Adaptive and Autonomous Aerospace Systems

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Part 1: Analysis of nonlinear and time-varying systems

Lect 3: Lyapunov methods

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Outline

- Background
- Lyapunov function candidates
- Lyapunov stability theorems for nonautonomous systems
- Invariance-like theorems
    - Barbalat’s Lemma
    - LaSalle/Yoshizawa theorem
- Recap on introductory examples: stability analysis.

Main advantage of Lyapunov approach:

Characterization of stability properties of equilibrium points without solving differential equations.

The Lyapunov method leverages the construction of a scalar-valued function with suitable properties.

- Generalization of the concept of energy for mechanical systems

Lyapunov function used in deriving MRAC laws

$$V(e, \Delta k_p, \Delta k_f) = \frac{1}{2} e^2 + \frac{1}{2\gamma_p} \Delta k_p^2 + \frac{1}{2\gamma_f} \Delta k_f^2$$

Positive definite functions

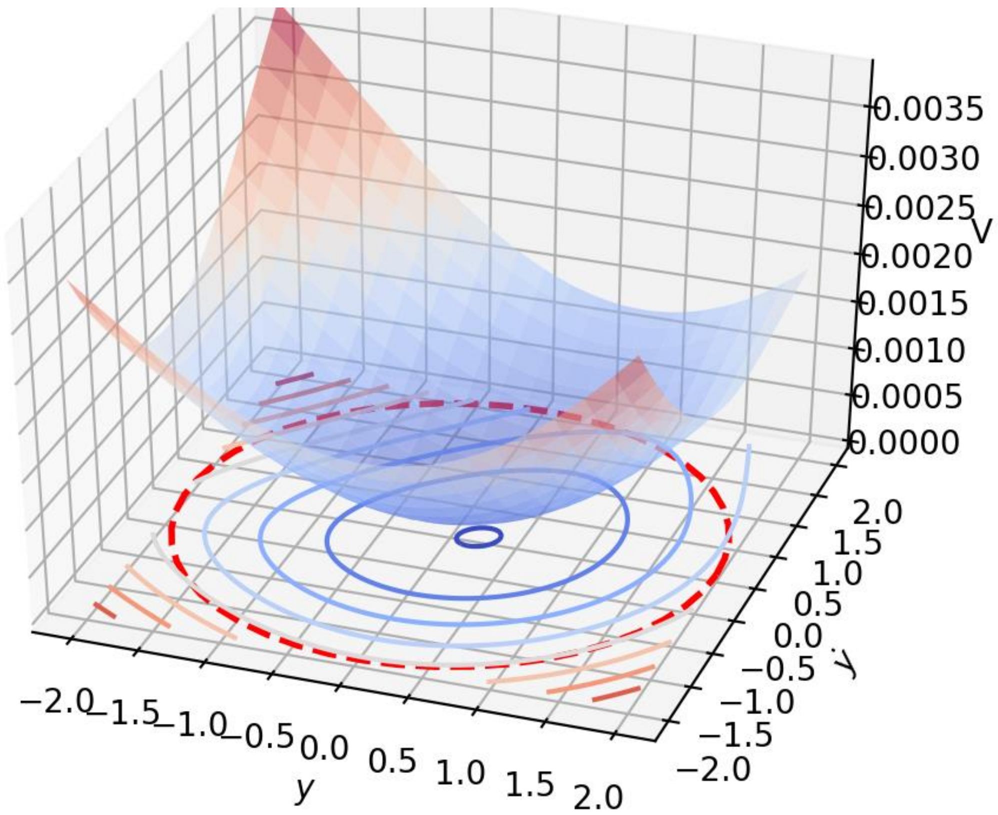
A function $V(x)$ is said to be Positive Definite (PD) if

$$V(x) \geq 0 \quad \forall x \in \mathbb{R}^n, \quad V(x) = 0 \iff x = 0$$

For a function to be a Lyapunov candidate, we need something more:

