Adaptive Aerospace Control

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Part 1: Analysis of nonlinear time-varying systems

Lect. 2: Stability definitions

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Outline

- Equilibrium point
- Stable, uniformly stable and unstable equilibrium point
- Attractivity and uniform attractivity
- Asymptotic and uniform asymptotic stability
- Exponential stability
- Global stability definitions

Def. Equilibrium point

A point $\bar{x} \in \mathbb{R}^n$ is said to be an equilibrium point for the state equation $\dot{x} = f(t,x)$ if $x(t_0) = \bar{x}$, $t_0 \geq 0 \implies x(t;t_0,\bar{x}) = \bar{x}$, $\forall t \geq t_0$.

- Without loss of generality, we can cast the stability analysis of a nontrivial solution to the state equation as the stability analysis of the origin.

Consider a nontrivial solution $t \mapsto \bar{x}(t;t_0,x_0)$. The change of coordinates

$$e = x - \bar{x}$$
$$\tau = t - t_0$$

gives

$$\frac{d}{d\tau}\tilde{e}(\tau) = \dot{x} - \dot{\bar{x}} = f(\tau + t_0, e(\tau + t_0) + \bar{x}(\tau + t_0)) - f(\tau + t_0, \bar{x}(\tau + t_0)) =: g(\tau, \tilde{e})$$

Then, $g(\tau, 0) = 0\ \forall \tau \geq 0$: $\bar{e} = 0$ is equilibrium point for the transformed (time-varying system)

Local stability definitions

Def. Stability in the sense of Lyapunov

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be

- stable if, $\forall \varepsilon > 0$ and $\forall t_0 \geq 0$, there is $\delta = \delta(t_0, \varepsilon)$ such that

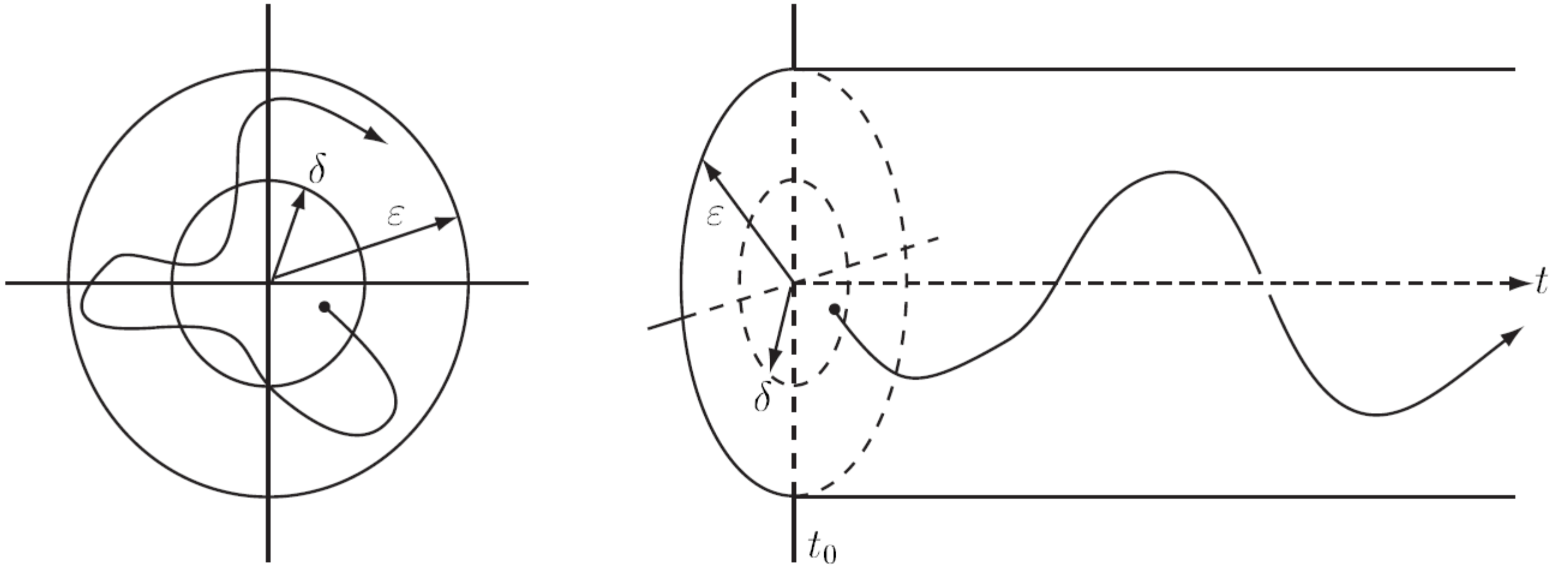
$$|x(t_0)| < \delta \implies |x(t; t_0, x(t_0))| < \varepsilon \quad \forall t \geq t_0;$$

- *uniformly* stable if, $\forall \varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ (independent of t_0) such that

$$|x(t_0)| < \delta,\ t_0 \geq 0 \implies |x(t; t_0, x(t_0))| < \varepsilon \quad \forall t \geq t_0;$$

- unstable if it is not stable.

