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EXERCISE 8 – EXTENDED KALMAN FILTER

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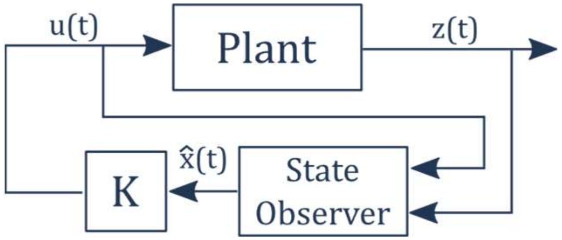
AGENDA

- THEORY REVIEW: Linearized Kalman Filter
- THEORY REVIEW: Extended Kalman Filter
- EXAMPLE
- HANDS-ON: Matlab

THEORY REVIEW: KBF

Real system:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} + L\boldsymbol{w}$$
$$\boldsymbol{z} = C\boldsymbol{x} + \boldsymbol{n}$$



Kalman-Bucy filter:

$$\dot{\hat{\boldsymbol{x}}} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + K_o(\boldsymbol{z} - C\hat{\boldsymbol{x}})$$

where:

$$K_o = PC^T R^{-1}$$
$$\dot{P} = A^T P + PA + Q - PCR^{-1}C^T P$$
$$P(t_0) = P_0$$

assumptions:

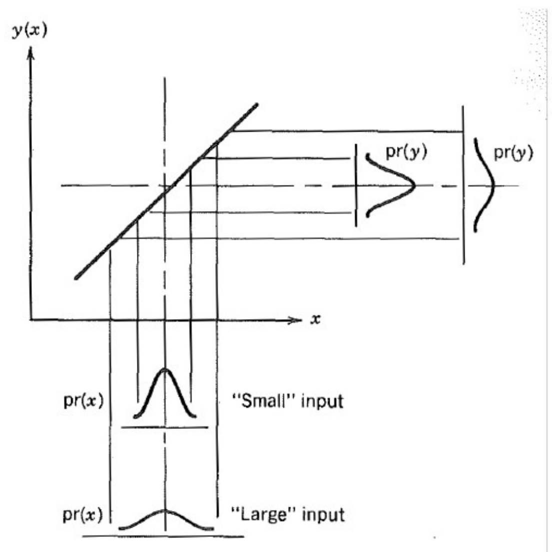
$$E[\boldsymbol{w}(t)] = 0 \qquad E[\boldsymbol{w}(t)\boldsymbol{w}(t)^T] = Q'$$
$$E[\boldsymbol{n}(t)] = 0 \qquad E[\boldsymbol{n}(t)\boldsymbol{n}(t)^T] = R$$
$$E[\boldsymbol{n}(t)\boldsymbol{w}(t)^T] = 0$$

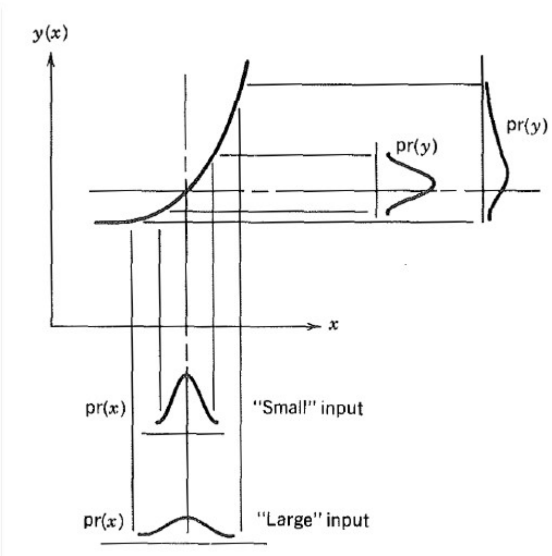
THEORY REVIEW: Linearized kalman filter

What if my real system is nonlinear? Are the assumptions used for the discrete/continuous Kalman filter still valid?

$$\boxed{\overline{\mathbf{x}}_{\mathbf{i}} = \Phi_{i-1}\overline{\mathbf{x}}_{\mathbf{i-1}} + \Gamma_{i-1}\overline{\mathbf{u}}_{\mathbf{i-1}}}$$

$$\boxed{P_i = \Phi_{i-1}P_{i-1}\Phi_{i-1}^\top + \underbrace{\Lambda_{i-1}Q'_{i-1}\Lambda_{i-1}^\top}_{Q_{i-1}}}$$





- Quasi-gaussian distribution of the output.
- CLT says that, if $\mathrm{N} \rightarrow \infty$, then **the sum of N quasi-Gaussian distributions tends to a Gaussian distribution.**
- for **additive and small** stochastic effects, Kalman filter can be modified to return a tending-to-optimal solution thanks to the **central limit theorem** (CLT).

THEORY REVIEW: Linearized kalman filter

We ALWAYS deal with nonlinear systems. Therefore, the state estimation must capture large variations of the state.

Real System

$$\dot{\boldsymbol{x}}(t) = f\big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)\big)$$

$$\boldsymbol{z}(t) = h\big(\boldsymbol{x}(t), \boldsymbol{n}(t)\big)$$

State Observer

$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t))\big)$$

$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

- How do we compute K_o for a non-linear observer?

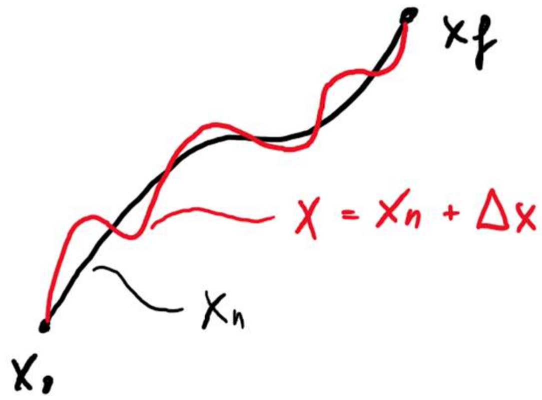
$$K_o(t) = P(t)C^T R^{-1}$$

THEORY REVIEW: Linearized kalman filter

Let us assume that the non-linear state and measurement equations are linearized around a known nominal trajectory $(\boldsymbol{x_N}, \boldsymbol{u_N}, \boldsymbol{w_N}, \boldsymbol{n_N})$.

