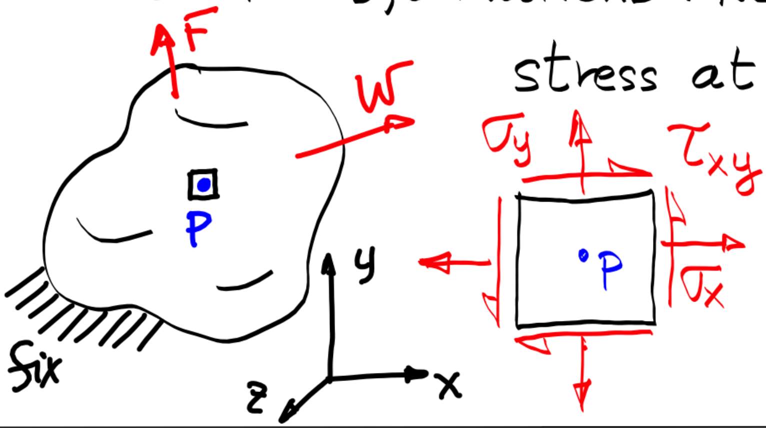
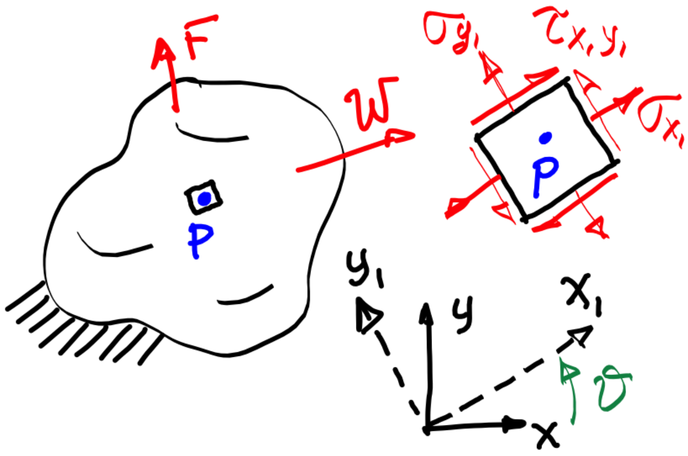
Stress Transformations: Recap

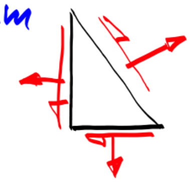
stress at P in $\left\{ \begin{matrix} (x\text{-}y) \\ \text{plane} \\ \text{stress} \end{matrix} \right\}$ special case



$$\sigma_P : \begin{Bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{Bmatrix}$$



from equilibrium of a "wedge"



↓ obtain

$\left.\begin{matrix} \sigma_{x_1} \\ \sigma_{y_1} \\ \tau_{x_1y_1} \end{matrix}\right\}$ in terms of $\left\{\begin{matrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \vartheta \end{matrix}\right.$

$$\sigma_{x_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\vartheta + \tau_{xy}\sin 2\vartheta$$

$$\sigma_{y_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\vartheta - \tau_{xy}\sin 2\vartheta$$

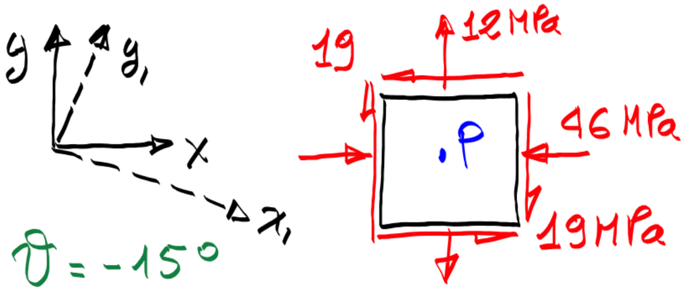
$$\tau_{x_1y_1}(\vartheta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\vartheta + \tau_{xy}\cos 2\vartheta$$

Example: at a point P of a loaded body, the stress on an element aligned w/ axes {x, y} is:

$$\left\{\begin{matrix} \sigma_x = -46 \text{ MPa} \\ \sigma_y = 12 \text{ MPa} \\ \tau_{xy} = -19 \text{ MPa} \end{matrix}\right\}$$