Stability



Remarks

- The concept of stability and *uniform* stability are the <u>same</u> for autonomous systems since changing the initial time does not affect the solutions.

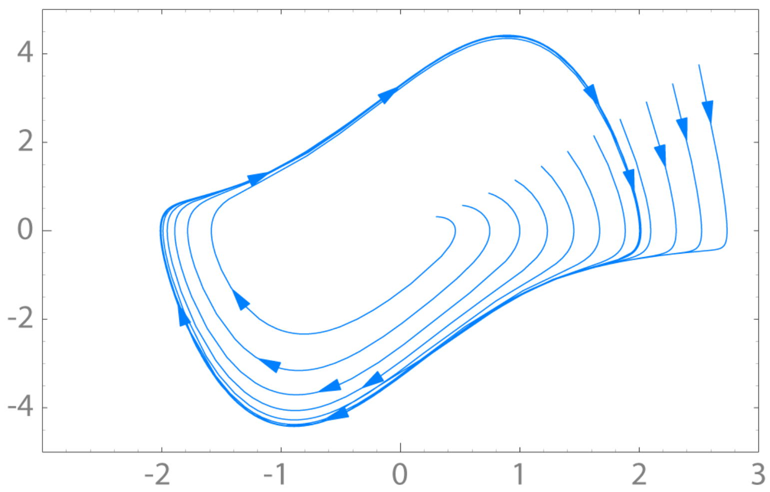
Example

$$\dot{x} = (6t\sin t - 2t)x \qquad x(t_0) = x_0$$

- Instability does not imply <u>unbounded</u> solutions.

Example: Van Der Pol oscillator.

The solutions starting arbitrarily close to the origin leave any circle strictly inside the limit cycle.



Local stability definitions

Def. Attractivity

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be

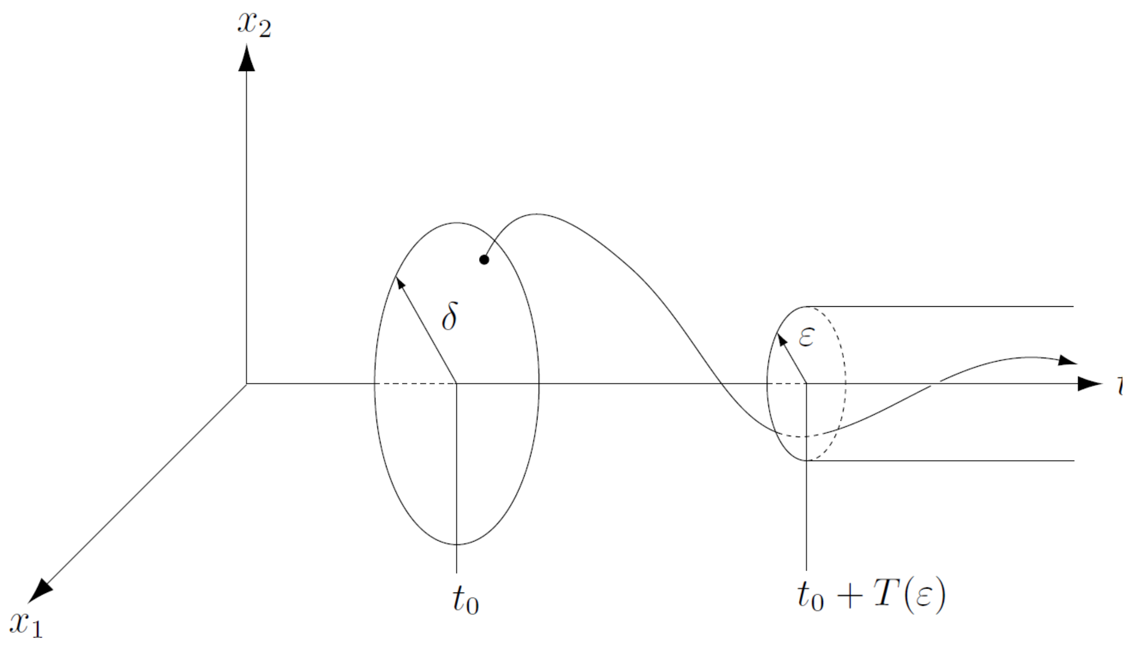
- attractive if, $\forall t_0 \geq 0$, there is $\eta(t_0) > 0$ such that