$$\dot{\boldsymbol{x}}_N(t) = f\big(\boldsymbol{x_N}(t), \boldsymbol{u_N}(t), \boldsymbol{w_N}(t)\big)$$

$$\boldsymbol{z}_N(t) = h\big(\boldsymbol{x}_N(t), \boldsymbol{n}_N(t)\big)$$



THEORY REVIEW: Linearized kalman filter

The linearized system equations are characterized by a nonlinear term, plus linear perturbation to the nominal trajectory:

$$\dot{\boldsymbol{x}}_{\boldsymbol{N}}(t) + \Delta\dot{\boldsymbol{x}} = f\big(\boldsymbol{x}_{\boldsymbol{N}}(t), \boldsymbol{u}_{\boldsymbol{N}}(t), \boldsymbol{w}_{\boldsymbol{N}}(t)\big) + \left.\frac{\partial f}{\partial x}\right|_N (\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{N}}) + \left.\frac{\partial f}{\partial u}\right|_N (\boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{N}}) + \left.\frac{\partial f}{\partial w}\right|_N (\boldsymbol{w} - \boldsymbol{w}_{\boldsymbol{N}})$$

$$\boldsymbol{z}_{\boldsymbol{N}}(t) + \Delta\boldsymbol{z} = h\big(\boldsymbol{x}_N(t), \boldsymbol{n}_N(t)\big) + \left.\frac{\partial h}{\partial \boldsymbol{x}}\right|_N (\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{N}}) + \left.\frac{\partial h}{\partial \boldsymbol{n}}\right|_N (\boldsymbol{n} - \boldsymbol{n}_{\boldsymbol{N}})$$

$$(\boldsymbol{w}_{\boldsymbol{N}} = \boldsymbol{E}[\boldsymbol{w}(\boldsymbol{t})] = 0)$$

$$(\boldsymbol{n}_{\boldsymbol{N}} = \boldsymbol{E}[\boldsymbol{n}(\boldsymbol{t})] = 0)$$

$$\left.\frac{\partial h}{\partial \boldsymbol{n}}\right|_N = I$$

Thus:

$$\dot{\boldsymbol{x}}_{\boldsymbol{N}}(t) + \Delta\dot{\boldsymbol{x}} = f\big(\boldsymbol{x}_{\boldsymbol{N}}(t), \boldsymbol{u}_{\boldsymbol{N}}(t), \boldsymbol{w}_{\boldsymbol{N}}(t)\big) + A(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{N}}) + B(\boldsymbol{u} - \boldsymbol{u}_{\boldsymbol{N}}) + L\boldsymbol{w}$$

$$\boldsymbol{z}_{\boldsymbol{N}}(t) + \Delta\boldsymbol{z} = h\big(\boldsymbol{x}_N(t), \boldsymbol{n}_N(t)\big) + C(\boldsymbol{x} - \boldsymbol{x}_{\boldsymbol{N}}) + \boldsymbol{n}$$

THEORY REVIEW: Linearized kalman filter

We can define a Kalman-Bucy Filter for the linearized system:

$$\Delta\boldsymbol{x}(t) = x - x_N \qquad \Delta\boldsymbol{z}(t) = z - z_N \qquad \Delta\boldsymbol{u}(t) = u - u_N$$

Linearized System

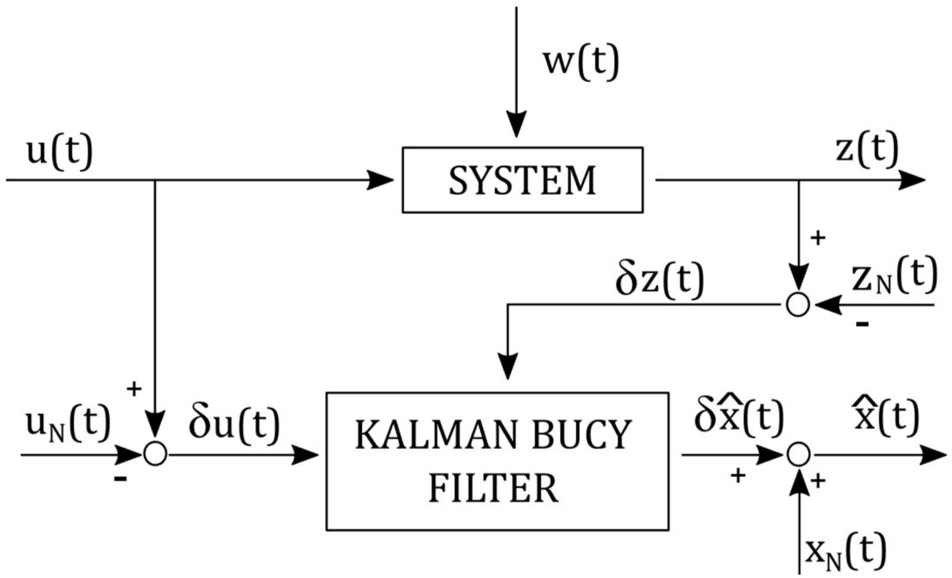
$$\Delta\dot{\boldsymbol{x}}(t) = A_N\Delta\boldsymbol{x}(t) + B_N\Delta\boldsymbol{u}(t) + L_N\Delta\boldsymbol{w}(\boldsymbol{t})$$

$$\Delta\boldsymbol{z}(t) = C_N\Delta\boldsymbol{x}(t) + \Delta\boldsymbol{n}(t)$$

Kalman-Bucy filter

$$\Delta\dot{\hat{\boldsymbol{x}}} = A_N\Delta\hat{\boldsymbol{x}} + B_N\Delta\boldsymbol{u} + K_o(\Delta\boldsymbol{z} - C_N\Delta\hat{\boldsymbol{x}})$$

$$\Delta\hat{\boldsymbol{z}} = C_N\Delta\hat{\boldsymbol{x}}$$



Where the observer gain matrix is evaluated solving the DRE:

$$\dot{P}(t) = Q + A_N P(t) + P(t)A_N^T - P(t)C_N^T R^{-1} C_N P(t)$$

$$K = P(t)C_N^T R^{-1}$$

$$Q = E[\Delta\boldsymbol{w}^T\Delta\boldsymbol{w}]$$

$$R = E[\Delta\boldsymbol{n}^T\Delta\boldsymbol{n}]$$

THEORY REVIEW: Linearized kalman filter

In summary, given the nominal trajectory, the linearized Kalman filter is defined by the following steps:

1. Compute the matrices of the linearized system.

$$\left.\frac{\partial f}{\partial x}\right|_N = A_N \qquad \left.\frac{\partial f}{\partial u}\right|_N = B_N \qquad \left.\frac{\partial f}{\partial w}\right|_N = L_N \qquad \left.\frac{\partial h}{\partial \boldsymbol{x}}\right|_N = C_N$$

2. Compute the covariance matrices.

$$Q = E[\Delta \boldsymbol{w}^T \Delta \boldsymbol{w}] \qquad R = E[\Delta \boldsymbol{n}^T \Delta \boldsymbol{n}]$$

3. Integrate the Kalman filter equations as:

$$\Delta\dot{\hat{\boldsymbol{x}}} = A_N \Delta\hat{\boldsymbol{x}} + B_N \Delta\boldsymbol{u} + K_o(\Delta\boldsymbol{z} - H_N \Delta\hat{\boldsymbol{x}}) \qquad K_o = P(t) C_N^T R^{-1}$$

$$\dot{P}(t) = Q + A_N P(t) + P(t) A_N^T - P(t) C_N^T R^{-1} C_N P(t)$$

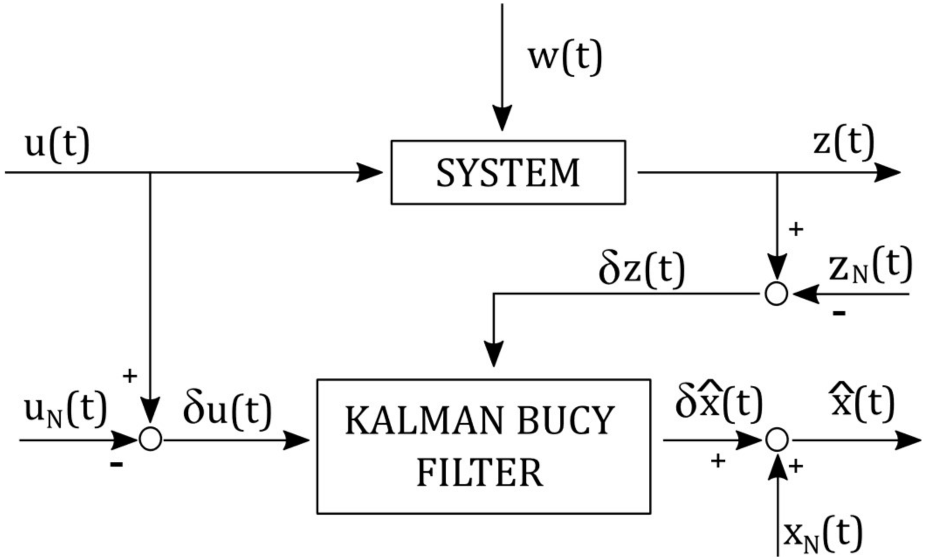
4. The estimated state (for the non-linear system) is given by: $\hat{\boldsymbol{x}} = \boldsymbol{x}_N + \Delta\hat{\boldsymbol{x}}$

THEORY REVIEW: Linearized kalman filter

This requires the knowledge of the nominal trajectory around which the dynamics is linearized.

- This assumption allows for the pre-evaluation of the linearized state matrices (less computational efforts required during cycle time).

- But the nominal trajectory is usually unknown.



THEORY REVIEW: Extended kalman filter

The extended Kalman filter takes as nominal trajectory the current state estimate.

$$\boldsymbol{x_N} = \widehat{\boldsymbol{x}}$$

Hence, the dynamic equations are written as:

Real System

$$\dot{\boldsymbol{x}}(t) = f\big(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)\big)$$

$$\boldsymbol{z}(t) = h\big(\boldsymbol{x}(t), \boldsymbol{n}(t)\big)$$

State Observer

$$\dot{\widehat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\widehat{\boldsymbol{x}}(t))\big)$$

$$\widehat{\boldsymbol{z}}(\boldsymbol{t}) = h(\widehat{\boldsymbol{x}}(t))$$

And the observer gain matrix K_o is evaluated considering:

$$\left.\frac{\partial f}{\partial x}\right|_{\widehat{\boldsymbol{x}},\boldsymbol{u}} = A_{\widehat{\boldsymbol{x}}} \qquad \left.\frac{\partial f}{\partial u}\right|_{\widehat{\boldsymbol{x}},\boldsymbol{u}} = B_{\widehat{\boldsymbol{x}}} \qquad \left.\frac{\partial f}{\partial w}\right|_{\widehat{\boldsymbol{x}},\boldsymbol{u}} = L_{\widehat{\boldsymbol{x}}} \qquad \left.\frac{\partial h}{\partial \boldsymbol{x}}\right|_{\widehat{\boldsymbol{x}},\boldsymbol{u}} = C_{\widehat{\boldsymbol{x}}}$$

I.e. solving the following DRE equation:

$$\dot{P}(t) = Q + A_{\hat{\boldsymbol{x}}}P(t) + P(t)A_{\hat{\boldsymbol{x}}}^T - P(t)C_{\hat{\boldsymbol{x}}}^T R^{-1} C_{\hat{\boldsymbol{x}}}P(t)$$

