Adaptive and Autonomous Aerospace Systems

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Part 2: Adaptive Control

Lect 4 : Robust adaptive control

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Outline

- Instabilities in standard MRAC designs
    - Parameters drift in the presence of bounded disturbances

- Robust adaptive control: modifications for robustness
    - The dead-zone modification
    - The σ-modification
    - The e-modification
    - The projection operator

Instabilities in standard MRAC designs

So far, we considered adaptive laws that achieved certain control objectives for a given class of nonlinear uncertain plants.

Some of the basic assumptions that we made in designing and analyzing these schemes are the following:

- The plant is free of noise, disturbances and unmodeled dynamics.
- The unknown parameters are constant.

Because in applications most of these assumptions will be violated, it is of interest from the practical point of view to understand the stability properties of the considered schemes when applied to more realistic scenarios.

Instabilities in standard MRAC designs

Let us consider the MIMO uncertain system

$$\dot{x} = A_{ref}x + B\Lambda(u_{ad} + \theta^\top \varphi(x)) + B_{ref}r + d(t)$$
$$y = C_{ref}x$$

which is behind the development of the adaptive MRAC augmentation approach.

Note that:

- The state x could contain both the plant and the baseline controller states
- The subscript e has been removed from the previous slides for shortness of notation.

The system operates in the presence of an unknown but bounded time-dependent disturbance $d(t)$

$$|d(t)| \leq d_M$$

where d_M is an upper bound.

Instabilities in standard MRAC designs

When selecting the adaptive component as

$$u_{ad} = -\hat{\theta}(t)^\top \varphi(x)$$
$$\dot{\hat{\theta}} = \Gamma_\theta \varphi(x) e^\top P B$$

The closed-loop system reads

$$\dot{x} = A_{ref} x - B\Lambda\Delta\theta^\top \varphi(x) + B_{ref} r + d(t)$$

where $\Delta\theta = \hat{\theta} - \theta \in R^{n_\theta \times m}$ is the matrix of parameter estimation errors.

Introducing as usual the reference model (RM) mismatch error $e \coloneqq x - x_{ref}$, we have

$$\dot{e} = \dot{x} - \dot{x}_{ref} = A_{ref} e - B\Lambda \left(\Delta\theta^\top \varphi(x)\right) + d(t)$$

Instabilities in standard MRAC designs

To study the stability of the closed-loop system under the adaptive control laws, we apply Lyapunov analysis.

Specifically, we consider the usual Lyapunov candidate

$$V\left(e, \Delta\Theta\right) = e^\top P e + \mathrm{trace}\left(\Delta\theta^\top \Gamma_\theta^{-1} \Delta\theta \Lambda\right)$$

where $P = P^\top > 0$ satisfies

$$P A_{ref} + A_{ref}^\top P = -Q, \quad Q = Q^\top > 0$$

Instabilities in standard MRAC designs

By computing the time-derivative of V along the closed-loop solutions, we obtain

$$\begin{aligned}\dot{V}\left(e, \Delta\theta\right) &= -e^\top Q e - 2e^\top PB\Lambda\left(\Delta\theta^\top \varphi\left(x\right)\right) + 2\,\mathrm{trace}\left(\Delta\theta^\top \Gamma_\theta^{-1}\dot{\hat{\theta}}\Lambda\right) \\ &= -e^\top Q e \underbrace{-2e^\top PB\Lambda\left(\Delta\theta^\top \varphi\left(x\right)\right) + 2\,\mathrm{trace}\left(\Delta\theta^\top \Gamma_\theta^{-1}\Gamma_\theta\varphi(x)e^\top PB\Lambda\right)}_{=0} + 2e^\top Pd(t)\end{aligned}$$

where the under-brace term is zero by design (the adaptive law $\dot{\hat{\theta}} = \Gamma_\theta\varphi(x)e^\top PB$ is designed to achieve this). Therefore, we have:

$$\dot{V}\left(e, \Delta_e\theta\right) = -e^\top Q e + 2e^\top Pd(t) \leq -\lambda_{min}(Q)|e|^2 + 2\lambda_{max}(P)d_M|e|$$

- In presence of disturbances, the negative semi-definite condition on the Lie derivative is lost.

