Reduced Basis Method and Variational Inequalities

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Introduction

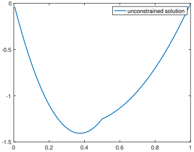

Figure 1: Material with different elasticity properties

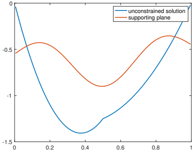

Figure 2: Adding a support plan

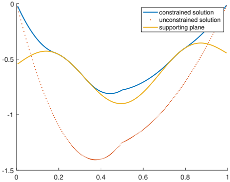

Figure 3: The desired solution

Parametric PDE

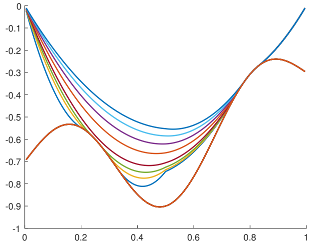

Figure 4: Solution associated with the elasticity with μ_1 uniformly distributed in $[0.05, 0.25]$ and a fixed obstacles

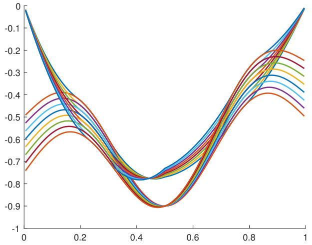

Figure 5: Solution associated with the obstacle with μ_2 uniformly distributed in $[-0.05, 0.5]$ and constant elasticity

Problem

Summary:

Constrained variational problem.

Must be solved in a short time or for a large number of parameters.

Our methods: [Haasdonk et al., 2012]

- Variational Inequality (VI) $\rightarrow$ saddle point formulation $\rightarrow$ QP problem
- Reduces Base Method: online/offline decomposition

Plan

1. Introduction
2. Theoretical Part
3. Numerical Experiment

Mathematical formulation

- Elliptic differential equation

$$\begin{cases} -\nabla \cdot (\nu \nabla u) = f & \text{in } (0,1) \\ u = 0 & \text{on } \{0,1\} \\ u \geq h & \end{cases}$$

Mathematical formulation

- Elliptic differential equation

$$\begin{cases} -\nabla \cdot (\nu \nabla u) = f & \text{in } (0,1) \\ u = 0 & \text{on } \{0,1\} \\ u \geq h \end{cases}$$

- Variational problem

$$\inf_{u \in H_0^1(\Omega)} \quad J(u) = \frac{1}{2}\int_0^1 \nu |\nabla u|^2 \mathrm{d}x - \int_0^1 f \,\mathrm{d}x \tag{1}$$

$$\text{s.t.} \quad u \geq h \tag{2}$$

Mathematical formulation

- Elliptic differential equation

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$$\text{s.t.} \quad u \geq h \tag{2}$$

- Saddle point problem

$$\begin{aligned} a(u(\mu), v; \mu) + b(v, \lambda(\mu)) &= f(v; \mu), & v \in V \\ b(u(\mu), \eta - \lambda(\mu)) &\leq g(\eta - \lambda(\mu); \mu), & \eta \in M, \end{aligned}$$

Reduced Basis Method

Original Problem

$$\begin{aligned} a(u(\mu), v; \mu) + b(v, \lambda(\mu)) &= f(v; \mu), & v \in V \\ b(u(\mu), \eta - \lambda(\mu)) &\leq g(\eta - \lambda(\mu); \mu), & \eta \in M \subset W, \end{aligned}$$

- $a(u, v; \mu) = \int_\Omega \nu(\mu)(x) \nabla u(x) \cdot \nabla v(x) dx \;, \quad u, v \in V$
- $b(u, \eta) = -\eta(u) \quad u \in V, \eta \in W$
- $f(v; \mu) = f(v) := -\int_\Omega v(x) dx, \quad v \in V.$
- $g(\eta; \mu) = \sum_{i=1}^{H} \underline{\eta}_i h(x_i; \mu) \quad$ for $\quad \eta = \sum_{i=1}^{H} \underline{\eta}_i \chi_i \in W$

Reduced Basis Method

Original Problem

$$
\begin{aligned}
a(u(\mu), v; \mu) + b(v, \lambda(\mu)) &= f(v; \mu), & v \in V \\
b(u(\mu), \eta - \lambda(\mu)) &\leq g(\eta - \lambda(\mu); \mu), & \eta \in M \subset W,
\end{aligned}
$$

