10. LINEAR PROGRAMMING

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Problem

Instance: a list of food sources $F_1, \ldots, F_n$; and for each source F_i:

- its price per gram p_i;
- the number of calories c_i per gram, and
- for each of 13 vitamins $V_1, \ldots, V_{13}$, the content $v_{i,j}$ in milligrams of vitamin V_j in one gram of food source f_i.

Task: find a combination of quantities of food sources such that:

- the total number of calories in all of the chosen food is equal to a recommended daily value of 2000 calories;
- for each $1 \leq j \leq 13$, the total intake of vitamin V_j is at least the recommended daily intake of w_j milligrams, and
- the price of all food per day is as low as possible.

Suppose we take x_i grams of each food source F_i for $1 \leq i \leq n$. Then the constraints are as follows.

- The total number of calories must satisfy

$$\sum_{i=1}^{n} c_i x_i = 2000;$$

- For each $1 \leq j \leq 13$, the total amount of vitamin V_j in all food must satisfy

$$\sum_{i=1}^{n} v_{i,j} x_i \geq w_j.$$

- Implicitly, all the quantities must be non-negative numbers, i.e. $x_i \geq 0$ for all $1 \leq i \leq n$.

- Our goal is to minimise the objective function, which is the total cost
$$y = \sum_{i=1}^{n} p_i x_i.$$
- Note that all constraints and the objective function are **linear**.

Problem

Instance: you are a politician and you want to ensure an election victory by making certain promises to the electorate. You can promise to build:

- bridges, each costing 3 billion;
- rural airports, each costing 2 billion, and
- Olympic swimming pools, each costing 1 billion.

Problem (continued)

You were told by your wise advisers that

- each bridge you promise brings you 5% of city votes, 7% of suburban votes and 9% of rural votes;
- each rural airport you promise brings you no city votes, 2% of suburban votes and 15% of rural votes;
- each Olympic swimming pool promised brings you 12% of city votes, 3% of suburban votes and no rural votes.

Problem (continued)

In order to win, you have to get at least 51% of each of the city, suburban and rural votes.

Task: decide how many bridges, airports and pools to promise in order to guarantee an election win at minimum cost to the budget.

- Let the number of bridges to be built be x_b, number of airports x_a and the number of swimming pools x_p.
- We now see that the problem amounts to minimising the objective $y = 3x_b + 2x_a + x_p$, while making sure that the following constraints are satisfied:

$$
\begin{aligned}
0.05x_b \qquad\qquad\quad + 0.12x_p &\geq 0.51 && \text{(city votes)}\\
0.07x_b + 0.02x_a + 0.03x_p &\geq 0.51 && \text{(suburban votes)}\\
0.09x_b + 0.15x_a \qquad\qquad\quad &\geq 0.51 && \text{(rural votes)}\\
x_b, x_a, x_p &\geq 0.
\end{aligned}
$$

- However, there is a very significant difference with the first example:
  - you can eat 1.56 grams of chocolate, but
  - you cannot promise to build 1.56 bridges, 2.83 airports and 0.57 swimming pools!
- The second example is an example of an **Integer Linear Programming problem**, which requires all the solutions to be integers.
- Such problems are MUCH harder to solve than the “plain” Linear Programming problems whose solutions can be real numbers.

Solving Linear Programming problems

- We won't see algorithms which solve LP problems in this lecture; we will only study the structure of these problems further.
- There are polynomial time algorithms for Linear Programming, including the ellipsoid algorithm.
- In practice we typically use the SIMPLEX algorithm instead; its worst case time complexity is exponential, but it is very efficient in the 'average' case.
- There is no known polynomial time algorithm for Integer Linear Programming!

In the **standard form** the *objective* to be maximised is given by

$$\sum_{i=1}^{n} c_i \, x_i$$

and the *constraints* are of the form

$$\sum_{i=1}^{n} a_{ij} x_i \leq b_j \qquad (1 \leq j \leq m);$$

$$x_i \geq 0 \qquad (1 \leq i \leq n).$$

- To get a more compact representation of linear programs, we use vectors and matrices.
- Let $\mathbf{x}$ represent a (column) vector,

$$\mathbf{x} = \langle x_1 \ldots x_n \rangle^T.$$

- Define a partial ordering on the vectors in $\mathbb{R}^n$ by $\mathbf{x} \leq \mathbf{y}$ if and only if the corresponding inequalities hold coordinate-wise, i.e., if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$.

