Adaptive and Autonomous Aerospace Systems

School of Industrial and Information Engineering - Aeronautical Engineering
Davide Invernizzi – Department of Aerospace Science and Technology

Part 2: Adaptive Control

Lect 2-3: MRAC for SISO and MIMO systems

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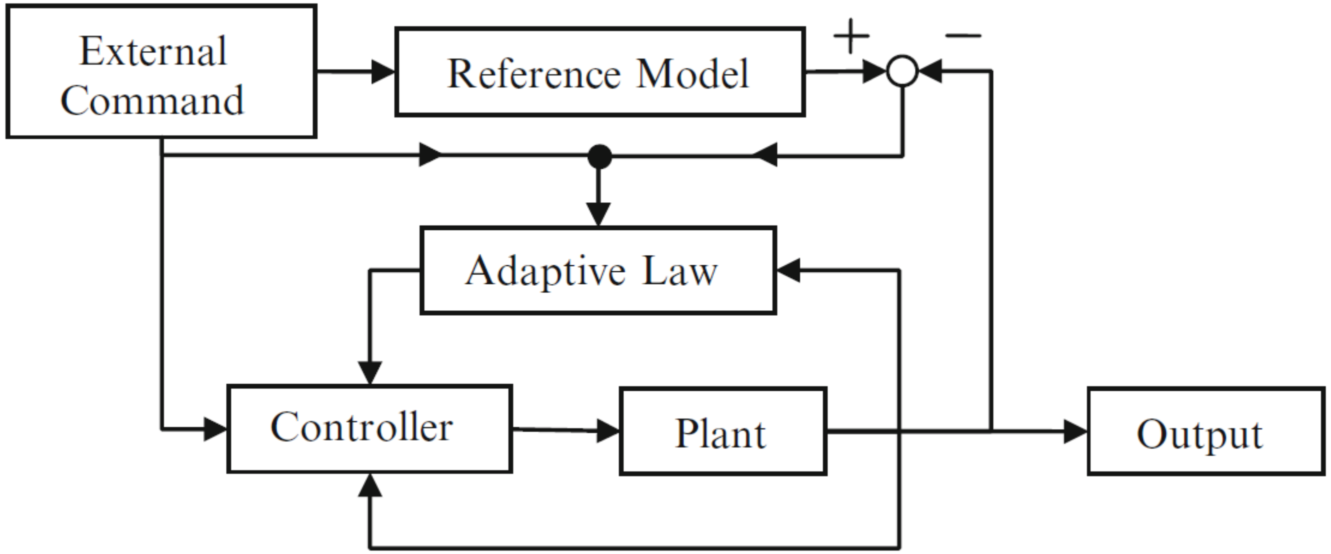


Introduction

A Model Reference Adaptive Controller (MRAC) is a controller whose parameters (gains) are updated in real-time using an adaptive law.

The adaptive law operates on the system output and on an external command (the reference input).

The command also drives a Reference Model (RM) that specifies the desired trajectories for the system to follow.

Introduction

The difference between the reference model output and the system output constitutes the tracking error (Reference Model mismatch error), which subsequently is sent to the adaptive law for online parameter adjustments.

Per design, the adaptive controller forces the system to follow the reference model dynamics while operating in the presence of the plant uncertainties.

The controller main objective is to maintain consistent performance of the closed-loop system in the presence of uncertainties and unknown variations in plant parameters.

MRAC for SISO systems:
Direct approach

MRAC for SISO systems

Recall the simplified helicopter pitch dynamics in hover from Lab 2:

$$\dot{q} = M_q q + M_\delta(\delta + f(q)) \qquad f(q) = \theta \tanh\left(\frac{360}{\pi} q\right)$$

Generalization of the model: consider the scalar uncertain systems of the form

$$\dot{x} = ax + b(u + \theta^\top \varphi(x))$$

Assumptions:

- $a, b \in R$, $\theta \in R^n$ are unknown but constant parameters.
- $\varphi(x): R \mapsto R^n$ is a known vector containing nonlinear basis functions (regressor)

$f(x) = \theta^\top \varphi(x)$ is an uncertain nonlinear term which is linear in the unknown parameters.

MRAC for SISO systems

Starting point: **Model Reference Control**

All parameters are assumed to be known and the following control law is considered:

$$u_{\text{ideal}} = k_x x + k_r r - \theta^T \Phi(x)$$

which comprises a feedback and feedforward term plus a term to cancel uncertainties. The corresponding closed-loop plant is

$$\dot{x} = (a + bk_x)\, x + bk_r r(t)$$

The gains of the control law are selected as follows (Matching Conditions)

$$\begin{aligned} a + bk_x &= a_{ref} \\ bk_r &= b_{ref} \end{aligned}$$

where a_{ref} and b_{ref} define the coefficients of the reference model, which describes the ideal desired response of the system to external commands $r(t)$.

With the ideal gains, the closed-loop system behaves as the reference model

$$\dot{x} = a_{ref} x + b_{ref} r$$

and the MRC problem is solved: the RM mismatch error

$$e := x - x_{ref}$$

converges to zero globally and exponentially, namely, $e = 0$ is GES:

$$\dot{e} = a_{ref} e, \qquad a_{ref} < 0$$

MRAC for SISO systems

However, in practice, one has only estimates $(\hat{a}, \hat{b}, \hat{\theta})$ of the true parameters and perfect model matching <u>cannot</u> be achieved in practice with MRC.

Using the matching conditions with the estimates, the corresponding gains $\hat{k}_x = \frac{a_{ref} - \hat{a}}{\hat{b}}$, $\hat{k}_r = \frac{b_{ref}}{\hat{b}}$ and MRC law $u_{MRC} = \hat{k}_x x + \hat{k}_r r - \hat{\theta}^\top \varphi(x)$ would result in the closed-loop system

$$\dot{x} = \left(a + b\hat{k}_x\right) x + b\left(\hat{k}_r r - (\hat{\theta} - \theta)^T \varphi(x)\right)$$

By adding $\pm b k_x x$ and $\pm b k_r r$ to the previous equation, where k_x and k_r are the ideal <u>unknown</u> gains, and then by introducing the matching conditions, one has

$$\dot{x} = \underbrace{a + bk_x}_{a_{ref}} x + \underbrace{bk_r}_{b_{ref}} r + b\underbrace{\left(\hat{k}_x - k_x\right)}_{\Delta k_x} x + b\underbrace{\left(\hat{k}_r - k_r\right)}_{\Delta k_r} r - b\underbrace{(\hat{\theta} - \theta)}_{\Delta\theta} \Phi(x)$$

where $\Delta k_x = \hat{k}_x - k_x, \quad \Delta k_r = \hat{k}_r - k_r, \quad \Delta\theta = \hat{\theta} - \theta$ are the gain and parameters errors.

MRAC for SISO systems

The closed-loop system looks like the reference model plus perturbations due to the non-perfect knowledge of the plant parameters.