- $V := \{v \in H_0^1(\Omega) | v_{|[x_k, x_{k+1}]} \in P_1, k = 0, \ldots, K-1\}$
- $\psi_i \in V$ the standard basis of V.
- $W = V'$.
- $M := \mathrm{span}_+\{\chi_i\}_{i=1}^H$, where $(\chi_i)_{i=1}^H : B = (b(\psi_i, \chi_j))_{i,j=1}^{H,H} = \mathrm{Id}$

Reduced Basis Method

Original Problem

$$
\begin{aligned}
a(u(\mu), v; \mu) + b(v, \lambda(\mu)) &= f(v; \mu), & v \in V \\
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\end{aligned}
$$

Reduced Problem

- $S = \{\mu_1, \ldots, \mu_N\} \subset \mathcal{P}$
- $W_N := \mathrm{span}\{\lambda(\mu_i)\}_{i=1}^N \subset W,$
- $M_N := \mathrm{span}_+\{\lambda(\mu_i)\}_{i=1}^N := \left\{\sum_{i=1}^N \alpha_i \lambda(\mu_i) | \alpha_i \geq 0\right\} \subset M,$
- $V_N := \mathrm{span}\{u(\mu_i)\}_{i=1}^N \subset V.$

Reduced Basis Method

Reduced Problem

Find $(u_N(\mu), \lambda_N(\mu)) \in V_N \times M_N$

$$\begin{aligned} a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) &= f(v_N; \mu), \quad v_N \in V_N && (3) \\ b(u_N(\mu), \eta_N - \lambda_N(\mu)) &\leq g(\eta_N - \lambda_N(\mu); \mu), \quad \eta_N \in M_N. && (4) \end{aligned}$$

Reduced Basis Method

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Find $(u_N(\mu), \lambda_N(\mu)) \in V_N \times M_N$

$$a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) = f(v_N; \mu), \quad v_N \in V_N \tag{3}$$

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What is a good choice of snapshots?

Reduced Basis Method

Reduced Problem

Find $(u_N(\mu), \lambda_N(\mu)) \in V_N \times M_N$

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What is a good choice of snapshots?

- POD approach
- Greedy approach

Reduced Basis Method

Reduced Problem

Find $(u_N(\mu), \lambda_N(\mu)) \in V_N \times M_N$

$$\begin{aligned} a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) &= f(v_N; \mu), \quad v_N \in V_N \qquad (3) \\ b(u_N(\mu), \eta_N - \lambda_N(\mu)) &\leq g(\eta_N - \lambda_N(\mu); \mu), \quad \eta_N \in M_N. (4) \end{aligned}$$

What is a good choice of snapshots?

- POD approach
- **Greedy approach**

Reduced Basis Method

Greedy Algorithm:
(u_1, λ_1) the true solution associated to a random value of μ.

- $\mu_n = \arg\max \Delta(\mu; V_{n-1} \times W_{n-1})$
- (u_n, λ_n) the true solution associated to μ_n.

Reduced Basis Method

Greedy Algorithm:
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True error: $\Delta_{\text{true}}(\mu) = e_N(\mu) = \|u(\mu) - u_N(\mu)\|_V + \|\lambda(\mu) - \lambda_N\|_W$

Reduced Basis Method

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True error: $\Delta_{\text{true}}(\mu) = e_N(\mu) = \|u(\mu) - u_N(\mu)\|_V + \|\lambda(\mu) - \lambda_N\|_W$

A posteriori estimate. $\Delta_{\text{a posteriori}}(\mu)$

A Posteriori Estimate

Residual

$$\begin{aligned} r(v;\mu) &:= f(v;\mu) - a(u_N(\mu), v;\mu) - b(v, \lambda_N(\mu)), \quad v \in V. \\ s(\eta;\mu) &:= b(u_N(\mu), \eta) - g(\eta;\mu), \quad \eta \in W. \end{aligned}$$

Note that

$$\begin{aligned} r(v;\mu) &= f(v) - a(u_N, v;\mu) - b(v, \lambda_N;\mu) - [f(v;\mu) - a(u, v;\mu) \\ &\quad - b(v, \lambda;\mu)] \\ &= a(u - u_N, v;\mu) + b(v, \lambda - \lambda_N;\mu) \end{aligned}$$