Write the coefficients in the objective function as

$$\mathbf{c} = \langle c_1 \ldots c_n \rangle^T \in \mathbb{R}^n,$$

the coefficients in the constraints as an $m \times n$ matrix

$$A = (a_{ij})$$

and the right-hand side values of the constraints as

$$\mathbf{b} = \langle b_1 \ldots b_m \rangle^T \in \mathbb{R}^m.$$

Then the standard form can be formulated simply as:

- maximize $\mathbf{c}^T\mathbf{x}$
- subject to the following two (matrix-vector) constraints:

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0}. \end{aligned}$$

Thus, a Linear Programming optimisation problem can be specified as a triplet $(A, \mathbf{b}, \mathbf{c})$, which is the form accepted by most standard LP solvers.

Translating other constraints to Standard Form

- The Standard Form doesn't immediately appear to handle the full generality of LP problems.

- LP problems could have:
  - equality constraints
  - unconstrained variables (i.e. potentially negative values x_i)
  - absolute value constraints

Equality constraints

- An LP problem may include equality constraints of the form

$$\sum_{i=1}^{n} a_{ij} x_i = b_j.$$

- Each of these can be replaced by two inequalities:

$$\sum_{i=1}^{n} a_{ij} x_i \geq b_j$$
$$\sum_{i=1}^{n} a_{ij} x_i \leq b_j.$$

- Thus, we can assume that all constraints are inequalities.

- In general, a "natural formulation" of a problem as a Linear Program does not necessarily require that all variables be non-negative.
- However, the Standard Form does impose this constraint.
- This poses no problem, because each occurrence of an unconstrained variable x_i can be replaced by the expression

$$x_i' - x_i^*$$

where x_i', x_i^* are new variables satisfying the inequality constraints

$$x_i' \geq 0, \; x_i^* \geq 0.$$

Absolute value constraints

- For a vector
$$\mathbf{x} = \langle x_1, \ldots, x_n \rangle^T,$$
we can define
$$|\mathbf{x}| = \langle |x_1|, \ldots, |x_n| \rangle^T.$$

- Some problems are naturally translated into constraints of the form
$$|A\mathbf{x}| \leq \mathbf{b}.$$

- This also poses no problem because we can replace such constraints with two linear constraints:
$$A\mathbf{x} \leq \mathbf{b} \text{ and } -A\mathbf{x} \leq \mathbf{b},$$
because $|x| \leq y$ if and only if $x \leq y$ and $-x \leq y$.

- Standard Form: maximize

$$\mathbf{c}^T\mathbf{x}$$

subject to

$$A\mathbf{x} \leq \mathbf{b}$$

and

$$\mathbf{x} \geq \mathbf{0}.$$

- Any vector $\mathbf{x}$ which satisfies the two constraints is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^T\mathbf{x}$ might be.

Sample linear program

As an example, let us consider the following optimisation problem.

Problem

$$\text{maximise} \qquad z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3 \tag{1}$$

subject to

$$x_1 + x_2 + 3x_3 \leq 30 \tag{2}$$

$$2x_1 + 2x_2 + 5x_3 \leq 24 \tag{3}$$

$$4x_1 + x_2 + 2x_3 \leq 36 \tag{4}$$

$$x_1, x_2, x_3 \geq 0 \tag{5}$$

How large can the value of the objective

$$z(x_1, x_2, x_3) = 3x_1 + x_2 + 2x_3$$

be, without violating the constraints?
We can achieve a crude bound by adding inequalities (2) and (3), to obtain

$$3x_1 + 3x_2 + 8x_3 \leq 54.$$

Since all variables are constrained to be non-negative, we are assured that

$$3x_1 + x_2 + 2x_3 \leq 3x_1 + 3x_2 + 8x_3 \leq 54,$$

i.e. the objective does not exceed 54. Can we do better?

We could try to look for coefficients $y_1, y_2, y_3 \geq 0$ to be used to form a linear combination of the constraints:

$$y_1(x_1 + x_2 + 3x_3) \leq 30y_1 \tag{6}$$
$$y_2(2x_1 + 2x_2 + 5x_3) \leq 24y_2 \tag{7}$$
$$y_3(4x_1 + x_2 + 2x_3) \leq 36y_3 \tag{8}$$

Then, summing up all these inequalities and factoring, we get

$$\begin{aligned}
&x_1(y_1 + 2y_2 + 4y_3) \\
&+ x_2(y_1 + 2y_2 + y_3) \\
&+ x_3(3y_1 + 5y_2 + 2y_3) \\
&\leq 30y_1 + 24y_2 + 36y_3.
\end{aligned}$$