$$|x(t_0)| < \eta(t_0) \implies \lim_{t \to \infty} |x(t; t_0, x(t_0))| = 0$$

- *uniformly* attractive if there is $\eta > 0$ (independent of t_0) such that $\forall \varepsilon > 0$, there is $T(\varepsilon)$ (independent of t_0) such that

$$|x(t_0)| < \eta,\ t_0 \geq 0 \implies |x(t; t_0, x(t_0))| < \varepsilon \quad \forall t \geq t_0 + T(\varepsilon).$$

Uniform attractivity



Local stability definitions

Stability and attractivity are <u>independent</u> notions: there are examples of system which have equilibrium points that are attractive but unstable or *viceversa*.

Therefore, by combining the two notions, two additional stability definitions are derived.

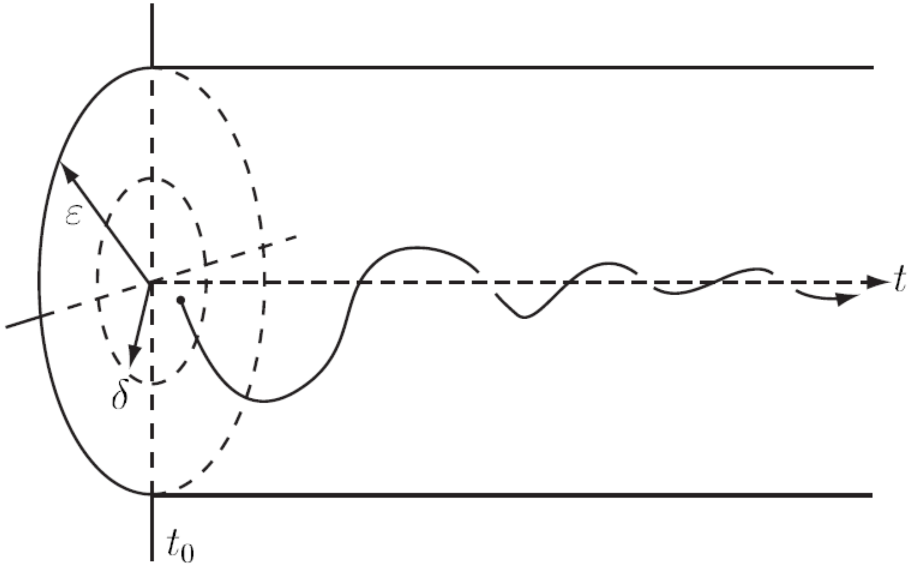
Def. (Local) Asymptotic Stability (LAS)