$$\lim_{|x| \to \infty} V(x) = +\infty$$

This property is called radial unboundedness



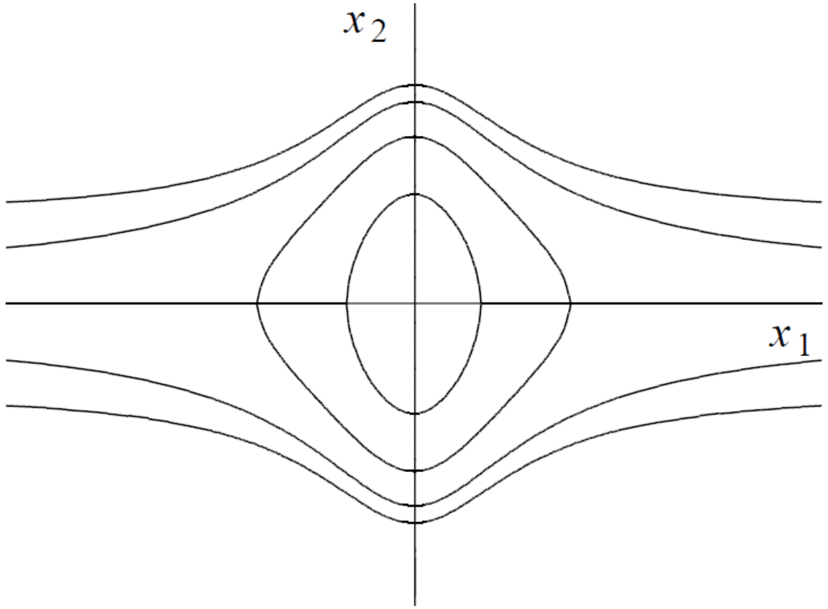
Radial unboundedness plays an important role in the global stability analysis as sublevel sets of the form

$$\Omega_c := \{x \in \mathbb{R}^n : V(x) \le c\}$$

are guaranteed to be bounded (and closed) for any $c > 0$ when $V(x)$ is radially unbounded.

Example

$$V(x) = \frac{x_1^2}{1 + x_1^2} + x_2^2$$



Lyapunov function candidates

A function is said to be positive (negative) semidefinite if

$$V(x) \geq (\leq) 0, \quad \forall x \in \mathbb{R}^n$$

Recall the MRAC example

$$\dot{V}(e, \Delta k_p, \Delta k_f) = a_r e^2 \leq 0 \quad \text{for } a_r < 0$$

The interest in comparison functions is also motivated by the following result.

Lemma. Properties of positive definite radially unbounded functions.

Let $V : \mathcal{D}_x \mapsto \mathbb{R}_{\geq 0}$ be a continuous, radially unbounded, positive definite function. Then, there exist class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|) \quad \forall x \in \mathbb{R}^n.$$

The lower bound establishes the fact that $V(x)$ is positive definite and radially unbounded.
The upper bound establishes the property that $V(x)$ is decrescent.

Example: $V(x) = x^\top P x$ with $P = P^\top > 0$ (symmetric positive definite)

Lyapunov function candidates

The ideas presented so far can be extended for nonautonomous systems, for which we use a function of the form

$$V(t,x) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}$$

Terminology: a function $V(t,x)$ is said to be

- positive semidefinite if $V(t,x) \geq 0$.
- positive definite and radially unbounded if $V(t,x) \geq \alpha_1(x)$ for some class-K_∞ function $\alpha_1(\cdot)$.
- decrescent if $V(t,x) \leq \alpha_2(x)$ for some class-K function $\alpha_2(\cdot)$.
- negative definite (semidefinite) if $-V(t,x)$ is positive definite (semidefinite)

Def. (Global) Lyapunov function candidate

A function $V(t,x) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$ is said to be a Lyapunov function candidate if

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \quad \forall (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$$

for some class-K_∞ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$.

Important property: sublevel sets of $V(t,x)$ are compact (closed and bounded sets).

The Lie derivative

How does the Lyapunov function change along the flow of the system?

Def.

Given a continuously differentiable function $V(t,x)$ in a domain $\mathcal{D}_t \times \mathcal{D}_x$, the expression

$$\mathcal{L}_f V(t,x) := \frac{\partial}{\partial t} V(t,x) + \langle \nabla_x V(x), f(t,x) \rangle \quad (t,x) \in \mathcal{D}_t \times \mathcal{D}_x$$

is called the *Lie derivative* of V.

Thm. Lyapunov theorem for nonautonomous systems

Let $x = 0$ be an equilibrium point for the system $\dot{x} = f(t, x)$, where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is locally Lipschitz in x, uniformly in t, and piecewise continuous in t. If there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, such that

$$\begin{aligned} &\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \\ &\mathcal{L}_f V(t, x) \leq 0 \qquad\qquad \forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \end{aligned}$$

for some class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, then $x = 0$ is UGS.

If the inequality is strengthened to

$$\mathcal{L}_f V(t, x) \leq -\rho(x) \qquad \forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$$

for some continuous positive definite function $\rho : \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, then $x = 0$ is UGAS.

Lyapunov stability theorem for nonautonomous systems

IMPORTANT

The conditions of Lyapunov's theorem are only sufficient.