→ calculate stress state on an element aligned w {x_1, y_1} rotated 15° CW from {x y}

- sketch stress state on original & rotated elements



$\theta = -15°$

$$\sigma_P = \begin{bmatrix} -46 & -19 \\ -19 & 12 \end{bmatrix}$$

$$\frac{\sigma_x + \sigma_y}{2} = -17 \text{ MPa}$$

$$\frac{\sigma_x - \sigma_y}{2} = -29 \text{ MPa}$$

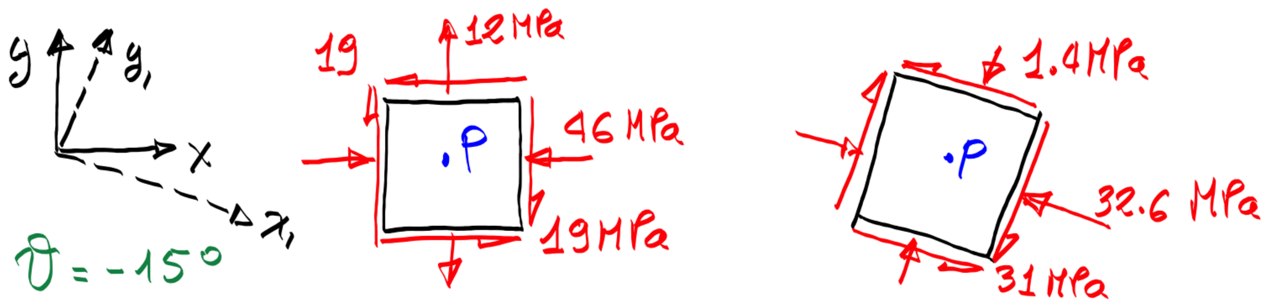
$$\sin 2\theta = \sin(-30) = -\tfrac{1}{2} = -0.5$$
$$\cos 2\theta = \cos(-30) = \sqrt{3}/2 = 0.866$$

$$\sigma_{x_1} = [+(-17) + (-29)(0.866) + (-19)(-0.5)] \text{ MPa} = -32.6 \text{ MPa}$$

$$\sigma_{y_1} = [+(-17) - (-29)(0.866) - (-19)(-0.5)] \text{ MPa} = -1.4 \text{ MPa}$$

$$\tau_{x_1 y_1} = [\quad -(-29)(-0.5) + (-19)(0.866)] \text{ MPa} = -31 \text{ MPa}$$

note: $\sigma_{x_1} + \sigma_{y_1} = -34 \text{ MPa} = \sigma_x + \sigma_y$



$\theta = -15°$

$$\sigma_P = \begin{bmatrix} -46 & -19 \\ -19 & 12 \end{bmatrix} \longleftarrow \text{WHAT IS THE "STRESS @ P"?} \longrightarrow \sigma_{P_1} = \begin{bmatrix} -32.6 & -31 \\ -31 & -1.4 \end{bmatrix}$$

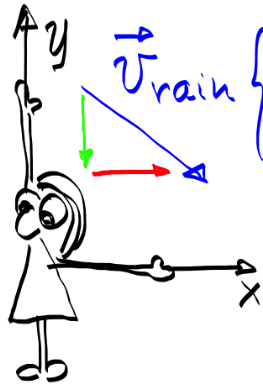
---

stress components depend on the orientation of the reference frame ☹ how do we:

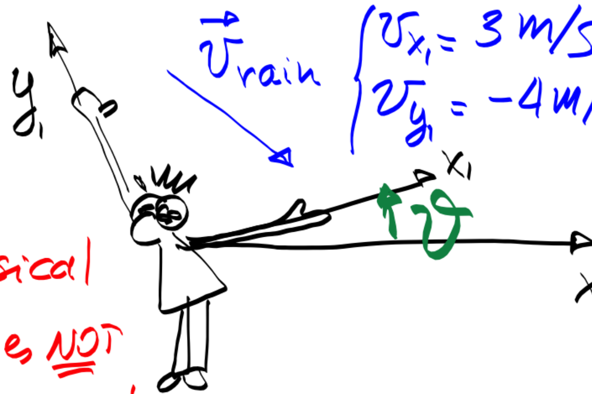
quantify the "magnitude" of stress to predict if the material will fail @ P?