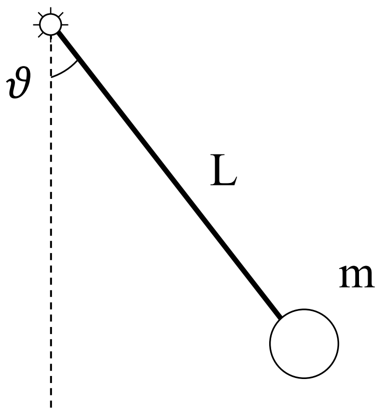
$$K = P(t)C_{\hat{\boldsymbol{x}}}^T R^{-1}$$

$$Q = E[\Delta\boldsymbol{w}^T \Delta\boldsymbol{w}]$$

$$R = E[\Delta\boldsymbol{n}^T \Delta\boldsymbol{n}]$$

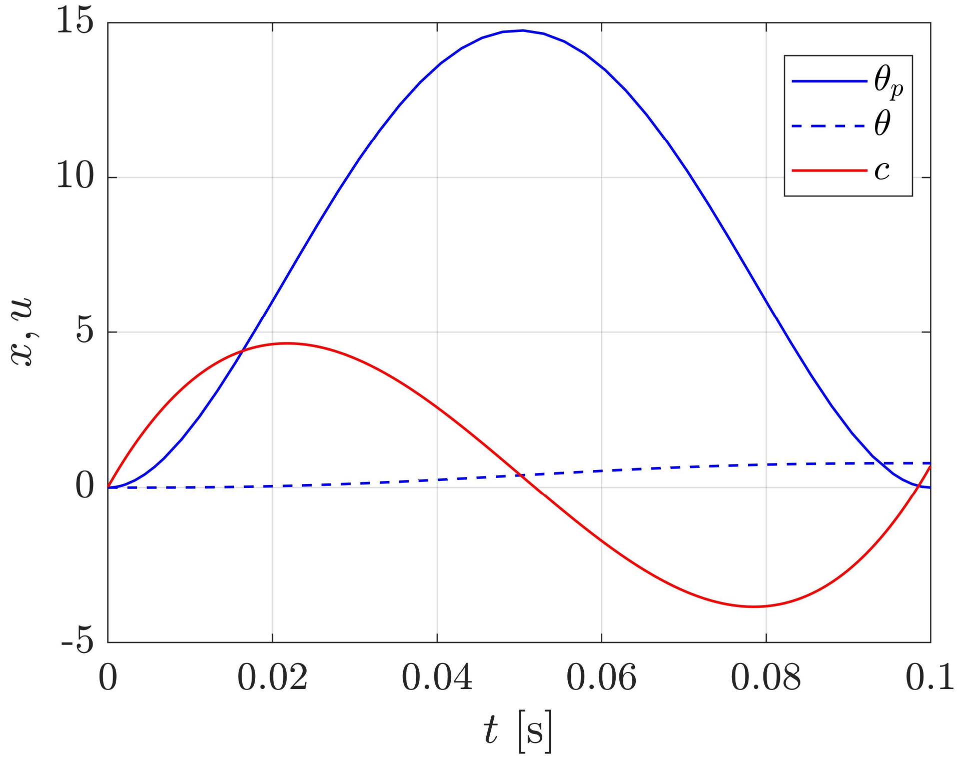
In summary, for any time instant the dynamics is linearized around the estimated state and the DRE follows.

<u>Goal</u>: Find the optimal state estimator for the nonlinear optimal trajectory *Optimal_Trajectory.mat studied in one of the previous exercises.*

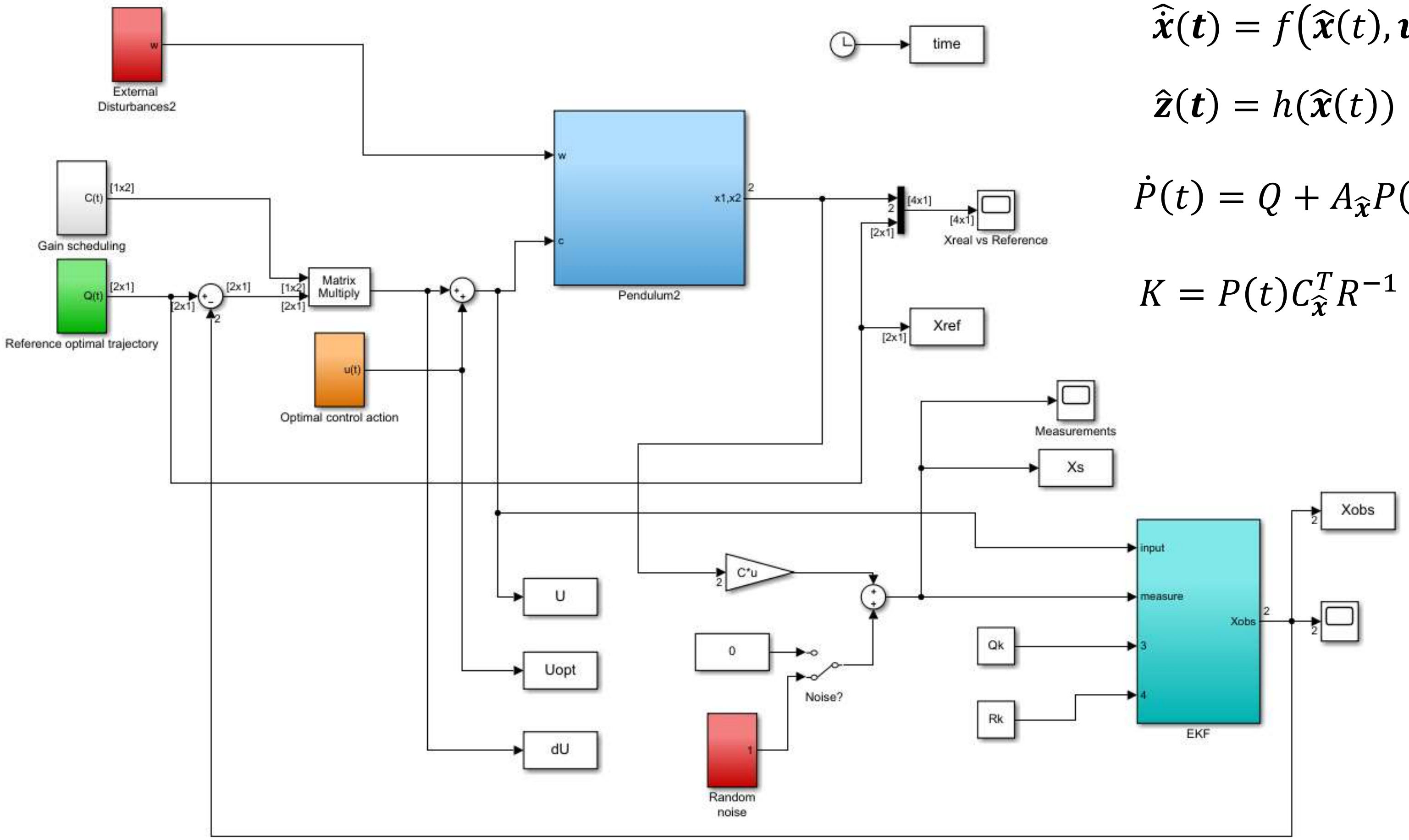


$$\begin{cases} \dot{x}_2 = -2\zeta\omega_0 x_2 - \omega_0^2 \sin(x_1) + \dfrac{c(t)}{mL^2} \\ \dot{x}_1 = x_2 \end{cases}$$

$$\boldsymbol{x} = [x_2, x_1]^T = \left[\dot{\theta}, \theta\right]^T \qquad \boldsymbol{x}_i = [0, 0]^T$$