A Posteiori Estimate

- for all $\mu \in \mathcal{P}$
  $\|\lambda(\mu) - \lambda_N(\mu)\|_W \leq \frac{1}{\beta} \left(\|r(\cdot;\mu)\|_V + \gamma_a(\mu) \, \|u(\mu) - u_N(\mu)\|_V \right).$

inf-sup stability of b

$$\exists \beta \text{ s.t. } \inf_{\eta \in W} \sup_{v \in V} \frac{b(v,\eta)}{\|v\|_V \|\eta\|_W} \geq \beta$$

$$\begin{aligned}
\|\lambda - \lambda_N\|_W &\leq \frac{1}{\beta} \sup_v \frac{b(v, \lambda - \lambda_N)}{\|v\|} \\
&= \frac{1}{\beta} \sup_v \frac{r(v) - a(u - u_N, v)}{\|v\|} \\
&\leq \frac{1}{\beta} \sup_v \frac{\|r\|\|v\| - \gamma_a \|u - u_N\|\|v\|}{\|v\|} \\
&= \frac{1}{\beta} (\|r\| - \gamma_a \|u - u_N\|)
\end{aligned}$$

A Posteriori Estimate

We define:

$$
\begin{aligned}
\delta_r(\mu) &:= \|r(\cdot;\mu)\|_{V'} = \|v_r(\mu)\|_V \\
\delta_{s1}(\mu) &:= \|\pi(\eta_s(\mu))\|_W \\
\delta_{s2}(\mu) &:= \langle \lambda_N(\mu), \pi(\eta_s(\mu)) \rangle_W .
\end{aligned}
$$

- The following inequality holds

$$
\alpha \|u - u_N\|_V^2 \le \delta_r \|u - u_N\|_V + \delta_{s1} \|\lambda - \lambda_N\|_W + \delta_{s2},
$$

$$
\begin{aligned}
\alpha \|u - u_N\|_V^2 &\le a(u - u_N, u - u_N) \\
&= r(u - u_N) - b(u - u_N, \lambda - \lambda_N) \\
&\le \delta_r \|u - u_N\|_V + b(u, \lambda_N - \lambda) + b(u_N, \lambda - \lambda_N) \\
&\le \delta_r \|u - u_N\|_V + g(\lambda_N - \lambda) + s(\lambda - \lambda_N) + g(\lambda - \lambda_N)
\end{aligned}
$$

A Posteriori Estimate

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$$
$$
b(u_N(\mu), \eta_N - \lambda_N(\mu)) \leq g(\eta_N - \lambda_N(\mu); \mu), \quad \eta_N \in M_N.
$$

$$
\begin{aligned}
\alpha \|u - u_N\|_V^2 &\leq \delta_r \|u - u_N\|_V + s(\lambda) - s(\lambda_N) \\
&= \delta_r \|u - u_N\|_V + \langle \lambda, \pi(\eta_s)\rangle_W + \langle \lambda, \eta_s - \pi(\eta_s)\rangle_W
\end{aligned}
$$

A Posteriori Estimate

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- The following inequality holds

$$\alpha \|u - u_N\|_V^2 \leq \delta_r \|u - u_N\|_V + \delta_{s1} \|\lambda - \lambda_N\|_W + \delta_{s2},$$

$$\begin{aligned}
\alpha \|u - u_N\|_V^2 &\leq \delta_r \|u - u_N\|_V + \langle \lambda, \pi(\eta_s)\rangle_W + \langle \lambda, \eta_s - \pi(\eta_s)\rangle_W \\
&\leq \delta_r \|u - u_N\|_V + \langle \lambda, \pi(\eta_s)\rangle_W \\
&= \delta_r \|u - u_N\|_V + \langle \lambda - \lambda_N, \pi(\eta_s)\rangle_W + \delta_{s2} \\
&\leq \delta_r \|u - u_N\|_V + \|\lambda - \lambda_N\|_W \delta_{s1} + \delta_{s2},
\end{aligned}$$