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be

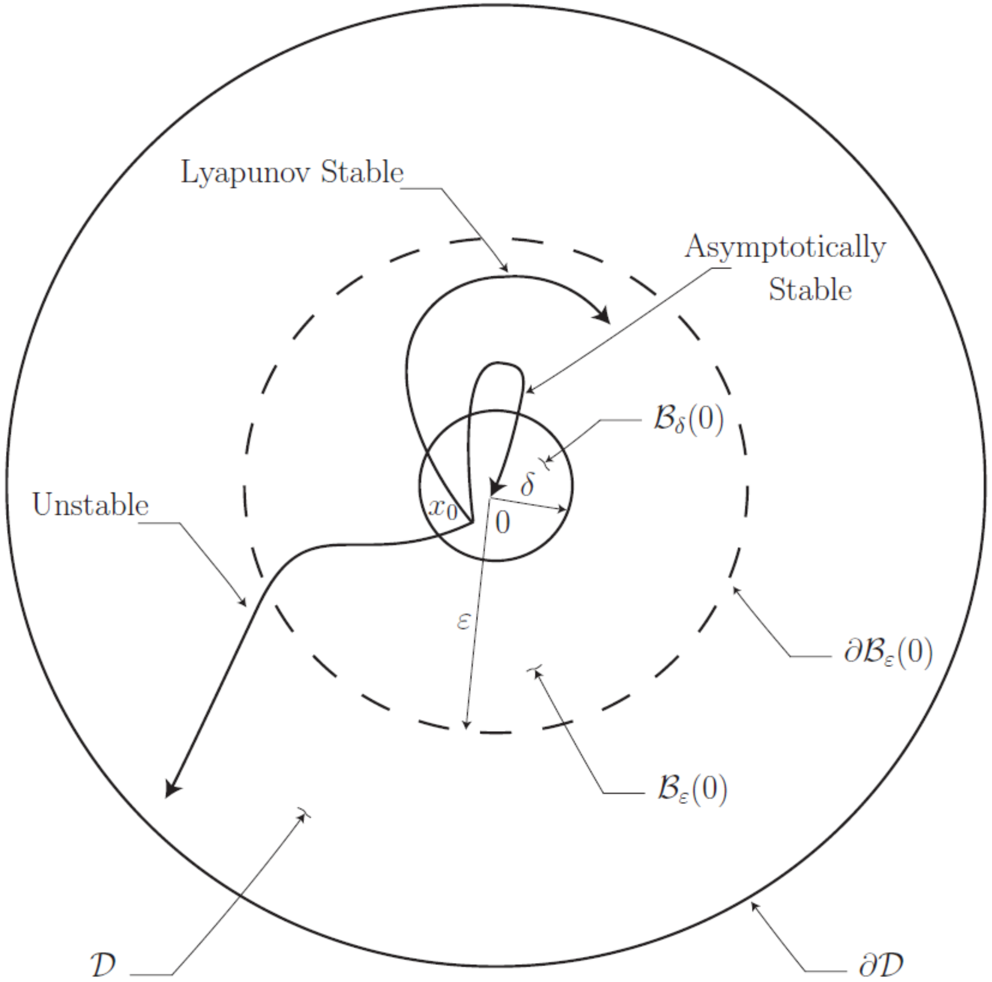
- Locally Asymptotically Stable (LAS) if it is stable and attractive;
- Uniformly Locally Asymptotically Stable (ULAS) if it is uniformly stable and attractive.

Example (UGS+GAT system) $\dot{x} = -\dfrac{1}{1+t}x$

Local Asymptotic Stability



Stability definitions



Local stability definitions

Def. (Local) Exponential Stability (LES)

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be Locally Exponentially Stable (LES) if there exist positive constants λ, c and $k \geq 1$ such that

$$|x(t_0)| < c,\ t_0 \geq 0 \implies |x(t; t_0, x(t_0))| \leq k|x(t_0)| \exp(-\lambda(t - t_0)) \quad \forall t \geq t_0.$$

Remark: exponential stability plays an important role in the analysis of perturbed systems.

Global stability definitions

Def. Uniform Global Stability

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be Uniformly Globally Stable (UGS) if $\forall \varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ (independent of t_0) such that $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$ and

$$|x(t_0)| < \delta,\, t_0 \geq 0 \implies |x(t; t_0, x(t_0))| < \varepsilon \quad \forall t \geq t_0;$$

Def. Uniform Global Attractivity

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be Uniformly Globally Attractive (UGAT) if $\forall \varepsilon > 0$ and $\forall \eta > 0$, there is $T(\eta, \varepsilon) > 0$ such that

$$|x(t_0)| < \eta,\, t_0 \geq 0 \implies |x(t; t_0, x(t_0))| < \varepsilon \quad \forall t \geq t_0 + T(\eta, \varepsilon);$$

Global stability definitions

Def. Uniform Global Asymptotic Stability (UGAS)

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be Uniformly Globally Asymptotically Stable (UGAS) if it is UGS and UGAT.

Remark: for an equilibrium to be UGAS, a necessary conditions is that it be the *only* equilibrium.

UGAS = UGS + UGAT

UGAS can be defined equivalently by combining UGAT with the notions of ULS and Uniform Global Boundedness (UGB).

Def. Uniform Global Boundedness (UGB)

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be Uniformly Globally Bounded if $\forall a > 0$, there is $b(a) > 0$ (independent of t_0) such that

$$|x(t_0)| < a,\ t_0 \geq 0 \implies |x(t; t_0, x(t_0))| < b \quad \forall t \geq t_0;$$

UGAS = ULS + UGB + UGAT

Finally, we recall the definition of **global exponential stability** which is a desirable stability property of all control systems, although hardly achievable in practice.