Exercise 1 – Extended Kalman filter



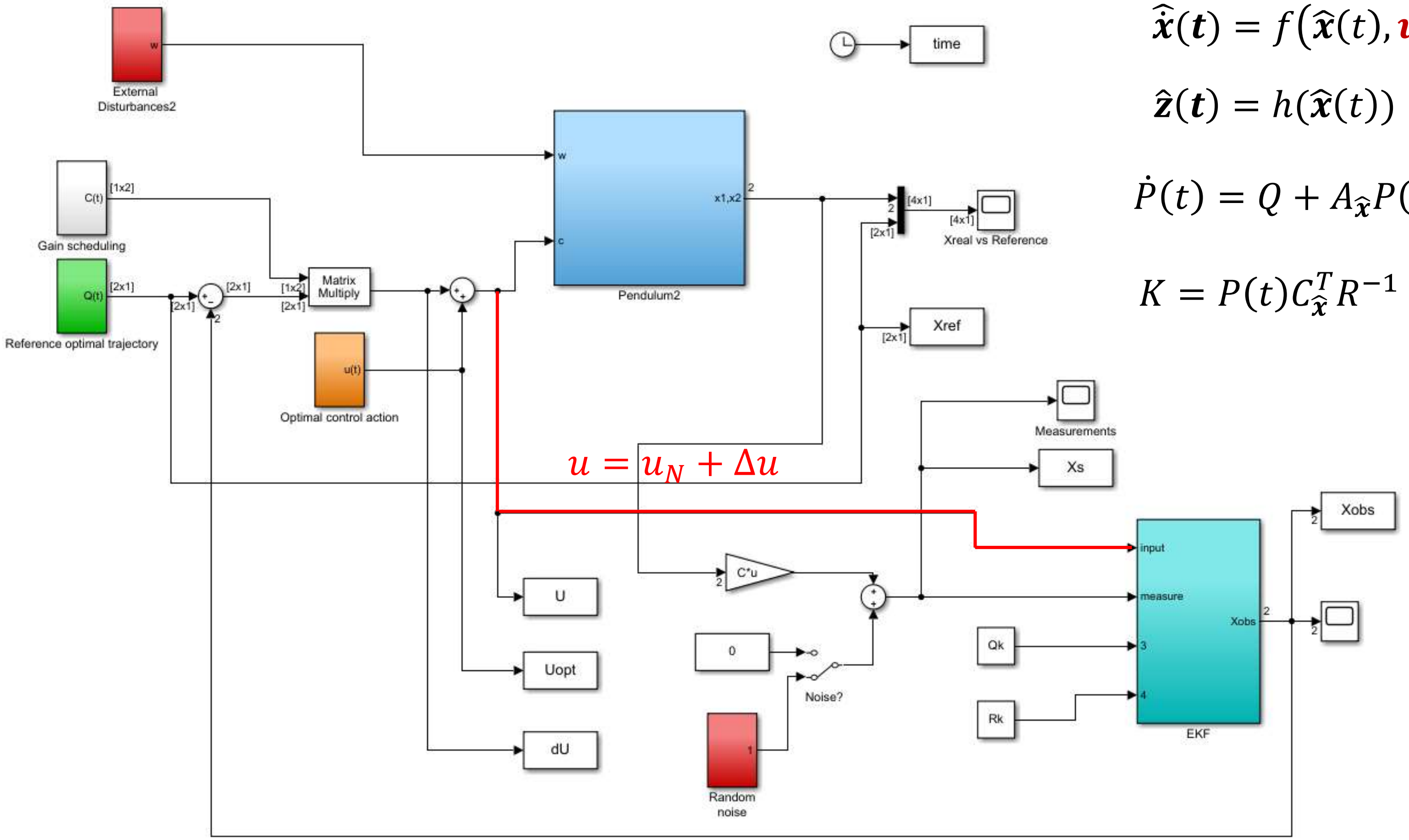
$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t))\big)$$

$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

$$\dot{P}(t) = Q + A_{\hat{x}}P(t) + P(t)A_{\hat{x}}^T - P(t)C_{\hat{x}}^T R^{-1} C_{\hat{x}} P(t)$$