Consider again the reference model mismatch error

$$e := x - x_{ref}$$

In the non-ideal case, the dynamics of e is given by

$$\dot{e} = a_{ref} e + b \left(\Delta k_x x + \Delta k_r r - \Delta \theta^T \varphi(x) \right)$$

In the presence of uncertain gains, the MRC <u>does not</u> achieve the design objective.

MRAC approach

The starting point of MRAC is the definition of a control input with the same form of the one in the MRC approach but in which the gains are time-varying estimates:

$$u_{MRAC} = \hat{k}_x(t)x + \hat{k}_r(t)r - \hat{\theta}^\top(t)\varphi(x)$$

In order to derive adaptive laws to compute in real-time the gains to achieve model reference matching, a Lyapunov design approach is employed

$$V\left(e, \Delta k_x, \Delta k_r, \Delta\theta\right) = \frac{1}{2}e^2 + \frac{|b|}{2}\left(\gamma_x^{-1}\Delta k_x^2 + \gamma_r^{-1}\Delta k_r^2 + \Delta\theta^T\Gamma_\theta^{-1}\Delta\theta\right)$$

The Lyapunov candidate is a quadratic positive definite function of the RM mismatch error and of the parameter estimation errors.

The scalars γ_x, γ_r and the matrix $\Gamma_\theta \in R^{n \times n}$ will be used as tunable parameters in the adaptive laws.

MRAC for SISO systems

By computing the time derivative of V along the closed-loop dynamics, one obtains

$$\begin{aligned}\dot{V}\left(e, \Delta k_x, \Delta k_r, \Delta\theta\right) = a_{ref} e^2 + 2|b|\left(\Delta k_x\left(xe\,\mathrm{sgn}(b) + \gamma_x^{-1}\dot{\hat{k}}_x\right)\right) \\ + |b|\left(\Delta k_r\left(re\,\mathrm{sgn}(b) + \gamma_r^{-1}\dot{\hat{k}}_r\right)\right) + |b|\Delta\theta^T\left(-\varphi(x)e\,\mathrm{sgn}(b) + \Gamma_\theta^{-1}\dot{\hat{\theta}}\right)\end{aligned}$$

The adaptive laws are derived by setting equal to zero the terms in the parentheses:

$$\begin{aligned}\dot{\hat{k}}_x &= -\gamma_x xe\,\mathrm{sgn}(b) \\ \dot{\hat{k}}_r &= -\gamma_r re\,\mathrm{sgn}(b) \\ \dot{\hat{\theta}} &= \Gamma_\theta \Phi(x)e\,\mathrm{sgn}(b)\end{aligned}$$

In this way, the Lie derivative becomes:

$$\mathcal{L}_f V\left(e, \Delta k_x, \Delta k_r, \Delta\theta\right) = a_{ref} e^2$$

which is negative semi-definite (recall that $a_{ref} < 0$ by design to have a stable RM).

By LaSalle/Yoshizawa theorem, we can conclude that:

- The equilibrium point $(e, \Delta k_x, \Delta k_r, \Delta\theta) = (0,0,0,0)$ is UGS
- The set $\{(e, \Delta k_x, \Delta k_r, \Delta\theta) \in R^{3+n} : e = 0\}$ is globally attractive, which means that

$$\lim_{t \to \infty} e(t) = 0$$

Remarks

- The attractivity property of the set is <u>not</u> uniform in general.
- The parameter estimation errors are <u>not</u> guaranteed to converge to zero and not even to converge to a constant.
- Under specific persistence of excitation conditions, one can prove that the origin of the closed-loop error system is UGAS.

MRAC for SISO systems:
Indirect approach

Indirect MRAC for SISO systems

We start considering a slightly modified uncertain scalar plant of the form

$$\dot{x} = ax + bu + \theta^\top \varphi(x)$$

Differently from the direct approach, the plant is not parametrized in terms of the control gains, but the adaptive laws are based on real-time estimates of the plant parameters:

$$\dot{x} = \hat{a}x + \hat{b}u + \hat{\theta}^\top \varphi(x) - \underbrace{(\hat{a} - a)}_{\Delta a} x - \underbrace{(\hat{b} - b)}_{\Delta b} u - \underbrace{(\hat{\theta}^\top \varphi(x) - \theta^\top \varphi(x))}_{\Delta \theta^\top \varphi(x)}$$

The goal of the indirect MRAC design is to derive adaptive laws for the *hat* variables to solve the RM control problem.

Indirect MRAC for SISO systems

The structure of the control law is based on the concept of "dynamic inversion", according to which the control input aims at replacing the system dynamics with the RM one:

$$u_{I-MRAC} = \frac{1}{\hat{b}}\left((a_{ref} - \hat{a})\,x + b_{\mathrm{ref}}\,r - \hat{\theta}^\top \varphi(x)\right)$$

From the above expression, one can define the time-varying gains

$$\hat{k}_x(\hat{a},\hat{b}) = \frac{a_{ref} - \hat{a}}{\hat{b}}, \qquad \hat{k}_r(\hat{b}) = \frac{b_{ref}}{\hat{b}}, \qquad \hat{k}_\theta(\hat{b},\hat{\theta}) = -\frac{\theta^\top}{\hat{b}}$$

Singularity when $\hat{b} = 0$

and rewrite the control law as:

$$u_{I-MRAC} = \hat{k}_x(\hat{a},\hat{b})x + \hat{k}_r(\hat{b}) + \hat{k}_\theta(\hat{b},\hat{\theta})\varphi(x)$$

The gains are computed from the plant parameters estimates using <u>algebraic</u> equations.

Indirect MRAC for SISO systems

The adaptive laws are derived again following a Lyapunov design approach.

First, the closed-loop dynamics is derived by substituting the indirect control law into the plant dynamics

$$\dot{x} = a_{ref}x + b_{ref}r - \Delta a x - \Delta b u - \Delta\theta^T \varphi(x)$$

and then the closed-loop error dynamics is derived as usual:

$$\dot{e} = a_{ref}e - \Delta a x - \Delta b u - \Delta\theta^\top \varphi(x)$$

The following Lyapunov candidate is considered

$$V(e, \Delta a, \Delta b, \Delta\theta) = \frac{1}{2}e^2 + \frac{1}{2\gamma_a}\Delta a^2 + \frac{1}{\gamma_b}\Delta b^2 + \frac{1}{2}\Delta\theta^T \Gamma_\theta^{-1} \Delta\theta$$

Indirect MRAC for SISO systems

The time derivative of the Lyapunov candidate along the closed-loop solutions reads:

$$\dot{V}(e, \Delta a, \Delta b, \Delta\theta) = a_{ref} e^2 + \Delta a \left(\gamma_a^{-1} \dot{\hat{a}} - xe\right) + \Delta b \left(\gamma_b^{-1} \dot{\hat{b}} - ue\right) + \Delta\theta^T \left(\Gamma_\theta^{-1} \dot{\hat{\theta}} - \varphi(x) e\right)$$

