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Adaptive and Autonomous Aerospace Systems

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Part 1: Analysis of nonlinear and time-varying systems

Lect 2: Preliminaries on nonlinear time-varying systems

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Outline

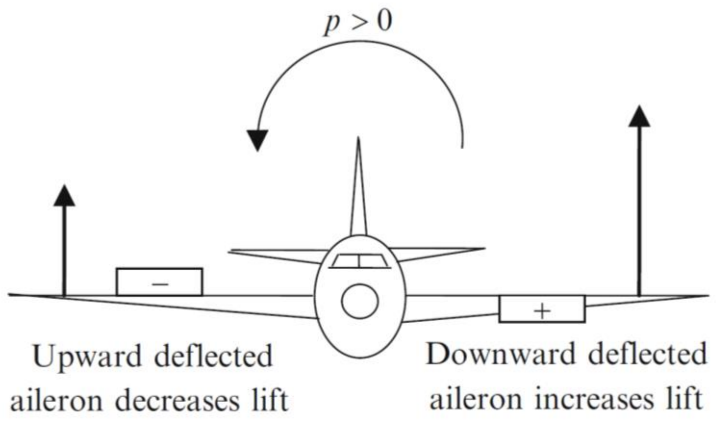
- Plan for next two lectures
  - Generalities on nonlinear systems
  - Stability definitions
  - Lyapunov methods for stability analysis (next lecture)

- Generalities on nonlinear systems
  - Models of nonlinear systems
  - From linear to nonlinear systems: essentially nonlinear phenomena
  - Solution concept and properties

Models from adaptive control

Closed-loop error system corresponding to MRAC applied to the aircraft roll rate dynamics

$$\begin{aligned}
\dot{e} &= (a_{ref} + L_{\delta_a}\Delta k_p)e + L_{\delta_a}(\Delta k_p p_{ref} + \Delta k_{p_{cmd}} p_{cmd}(t)) \\
\dot{\Delta k}_p &= -\gamma_p sign(L_{\delta_a})(e + p_{ref})e \\
\dot{\Delta k}_{p_{cmd}} &= -\gamma_{p_{cmd}} sign(L_{\delta_a}) p_{cmd}(t) e \\
\dot{p}_{ref} &= a_{ref} p_{ref} + b_{ref} p_{cmd}(t)
\end{aligned}$$

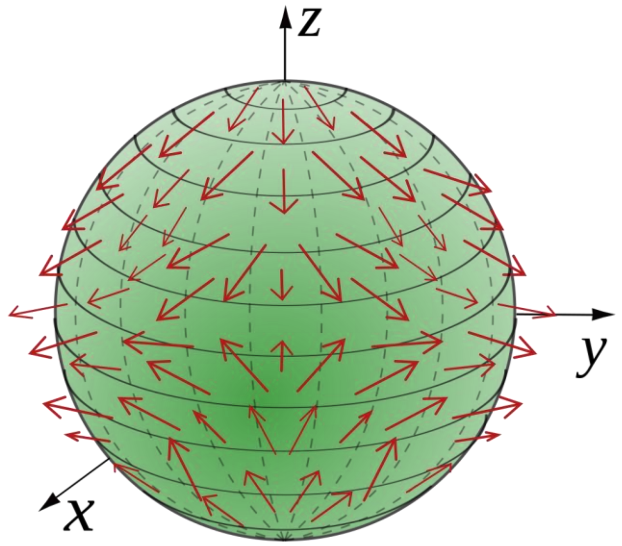


- When dealing with adaptive control, the closed-loop system is nonlinear time-varying even when the platform to be controlled is described by linear time-invariant model.

Nonlinear differential equations

In this course we consider models described by first-order ODE

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(t_0) = x_0$$
$$y(t) = h(t, x(t), u(t))$$



- $x \in D_x$ is the state
- $u \in D_u$ is the input
- $f(\cdot,\cdot,\cdot): D_t \times D_x \times D_u \mapsto T_x D_x$ is the vector field
- $h(\cdot,\cdot,\cdot): D_t \times D_x \times D_u \mapsto D_y$ is the output map
- x_0 is the initial state

Special cases

Unforced state equation

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0$$

N.B.: this model does not necessarily arise only by setting $u(t) = 0$.

- It is the model encountered in the stability analysis of equilibria for closed-loop systems under state feedback control $u = \gamma(t, x)$.

Autonomous vs nonautonomous systems

$$\dot{x}(t) = f(x(t)), \qquad x(t_0) = x_0$$
$$y(t) = h(x(t))$$

Autonomous differential equation = time-invariant

- There is no explicit dependence on time in both the vector field and in the output map.

Important property: solutions to autonomous differential equations depend only on the time elapsed and not on the initial time (time-shifted solutions are also solutions)

Without loss of generality, we can assume $t_0 = 0$.