Instabilities in standard MRAC designs

The condition $\dot{V}(t) < 0$ is valid only outside the set

$$\Omega_d = \left\{ (e, \Delta\theta) : |e| \leq 2 \frac{\lambda_{max}(P)}{\lambda_{min}(Q)} d_M =: e_d \right\}$$

The set Ω_d is not compact in the $e, \Delta\theta$ space (it is unbounded along $\Delta\theta$).

Inside Ω_d, $\dot{V}(t)$ can become positive and therefore parameter estimates ($\hat{\theta}$) can drift.

- Albeit stable, the adaptive laws that we have derived thus far are not robust to arbitrarily small perturbations.
- To enforce robustness of MRAC laws, several solutions have been devised in the literature, in the following we will discuss the most relevant ones.

1) Dead-zone modification

The standard MRAC adaptive law is modified according to the following expression:

$$\dot{\hat{\theta}} = \begin{cases} -\Gamma_\theta \varphi(x) e^\top PB, & \text{if } ||e|| > e_0 \\ 0, & \text{otherwise} \end{cases}$$

Adaptation occurs as long as $\dot{V} < 0$, preventing adaptive parameters from drifting.

Whenever, $|e| \leq e_0$, the adaptation process is stopped, and the parameters are frozen:

$$\hat{\theta}(t) = \hat{\theta}(\bar{t}) \qquad \forall t \geq \bar{t}$$

where $\bar{t}$ is the first time instant for which $|e(\bar{t}\,)| \leq e_0$.

Remarks

- The dead-zone modification is not Lipschitz continuous, and therefore may cause chattering when $|e| \approx e_0$.
- One needs to known the upper bound d_M to compute e_0, which might introduce conservatism.
- Once frozen, the parameters are not updated even if the disturbance disappears.
- The modified laws guarantees that all the signals in the closed-loop system are Uniformly Globally Bounded (UGB) and Globally Uniformly Ultimately Bounded (GUUB)

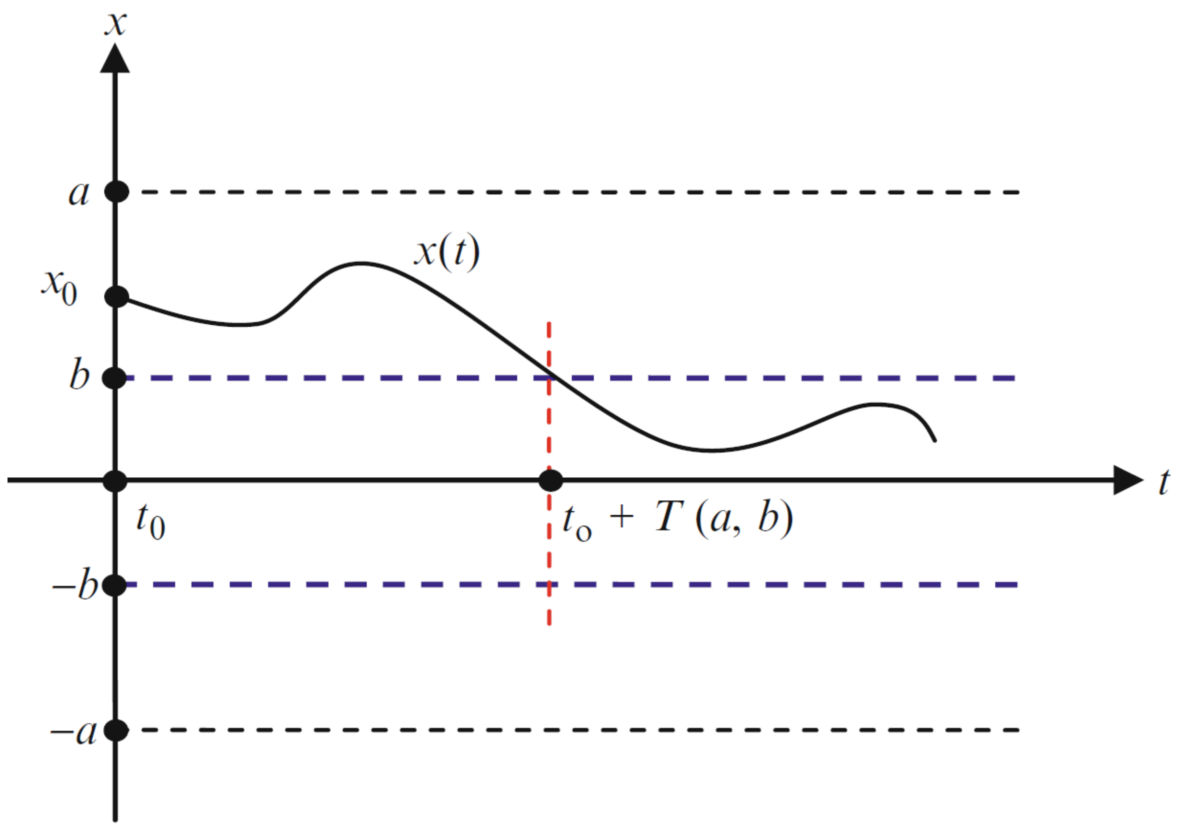
While we already encountered the definition of UGB, the definition of UUB goes as follows:

The solutions of $\dot{x} = f(t, x)$ are said to be Uniformly Ultimately Bounded with ultimate bound b if there exist positive scalars b and c, independent of $t_0 \geq 0$, and $\forall a \in (0, c)$, there exists $T(a, b) > 0$ (independent of t_0), such that

$$|x(t_0)| < a,\ t_0 \geq 0 \implies |x(t; t_0, x(t_0))| < b \quad \forall t \geq t_0 + T(a, b);$$

The solutions are said to be Globally UUB (GUUB) if the above definition holds for arbitrarily large a.

Graphical interpretation of UUB for nonautonomous systems



2) σ-modification

The adaptive law is modified as follows

$$\dot{\hat{\theta}} = \Gamma_\theta(\varphi(x)e^\top PB - \sigma\hat{\theta})$$

In essence, this modification adds damping to the ideal adaptive law.

There are performance-related drawbacks in applying the σ –modification:

- when the tracking error becomes small $(e \approx 0)$, the adaptive law is approximated by $\dot{\hat{\theta}} = -\sigma\Gamma_\theta\hat{\theta}$
- In this case, the adaptive parameters tend to return to the origin, they "unlearn" the gain values that caused the RM error to become small in the first place.

3) e-modification

The main idea behind this modification is to replace the constant damping gain σ with a term proportional to a linear combination of the system RM mismatch error:

$$\dot{\hat{\theta}} = \Gamma_\theta(\varphi(x)e^\top PB - \sigma|e^\top PB|\hat{\theta})$$

The modification allows avoiding the unlearning effect of the σ-modification while achieving the UGB and GUUB of all the closed-loop signals.

4) Projection Operator

The mechanisms behind the modifications considered thus far prevent drifting of the parameters by essentially slowing down the adaptation rate.

Such an effect is considered detrimental since it contradicts the goal of reducing the error as fast as possible.

A modification that avoids these issues while enforcing UGB and GUUB of all the closed-loop signals is the Projection Operator.

The Projection Operator (short, PO) is a “smart” integrator defined by Lipschitz continuous function that keeps the adaptation parameters bounded while making the Lie Derivative of the Lyapunov function negative semi-definite in ideal conditions.

Given $(\theta, y) \in R^{n_\theta} \times R^{n_\theta}$, the projection operator is defined as follows

$$\mathrm{proj}(\theta, y) = \begin{cases} y - f(\theta)\dfrac{\nabla f(\theta)(\nabla f(\theta))^\top}{\|\nabla f(\theta)\|^2} y, & \text{if } \left[f(\theta) > 0 \wedge y^\top \nabla f(\theta) > 0\right] \\ y & , \text{ otherwise} \end{cases}$$

where $f(\theta): R^{n_\theta} \to R$ is a convex C^1 function (to be designed) for which the (convex) sublevel sets

$$\Omega_0 := \{\theta \in \mathbb{R}^{n_\theta} : f(\theta) \leq 0\}, \quad \Omega_1 := \{\theta \in \mathbb{R}^{n_\theta} : f(\theta) \leq 1\}$$

are such that $\emptyset \neq \Omega_0 \subset \Omega_1$.