By selecting the adaptive laws to make the terms in the parentheses vanish, namely, by considering

$$\dot{\hat{a}} = \gamma_a x e$$

$$\dot{\hat{b}} = \gamma_b u e$$

$$\dot{\hat{\theta}} = \Gamma_\theta \varphi(x) e$$

the Lie derivative becomes negative semi-definite

$$\mathcal{L}_f V(e, \Delta a, \Delta b, \Delta\theta) = a_{ref} e^2 \leq 0$$

Provided that $\hat{b}(t) \neq 0 \; \forall t \geq 0$ (which would make u singular!), by LaSalle/Yoshizawa theorem, one can conclude that:

- The equilibrium point $(e, \Delta k_x, \Delta k_r, \Delta\theta) = (0,0,0,0)$ is UGS
- The set $\{(e, \Delta k_x, \Delta k_r, \Delta\theta) \in R^{3+n}: e = 0\}$ is globally attractive, which means that

$$\lim_{t \to \infty} e(t) = 0$$

for all initial conditions in R^{3+n}.

The singularity condition for $\hat{b} = 0$, can be avoided by implementing a smart integrator:

$$\dot{\hat{b}} = \begin{cases} \gamma_b u e, & \text{if } |\hat{b}| > b_{\min} \vee \left[\hat{b} = b_{\min} \operatorname{sgn} b \wedge (ue) \operatorname{sgn} b > 0\right] \\ 0, & \text{if } |\hat{b}| = b_{\min} \wedge (ue) \operatorname{sgn} b < 0 \end{cases}$$

Indirect MRAC for SISO systems: predictor-based approach

The indirect approach can be extended to achieve better transient performance by including the prediction dynamics

$$\dot{\hat{x}} = \hat{a}x + \hat{b}u + \hat{\theta}^\top \varphi(x) + \ell(x - \hat{x})$$

together with the following adaptive laws

$$\begin{aligned}
\dot{\hat{a}} &= \gamma_a x \hat{e} \\
\dot{\hat{b}} &= \gamma_b u \hat{e} \\
\dot{\hat{\theta}} &= \Gamma_\theta \varphi(x) \hat{e}
\end{aligned}$$

where $\hat{e} = x - \hat{x}$.

For stability analysis under these adaptive laws, consider the Lyapunov function

$$V(e, \hat{e}, \Delta a, \Delta b, \Delta\theta) = \frac{1}{2}e^2 + \lambda\left(\frac{1}{2}\hat{e}^2 + \frac{1}{2\gamma_a}\Delta a^2 + \frac{1}{\gamma_b}\Delta b^2 + \frac{1}{2}\Delta\theta^T\Gamma_\theta^{-1}\Delta\theta\right)$$

where $e = x - x_{ref}$ is the usual RM mismatch error and $\lambda > 0$ is a positive constant. The time derivative of V along the closed-loop solutions reads

$$\dot{V}(e, \hat{e}, \Delta a, \Delta b, \Delta\theta) = a_{ref}e^2 - \ell\hat{e} - \ell\lambda\hat{e}^2 = -\begin{bmatrix} e \\ \hat{e} \end{bmatrix}^\top \begin{bmatrix} -a_{ref} & \frac{\ell}{2} \\ \frac{\ell}{2} & \lambda\ell \end{bmatrix} \begin{bmatrix} e \\ \hat{e} \end{bmatrix}$$

Since λ is an arbitrary positive constant, we can set it as

$$\lambda > -\frac{\ell}{4a_{ref}}$$

to ensure that the Lie derivative is negative semi-definite $(\mathcal{L}_f V \leq 0)$ globally.

Indirect MRAC for SISO systems: predictor-based approach

The previous results, leveraging once again LaSalle/Yoshizawa, allow one to conclude that:

- the equilibrium point $(e, \hat{e}, \Delta k_x, \Delta k_r, \Delta\theta) = (0,0,0,0,0)$ is UGS;
- the set $\{(e, \hat{e}, \Delta k_x, \Delta k_r, \Delta\theta) \in R^{3+n}: e = 0, \hat{e} = 0\}$ is globally attractive, which means that

$$\lim_{t\to\infty} e(t) = 0$$

$$\lim_{t\to\infty} \hat{e}(t) = 0$$

  globally.

Hence, the MRC problem is solved <u>regardless</u> of the <u>considered</u> uncertainties.

Conjecture: the indirect predictor-based MRAC achieve smoother transients than the standard MRAC.

MRAC for MIMO systems: Direct approach

MRAC for MIMO systems: direct approach

Recall the example about delta wing aircraft roll dynamics from Lab 3:

$$\underbrace{\begin{pmatrix} \dot{\varphi} \\ \dot{p} \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{pmatrix} 0 & 1 \\ \theta_1 & \theta_2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \varphi \\ p \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} \underbrace{\theta_6}_{\Lambda} (\underbrace{\delta_a}_{u} + \underbrace{\frac{1}{\theta_6} \left(\left(\theta_3 |\varphi| + \theta_4 |p| \right) p + \theta_5 \varphi^3 \right))}_{f(x) = \theta^T \varphi(x)}$$

Generalization: consider the uncertain systems of the form:

$$\dot{x} = Ax + B\Lambda(u + \theta^\top \varphi(x))$$

Assumptions:

- $A \in R^{n\times n}, \Lambda \in R^{m\times m}$, $\theta \in R^{n_\theta}$ are unknown matrices with constant elements
- $\Lambda \in R^{m\times m}$ is positive definite.
- $B \in R^{n\times m}$ is a known input matrix.
- $\varphi(x): R^n \mapsto R^{n_\theta}$ known vector containing nonlinear basis functions (regressor).
- The pair $(A, B\ \Lambda)$ is assumed to be controllable.

$f(x) = \theta^\top \varphi(x)$ is an uncertain nonlinear term which is linear in the unknown parameters.

MRAC for MIMO systems: direct approach

Starting point: **Model Reference Control**

All parameters are assumed to be known and the following control law is considered:

$$u_{\text{ideal}} = K_x^\top x + K_r^\top r - \theta^\top \varphi(x)$$

The corresponding closed-loop plant is

$$\dot{x} = \left(A + B\Lambda K_x^\top\right) x + B K_r^\top r$$

The gains of the control law are selected as follows (Matching Conditions)

$$\begin{aligned} A + B\Lambda K_x^\top &= A_{ref} \\ B K_r^\top &= B_{ref} \end{aligned}$$

where A_{ref} and B_{ref} define the matrices of the reference model, which describes the ideal desired response of the system to external commands $r(t)$.

MRAC for MIMO systems: direct approach

A necessary condition to have a solution of the model matching equations is the pair $(A, B\Lambda)$ be controllable.

Given $(A, B, \Lambda, A_{ref}, B_{ref})$, we assume in the following that at least a solution to the matching equations exists.