↳ go back to easier quantities (vectors) to figure out a plan to deal with this..

vector



$\vec{v}_{rain} \begin{cases} v_x = 4\,m/s \\ v_y = -3\,m/s \end{cases}$



$\vec{v}_{rain} \begin{cases} v_{x_1} = 3\,m/s \\ v_{y_1} = -4\,m/s \end{cases}$

$\vec{v}_{rain}$ is a physical quantity: it does <u>NOT</u> depend on observer!

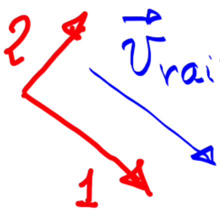
Note 1 For <u>all</u> vectors can use these relations to transform components:

$$\begin{cases} v_{x_1} = v_x \cos\vartheta + v_y \sin\vartheta \\ v_{y_1} = -v_x \sin\vartheta + v_y \cos\vartheta \end{cases}$$

[if you know matrix algebra can write as:

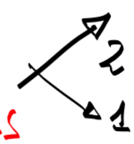
$$\begin{Bmatrix} v_{x_1} \\ v_{y_1} \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}}_{[Q]\text{: rotation matrix}} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix} \qquad \begin{bmatrix} \hat{e}_{x_1}\cdot\hat{e}_x & \hat{e}_{x_1}\cdot\hat{e}_y \\ \hat{e}_{y_1}\cdot\hat{e}_x & \hat{e}_{y_1}\cdot\hat{e}_y \end{bmatrix}$$
]

Note 2 There are "special" orientations of the axes along which
{ some components assume their maximum values
{ some components are zero



$\vec{v}_{rain} = \begin{Bmatrix} v_1 = 6 \\ v_2 = 0 \end{Bmatrix}$

these are called { principal values / principal directions



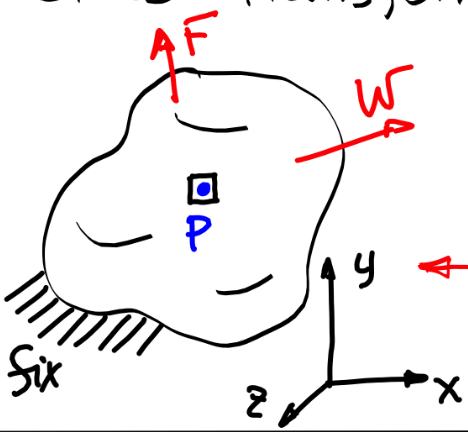
Note 3 there are "characteristics" of $\vec{v}_{rain}$ on which all observers agree : <u>INVARIANT</u>

$$\sqrt{v_x^2 + v_y^2} = v \text{ (magnitude)} = \sqrt{v_{x_1}^2 + v_{y_1}^2}$$

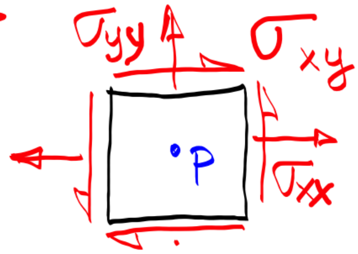
for a vector the magnitude is invariant

in <u>any</u> coordinates plug in components
$\Rightarrow$ get same value for invariant

Stress Transformations : Recap



stress at P in $\left\{ \begin{matrix} (x\text{-}y) \\ \text{plane} \\ \text{stress} \end{matrix} \right\}$ special case



$\underset{\sim}{\sigma_P} : \begin{Bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{Bmatrix}$ CARTESIAN COMPONENTS

STRESS @ a point is a TENSOR

$\boxed{\text{NOTE 1}}$ ⇒ Transformation equations apply To $\underline{\text{ANY}}$ Tensor

STRAIN IS ALSO A TENSOR : $\underset{\sim}{\varepsilon_P} : \begin{Bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{Bmatrix}$

⇒ it can be Transformed w/ same equations!