Remark: the variable y should be interpreted as the vector field of a (vector) differential equation for variable θ, namely, $\dot{\theta} = y$.

Recap: convex sets and functions

Def. Convex set

A set $E \subset R^n$ is said to be convex if

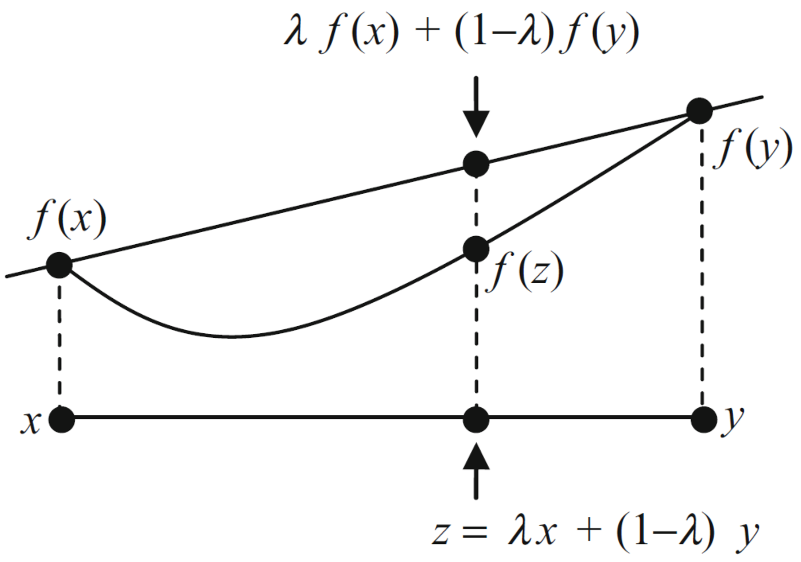
$$(1-\lambda)x_1 + \lambda x_2 \in E \quad \forall x_1, x_2 \in E, \forall \lambda \in [0,1]$$

Essentially, a convex set has the following property. For any two points x, y ∈ E where E is convex, all the points on the connecting line from x to y are also in E.

Def. Convex function

A function $f: R^n \rightarrow R$ is said to be convex if

$$f\left((1-\lambda)x_1 + \lambda x_2\right) \leq (1-\lambda)f(x_1) + \lambda f(x_2) \quad \forall x_1, x_2 \in \mathbb{R}^n \forall \lambda \in [0,1]$$



Graphical interpretation of convex functions

Lemma. Convex sets from convex functions

Let function $f: R^n \to R$ be convex. The, for any $c > 0$ the sublevel set

$$\Omega_c := \{x \in \mathbb{R}^n : f(x) \leq c\}$$

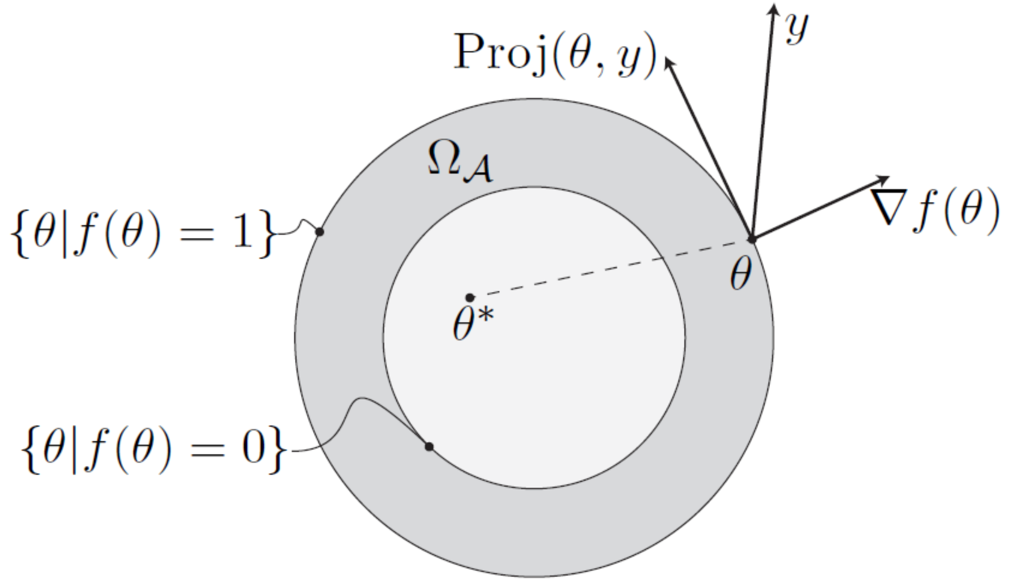
is convex.

