Exercise: Flow through a Cut

COMP3121/9101 23T1

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This document presents the solution to an exercise discussed in Lecture 9.

Problem

Let f be a flow, and let (S, T) be an s-t cut. Prove that $f(S, T) = |f|$.

Solution

Recall the definition of $f(S, T)$, the net flow across the cut. It counts the flow along edges from S to T in the positive and flow along edges from T to S in the negative, so we obtain:

$$f(S,T) = \sum_{\substack{(v,w)\in E \\ v\in S, w\in T}} f(v,w) - \sum_{\substack{(u,v)\in E \\ u\in T, v\in S}} f(u,v). \tag{1}$$

First, we will regroup the right-hand side terms by v. For each $v \in S$, we gather the terms in (1) relating to edges from v to some $w \in T$ and edges from some $u \in T$ to v.

$$f(S,T) = \sum_{v\in S} \left[\underbrace{\sum_{\substack{(v,w)\in E \\ w\in T}} f(v,w)}_{X_v} - \underbrace{\sum_{\substack{(u,v)\in E \\ u\in T}} f(u,v)}_{Y_v} \right]. \tag{2}$$

Now X_v consists of the outgoing flow from v along only those edges which cross the partition. This can be rewritten as the total outgoing flow less the outgoing flow from v to other vertices of S, i.e.

$$X_v = \sum_{(v,w)\in E} f(v,w) - \sum_{\substack{(v,w)\in E \\ w\in S}} f(v,w) \tag{3}$$

and similarly

$$Y_v = \sum_{(u,v)\in E} f(u,v) - \sum_{\substack{(u,v)\in E \\ u\in S}} f(u,v). \tag{4}$$

Substituting (3) and (4) into (2) gives

$$f(S,T) = \sum_{v\in S} \left[\left(\underbrace{\sum_{(v,w)\in E} f(v,w)}_{A_v} - \underbrace{\sum_{\substack{(v,w)\in E \\ w\in S}} f(v,w)}_{B_v} \right) - \left(\underbrace{\sum_{(u,v)\in E} f(u,v)}_{C_v} - \underbrace{\sum_{\substack{(u,v)\in E \\ u\in S}} f(u,v)}_{D_v} \right) \right]. \tag{5}$$

Expanding the round brackets gives $A_v - B_v - C_v + D_v$, so we can rearrange as $(A_v - C_v) + (D_v - B_v)$. Splitting into

$$\sum_v (A_v - C_v) \text{ and } \sum_v (D_v - B_v),$$

we have

$$f(S,T) = \sum_{v\in S} \left[\sum_{(v,w)\in E} f(v,w) - \sum_{(u,v)\in E} f(u,v) \right] + \sum_{v\in S} \left[\sum_{\substack{(u,v)\in E \\ u\in S}} f(u,v) - \sum_{\substack{(v,w)\in E \\ w\in S}} f(v,w) \right]. \tag{6}$$

Now, each term of the first sum is the outgoing flow from v less the incoming flow to v. By the *flow conservation* property, this value is 0 for $v \in V \setminus \{s,t\}$. As $v \in S$ and S includes s but *not* t, the sum has a term where $v = s$, and does *not* have a term where $v = t$. The outgoing flow from s is equal to the value of the flow $|f|$ (by definition) and the incoming flow to s is zero. Therefore the entire first sum is $|f|$.

$$f(S,T) = |f| + \sum_{v \in S} \left[\sum_{\substack{(u,v) \in E \\ u \in S}} f(u,v) - \sum_{\substack{(v,w) \in E \\ w \in S}} f(v,w) \right]. \tag{7}$$

Next, we split up the second sum into the positive terms and negative terms. Earlier, we grouped these terms by v; we will now ungroup them.

$$f(S,T) = |f| + \sum_{\substack{(u,v) \in E \\ u,v \in S}} f(u,v) - \sum_{\substack{(v,w) \in E \\ v,w \in S}} f(v,w). \tag{8}$$

Both sums take the total flow through all edges within S, so they are equal. Cancelling out, we finally obtain the required result

$$f(S,T) = |f|.$$