(if you know matrix algebra can write as : $\begin{bmatrix} \sigma_{x_1} & \tau_{x_1y_1} \\ \tau_{x_1y_1} & \sigma_{y_1} \end{bmatrix} = [Q] \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} [Q]^T$)

Want insight of how stress on a face (y_1) varies w/ ϑ

$$\sigma_{x_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\vartheta + \tau_{xy} \sin 2\vartheta$$

$$\sigma_{y_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\vartheta - \tau_{xy} \sin 2\vartheta$$

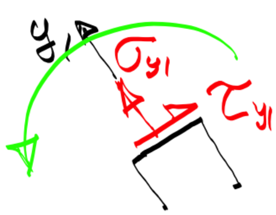
$$\tau_{x_1y_1}(\vartheta) = - \frac{\sigma_x - \sigma_y}{2} \sin 2\vartheta + \tau_{xy} \cos 2\vartheta$$

$\boxed{\text{Note 3}}$ $\frac{\sigma_x + \sigma_y}{2} = \bar{\sigma}$ is an INVARIANT ⟶ $\left(\frac{\sigma_x' + \sigma_y'}{2} = \frac{\sigma_x + \sigma_y}{2} \right)$

Want to be able to "visualize" how the stress components change as we rotate the coordinate axes

Mohr had an idea

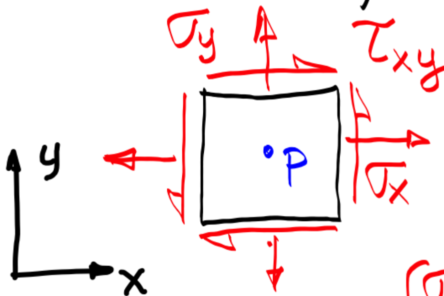
do this next time ...

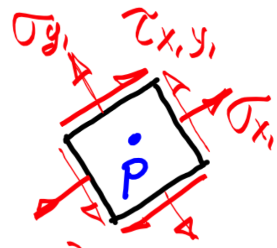


let's focus on the "y" face..

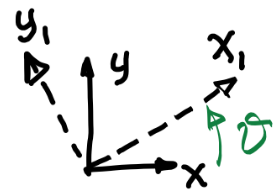
how do σ_{y_1} & $\tau_{x_1y_1}$ change?

Stress Transformations : Recap from Monday





$$\underset{\sim}{\sigma_p} : \begin{Bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{Bmatrix} \quad \begin{Bmatrix} \sigma_{x_1} & \tau_{x_1y_1} \\ \tau_{x_1y_1} & \sigma_{y_1} \end{Bmatrix} : \underset{\sim}{\sigma_p}$$



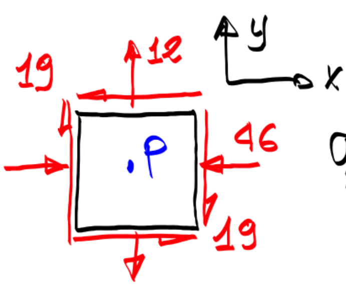
$$\sigma_{x_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\vartheta + \tau_{xy}\sin 2\vartheta$$

$$\sigma_{y_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\vartheta - \tau_{xy}\sin 2\vartheta$$

$$\tau_{x_1y_1}(\vartheta) = -\frac{\sigma_x - \sigma_y}{2}\sin 2\vartheta + \tau_{xy}\cos 2\vartheta$$

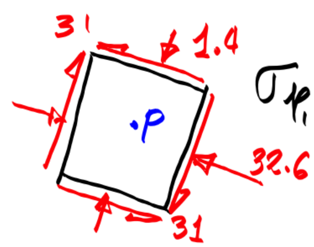
Note: $\frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_{x_1} + \sigma_{y_1}}{2} = \bar{\sigma}$ is an <u>INVARIANT</u> of $\underset{\sim}{\sigma}$

are there others ?