A posteriori Estimate

$$
\begin{aligned}
\|u(\mu) - u_N(\mu)\|_V &\leq \Delta_u(\mu) := c_1(\mu) + \sqrt{c_1(\mu)^2 + c_2(\mu)}, \\
\|\lambda(\mu) - \lambda_N(\mu)\|_W &\leq \Delta_\lambda(\mu) := \frac{1}{\beta}\left(\delta_r(\mu) + \gamma_a(\mu)\Delta_u(\mu)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
c_1(\mu) &:= \frac{1}{2\alpha(\mu)}\left(\delta_r(\mu) + \frac{\delta_{s1}(\mu)\gamma_a(\mu)}{\beta}\right), \\
c_2(\mu) &:= \frac{1}{\alpha(\mu)}\left(\frac{\delta_{s1}(\mu)\delta_r(\mu)}{\beta} + \delta_{s2}(\mu)\right).
\end{aligned}
$$

Implementation

Solvers for the main problem

- Active Set Method
- Primal-dual inner point method

Finding Snapshots

- Greedy Algorithm
    - True error.
    - A posteriori estimate.

Solvers for the reduced problem

- Primal-dual inner point method

Reduced Basis Solution

Reduced Problem

$$a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) = f(v_N; \mu), \quad v_N \in V_N$$
$$b(u_N(\mu), \eta_N - \lambda_N(\mu)) \leq g(\eta_N - \lambda_N(\mu); \mu), \quad \eta_N \in M_N.$$

$$\overline{u_N}^T = (\alpha_1, \dots, \alpha_N) \quad u_N = \sum_i \alpha_i u_i$$

$$\overline{\lambda_N}^T = (\beta_1, \dots, \beta_N) \quad \lambda_N = \sum_i \beta_i \lambda_i$$

$$\sum_i \alpha_i a(u_i, u_j) + \sum_i \beta_i b(u_j, \lambda_i) = f(u_j)$$

Reduced Basis Solution

Reduced Problem

$$a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) = f(v_N; \mu), \quad v_N \in V_N$$
$$b(u_N(\mu), \eta_N - \lambda_N(\mu)) \leq g(\eta_N - \lambda_N(\mu); \mu), \quad \eta_N \in M_N.$$

$$\overline{u_N}^T = (\alpha_1, \dots, \alpha_N) \quad u_N = \sum_i \alpha_i u_i$$

$$\overline{\lambda_N}^T = (\beta_1, \dots, \beta_N) \quad \lambda_N = \sum_i \beta_i \lambda_i$$

$$\sum_i \alpha_i a(u_i, u_j) + \sum_i \beta_i b(u_j, \lambda_i) = f(u_j)$$

$$\overline{A}_N(\mu)\overline{u}_N(\mu) + \overline{B}_N \overline{\lambda}_N(\mu) = \overline{f}_N(\mu)$$

$$\overline{A}_N(\mu) = (a(u_j, u_i; \mu)), \ \overline{B}_N = (b(u_i, \lambda_j)), \overline{f}_N = (f(u_i, \mu))$$

Reduced Basis Solution

Reduced Problem

$$a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) = f(v_N; \mu), \quad v_N \in V_N$$
$$b(u_N(\mu), \eta_N - \lambda_N(\mu)) \leq g(\eta_N - \lambda_N(\mu); \mu), \quad \eta_N \in M_N.$$

$$\overline{A}_N(\mu)\overline{u}_N(\mu) + \overline{B}_N\overline{\lambda}_N(\mu) = \overline{f}_N(\mu)$$

$$b(\sum_i \alpha_i u_i, \sum_j \beta_j \lambda_j) = g(\sum_j \beta_j \lambda_j; \mu)$$

$$\sum_j \beta_j (\sum_i \alpha_i b(u_i, \lambda_j) - g(\lambda_j)) = 0$$

$$\overline{\lambda}_N^T(\overline{g}_N - \overline{B}_N^T \overline{u}_N) = 0$$

$$\overline{g}_N = (g(\lambda_i; \mu))_{N\times 1}$$

Reduced Basis Solution

Reduced Problem

$$a(u_N(\mu), v_N; \mu) + b(v_N, \lambda_N(\mu)) = f(v_N; \mu), \quad v_N \in V_N$$
$$b(u_N(\mu), \eta_N - \lambda_N(\mu)) \leq g(\eta_N - \lambda_N(\mu); \mu), \quad \eta_N \in M_N.$$

$$\begin{aligned}
\overline{A}_N(\mu)\overline{u}_N(\mu) + \overline{B}_N\overline{\lambda}_N(\mu) &= \overline{f}_N(\mu) \\
\overline{\lambda}_N(\mu) &\geq 0 \\
\overline{g}_N(\mu) - \overline{B}_N^T\overline{u}_N(\mu) &\geq 0 \\
\overline{\lambda}_N(\mu)^T(\overline{g}_N(\mu) - \overline{B}_N^T\overline{u}_N(\mu)) &= 0.
\end{aligned}$$