2D Graphical illustration

The PO does not alter the vector y if y belongs to the convex set Ω_0.

In the annulus set $\Omega_A \coloneqq \Omega_1 \setminus \Omega_0$, the PO subtracts a vector normal to the boundary $\{\theta \in R^{n_\theta}: f(\theta) = \lambda,\ \lambda \in (0,1)\}$ from y.

As a result, we get a smooth transformation from the original vector field y to the tangent to the boundary for $\lambda = 1$.



Lemma 1

Given the Projection Operator $proj: R^{n_\theta} \times R^{n_\theta} \to R^{n_\theta}$, for any $\theta^* \in \Omega_0$, the following inequality is satisfied

$$(\theta - \theta^*)^\top (\mathrm{proj}(\theta, y) - y) \leq 0$$

Proof

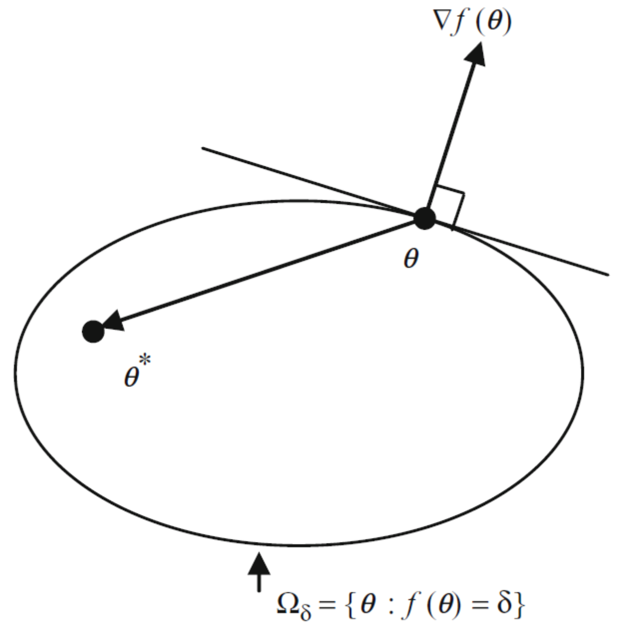
Note that $(\theta - \theta^*)^\top (\mathrm{proj}(\theta, y) - y) = (\theta^* - \theta)^\top (y - \mathrm{proj}(\theta, y))$

Case 1) If $f(\theta) > 0 \wedge y^\top \nabla f(\theta) > 0$, then

$$(\theta^* - \theta)^\top \left(y - \left(y - \frac{\nabla f(\theta)(\nabla f(\theta))^\top}{\|\nabla f(\theta)\|^2} y f(\theta) \right) \right)$$

Robust adaptive control: modifications for robustness

Since for a convex, C^1 function it holds that (see the picture)



$$(\theta^* - \theta)^\top \nabla f(\theta) \leq 0$$

then,

$$\underbrace{(\theta^* - \theta)^\top \nabla f(\theta)}_{\leq 0} \underbrace{(\nabla f(\theta))^\top y}_{>0} \quad \underbrace{f(\theta)}_{\geq \nabla f(\theta)\|^2} \quad \leq 0$$

Case 2) Otherwise, $proj(\theta, y) = y$ and

$$(\theta - \theta^*)^\top (\mathrm{proj}(\theta, y) - y) = (\theta - \theta^*)^\top (\mathrm{proj}(y - y) = 0$$

which concludes the proof.