All # in MPa

$$\underset{\sim}{\sigma_p} = \begin{Bmatrix} -46 & -19 \\ -19 & 12 \end{Bmatrix} \xrightarrow{\vartheta = -15^\circ}$$



$$\sigma_{p_1} = \begin{Bmatrix} -32.6 & -31 \\ -31 & -1.4 \end{Bmatrix}$$

Mohr (Christian Otto)

$$\underbrace{\left(\sigma_{y_1}(\vartheta) - \bar{\sigma}\right)^2}_{(\sigma - \bar{\sigma})^2} + \underbrace{\left(\tau_{x_1y_1}(\vartheta)\right)^2}_{(\tau)^2} = \underbrace{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}_{R^2}$$

$$(\sigma - \bar{\sigma})^2 + (\tau)^2 = R^2$$

sign convention

Note: R^2 is also an INVARIANT

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\vartheta + \tau_{xy}\sin 2\vartheta$$

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\vartheta - \tau_{xy}\sin 2\vartheta$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\vartheta + \tau_{xy}\cos 2\vartheta$$

Proof that R is invariant ∀

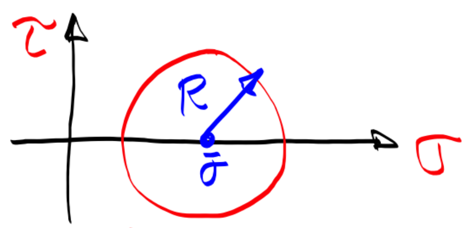
$$\left(\frac{\sigma_{x_1} - \sigma_{y_1}}{2}\right)^2 + \left(\tau_{x_1 y_1}\right)^2 = \left(\frac{\sigma_x - \sigma_y}{2}\cos 2\vartheta + \tau_{xy}\sin 2\vartheta\right)^2 + \left(-\frac{\sigma_x - \sigma_y}{2}\sin 2\vartheta + \tau_{xy}\cos 2\vartheta\right)^2$$

$$= \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2$$

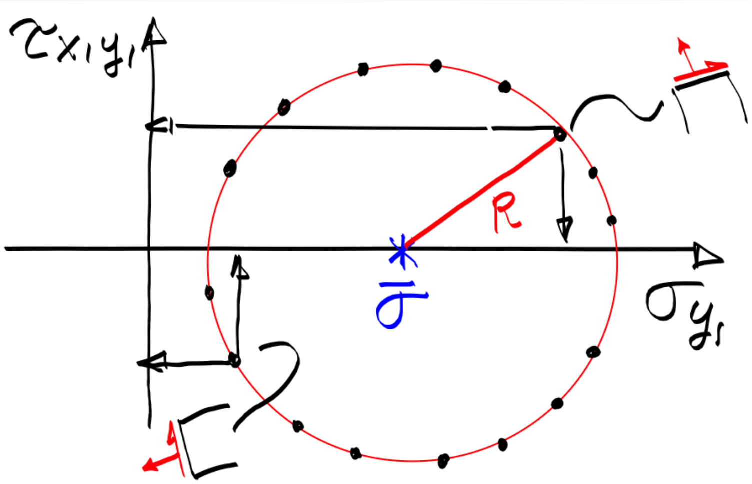
$$= R^2 \text{ : invariant!}$$

Mohr circle of stress

$$(\sigma - \bar{\sigma})^2 + (\tau)^2 = R^2$$



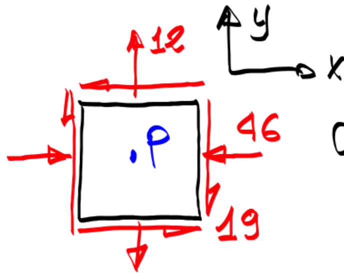
Equation of a CIRCLE in the (σ, τ) plane
with center @ $(\bar{\sigma}, 0)$
and radius $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$



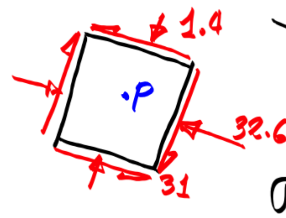
each point corresponds to a different y_1 face orientation (ϑ)

The "coordinates" of that point are the stress on that face.