If we compare this with our objective (1) we see that if we choose y_1, y_2 and y_3 so that:

$$\begin{aligned} y_1 + 2y_2 + 4y_3 &\geq 3 \\ y_1 + 2y_2 + y_3 &\geq 1 \\ 3y_1 + 5y_2 + 2y_3 &\geq 2 \end{aligned}$$

then

$$\begin{aligned} 3x_1 + x_2 + 2x_3 \leq\ & x_1(y_1 + 2y_2 + 4y_3) \\ & + x_2(y_1 + 2y_2 + y_3) \\ & + x_3(3y_1 + 5y_2 + 2y_3). \end{aligned}$$

Combining this with (6) – (8) we get

$$30y_1 + 24y_2 + 36y_3 \geq 3x_1 + x_2 + 2x_3 = z(x_1, x_2, x_3).$$

Consequently, in order to find a tight upper bound for our objective $z(x_1, x_2, x_3)$ in the original problem P, we have to find y_1, y_2, y_3 which solve problem P^*:

$$\text{minimise:} \qquad z^*(y_1, y_2, y_3) = 30y_1 + 24y_2 + 36y_3 \tag{9}$$

subject to:

$$y_1 + 2y_2 + 4y_3 \geq 3 \tag{10}$$

$$y_1 + 2y_2 + y_3 \geq 1 \tag{11}$$

$$3y_1 + 5y_2 + 2y_3 \geq 2 \tag{12}$$

$$y_1, y_2, y_3 \geq 0 \tag{13}$$

Sample linear program

Then

$$\begin{aligned} z^*(y_1, y_2, y_3) &= 30y_1 + 24y_2 + 36y_3 \\ &\geq 3x_1 + x_2 + 2x_3 \\ &= z(x_1, x_2, x_3) \end{aligned}$$

will be a tight upper bound.

The new problem P^* is called the *dual problem* of P.

Let us now repeat the whole procedure in order to find the dual of P^*, which will be denoted $(P^*)^*$.
We are now looking for $z_1, z_2, z_3 \geq 0$ to multiply inequalities (10)–(12) and obtain

$$z_1(y_1 + 2y_2 + 4y_3) \geq 3z_1$$
$$z_2(y_1 + 2y_2 + y_3) \geq z_2$$
$$z_3(3y_1 + 5y_2 + 2y_3) \geq 2z_3$$

Summing these up and factoring produces

$$\begin{aligned} & y_1(z_1 + z_2 + 3z_3) \\ & + y_2(2z_1 + 2z_2 + 5z_3) \\ & + y_3(4z_1 + z_2 + 2z_3) \\ & \geq 3z_1 + z_2 + 2z_3 \end{aligned} \tag{14}$$

If we choose multipliers z_1, z_2, z_3 so that

$$\begin{aligned} z_1 + z_2 + 3z_3 &\leq 30 \\ 2z_1 + 2z_2 + 5z_3 &\leq 24 \\ 4z_1 + z_2 + 2z_3 &\leq 36 \end{aligned}$$

we will have:

$$\begin{aligned} & y_1(z_1 + z_2 + 3z_3) \\ & + y_2(2z_1 + 2z_2 + 5z_3) \\ & + y_3(4z_1 + z_2 + 2z_3) \\ & \leq 30y_1 + 24y_2 + 36y_3 \end{aligned}$$

Combining this with (14) we get

$$3z_1 + z_2 + 2z_3 \leq 30y_1 + 24y_2 + 36y_3.$$

Consequently, finding the double dual program $(P^*)^*$ amounts to maximising the objective $3z_1 + z_2 + 2z_3$ subject to the constraints

$$
\begin{aligned}
z_1 + z_2 + 3z_3 &\leq 30 \\
2z_1 + 2z_2 + 5z_3 &\leq 24 \\
4z_1 + z_2 + 2z_3 &\leq 36 \\
z_1, z_2, z_3 &\geq 0
\end{aligned}
$$

This is exactly our starting program P, with only the variable names changed! Thus, the double dual program $(P^*)^*$ is just P itself.

- It appeared at first that looking for the multipliers y_1, y_2, y_3 did not help much, because it only reduced a maximisation problem to an equally hard minimisation problem.

- It is useful at this point to remember how we proved that the Ford-Fulkerson algorithm produces a **maximal flow**, by showing that it terminates only when we reach the capacity of a **minimal cut**.