Failure of a Lyapunov function candidate to satisfy the conditions for stability or asymptotic stability does not mean that the equilibrium point is not stable or asymptotically stable.

It only means that such stability property cannot be established by using this Lyapunov function candidate.

Invariance-like theorems

Many control systems render the Lie derivative of certain Lyapunov function candidates only at most negative semi-definite by design.

For autonomous systems, this case can be tackled by resorting to the well-known **LaSalle**'s invariance set theorem and the theorem of **Krasovskii** and **Barbashin**.

When dealing with nonautonomous systems, the situation is much more involved, and the available results are in general much weaker.

The following lemma establishes convergence properties for uniformly continuous integrable function and is instrumental in deriving invariance-like theorems.

Lemma (Barbalat)

Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t\to\infty} \int_0^t \phi(\tau) d\tau$ exists and is finite. Then, $\lim_{t\to\infty} \phi(t) = 0$.

Thm. (LaSalle/Yoshizawa).

Let $x = 0$ be an equilibrium point for the system $\dot{x} = f(t, x)$, where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x, uniformly in t, on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$. If there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, such that

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|)$$
$$\mathcal{L}_f(t, x) \leq \rho(x) \quad \forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$$

for some class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ and a continuous positive semi-definite $\rho(\cdot)$ on $\mathbb{R}^n$. Then, $x = 0$ is UGS and the solution $x(t; t_0, x_0)$ satisfies

$$\lim_{t \to \infty} \rho(x(t; t_0, x_0)) = 0 \quad \forall t_0 \geq 0, \forall x_0 \in \mathbb{R}^n.$$

Invariance-like theorems

The result of the previous theorem is much weaker result than its counterpart for autonomous systems.

- The trajectories do not converge in general to an invariant set contained in $\mathrm{E} := \{x \in R^n : \rho(x) = 0\}$.
- The set E is not guaranteed to be uniformly attractive.

To prove that the origin is UGAS even in case $\dot{V} \leq 0$, additional uniform restrictions on the change of V over a finite interval of time must be imposed.

Simple algebraic system

Error dynamics

$$e_\theta = \theta - \bar{\theta}$$

$$\dot{e}_\theta = -\gamma u(t)^2 e_\theta$$

Equilibrium condition (constant reference $u(t) = \bar{u}$)

$$-\gamma \bar{u} \bar{e}_\theta = 0 \rightarrow \bar{e}_\theta = 0$$

- The origin is the unique equilibrium for any $\bar{u} \neq 0$.

Stability analysis

Consider the Lyapunov candidate

$$V(e_\theta) = \frac{1}{2\gamma} e_\theta^2$$

The Lie derivative reads

$$\mathcal{L}_f V(t, e_\theta) = -\gamma \bar{u}^2 e_\theta^2 < 0 \forall (t, e_\theta) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \setminus \{0\}$$

By Lyapunov theorem, the origin is GAS. Since the system is LTI, it is automatically GES.

What happens when $u = u(t)$ is not just a constant signal?

The origin is still an equilibrium point, but note that $\dot{e}_\theta$ can be zero also when $e_\theta \neq 0$ whenever the input is zero.

Consider the Lyapunov candidate

$$V(e_\theta) = \frac{1}{2\gamma} e_\theta^2$$

The Lie derivative reads

$$\mathcal{L}_f V(t, e_\theta) = -\gamma u^2(t) e_\theta^2 \leq 0 \quad \forall (t, e_\theta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$$

By Lyapunov theorem, the origin is UGS but we cannot conclude attractivity. Further analysis is required.

Because $V(e_\theta(t))$ is nonincreasing as a function of time and it is lower bounded (by zero), from calculus we know that

$$\lim_{t\to\infty} V(e_\theta(t)) = V_\infty$$

Hence, V will converge to a finite limit (no "uniform" property is guaranteed).

By exploiting the definition of V, we also know that

$$\lim_{t\to\infty} |e_\theta(t)| = \pm\sqrt{2\gamma V_\infty} \implies \lim_{t\to\infty} \theta(t) = \bar{\theta} \pm \sqrt{2\gamma V_\infty}$$

Therefore, the parameters are guaranteed to converge to a constant value.

To achieve parameter convergence to the ideal value $(V_\infty = 0)$, a persistence of excitation condition must be considered for the input signal.