Example (all MPa)



$$\sigma_P = \begin{Bmatrix} -46 & -19 \\ -19 & 12 \end{Bmatrix} \qquad \vartheta = -15°$$



$$\sigma_{P_1} = \begin{Bmatrix} -32.6 & -31 \\ -31 & -1.4 \end{Bmatrix}$$

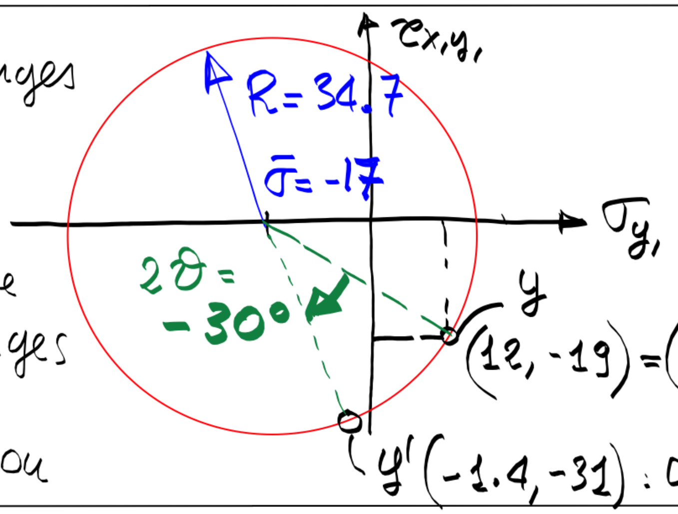
$$\frac{\sigma_x + \sigma_y}{2} = -17 = \bar{\sigma} \qquad \frac{\sigma_x - \sigma_y}{2} = -29$$

$$\tau_{xy} = -19 \qquad R = \sqrt{29^2 + 19^2} = 34.7$$

(INVARIANT)

$$R = \sqrt{15.6^2 + 31^2} = 34.7$$

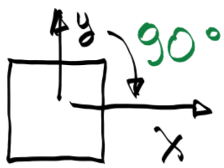
Angle changes on Mohr's circle are **double** the angle changes in surface orientation



⊕ direction for stresses on circle

Note #1: can find the stress on any face @ ϑ from y face by "walking 2ϑ around circle" ☺

$\Rightarrow$ stress on x-face is @ 180° from y-face



careful w/ signs!

an x-face with a negative τ_{xy} has a positive τ on the circle

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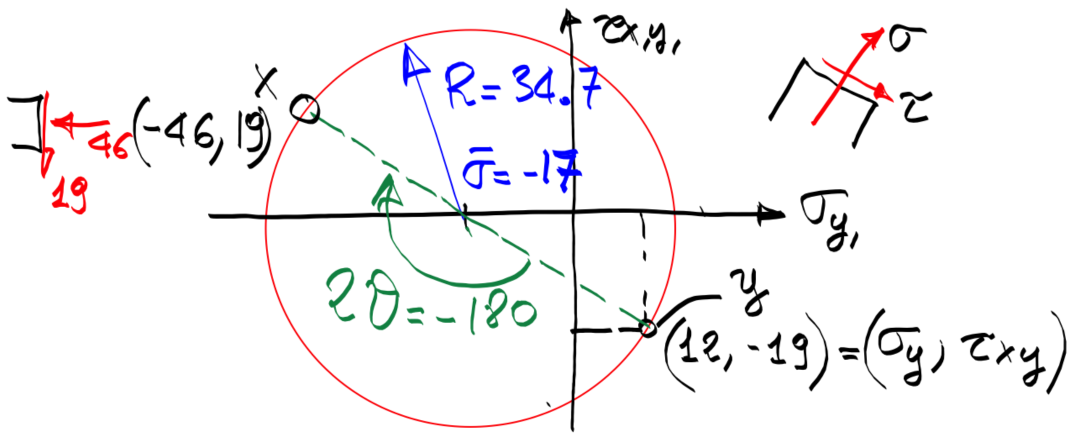
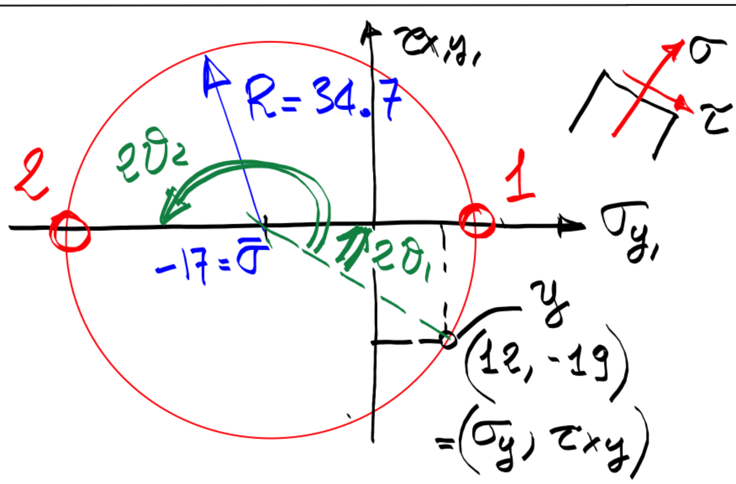
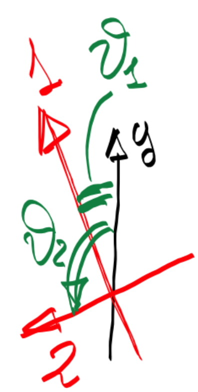