In practice, the structure of A, B and Λ is known and then the reference model matrices A_{ref} and B_{ref} are chosen so that there is a solution to the matching conditions.

Otherwise, the model reference control problem <u>cannot</u> be solved.

MRAC for MIMO systems: direct approach

A necessary condition to have a solution of the model matching equations is the pair $(A, B\Lambda)$ be controllable.

With the ideal gains, the closed loop system behaves as the reference model

$$\dot{x} = A_{ref}x + B_{ref}r$$

and the MRC problem is solved, in particular, the RM mismatch error

$$e := x - x_{ref}$$

converges to zero globally and exponentially, namely, $e = 0$ is GES:

$$\dot{e} = A_{ref}e, \qquad A_{ref} < 0 \quad \text{(Negative definite)}$$

MRAC for MIMO systems: direct approach

However, in practice, one has only estimates $(\hat{A}, \widehat{\Lambda}, \hat{\theta})$ of the parameters and this ideal result cannot be achieved in practice.

Using this estimates to compute the gains according to the matching conditions would result in the following closed-loop system

$$u_{MRC} = \hat{K}_x x + \hat{K}_r r - \hat{\theta}^\top \varphi(x)$$

$$\dot{x} = \left(A + B\Lambda \hat{K}_x^\top\right) x + B\Lambda\left(\hat{K}_r^\top r - (\hat{\theta} - \theta)^\top \varphi(x)\right)$$

By adding $\pm B\Lambda K_x^\top x$ and $\pm B\Lambda K_r^\top r$ in the previous equation, where K_x and K_r are the ideal unknown gains, and then by introducing the matching conditions, one has

$$\dot{x} = \underbrace{A + B\Lambda K_x^\top}_{A_{ref}} x + \underbrace{B\Lambda K_r^\top}_{B_{ref}} r + B\Lambda \underbrace{\left(\hat{K}_x - \hat{K}_x\right)^\top}_{\Delta K_x^\top} x + B\Lambda \underbrace{\left(\hat{K}_r - K_r\right)^\top}_{\Delta K_r^\top} r - B\Lambda \underbrace{(\hat{\theta} - \theta)^\top}_{\Delta\theta^\top} \varphi(x)$$

where $\Delta K_x = \hat{K}_x - K_x,\ \Delta K_r = \hat{K}_r - K_r,\ \Delta\theta = \hat{\theta} - \theta$ are the gain and parameters errors.

MRAC for MIMO systems: direct approach

The closed-loop system looks like the reference model plus perturbations due to the non-perfect knowledge of the plant parameters.

Consider again the RM mismatch error

$$e := x - x_{ref}$$

In the non ideal case, the dynamics of e is given by

$$\dot{e} = A_{ref} e + B\Lambda \left(\Delta K_x^\top x + \Delta K_r^\top r - \Delta\theta^T \varphi(x) \right)$$

In the presence of uncertain gains, the MRC does not achieve the design objective.

MRAC approach

The starting point of MRAC is the definition of a control input with the <u>same form</u> of the one used in the MRC approach but in which the gains are time-varying estimates generated:

$$u_{MRAC} = \hat{K}_x(t)x + \hat{K}_r(t)r - \hat{\theta}^\top(t)\varphi(x)$$

In order to derive adaptive laws to compute in real-time the gains to achieve model reference matching, a Lyapunov design approach is employed

$$V\left(e, \Delta K_x, \Delta K_r, \Delta\Theta\right) = e^\top Pe + \mathrm{tr}\left(\left[\Delta K_x^\top \Gamma_x^{-1} \Delta K_x + \Delta K_r^\top \Gamma_r^{-1} \Delta K_r + \Delta\Theta^\top \Gamma_\Theta^{-1} \Delta\Theta\right] \Lambda\right)$$

This function is a quadratic positive definite function of the RM mismatch error and of the parameter estimation errors

The fact that V is a Lyapunov function candidate has been proven in class exploiting the property of the trace operator.

MRAC for MIMO systems: direct approach

The matrices $\Gamma_x, \Gamma_r, \Gamma_\theta$ will be used as tunable parameters in the adaptive laws and $P = P^\top > 0$ satisfies the Lyapunov equation:

$$P A_{ref} + A_{ref}^\top P = -Q, \quad Q = Q^\top > 0$$

By computing the time derivative of V along the closed-loop solutions, one obtains

$$\begin{aligned}
\dot{V} = \dot{e}^T P e + e^T P \dot{e} + 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta\Theta^T \Gamma_\Theta^{-1} \dot{\Theta}\right] \Lambda\right) \\
= \left(A_{\text{ref}}\, e + B\Lambda \left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right)\right)^T P e \\
+ e^T P \left(A_{\text{ref}}\, e + B\Lambda \left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right)\right) \\
+ 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta\Theta^T \Gamma_\Theta^{-1} \dot{\Theta}\right] \Lambda\right) \\
= e^T \left(A_{\text{ref}}\, P + P A_{ref}\right) e + 2 e^T P B \Lambda \left(\Delta K_x^T x + \Delta K_r^T r - \Delta\Theta^T \Phi(x)\right) \\
+ 2\,\mathrm{tr}\left(\left[\Delta K_x^T \Gamma_x^{-1} \dot{\hat{K}}_x + \Delta K_r^T \Gamma_r^{-1} \dot{\hat{K}}_r + \Delta\Theta^T \Gamma_\Theta^{-1} \dot{\Theta}\right] \Lambda\right)
\end{aligned}$$

By using the Lyapunov equation, the expression can be further simplified:

$$\begin{aligned}\dot{V} = -e^T Q e &+ \left[2e^T PB\Lambda\Delta K_x^T x + 2\,\mathrm{tr}\left(\Delta K_x^T \Gamma_x^{-1}\dot{\hat{K}}_x\Lambda\right)\right] \\ &+ \left[2e^T PB\Lambda\Delta K_r^T r + 2\,\mathrm{tr}\left(\Delta K_r^T \Gamma_r^{-1}\dot{\hat{K}}_r\Lambda\right)\right] \\ &+ \left[-2e^T PB\Lambda\Delta\Theta^T\Phi(x) + 2\,\mathrm{tr}\left(\Delta\Theta^T \Gamma_\Theta^{-1}\dot{\hat{\Theta}}\Lambda\right)\right]\end{aligned}$$

Via the vector trace identity $a^\top b = tr(a^\top b) = tr(ba^\top)$, the following identities hold

$$\underbrace{e^T PB\Lambda}_{a^T}\underbrace{\Delta K_x^T x}_{b} = \mathrm{tr}(\underbrace{\Delta K_x^T x}_{b}\underbrace{e^T PB\Lambda}_{a^T})$$

$$\underbrace{e^T PB\Lambda}_{a^T}\underbrace{\Delta K_r^T r}_{b} = \mathrm{tr}(\underbrace{\Delta K_r^T r}_{b}\underbrace{e^T PB\Lambda}_{a^T})$$

$$\underbrace{e^T PB\Lambda}_{a^T}\underbrace{\Delta\Theta^T\Phi(x)}_{b} = \mathrm{tr}(\underbrace{\Delta\Theta^T\Phi(x)}_{b}\underbrace{e^T PB\Lambda}_{a^T})$$