$$K = P(t)C_{\hat{x}}^T R^{-1}$$

Exercise 1 – Extended Kalman filter

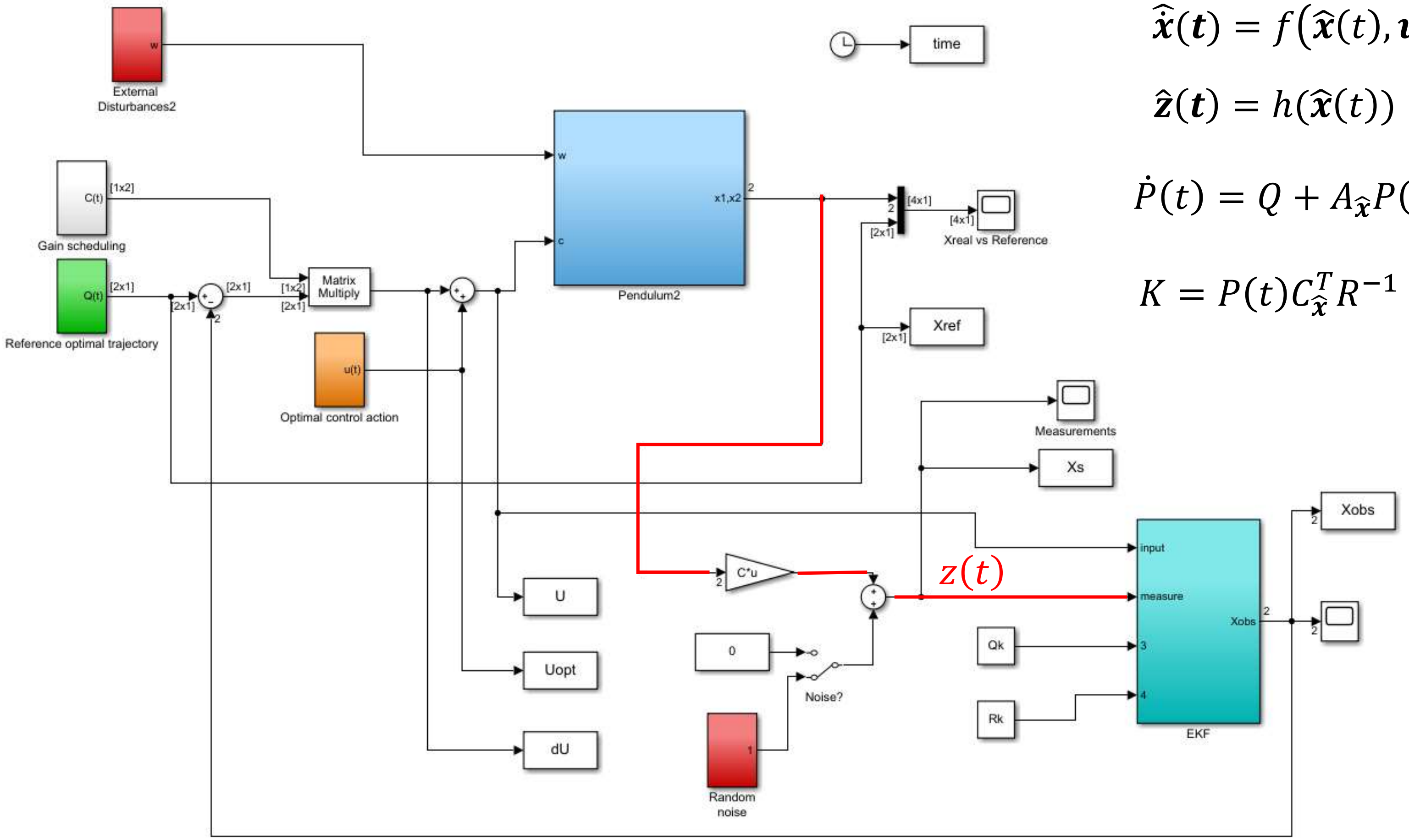


$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)) + K_o(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t)))$$

$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

$$\dot{P}(t) = Q + A_{\hat{\boldsymbol{x}}}P(t) + P(t)A_{\hat{\boldsymbol{x}}}^T - P(t)C_{\hat{\boldsymbol{x}}}^T R^{-1} C_{\hat{\boldsymbol{x}}} P(t)$$

$$K = P(t)C_{\hat{\boldsymbol{x}}}^T R^{-1}$$



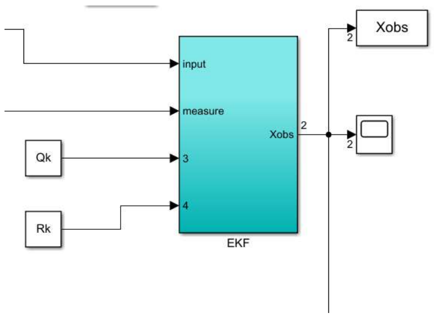
$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t))\big)$$

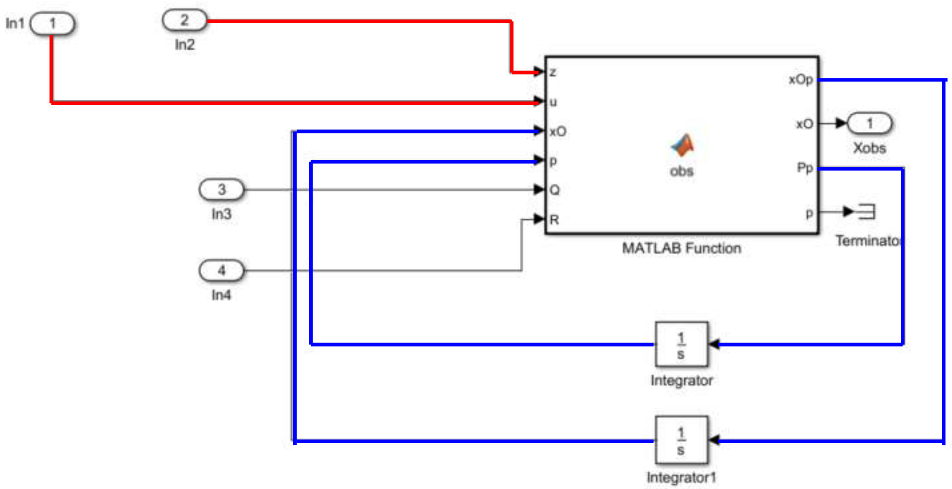
$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

$$\dot{P}(t) = Q + A_{\hat{x}}P(t) + P(t)A_{\hat{x}}^T - P(t)C_{\hat{x}}^T R^{-1} C_{\hat{x}} P(t)$$

$$K = P(t)C_{\hat{x}}^T R^{-1}$$

Exercise 1 – Extended Kalman filter





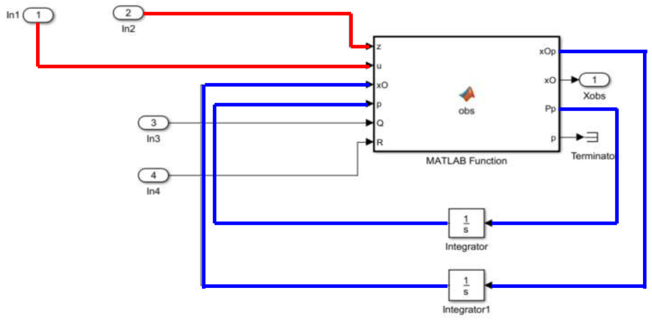
$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t))\big)$$