Error dynamics of MRAC for roll rate dynamics example

$$\dot{e} = (a_{ref} + L_{\delta_a}\Delta k_p)e + L_{\delta_a}(\Delta k_p(e_{ref} + p_{cmd}) + \Delta k_{p_{cmd}} p_{cmd})$$

Reference model mismatch

$$\dot{\Delta k}_p = -\gamma_p sign(L_{\delta_a})(e + e_{ref} + p_{cmd})e$$

$$\dot{\Delta k}_{p_{cmd}} = -\gamma_{p_{cmd}} sign(L_{\delta_a}) p_{cmd} e$$

Gain estimation error dynamics

$$\dot{e}_{ref} = a_{ref} e_{ref}$$

Reference model error dynamics

N.B. $e \coloneqq p - p_{ref}$; $e_{ref} = p_{ref} - p_{cmd}$; $a_{ref} = b_{ref}$

Equilibrium conditions (constant commands $p_{cmd} = \bar{p}_{cmd}$).

$$\begin{aligned}(a_{ref} + L_{\delta_a}\overline{\Delta k}_p)(\bar{e} + L_{\delta_a}(\overline{\Delta k}_p(\bar{e}_{ref} + \bar{p}_{cmd}) + \overline{\Delta k}_{p_{cmd}}\bar{p}_{cmd}) &= 0 \\ -\gamma_p sign(L_{\delta_a})(\bar{e} + \bar{e}_{ref} + \bar{p}_{cmd})\bar{e} &= 0 \\ -\gamma_{p_{cmd}} sign(L_{\delta_a})\bar{p}_{cmd}\bar{e} &= 0 \\ \bar{e}_{ref} &= 0\end{aligned}$$

$$\begin{cases} \bar{e} = 0 \\ \overline{\Delta k}_p = -\overline{\Delta k}_{p_{cmd}} \\ \overline{\Delta k}_{p_{cmd}} = \overline{\Delta k}_{p_{cmd}} \\ \bar{e}_{ref} = 0 \end{cases}$$

2D subspace

Stability analysis

The origin (the desired equilibrium) is <u>not the unique equilibrium point</u>: there is a continuum of equilibria (a 2D subspace of R^4).

➢ For constant commands, this fact <u>forbids</u> achieving global asymptotic results.

Define $x := \begin{bmatrix} e & \Delta k_p & \Delta k_{p_{cmd}} & e_{ref} \end{bmatrix}^\top$ and consider the Lyapunov candidate

$$V(e, \Delta k_p, \Delta k_f, e_{ref}) = \frac{1}{2}e^2 + \frac{1}{2\gamma_p}\Delta k_p^2 + \frac{1}{2\gamma_f}\Delta k_f^2 + \frac{1}{2}e_{ref}^2$$

which satisfies $\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|)$ where

$$\alpha_1(|x|) = \frac{1}{2}\min\left(1, \frac{1}{\gamma_p}, \frac{1}{\gamma_f}\right)|x|^2 \qquad \alpha_1(|x|) = \frac{1}{2}\max\left(1, \frac{1}{\gamma_p}, \frac{1}{\gamma_f}\right)|x|^2$$

Recap: Introductory examples in Adaptive Control

The Lie derivative is Negative Semi-Definite (NDS) (by design)

$$\mathcal{L}_f V(e, \Delta k_p, \Delta k_f, e_{ref}) = a_{ref} e^2 - e_{ref}^2 = \rho(x) \leq 0, \qquad a_{ref} < 0$$

According to Lyapunov theorem, $x = 0$ is UGS, namely, $|x(t)| \leq \gamma(|x_0|)\ \forall x \in R^4, \forall t \geq 0$, with a class K_∞ function

$$\gamma(|x|) = \alpha_1^{-1}(\alpha_2(|x|)) = \sqrt{\frac{\max\left(1, \frac{1}{\gamma_p}, \frac{1}{\gamma_f}\right)}{\min\left(1, \frac{1}{\gamma_p}, \frac{1}{\gamma_f}\right)}}|x|$$

By UGS, all the signals in the closed-loop system are uniformly globally bounded ($x(t) \in L_\infty$):

$$\forall a > 0, \text{ there exists } b(a) > 0 \text{ such that } |x_0| < a, \; |x(t)| < b(a) \quad \forall t \geq 0$$

with $b(a) = \gamma(a)$.