Substituting the previous identities in the expression for $\dot{V}(t)$, one obtains

$$\dot{V}(t) = -e^T Q e + 2\,\mathrm{tr}\left(\Delta K_x^T \left[\Gamma_x^{-1}\dot{\hat{K}}_x + x e^T P B\right]\Lambda\right) \\ + 2\,\mathrm{tr}\left(\Delta K_r^T \left[\Gamma_r^{-1}\dot{\hat{K}}_r + r e^T P B\right]\Lambda\right) + 2\,\mathrm{tr}\left(\Delta \Theta^T \left[\Gamma_\Theta^{-1}\dot{\Theta} - \Phi(x) e^T P B\right]\Lambda\right)$$

The adaptive laws are selected to cancel the terms under the trace operator:

$$\dot{\hat{K}}_x = -\Gamma_x x e^\top P B$$

$$\dot{\hat{K}}_r = -\Gamma_r r(t) e^\top P B$$

$$\dot{\hat{\theta}} = \Gamma_\theta \varphi(x) e^\top P B$$

In this way, the time derivative of $V(t)$ along the closed-loop system solutions becomes

$$\dot{V}(t) = -e(t)^\top Q e(t), \quad Q = Q^\top > 0$$

MRAC for MIMO systems: direct approach

Hence, the corresponding Lie derivative

$$\mathcal{L}_f V\left(e, \Delta k_x, \Delta k_r, \Delta\theta\right) = -e^\top Q e$$

is negative semi-definite.

By LaSalle/Yoshizawa theorem, we can conclude that:

- The equilibrium point $(e, \Delta K_x, \Delta K_r, \Delta\theta) = (0,0,0,0)$ is UGS
- The set $\{(e, \Delta K_x, \Delta K_r, \Delta\theta) \in R^n \times R^{n\times m} \times R^{m\times m} \times R^{n_\theta \times m}: e = 0\}$ is globally attractive, which means that

$$\lim_{t\to\infty} e(t) = 0$$

Remarks

The same remarks for the SISO case apply also to the MIMO case:

- The attractivity property of the set is <u>not</u> uniform in general.
- The parameter estimation errors are <u>not</u> guaranteed to converge to zero and not even to converge to a constant.
- Under specific persistence of excitation conditions, one can prove that the origin of the closed-loop error system is UGAS.

MRAC for MIMO systems:
Direct approach with integral feedback connections

MRAC for MIMO systems: direct approach with integral feedback connections

We consider scalar uncertain systems of the form

$$\dot{x}_p = A_p x_p + B_p \Lambda (u + \theta^\top \varphi(x_p))$$

Assumptions:

- $A_p \in R^{n\times n}, \Lambda \in R^{m\times m}$, $\theta \in R^{n_\theta \times m}$ are unknown matrices with constant elements.
- $\Lambda \in R^{m\times m}$ is positive definite.
- $B_p \in R^{n\times m}$ is a known input matrix with constant elements.
- $\varphi(x_p)$: $R^n \mapsto R^{n_\theta}$ is a known vector containing nonlinear basis functions (regressor).
- The pair $(A_p, B_p\Lambda)$ is assumed to be controllable.

As before, $\theta^\top \varphi(x)$ is an uncertain nonlinear term which is linear in the unknown parameters (matched uncertainty).

The **control goal** of interest is bounded command tracking:

find u such that the system regulated output

$$y = C_p x_p \in \mathbb{R}^m$$

tracks any bounded possibly time-varying command $\mathrm{r} = y_{cmd}(t) \in R^m$ with bounded errors in the presence of uncertainties in A_p, Λ, θ.

$C_p \in R^{m \times n_p}$ is assumed to be known and constant.

To measure the controller performance, the system output tracking error is defined:

$$e_y = y - r$$

It is well known that in order to track constant references, a possible approach is to augment the system with an additional state corresponding to the integral of the output error, namely,

$$x_I(t) = \int_0^t e_y(\tau) d\tau$$

which is generated in real-time by integrating the differential equation

$$\dot{x}_I = e_y$$

We can now proceed in building the augmented open-loop dynamics by defining the augmented state vector

$$x := \begin{bmatrix} x_I^\top & x_p^\top \end{bmatrix}^\top \in \mathbb{R}^{m+n_p}$$

The corresponding augmented open-loop dynamics is

$$\dot{x} = Ax + B\Lambda\left(u + f\left(x_p\right)\right) + B_{ref}y_{cmd}$$

where

$$A = \begin{bmatrix} 0_{m\times m} & C_p \\ 0_{n_p\times m} & A_p \end{bmatrix}, \quad B = \begin{bmatrix} 0_{m\times m} \\ B_p \end{bmatrix}, \quad B_{ref} = \begin{bmatrix} -I_{m\times m} \\ 0_{n_p\times m} \end{bmatrix}$$

and

$$y = \underbrace{\begin{bmatrix} 0_{m\times m} & C_p \end{bmatrix}}_{C} x = Cx$$

is the augmented system-controlled output.

MRAC for MIMO systems: direct approach with integral feedback connections

In order to go on with the control design, we need to assume controllability for the extended system matrices.

A necessary and sufficient condition for this to happen is the original pair $(A_p, B_p\Lambda)$ be controllable and

$$\det\left(\begin{bmatrix} A_p & B_p\Lambda \\ C_p & 0_{m\times m} \end{bmatrix}\right) \neq 0$$

(This technical condition ensures that the open-loop transfer function does not have zeros at the origin).

Model matching assumption

Given a Hurwitz matrix A_{ref} and an unknown positive-definite matrix Λ, there exists at least one ideal gain matrix $K_x \in R^{n\times m}$ such that

$$A + B\Lambda K_x^\top = A_{ref}$$

MRAC for MIMO systems: direct approach with integral feedback connections

Based on this, by adding $\pm B\Lambda K_x^\top$ to the augmented open-loop dynamics, we obtain

$$\dot{x} = A_{ref}x + B\Lambda\left(u - K_x^\top x + \theta^\top \varphi\left(x_p\right)\right) + B_{ref}r$$

A natural choice for control design is to select the input u as follows

$$u = \hat{K}_x^T x - \hat{\theta}^\top \varphi\left(x_p\right)$$

The control law comprises a feedback term $(\hat{K}_x^\top x)$ and a term to compensate for matched uncertainties $\left(-\hat{\theta}^\top \varphi(x_p)\right)$.