When moving from *linear* to *nonlinear* systems, the well-known superposition principle and all the nice results empowered by linear algebra do not hold any more.

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t).$$

General solution to Linear Time-Varying (LTV) systems

$$y(t) = C\phi(t, t_0)x_0 + C\int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

State transition matrix

$$\frac{\partial\phi(t, t_0)}{\partial t} = A(t)\phi(t, t_0) \quad \text{with } \phi(t_0, t_0) = I_n$$
$$\phi(t, t) = \phi(t_0, t_0) = I_n$$
$$\phi^{-1}(t, t_0) = \phi(t_0, t)$$
$$\phi(t, t_0) = \phi(t, t_1)\phi(t_1, t_0)$$

Essentially nonlinear phenomena

First attempt to control design for nonlinear systems: linearization about an equilibrium.

Two main limitations:

- The results are valid locally (how far from the desired equilibrium?)
- There are essentially nonlinear phenomena that do not occur for linear systems
    - Multiple equilibria
    - Finite escape time
    - Limit cycles
    - Subharmonic regimes
    - Chaotic motion
    - Bifurcation

Multiple equilibria

Linear systems can have just one equilibrium point or a continuum of equilibria.

- if x_a and x_b are two equilibrium points, then by linearity any point on the line $\theta x_a + (1 - \theta x_b)$ will be an equilibrium point.

Nonlinear systems can have multiple isolated equilibria.

Example: pendulum with friction.

$$\dot{\theta} = q$$
$$\dot{q} = -cq - k\sin(\theta)$$

Finite escape time

The state of an unstable linear system can go to infinity as times approaches infinity.

- For a nonlinear systems, solutions might blow up in finite time.

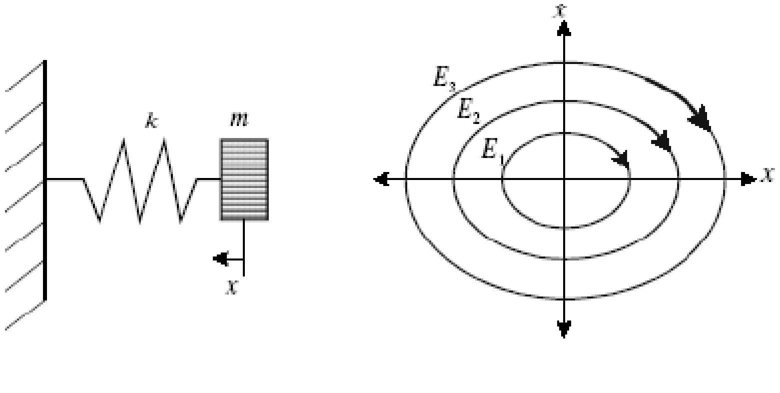
Example

$$\dot{x} = -x^2, \quad x(0) = -1$$

In linear systems, steady-state oscillations can occur with a pair of purely imaginary poles.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -\beta\\ \beta & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$x_1(t) = x_1(0)\cos(\beta t) + x_2(0)\sin(\beta t)$$
$$x_2(t) = x_1(0)\sin(\beta t) - x_2(0)\cos(\beta t)$$



(a) (b)

- The amplitude of the oscillations depends on the initial conditions and they are destroyed by small perturbations.

Instead, nonlinear systems can achieve robust steady-state oscillations.

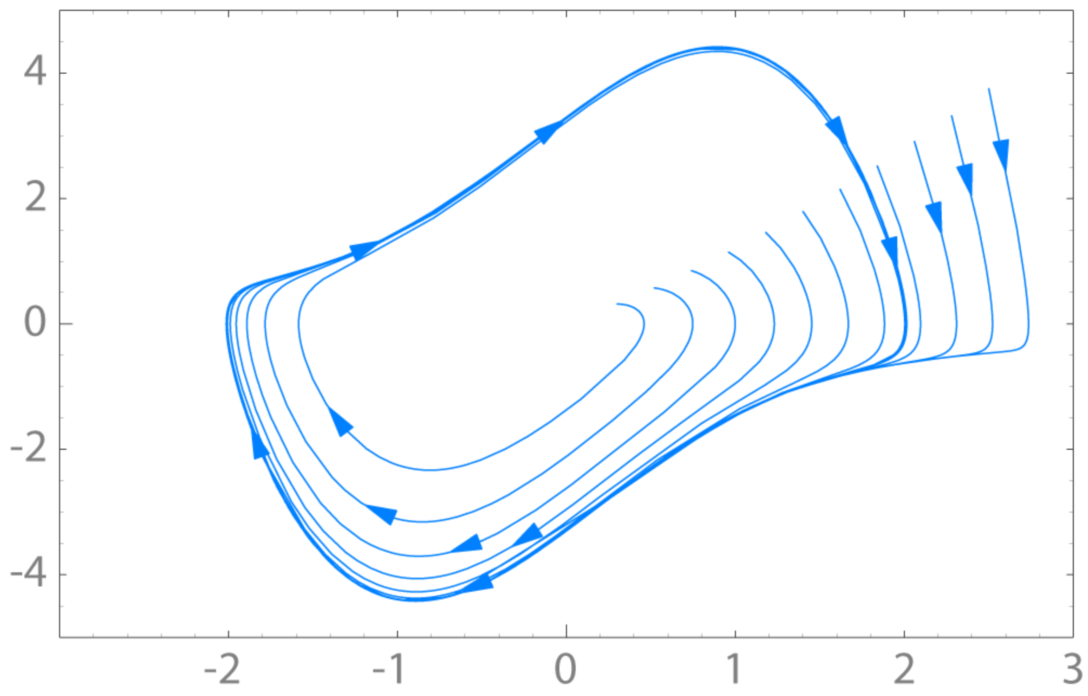
Example: Van der Pol oscillator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2$$