Moreover, since all the conditions of LaSalle/Yoshizawa theorem are satisfied, the solutions converge to the set

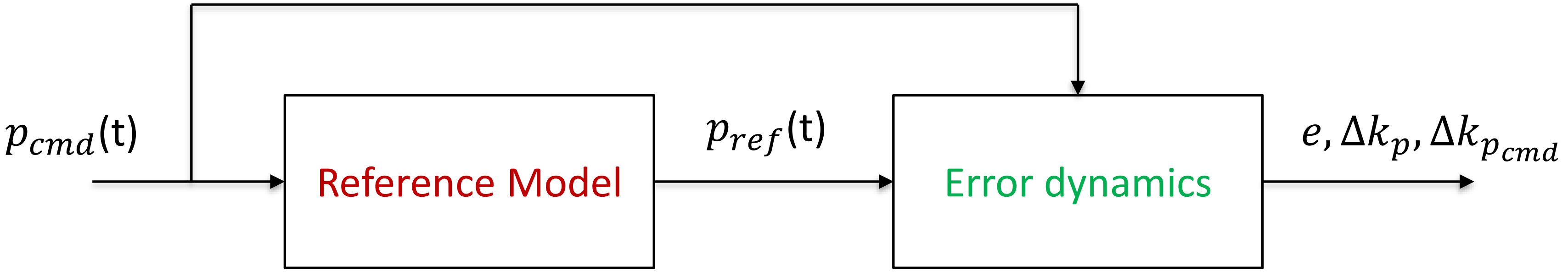
$$E := \left\{x \in \mathbb{R}^4 : \rho(x) = 0\right\} = \{(e, \Delta k_p, \Delta k_{p_{cmd}}, e_{ref}) : e = e_{ref} = 0\}$$

which coincides with the equilibrium set.

In case of constant commands, the model mismatch error e converges to zero while the gain estimation errors do not.

What happens when $p_{cmd} = p_{cmd}(t)$ is not just a constant signal?

The overall system can be seen as a **cascade** with the reference model as upper subsystem and the error dynamics as the lower subsystem.



Upper subsystem

$$\dot{p}_{ref} = a_{ref} p_{ref} + b_{ref} p_{cmd}(t)$$

Lower subsystem

$$\dot{e} = (a_{ref} + L_{\delta_a} \Delta k_p) e + \\ + L_{\delta_a} (\Delta k_p (p_{ref}(t)) + \Delta k_{p_{cmd}} p_{cmd}(t))$$

$$\dot{\Delta k}_p = -\gamma_p sign(L_{\delta_a})(e + p_{ref}) e$$

$$\dot{\Delta k}_{p_{cmd}(t)} = -\gamma_{p_{cmd}(t)} sign(L_{\delta_a}) p_{cmd}(t) e$$

Recap: Introductory examples in Adaptive Control

The lower subsystem involving the dynamics of $e, \Delta k_p, \Delta k_{p_{cmd}}$ has an equilibrium point at the origin $\left(e, \Delta k_p, \Delta k_{p_{cmd}}\right) = (0,0,0)$.

Applying LaSalle/Yoshizawa theorem by leveraging the Lyapunov function candidate

$$V(e, \Delta k_p, \Delta k_f) = \frac{1}{2}e^2 + \frac{1}{2\gamma_p}\Delta k_p^2 + \frac{1}{2\gamma_f}\Delta k_f^2$$

one has

$$\mathcal{L}_f V(e, \Delta k_p, \Delta k_f) = a_{ref}e^2 = \rho(x) \leq 0, \qquad a_{ref} < 0,$$

from which one also deduces that the origin is UGS and that $e(t)$ converges to zero along all solutions (the set $E \coloneqq \left\{\left(e, \Delta k_p, \Delta k_{p_{cmd}}\right) \in R^3 : e = 0\right\}$ is attractive).

- Asymptotic model matching is <u>always guaranteed</u> but the error convergence need not be uniform.

Recap: Introductory examples in Adaptive Control

UGAS of the origin can be concluded for certain command signals by relying on more advanced stability results.

As previously mentioned, persistence of excitation plays a fundamental role in achieving such a desirable result.

For the case at hand, it can be shown that a sinusoidal reference input is enough to guarantee parameter convergence, meaning that in such a case $\Delta k_p(t), \Delta k_{p_{cmd}}(t) \to 0$ as $t \to \infty$.

(READ) The following result allows concluding UGAS

Thm. (Anderson&Moore):

Let $x = 0$ be an equilibrium point for the system $\dot{x} = f(t, x)$, where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x, uniformly in t, on $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$. Suppose that there exists a continuously differentiable function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}_{\geq 0}$, such that

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|)$$
$$\mathcal{L}_f V(t, x) \leq 0$$
$$V(t + \delta, x(t + \delta; t, x)) - V(t, x) \leq -\lambda V(t, x), \quad 0 < \lambda < 1$$

$\forall (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and for some $\delta > 0$, where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$ are class-$\mathcal{K}_\infty$ functions and $x(t + \delta; t, x))$ denotes the solution starting at time t in x and evaluated at time $t + \delta$. Then, the equilibrium point $x = 0$ is UGAS.