The closed-loop system then becomes

$$\dot{x} = A_{ref}x + B\Lambda(\underbrace{(\hat{K}_x - K_x)^\top}_{\Delta K_x^\top} x - \underbrace{(\hat{\theta} - \theta)^\top}_{\Delta\theta^\top} \varphi(x_p)) + B_{ref}r$$

$$= A_{ref}x + B\Lambda\left(\Delta K_x^\top x - \Delta\theta^\top \varphi\left(x_p\right)\right) + B_{ref}r$$

MRAC for MIMO systems: direct approach with integral feedback connections

At this point, it is natural to define the reference model dynamics as

$$\dot{x}_{ref} = A_{ref} x_{ref} + B_{ref} y_{cmd}, \quad y_{ref} = C_{ref} x_{ref}$$

It is possible to verify that the transfer function $G_{ref}(s)$ from r to y_{ref}

$$y_{ref} = \underbrace{\left[C_{ref} \left(s I_{n \times n} - A_{ref} \right)^{-1} B_{ref} \right]}_{G_{ref}(s)} r$$

has unit DC gain, namely, that

$$G_{ref}(s = 0) = I_m$$

By means of the final value theorem, this imply that for a constant command $r(t) = \bar{y}_{cmd}$, $\lim_{t \to \infty} y_{ref}(t) = \bar{y}_{cmd}$.

MRAC for MIMO systems: direct approach with integral feedback connections

Proof of asymptotic tracking of constant references

By definition $A_{ref} = A + B\Lambda K_x^\top$ where matrix K_x can be split in two blocks, one for the state of plant and one for the state of the integrator:

$$K_x^\top = \begin{bmatrix} K_p^\top \\ K_I^\top \end{bmatrix}$$

The reference model dynamics is $\dot{x}_{ref} = A_{ref}x_{ref} + B_{ref}r$. Using the definitions

$$A = \begin{bmatrix} 0_{m\times m} & C_p \\ 0_{n_p\times m} & A_p \end{bmatrix}, \quad B = \begin{bmatrix} 0_{m\times m} \\ B_p \end{bmatrix}, \quad B_{ref} = \begin{bmatrix} -I_{m\times m} \\ 0_{n_p\times m} \end{bmatrix}$$

The system dynamics can be written as a cascade (thanks to the triangular structure of A_{ref})

$$\begin{bmatrix} \dot{x}_{ref_I} \\ \dot{x}_{ref_p} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & C_p \\ B_p\Lambda K_I^\top & A_p + B_p\Lambda K_p^\top \end{bmatrix}}_{A_{ref}} \begin{bmatrix} x_{ref_I} \\ x_{ref_p} \end{bmatrix} + \begin{bmatrix} -I \\ 0 \end{bmatrix} r$$

MRAC for MIMO systems: direct approach with integral feedback connections

In transfer function form, the reference model dynamics is described by:

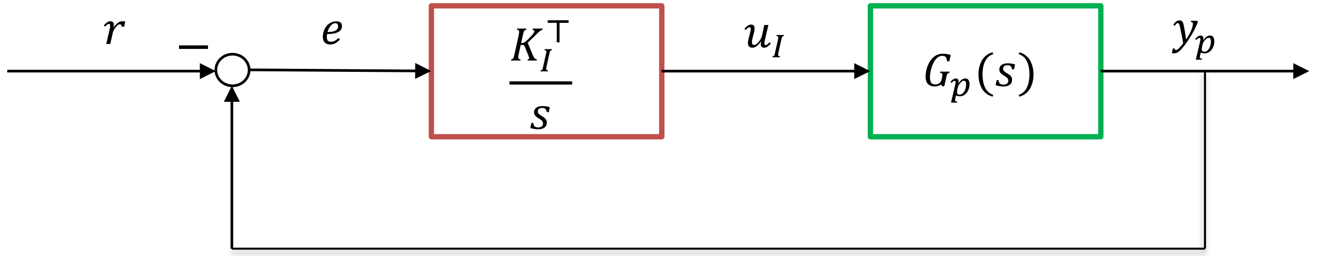
- Plant with state feedback (lower subsystem in the cascade)

$$y_P = G_p(s)u_I \qquad G_p(s) = C_p(sI - A_p - B_p\Lambda K_p^\top)^{-1}B_p\Lambda$$

- Integrator (upper subsystem in the cascade)

$$u_I = G_I(s)e \qquad G_I(s) = \frac{K_I^\top}{s}$$

- Interconnection $e = y_p - r$



MRAC for MIMO systems with integral feedback connections

- State feedback does not alter the location of the zeros of the open-loop system.
- The condition

$$\det\left(\begin{bmatrix} A_p & B_p\Lambda \\ C_p & 0_{m\times m}\end{bmatrix}\right) \neq 0$$

  guarantees that the open-loop transfer function does not have zeros at the origin.

Then, the loop function

$$L(s) = G_I(s)G_p(s) = \frac{K_I^\top}{s}G_p(s)$$

will have a pole at the origin (no cancellation can occur). Hence, the closed-loop transfer function

$$T(s) = -(I - L(s))^{-1}L(s) = -\left(I - \frac{K_I^\top}{s}G_p(s)\right)^{-1}\frac{K_I^\top}{s}G_p(s) = -(sI - K_I^\top G_p(s))^{-1}K_I^\top G_p(s)$$

will have unit DC gain (note that the system has positive feedback interconnection)

$$T(s=0) = -(-K_I^\top G_p(s))^{-1}K_I^\top G_p(s) = I$$

End of the proof.

MRAC for MIMO systems with integral feedback connections

Introducing as usual the reference model mismatch error $e := x - x_{ref}$, we have

$$\dot{e} = A_{ref}e + B\Lambda\left(\Delta K_x^\top x - \Delta\theta^\top \varphi\left(x_p\right)\right)$$

We now proceed with the Lyapunov-based design, eventually leading to the design of stable adaptive laws and a verifiable closed-loop system tracking performance.

Toward that end, let us consider a quadratic Lyapunov function candidate

$$V\left(e, \Delta K_x, \Delta\Theta\right) = e^\top P e + \mathrm{trace}\left(\Delta K_x^\top \Gamma_x^{-1} \Delta K_x \Lambda\right) + \mathrm{trace}\left(\Delta\theta^\top \Gamma_\theta^{-1} \Delta\theta\Lambda\right)$$

where $\Gamma_x = \Gamma_x^\top > 0, \Gamma_\theta = \Gamma_\theta^\top > 0$ are adaptive gain matrix and $P = P^\top > 0$ satisfies the Lyapunov equation

$$PA_{ref} + A_{ref}^\top P = -Q, \quad Q = Q^\top > 0$$

MRAC for MIMO systems: direct approach with integral feedback connections

Proceeding as usual, we compute the time-derivative of V along the closed-loop solutions:

$$\begin{aligned}\dot{V}\left(e, \Delta K_x, \Delta\theta\right) = &-e^\top Q e + 2e^\top P B \Lambda \left(\Delta K_x^\top x - \Delta\theta^\top \varphi\left(x_p\right)\right) \\ &+ 2\,\mathrm{trace}\left(\Delta K_x^\top \Gamma_x^{-1} \dot{K}_x \Lambda\right) + 2\,\mathrm{trace}\left(\Delta\theta^\top \Gamma_\theta^{-1} \dot{\hat{\theta}} \Lambda\right)\end{aligned}$$