Primal and dual linear programs

In general, the *primal* Linear Program P and its *dual* P^* are:

$$
\begin{array}{lll}
P: & \text{maximize} & z(\mathbf{x}) = \sum_{i=1}^{n} c_i x_i, \\
 & \text{subject to} & \sum_{i=1}^{n} a_{ij} x_i \le b_j \qquad (1 \le j \le m) \\
 & \text{and} & x_1, \ldots, x_n \ge 0; \\
P^*: & \text{minimize} & z^*(\mathbf{y}) = \sum_{j=1}^{m} b_j y_j, \\
 & \text{subject to} & \sum_{j=1}^{m} a_{ij} y_j \ge c_i \qquad (1 \le i \le n) \\
 & \text{and} & y_1, \ldots, y_m \ge 0.
\end{array}
$$

We can equivalently write P and P^* in matrix form:

$$\begin{array}{rll} P: & \text{maximize} & z(\mathbf{x}) = \mathbf{c}^T\mathbf{x}, \\ & \text{subject to} & A\mathbf{x} \leq \mathbf{b} \\ & \text{and} & \mathbf{x} \geq 0; \\ P^*: & \text{minimize} & z^*(\mathbf{y}) = \mathbf{b}^T\mathbf{y}, \\ & \text{subject to} & A^T\mathbf{y} \geq \mathbf{c} \\ & \text{and} & \mathbf{y} \geq 0. \end{array}$$

Recall that any vector $\mathbf{x}$ which satisfies the two constraints $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ is called a *feasible solution*, regardless of what the corresponding objective value $\mathbf{c}^T\mathbf{x}$ might be.

Theorem

If $\mathbf{x} = \langle x_1 \ldots x_n \rangle$ is any feasible solution for P and $\mathbf{y} = \langle y_1 \ldots y_m \rangle$ is any feasible solution for P^*, then:

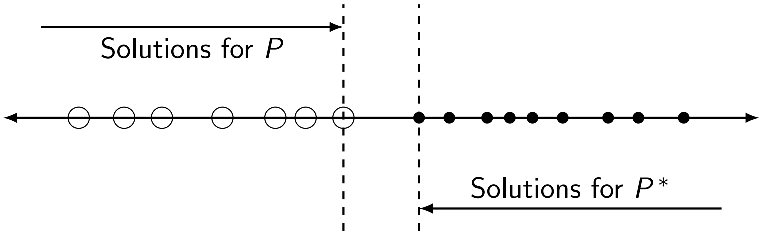
$$z(\mathbf{x}) = \sum_{i=1}^{n} c_i x_i \leq \sum_{j=1}^{m} b_j y_j = z^*(\mathbf{y})$$

Proof

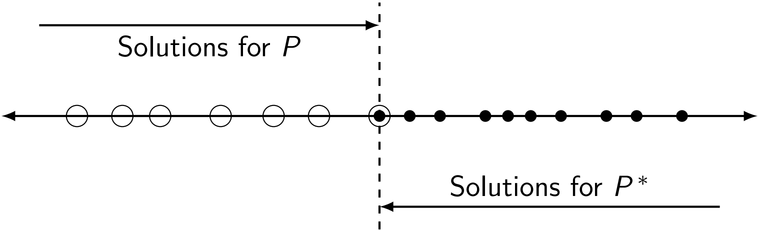
Since **x** and **y** are feasible solutions for P and P^* respectively, we can use the constraint inequalities, first from P^* and then from P to obtain

$$
\begin{aligned}
z(\mathbf{x}) = \sum_{i=1}^{n} c_i x_i &\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} y_j \right) x_i \\
&= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} a_{ij} x_i \right) y_j \leq \sum_{j=1}^{m} b_j y_j \\
&= z^*(\mathbf{y}).
\end{aligned}
$$

Thus, the value of (the objective of P^* for) any feasible solution of P^* is an upper bound for the set of all values of (the objective of P for) all feasible solutions of P, and every feasible solution of P is a lower bound for the set of feasible solutions for P^*.



If we find a feasible solution for P which is equal to a feasible solution to P^*, this common value must be the maximal feasible value of the objective of P and the minimal feasible value of the objective of P^*.



- If we use a search procedure to find an optimal solution for P we know when to stop: when such a value is also a feasible solution for P^*.
- This is why the most commonly used LP solving method, the SIMPLEX method, produces an optimal solution for P: because it stops at a value of the primal objective which is also a value of the dual objective.
- See the supplemental notes for the details and an example of how the SIMPLEX algorithm runs.

There are five sisters in a house.

- Sharon is reading a book.
- Jennifer is playing chess.
- Catherine is cooking.
- Anna is doing laundry.

What is Helen, the fifth sister, doing?

That's All, Folks!!