Note #2 there are special orientations where
- The shear stress is zero
- The normal stress is max 1, min 2

} 1, 2

these orientations are at 90° (180 on circle) and they are called the Principal directions (axes)



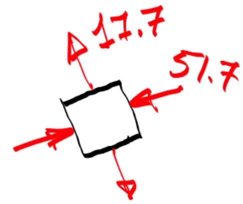


Note #3 the (normal) stresses acting on faces normal to principal axes are the

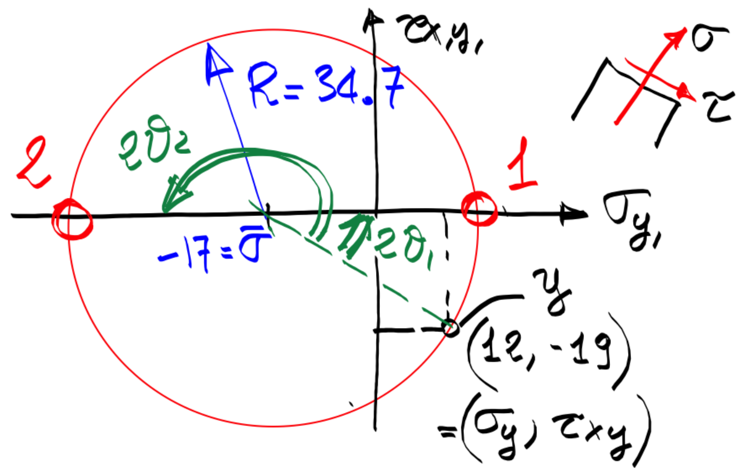
principal stresses : σ_1 (max)
σ_2 (min)

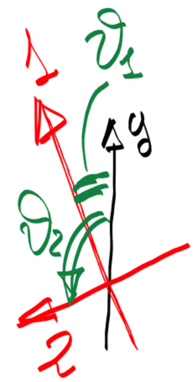
$$\sigma_1 = \bar{\sigma} + R = -17 + 34.7 = 17.7$$

$$\sigma_2 = \bar{\sigma} - R = -17 - 34.7 = -51.7$$



(we like )





Note #4 can determine the orientation of principal axes from geometry :

$$2\theta_1 = \tan^{-1}\left(\frac{19}{12+17}\right) = \tan^{-1}\left(\frac{-\tau_{xy}}{(\sigma_y - \bar{\sigma})}\right) = \tan^{-1}\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 2\theta_1$$

$$2\theta_2 = 2\theta_1 + 180^\circ \longrightarrow \theta_2 = \theta_1 + 90^\circ$$