(note that $\Delta\dot{\theta} = \dot{\hat{\theta}} - \dot{\theta} = \dot{\hat{\theta}}$ since θ is constant by assumption)

Applying the vector trace identity $a^\top b = trace(ba^\top)$, we obtain

$$\begin{aligned}\dot{V}\left(e, \Delta K_x, \Delta\theta\right) = &-e^\top Q e + 2\,\mathrm{trace}\left(\Delta K_x^\top \left\{\Gamma_x^{-1} \dot{\hat{K}}_x + x e^\top P B\right\} \Lambda\right) \\ &+ 2\,\mathrm{trace}\left(\Delta\theta^\top \left\{\Gamma_\theta^{-1} \dot{\hat{\theta}} - \varphi\left(x_p\right) e^\top P B\right\} \Lambda\right)\end{aligned}$$

MRAC for MIMO systems: direct approach with integral feedback connections

If the adaptive laws are chosen such that

$$\dot{\hat{K}}_x = -\Gamma_x x e^\top P B$$
$$\dot{\hat{\theta}} = \Gamma_\theta \varphi(x_p) e^\top P B$$

then, we would obtain

$$\mathcal{L}_f V\left(e, \Delta K_x, \Delta\theta\right) = -e^\top Q e \leq 0$$

By LaSalle/Yoshizawa theorem, we can conclude that:

- The equilibrium point $(e, \Delta K_x, \Delta\theta) = (0,0,0)$ is UGS
- The set $\{(e, \Delta K_x, \Delta\theta) \in R^n \times R^{n\times m} \times R^{m\times m} \times R^{n_\theta \times m} : e = 0\}$ is globally attractive, which means that $\lim_{t\to\infty} e(t) = 0$.

MRAC for MIMO systems: direct approach with integral feedback connections

Since the $e(t)$ converges to zero, then $x(t) \to x_{ref}(t)$ as $t \to \infty$ (asymptotic convergence).

This in turn means that the system output $y(t) = Cx(t) \to y_{ref}(t) = Cx_{ref}(t)$ asymptotically.

Since $y_{ref}(t)$ tracks any bounded time-varying command $y_{cmd}(t)$ with bounded errors, so does $y(t)$: the regulated output tracks bounded references with bounded errors.

Finally, because $y_{ref}(t)$ tracks with zero steady-state errors constant commands $\bar{y}_{cmd}$, the proposed MRAC law achieves regulation of constant commands regardless of the considered parametric uncertainties.

Comparison

Plant $\dot{x}_p = A_p x_p + B_p \Lambda (u + \theta^\top \varphi(x_p))$

Standard MRAC law

$$u_{MRAC} = \hat{K}_x(t) x_p + \hat{K}_r(t) r - \hat{\theta}^\top(t) \varphi(x_p)$$

$$\dot{\hat{K}}_x = -\Gamma_x x_p e^\top P B$$

$$\dot{\hat{K}}_r = -\Gamma_r r(t) e^\top P B$$

$$\dot{\hat{\theta}} = \Gamma_\theta \varphi(x_p) e^\top P B$$

MRAC with integral feedback interconnection

$$x = \begin{bmatrix} x_p^\top & x_I^\top \end{bmatrix}^\top$$

$$u_{MRAC-IF} = \hat{K}_x^T x - \hat{\theta}^\top \varphi(x_p)$$

$$\dot{K}_x = -\Gamma_x x e^\top P B$$

$$\dot{\hat{\theta}} = \Gamma_\theta \varphi(x_p) e^\top P B$$

$$\dot{x}_I = e_y$$

MRAC for MIMO systems:
Direct augmentation approach

MRAC for MIMO systems: direct augmentation approach

The adaptive augmentation approach is based on the development of adaptive laws to be added on top of an existing controller designed on the nominal plant (the plant free of uncertainties)

Starting from the plant $\dot{x}_p = A_p x_p + B_p \Lambda (u + \theta^\top \varphi(x_p))$, we consider that

- $\Lambda \in R^{m\times m}$, $\theta \in R^{n_\theta}$ are unknown matrices with constant elements.
- A_p is a known matrix with constant elements (main difference w.r.t. previous approach)
- $\Lambda \in R^{m\times m}$ is positive definite.
- $B_p \in R^{n\times m}$ is a known input matrix.
- $\varphi(x): R^n \mapsto R^{n_\theta}$ is a known vector containing nonlinear basis functions (regressor).
- The pair $(A_p, B_p\Lambda)$ is assumed to be controllable.

The assumption A_p be known can be relaxed under matching conditions, namely, by assuming that there exists a (unknown) matrix θ_A such that

$$A_p = \bar{A}_p + B\Lambda\theta_A$$

where $\bar{A}_p$ is the nominal state matrix used to design the baseline controller.

Then, the approach that we will present next can be carried out by embedding θ_A into θ and by working with $\bar{A}_p$ instead of A_p.

MRAC for MIMO systems: direct augmentation approach

We also assume that there exist a baseline controller

$$\dot{x}_c = A_c x_c + B_{cy} y_p + B_{cr} r$$
$$u_{bl} = C_c x_c + D_{cy} y_p + D_{cr} r$$

which stabilizes and achieves desired control objectives for the plant

$$\dot{x}_p = A_p x_p + B_p u$$

namely, the uncertainty-free plant $(\Lambda = I, \theta = 0)$.

MRAC for MIMO systems: direct augmentation approach

Using $u = u_{bl}$, the closed-loop nominal dynamics is described by

- Plant dynamics

$$\begin{aligned}\dot{x}_p &= A_p x_p + B_p u_{bl} = A_p x_p + B_p C_c x_c + B_p D_{cy} y_p + B_p D_{cr} r \\ &= A_p x_p + B_p C_c x_c + B_p D_{cy} C_p x_p + B_p D_{cr} r = (A_p + B_p D_{cy} C_p) x_p + B_p C_c x_c + B_p D_{cr} r\end{aligned}$$

- Controller dynamics

$$\dot{x}_c = A_c C_c + B_{cy} C_p x_p + B_{cr} r$$

Combining the plant and controller state $x = \begin{bmatrix} x_p^\top & x_c^\top \end{bmatrix}^\top$, the augmented closed-loop (nominal) dynamics is:

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_p + B_p D_{cy} C_p & B_p C_c \\ B_{cy} C_p & A_c \end{bmatrix} \begin{bmatrix} x_p \\ x_c \end{bmatrix} + \begin{bmatrix} B_p D_{cr} \\ B_{cr} \end{bmatrix} r$$

