x'' = x - x'$:

$$\phi_L(x) = \Pi\left(\frac{x}{L}\right)\int_{-\infty}^{+\infty}\rho(x')\,G_{2LR}(x-x')dx' \tag{69}$$

This is a cyclic convolution, so we can proceed as followd. We split the integral:

$$\phi_L(x) = \Pi\left(\frac{x}{L}\right) \sum_{n=-\infty}^{+\infty} \int_{-L+2nL}^{L+2nL} \rho(x')\, G_{2LR}(x - x')dx' \tag{70}$$

In each term I replace $x''' = x' + 2nL$:

$$\phi_L(x) = \Pi\left(\frac{x}{L}\right) \sum_{n=-\infty}^{+\infty} \int_{-L}^{L} \rho(x''' - 2nL)\, G_{2LR}(x - x''' - 2nL)dx' \tag{71}$$

But G_{2LR} is periodic:

$$\begin{aligned} \phi_L(x) &= \Pi\left(\frac{x}{L}\right) \sum_{n=-\infty}^{+\infty} \int_{-L}^{L} \rho(x''' - 2nL)\, G_{2LR}(x - x''')dx''' \\ &= \Pi\left(\frac{x}{L}\right) \int_{-L}^{L} \sum_{n=-\infty}^{+\infty} \rho(x''' - 2nL)\, G_{2LR}(x - x''')dx''' \end{aligned} \tag{72}$$

I can define a replicated version of $\rho(x)$:

$$\rho_{2LR}(x) = \sum_{n=-\infty}^{+\infty} \rho(x - 2nL) \tag{73}$$

obtaining: But G_{2LR} is periodic:

$$\phi_L(x) = \Pi\left(\frac{x}{L}\right) \int_{-L}^{L} \sum_{n=-\infty}^{+\infty} \rho_{2LR}(x')\, G_{2LR}(x - x')dx' \tag{74}$$

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