Lemma 2

Consider the following initial value problem:

- $\dot{\theta} = \text{proj}(\theta, y)$
- $\theta(t_0) = \theta_0 \in \Omega_1 = \{\theta \in \mathbb{R}^{n_\theta} \mid f(\theta) \leq 1\}, t_0 \geq 0$
- $f(\theta) : \mathbb{R}^{n_\theta} \to \mathbb{R}$ is convex

Then, $\theta(t) \in \Omega_1 \; \forall t \geq t_0$.

The lemma states that applying the PO to the vector field of a differential equation ensures that solutions starting inside a desired convex set (specified by using a suitable convex function f) will be trapped inside of it for all future times.

Generalization for MRAC laws

To include gains in the adaptive law, the PO can be extended as follows

$$\mathrm{proj}(\theta, \Gamma y) = \begin{cases} \Gamma y - f(\theta)\Gamma \dfrac{\nabla f(\theta)(\nabla f(\theta))^\top}{\|\nabla f(\theta)\|_\Gamma^2}\Gamma y & \text{if } \left[f(\theta) > 0 \wedge y^\top \Gamma \nabla f(\theta) > 0\right] \\ \Gamma y & \text{otherwise} \end{cases}$$

where $\Gamma \in R^{n_\theta \times m}$ is a symmetric positive-definite matrix.

Recall that in the adaptive law, m represents the number of inputs, $u \in R^m$.

Robust adaptive control: modifications for robustness

In the most general form, the uncertain parameters are collected in a matrix $\theta \in R^{n_\theta \times n}$ such that the uncertain nonlinear term in the plant dynamics is given by $\theta^\top \varphi(x)$.

The projection operator can be extended to deal with a matrix of parameters as follows

$$\mathrm{Proj}(\theta, \Gamma Y) = \left(\mathrm{proj}\left(\theta e_1, \Gamma Y e_1\right) \quad \ldots \quad \mathrm{proj}\left(\theta e_{n_\theta}, \Gamma Y e_{n_\theta}\right)\right)$$

where $e_i = [0 \cdots 1 \cdots 0]^\top \in R^m$, $Y \in R^{n_\theta \times m}$ is a matrix-valued vector field and $\theta e_i, Y e_i$ are the *i-th* columns of θ and Y, respectively.

Note that the matrix PO satisfies the following inequality (which will be exploited in the Lyapunov analysis):

$$\mathrm{trace}\left(\Delta\theta^\top \left(\Gamma^{-1}\,\mathrm{Proj}(\hat{\theta}, \Gamma Y) - Y\right)\right) = \sum_{j=1}^{n_\theta} \underbrace{(\hat{\theta} - \theta)_j^\top \left(\Gamma^{-1}\,\mathrm{Proj}\left(\hat{\theta}, \Gamma Y_j\right) - Y_j\right)}_{\leq 0} \leq 0$$

Projection-based MRAC

Consider the usual Lyapunov candidate

$$V\left(e, \Delta\Theta\right) = e^\top P e + \operatorname{trace}\left(\Delta\theta^\top \Gamma_\theta^{-1} \Delta\theta \Lambda\right)$$

Its time-derivative along the system trajectories reads

$$\begin{aligned}
\dot{V}\left(e, \Delta\theta\right) &= -e^\top Q e - 2e^\top P B \Lambda \left(\Delta\theta^\top \varphi\left(x\right)\right) + 2 \operatorname{trace}\left(\Delta\theta^\top \Gamma_\theta^{-1} \dot{\hat{\theta}} \Lambda\right) \\
&= -e^\top Q e - 2 \operatorname{trace} \Delta\theta^\top \varphi(e^\top P B \Lambda \left(x\right) + 2 \operatorname{trace}\left(\Delta\theta^\top \Gamma_\theta^{-1} \dot{\hat{\theta}} \Lambda\right) + 2e^\top P d(t) \\
&= -e^\top Q e + 2 \operatorname{trace}\left(\Delta\theta^\top \left(\Gamma_\theta^{-1} \dot{\hat{\theta}} - \varphi(x) e^\top P B\right) \Lambda\right) + 2e^\top P d(t)
\end{aligned}$$