MRAC for MIMO systems: direct augmentation approach

Such a dynamics plays the role of the reference model in the standard MRAC approach:

$$A_{ref} := \begin{bmatrix} A_p + B_p D_{cy} C_p & B_p C_c \\ B_{cy} C_p & A_c \end{bmatrix} \qquad B_{ref} = \begin{bmatrix} B_p D_{cr} \\ B_{cr} \end{bmatrix} \qquad C_{ref} = \begin{bmatrix} C_p & 0 \end{bmatrix}$$

The question the we address now is: <u>how can we restore the baseline controller performance for the original uncertain plant?</u>

The adaptive augmentation approach splits the control input into two contributions

$$u = u_{bl} + u_{ad}$$

where

- u_{bl} is the output of the baseline controller
- u_{ad} is an adaptive component to be designed (to counteract uncertainties)

MRAC for MIMO systems: direct augmentation approach

The design of u_{ad} is as follow. Consider the uncertain plant

$$\dot{x}_p = A_p x_p + B_p \Lambda \left(u_{bl} + u_{ad} + \theta^\top \varphi\left(x_p\right)\right)$$

Add and subtract from the term in the parenthesis $\pm\Lambda^{-1} u_{bl}$ to obtain

$$\begin{aligned}
\dot{x}_p &= A_p x_p + B_p \Lambda \left(u_{bl} \pm \Lambda^{-1} u_{bl} u_{ad} + \theta^\top \varphi\left(x_p\right)\right) \\
&= A_p x_p + B_p \Lambda \Lambda^{-1} u_{bl} + B_p \Lambda \left((I - \Lambda^{-1}) u_{bl} + u_{ad} + \theta^\top \varphi(x_p)\right) \\
&= A_p x_p + B_p u_{bl} + B_p \Lambda \left(K_u^\top u_{bl} + u_{ad} + \theta^\top \varphi(x_p)\right)
\end{aligned}$$

where we defined the uncertain gain matrix

$$K_u^\top = (I - \Lambda^{-1})$$

MRAC for MIMO systems: direct augmentation approach

The system now can be written in compact form as

$$\dot{x}_p = A_p x_p + B_p u_{bl} + B_p \Lambda \left(u_{ad} + \theta_e^\top \varphi_e(x_p, x_c) \right)$$

where

$$\theta_e^\top := \begin{bmatrix} K_u^\top & \theta^\top \end{bmatrix} \qquad \varphi_e(x_p, x_c) := \begin{bmatrix} u_{bl}(x_p, x_c) \\ \varphi(x_p) \end{bmatrix}$$

are the extended vector of unknown parameters and the extended regressor, respectively.

At this point, the adaptive component is selected as

$$u_{ad} = -\hat{\theta}_e(t)^\top \varphi_e(x_p, x_c)$$

where $\hat{\theta}_e(t)$ is the extended parameter estimate at time t generated in real-time by the adaptive law.

MRAC for MIMO systems: direct augmentation approach

To derive stable adaptive laws, we follow the usual Lyapunov-based design, starting from the augmented plant in which the baseline controller dynamics has been included

$$\begin{bmatrix} \dot{x}_p \\ \dot{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A_p + B_p D_{cy} C_p & B_p C_c \\ B_{cy} C_p & A_c \end{bmatrix}}_{A_{ref}} \begin{bmatrix} x_p \\ x_c \end{bmatrix} - \begin{bmatrix} B_p \Lambda \Delta\theta_e^\top \varphi_e(x_p, x_c) \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} B_p D_{cr} \\ B_{cr} \end{bmatrix}}_{B_{ref}} r$$

where $\Delta\theta_e = \hat{\theta}_e - \theta$ is the parameter estimation error and where the reference model matrices are highlighted.

Using the augmented state $x = \left[x_p^\top \;\; x_c{}^\top\right]^\top$, the dynamics is compactly written as

$$\dot{x} = A_{ref} x - B_e \Lambda \Delta\theta_e^\top \varphi_e(x) + B_{ref} r \qquad\qquad B_e = \begin{bmatrix} B_p \\ 0 \end{bmatrix}$$

MRAC for MIMO systems: direct augmentation approach

Introducing as usual the reference model mismatch error $e \coloneqq x - x_{ref}$, we have

$$\dot{e} = \dot{x} - \dot{x}_{ref} = A_{ref}e - B_e\Lambda\left(\Delta\theta_e^\top \varphi_e(x)\right)$$

We now proceed with the Lyapunov design to derive the adaptive laws.

Let us consider a quadratic Lyapunov function candidate

$$V\left(e, \Delta\Theta\right) = e^\top Pe + \mathrm{trace}\left(\Delta\theta_e^\top \Gamma_{\theta_e}^{-1} \Delta\theta_e \Lambda\right)$$

where $\Gamma_{\theta_e} = \Gamma_{\theta_e}^\top > 0$ is the adaptive gain matrix and $P = P^\top > 0$ satisfies the Lyapunov equation

$$PA_{ref} + A_{ref}^\top P = -Q, \quad Q = Q^\top > 0$$

where Q is a tunable matrix.

MRAC for MIMO systems: direct augmentation approach

We can compute the time-derivative of V along the closed-loop solutions:

$$\dot{V}\left(e, \Delta\theta\right) = -e^\top Q e - 2 e^\top P B_e \Lambda \left(\Delta\theta_e^\top \varphi\left(x\right)\right) + 2\,\mathrm{trace}\left(\Delta\theta_e^\top \Gamma_{\theta_e}^{-1} \dot{\hat{\theta}}_e \Lambda\right)$$

Applying the vector trace identity $a^\top b = trace(b a^\top)$, we obtain

$$\dot{V}\left(e, \Delta\theta_e\right) = -e^\top Q e + 2\,\mathrm{trace}\left(\Delta\theta_e^\top \left\{\Gamma_{\theta_e}^{-1} \dot{\hat{\theta}}_e - \varphi_e\left(x\right) e^\top P B_e\right\} \Lambda\right)$$

If the adaptive laws were chosen as

$$\dot{\hat{\theta}}_e = \Gamma_{\theta_e} \varphi_e(x) e^\top P B_e$$

then, we would obtain

$$\dot{V}\left(e, \Delta_e\theta\right) = -e^\top Q e \leq 0$$

MRAC for MIMO systems: direct augmentation approach

By LaSalle/Yoshizawa theorem, we can conclude that

- The equilibrium point $(e, \Delta\theta) = (0{,}0)$ is UGS
- The set $\{(e, \Delta\theta_e) \in R^n \times R^{(n_\theta+m)\times 2m}: e = 0\}$ is globally attractive, which means that $\lim_{t\to\infty} e(t) = 0$.

Remarks

- The adaptive augmentation approach allows one to seamlessly combine a baseline controller with an adaptation law.
- Nominal closed-loop performance are recovered in the presence of the considered class of uncertainty.
- The adaptation law can be easily turned-off during the operation of the plant (with some care)