Limit cycle: nontrivial periodic solution

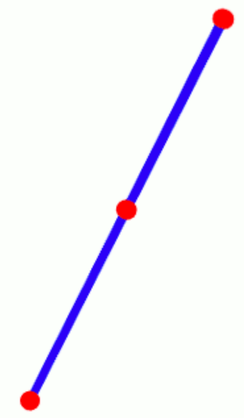
$$\bar{x}(t+T) = \bar{x}(t) \quad \forall t \geq 0$$

$$\Omega := \{x \in \mathbb{R}^n : x = \bar{x}(t), 0 \leq t \leq T\}$$



Chaotic motion "deterministic chaos"

Chaos theory is a branch of mathematics focusing on the study of dynamical systems whose apparently chaotic motion is governed by deterministic laws that are highly sensitive to initial conditions.



Solution concept and fundamental properties

Def. "classical solutions".

A function $\bar{x} : \mathcal{D}_t \supseteq \mathcal{I} \mapsto \mathbb{R}^n$, where $\mathcal{I}$ is an open interval, is called a solution of the dynamical system $\dot{x} = f(t, x)$, $x(t_0) = x_0$ if it is continuously differentiable in $\mathcal{I}$, $t_0 \in \mathcal{I}$, and satisfies

$$\begin{aligned} \dot{\bar{x}}(t) &= f(t, \bar{x}(t)) \quad \forall t \in \mathcal{I} \\ \bar{x}(t_0) &= x_0. \end{aligned}$$

- When the vector field is continuous in both arguments, a classical solution exists but it is not guaranteed to be unique (*Peano*).
- To deal with discontinuous reference signals (such as *steps* or *pulses*), we must refer to solutions which are continuously differentiable only in a piecewise sense.

Existence and uniqueness

Uniqueness is ensured by considering Lipschitz continuous vector fields.

Def. Locally Lipschitz function

A function $f = f(t, x)$ is locally Lipschitz in x, uniformly in t, on $[t_0,\, t_1] \times \mathcal{D}_x$, if any point $x_0 \in \mathcal{D}_x$ has a neighborhood $\mathcal{N}_0$ in which there exists a constant L_0

$$|f(t, y) - f(t, x)| \leq L_0 |y - z| \quad \forall y, z \in \mathcal{N}_0, \quad \forall t \in [t_0,\, t_1]$$

The following theorem establishes local existence and uniqueness of solutions.

Thm. Local Existence & Uniqueness (E&U)

Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x, uniformly in t. Then, there exists a scalar $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, $x(t_0) = x_0$ has a unique piecewise $\mathcal{C}^1$ solution in the interval $[t_0, t_0 + \delta]$.

The local Lipschitz conditions is guaranteed under regularity conditions on the vector field.

Lemma. Sufficient conditions for local lipschitzness.

If function $f(t, x)$ and all its partial derivatives $\frac{\partial f_i}{\partial x_j}$, $i, j = 1, \dots, n$, are continuous on $[t_0,\, t_1] \times \mathcal{D}_x$, then $f(t, x)$ is locally Lipschtiz in x.

Remark

- Under a global Lipschitz condition, the solution is unique and exists $\forall t \geq t_0$.
  - Easily verified for linear systems
  - restrictive condition for nonlinear systems (e.g., $\dot{x} = -x^3, x(0) = x_0$)

The following theorem states that only local lipschitzness is required for E&U provided one knows something more about the solutions if the system.

Thm. Global E&U

Let $f(t,x)$ be piecewise continuous in t, locally Lipschitz in x, uniformly in t, $\forall t \geq t_0$ and $\forall x$ in a domain $\mathcal{D}_x \subseteq \mathbb{R}^n$. Let W be a compact subset of $\mathcal{D}_x$, *i.e.*, a closed and bounded set, $x_0 \in W$, and suppose that it is known that every solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$ lies entirely in W. Then, there is a unique solution defined $\forall t \geq t_0$.