Selecting the adaptive law as

$$\dot{\hat{\theta}} = \mathrm{Proj}\left(\hat{\theta}, \Gamma_\theta \varphi(x) e^\top P B\right)$$

the following chain of inequalities for $\dot{V}$ can be derived (Λ is assumed to be <u>diagonal</u> PD):

$$\begin{aligned}\dot{V}(e, \Delta\theta) &\le -e^\top Q e + 2e^\top P d(t) \le -\lambda_{\min}(Q)|e|^2 + 2|e|\lambda_{\max}(P) d_M \\ &= -\lambda_{\min}(Q)|e|\left(|e| - 2\frac{\lambda_{\max}(P) d_M}{\lambda_{\min}(Q)}\right)\end{aligned}$$

Therefore, the Lie derivative is negative outside the set

$$\Omega_d = \left\{(e, \Delta\theta) : |e| \le 2\frac{\lambda_{max}(P)}{\lambda_{min}(Q)} d_M =: e_d\right\}$$

Robust adaptive control: modifications for robustness

Moreover, for any initial conditions

$$|\hat{\theta}(t_0)e_i| \leq \theta_{M_i}$$

by leveraging the properties of the PO, we have

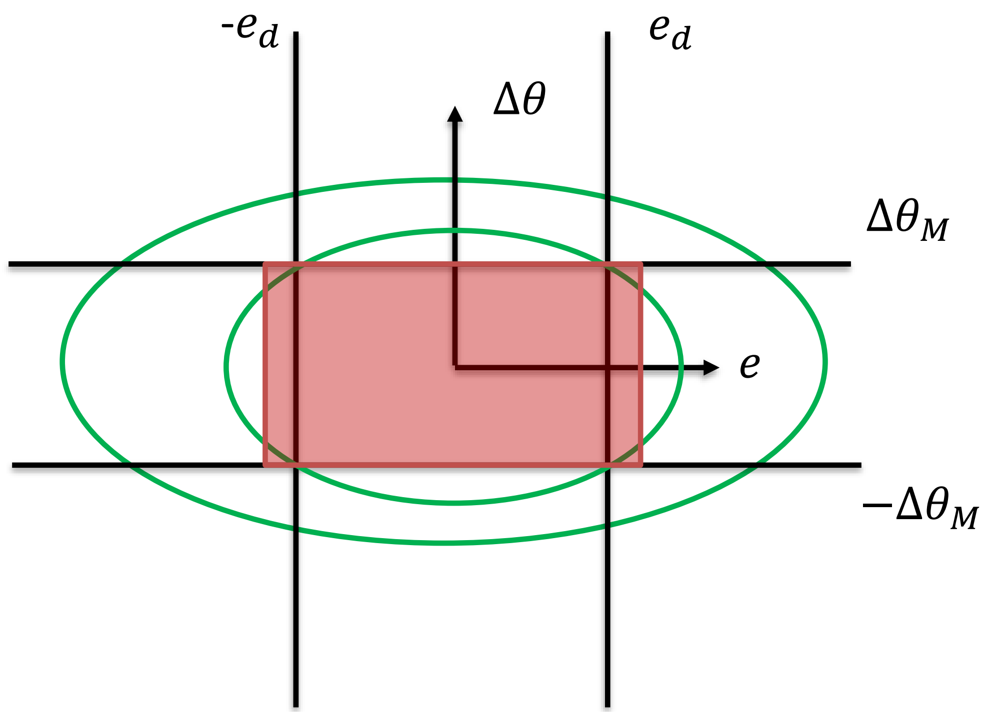
$$|\hat{\theta}(t)e_i| \leq \theta_{M_i} \quad \forall i = 1, \ldots, n_\theta$$

Therefore, all solutions starting in the set $\{(e, \Delta\theta\ ) \in R^n \times R^{n_\theta \times m}\colon |\Delta\theta| \leq \Delta\theta_M\}$, where $\Delta\theta_M = 2\max\{\theta_{M_1}, \ldots, \theta_{M_{n_\theta}}\}$, will be trapped after a finite time in a compact set which contains the set

$$\left\{(e, \Delta\theta) \in \mathbb{R}^n \times \mathbb{R}^{n_\theta \times n} : |e| \leq e_d, \quad |\Delta\theta| \leq \Delta\theta_M\right\}$$

2D graphical representation

- The green ellipses represent level curves of the Lyapunov candidate
$$V = e^2 + \frac{\Delta\theta^2}{\gamma_\theta}$$
- The red rectangle represent the set in which the closed-loop trajectories are ultimately confined.