Def. Global Exponential Stability (GES)

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is said to be Globally Exponentially Stable (GES) if there exist positive constants λ, c and $k \geq 1$ such that

$$|x(t; t_0, x(t_0)| \leq k|x(t_0)| \exp(-\lambda(t - t_0)) \quad \forall t \geq t_0, \forall x_0 \in \mathbb{R}^n.$$

Stability definitions in terms of comparison functions

Stability definitions involving uniformity take a remarkably simple form when making use of special functions, which are called comparison functions (*Massera,1956*).

Def. Class-$\mathcal{K}$ functions

A continuous function $\alpha : [0, a) \mapsto \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$

Example: $\alpha(r) = \tan^{-1}(r)$

Def. Class-$\mathcal{K}_\infty$ functions

A continuous function $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and $\lim_{r\to\infty} \alpha(r) = \infty$.

Example: $\alpha(r) = r^c$ for $c > 0$.

Def. Class-$\mathcal{KL}$ functions

A continuous function $\beta : [0, a) \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ if the mapping $r \mapsto \beta(r, s)$ is of class-$\mathcal{K}$, while the mapping $s \mapsto \beta(r, s)$ is strictly decreasing and $\lim_{s\to\infty} \beta(r, s) = 0$.

Examples

$$\beta(r, s) = \frac{r}{ksr + 1}, \quad k > 0$$

$$\beta(r, s) = r^c e^{-as}, \quad a, c > 0$$

Local stability definitions revisited

Lemma: stability in terms of comparison functions

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- ULS if and only if there exist a class-$\mathcal{K}$ function $\alpha(\cdot)$ and a scalar $c > 0$ (independent of t_0) such that

$$|x(t; t_0, x(t_0))| \leq \alpha(|x(t_0)|), \quad \forall t \geq t_0 \geq 0, \, \forall |x(t_0)| < c$$

- ULAS if and only if there exist a class-$\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a scalar $c > 0$ (independent of t_0) such that

$$|x(t; t_0, x(t_0))| \leq \beta(|x(t_0)|, t - t_0), \quad \forall t \geq t_0 \geq 0, \, \forall |x(t_0)| < c$$

Global stability definitions revisited

Lemma: stability in terms of comparison functions

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- UGS if and only if there exist a class-$\mathcal{K}_\infty$ function $\gamma(\cdot)$ such that

$$|x(t; t_0, x(t_0))| \leq \gamma(|x(t_0)|), \quad \forall t \geq t_0 \geq 0, \, \forall x(t_0) \in \mathbb{R}^n$$

- UGAS if and only if there exist a class-$\mathcal{KL}$ function $\beta(\cdot, \cdot)$ such that

$$|x(t; t_0, x(t_0))| \leq \beta(|x(t_0)|, t - t_0), \quad \forall t \geq t_0 \geq 0, \, \forall x(t_0) \in \mathbb{R}^n$$

Stability of the estimator for the simple algebraic system

General solution of the gain estimation error

$$e_\theta(t) = \exp\left(-\gamma \int_{t_0}^{t} u(\tau)^2 d\tau\right) e_\theta(t_0)$$

Since

$$\left|\exp\left(-\gamma \int_{t_0}^{t} u(\tau)^2 d\tau\right)\right| \leq 1, \quad \forall t \geq t_0$$

we have UGS, i.e., $\forall t \geq t_0$,

$$|e_\theta(t)| \leq |e_\theta(t_0)| = \gamma(|e_\theta(t_0)|) \rightarrow \gamma(r) = r, \quad \gamma(\cdot) \in \mathcal{K}_\infty$$

Solution for PE input signals: GES

$$|e_\theta(t)| \leq \exp(-\gamma(t-t_0))|e_\theta(t_0)| = \beta(|e_\theta(t_0)|, t-t_0), \quad \beta(r,s) = r\exp(-\lambda s), \quad \beta(\cdot,\cdot) \in \mathcal{KL}$$

Key points

- There are a lot of ways of characterizing stability…
- Stability with respect to a trajectory can be always characterized as stability with respect to an equilibrium point (for a time-vary system).
- When dealing with a nonlinear model, we cannot talk about stability of the system but we must refer to the stability of a specific equilibrium point of the system.
- An unstable equilibrium point can have bounded solutions in its neighborhood.
- An equilibrium point can be unstable yet attractive!

Key points

- When dealing with time-varying systems things become more complicated…
- Uniformity is important!
- ULAS is nice but exponential stability is more robust.
- Achieving uniformity in adaptive systems is not straightforward…
- I cannot have global asymptotic stability with more than one equilibrium point.
- There are special “comparison” functions that help me characterize stability notions in a compact manner.