$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

$$\dot{P}(t) = Q + A_{\hat{\boldsymbol{x}}}P(t) + P(t)A_{\hat{\boldsymbol{x}}}^T - P(t)C_{\hat{\boldsymbol{x}}}^T R^{-1} C_{\hat{\boldsymbol{x}}} P(t)$$

$$K = P(t)C_{\hat{\boldsymbol{x}}}^T R^{-1}$$

Exercise 1 – Extended Kalman filter



$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t))\big)$$

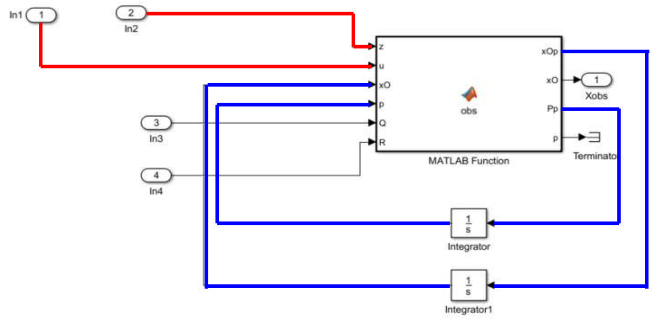
$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

$$\dot{P}(t) = Q + A_{\hat{\boldsymbol{x}}} P(t) + P(t) A_{\hat{\boldsymbol{x}}}^T - P(t) C_{\hat{\boldsymbol{x}}}^T R^{-1} C_{\hat{\boldsymbol{x}}} P(t)$$

$$K = P(t) C_{\hat{\boldsymbol{x}}}^T R^{-1}$$

```
function [xOp,xO,Pp,p]  = obs(z,u,xO,p,Q,R)

Nstates = size(xO,1);

xOp = zeros(Nstates,1);
Pp = zeros(Nstates^2,1);

%% Linearization around the estimated position

%% Differential Riccati equation

%% State estimate
```

Exercise 1 – Extended Kalman filter



$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t))\big)$$

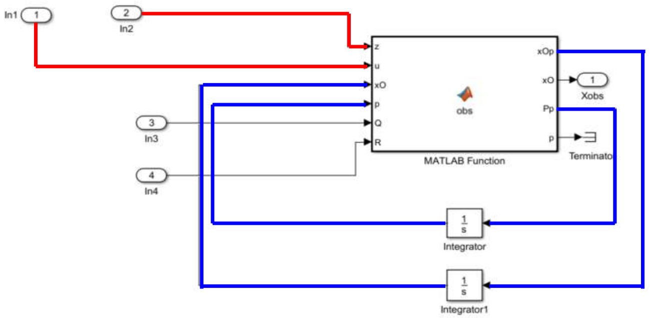
$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

$$\dot{P}(t) = Q + A_{\hat{x}}P(t) + P(t)A_{\hat{x}}^T - P(t)C_{\hat{x}}^T R^{-1} C_{\hat{x}} P(t)$$

$$K = P(t)C_{\hat{x}}^T R^{-1}$$

```
%% Linearization around the estimated position

% Jacobian of the dynamics
A = [ - 2*zeta*w0, - w0^2*cos(xO(2));
                1,                 0];
Lm = [ 1/m/L^2; 0];

% observation matrix
C = [0,1];
```

Exercise 1 – Extended Kalman filter



$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t))\big)$$

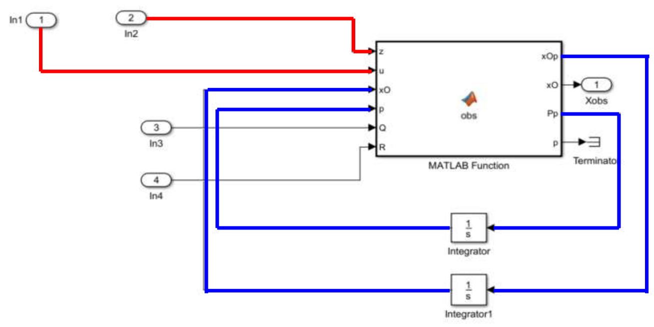
$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

$$\dot{P}(t) = Q + A_{\hat{x}}P(t) + P(t)A_{\hat{x}}^T - P(t)C_{\hat{x}}^T R^{-1} C_{\hat{x}} P(t)$$

$$K = P(t)C_{\hat{x}}^T R^{-1}$$

```
%% Differential Riccati equation

% transformation vector -> matrix
P = zeros(Nstates,Nstates);
P(1:end) = p(1:end,1);

% DRE
Out = (A*P + P*A' + Lm*Q*Lm' - P*C'*R^-1*C*P);

K = P*C'*R^-1;

% transformation matrix -> vector
Pp(1:end,1) = Out(1:end)';
```

Exercise 1 – Extended Kalman filter



$$\dot{\hat{\boldsymbol{x}}}(\boldsymbol{t}) = f\big(\hat{\boldsymbol{x}}(t), \boldsymbol{u}(t)\big) + K_o\big(\boldsymbol{z}(t) - h(\hat{\boldsymbol{x}}(t))\big)$$