Remark

The initial condition $\hat{\theta}(t_0)$ must be inside the set $\left|\hat{\theta}(t_0)\right| \leq \theta_M$.

Practical implementation of the PO

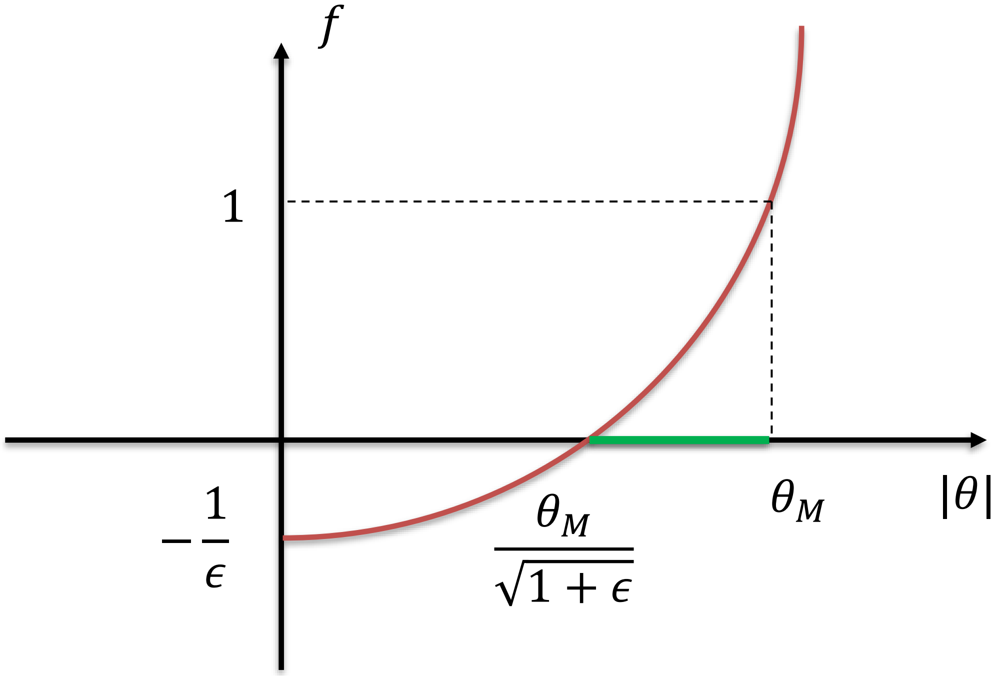
A common selection for the convex function f when $\theta \in R^{n_\theta}$

$$f(\theta) = \frac{(1+\epsilon)|\theta|^2 - \theta_M^2}{\epsilon\theta_M^2}$$

where $\epsilon > 0$. For this function, we have

$$\Omega_0 = \left\{\theta \in \mathbb{R}^{n_\theta} : |\theta| \le \frac{\theta_M}{\sqrt{1+\epsilon}}\right\}$$

$$\Omega_1 = \{\theta \in \mathbb{R}^{n_\theta} : |\theta| \le \theta_M\}$$



- The PO is active in the green interval $(\Omega_1 \setminus \Omega_0)$.

The gradient of f is given by

$$\nabla f = \frac{2(1+\epsilon))}{\epsilon\theta_M}\theta$$

Robust adaptive control: modifications for robustness

In the previous development of the PO, we assumed to know an upper bound on each column of the uncertain matrix θ.

This approach might be conservative as different upper bounds can exist for each element.

The approach seen so far can be easily extended to deal with individual bounds by specifying a function

$$f(\theta_{ij}) = \frac{(1+\epsilon_{ij})\theta_{ij}^2 - \theta_{M_{ij}}^2}{\epsilon_{ij}\theta_{M_{ij}}^2}$$

for each $\theta_{ij}, i = 1, \dots n_\theta, j = 1, \dots, m$.