Plan

1 Introduction

2 Theoretical Part

3 Numerical Experiment

Numerical Experiment

Perform Greedy Algorithm (with true error)

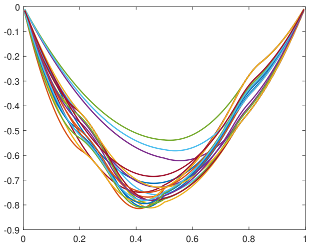


Figure 6: Primal Snapshots

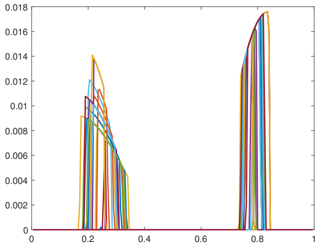


Figure 7: Dual Snapshots

Numerical Experiment

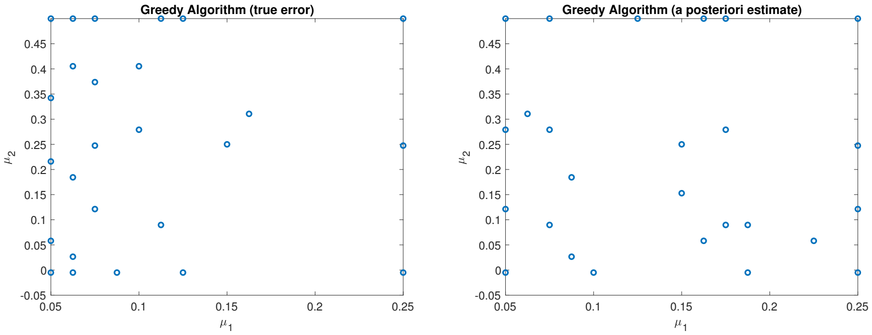
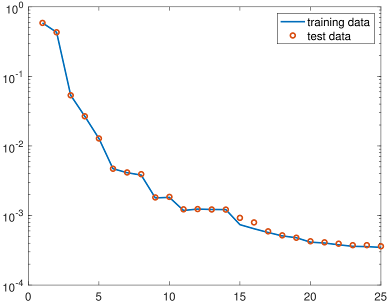


Figure 8: Parameters associated with Snapshots

Numerical Experiment



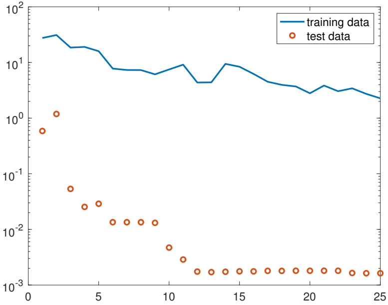


Figure 9: Greedy algorithm with Δ_{true} and $\Delta_{\text{a posteriori}}$ over time, tested with Δ_{true}

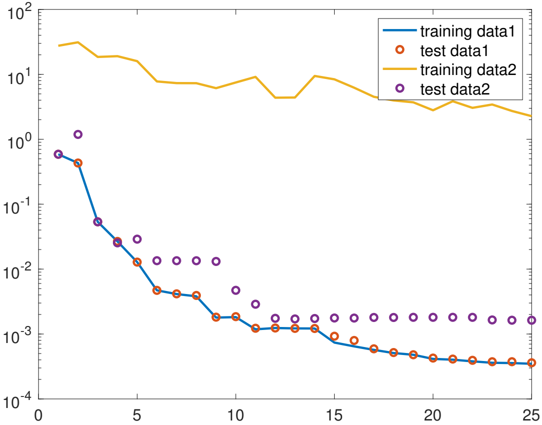


Figure 10: Greedy algorithm with Δ_{true} and $\Delta_{\text{a posteriori}}$, tested with Δ_{true}

Reference I

Haasdonk, B., Salomon, J., and Wohlmuth, B. (2012).
A reduced basis method for parametrized variational inequalities.
SIAM Journal on Numerical Analysis, 50(5):2656–2676.