$$\hat{\boldsymbol{z}}(\boldsymbol{t}) = h(\hat{\boldsymbol{x}}(t))$$

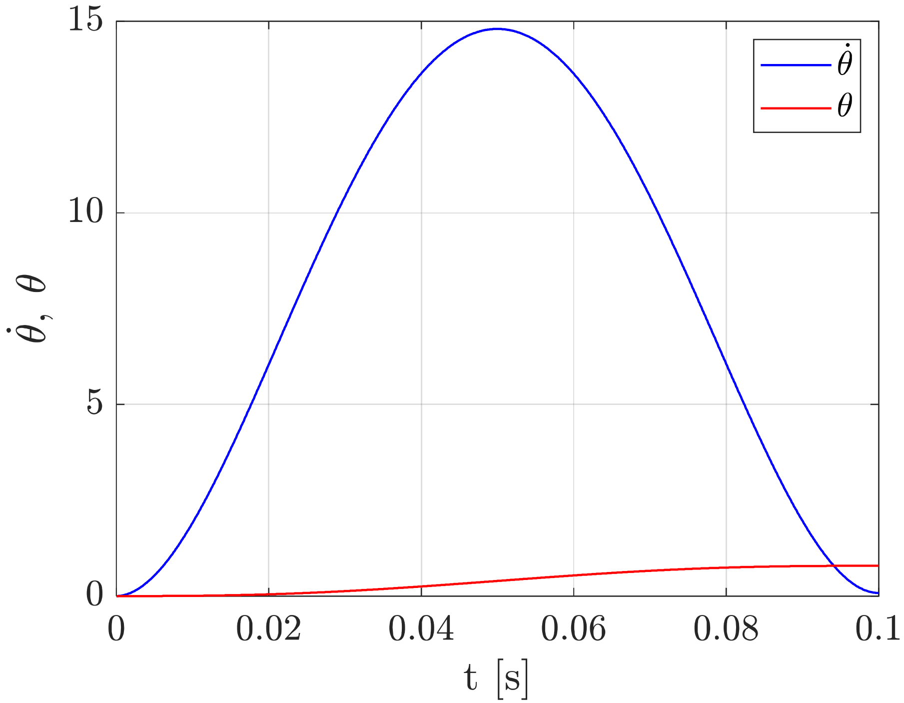
$$\dot{P}(t) = Q + A_{\hat{x}}P(t) + P(t)A_{\hat{x}}^T - P(t)C_{\hat{x}}^T R^{-1} C_{\hat{x}} P(t)$$

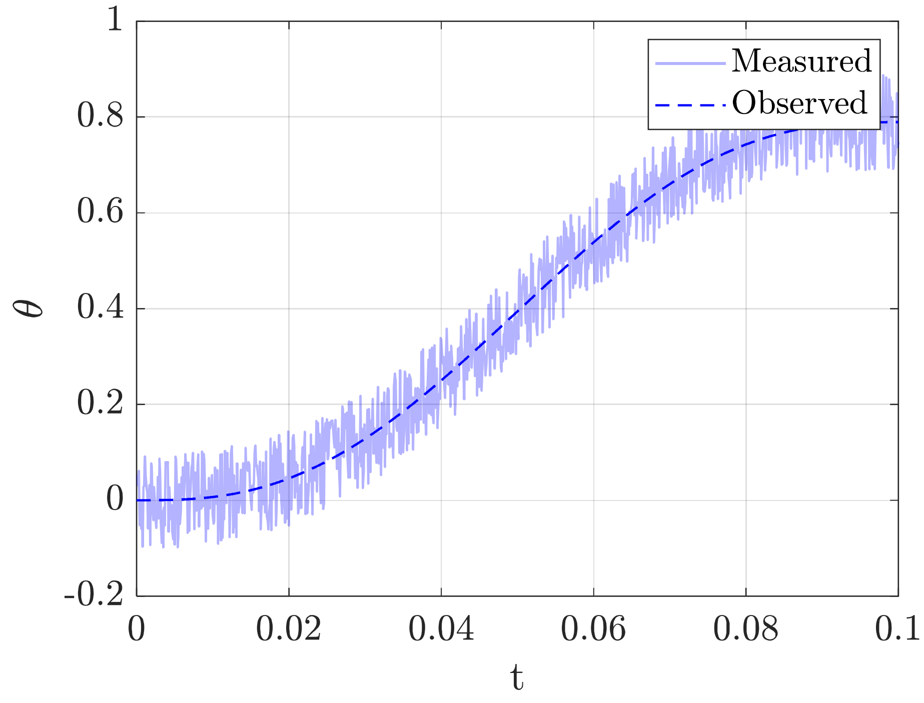
$$K = P(t)C_{\hat{x}}^T R^{-1}$$

```
%% State Estimate
```

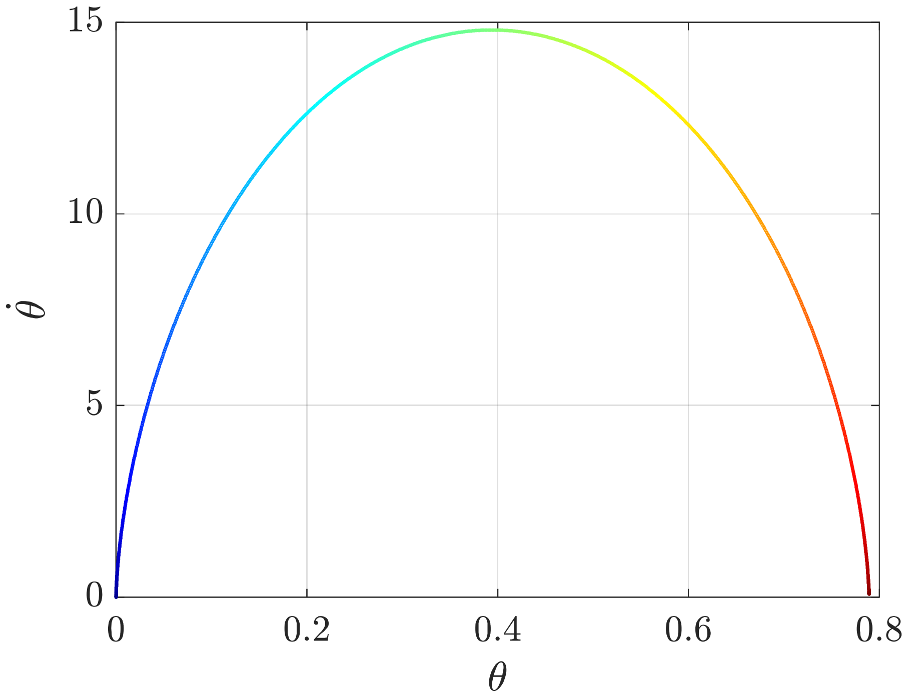
```
xOp(:,1) = [ - 2*zeta*w0*xO(1) - w0^2*sin(xO(2)) + u/m/L^2;
                                                  xO(1)];

xOp = xOp + K*(z - C*xO);
```

Exercise 1 – Extended Kalman filter





Exercise 1 – Extended Kalman filter



HANDS-ON