(can also get this by finding θ that maximizes σ_{x_1})

alternative derivation of θ_1, σ_1

to find $\begin{cases}\max \\ \min\end{cases} \sigma_{y_1} \Rightarrow \frac{d\sigma_{y_1}}{d\theta} = 0$ for $\theta = \theta_1$

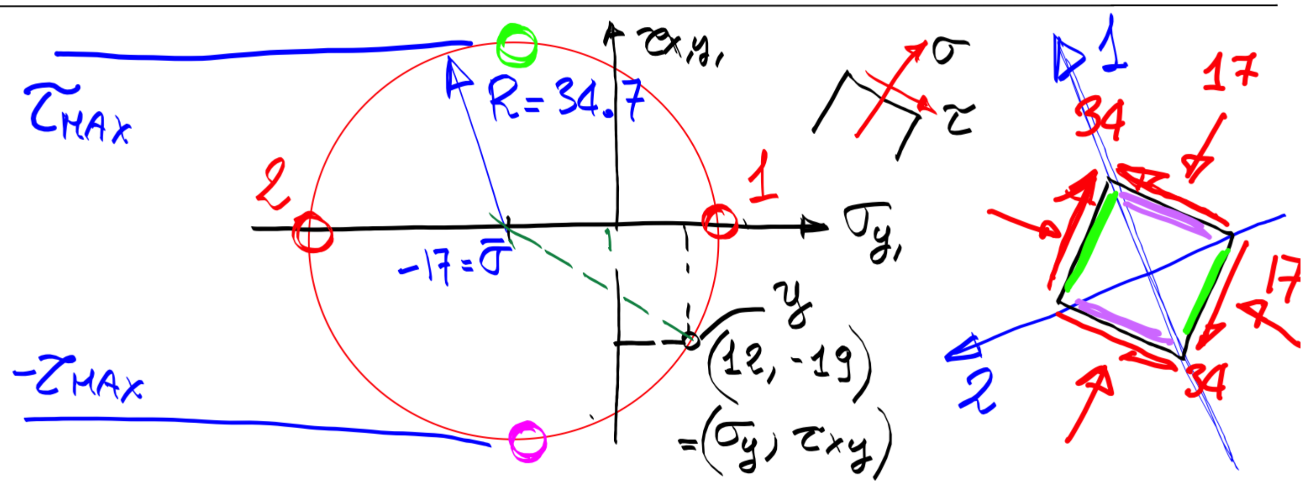
$$\frac{d\sigma_{y_1}}{d\theta} = 2\frac{\sigma_x - \sigma_y}{2}\sin 2\theta_1 - 2\tau_{xy}\sin 2\theta_1 = 0$$

$\tan(2\theta_1) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \Rightarrow$ sub into σ_{y_1} to get $\sigma_1 = \bar{\sigma} + R$

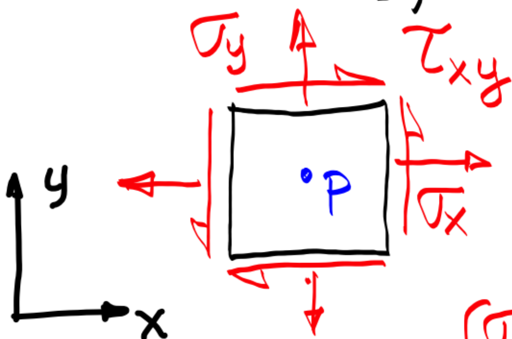
Note # 5 If we look @ element oriented @ 45° from principal axes (90° on circle) we find the maximum shear stress, τ_{MAX}

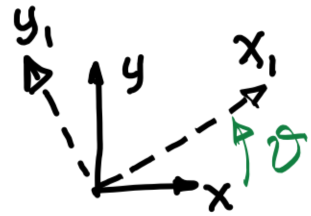
$\tau_{MAX} = R = \frac{\sigma_1 - \sigma_2}{2}$ (also invariant!)

faces with τ_{MAX} also have normal stress $= \bar{\sigma}$



Stress Transformations : Recap from Monday





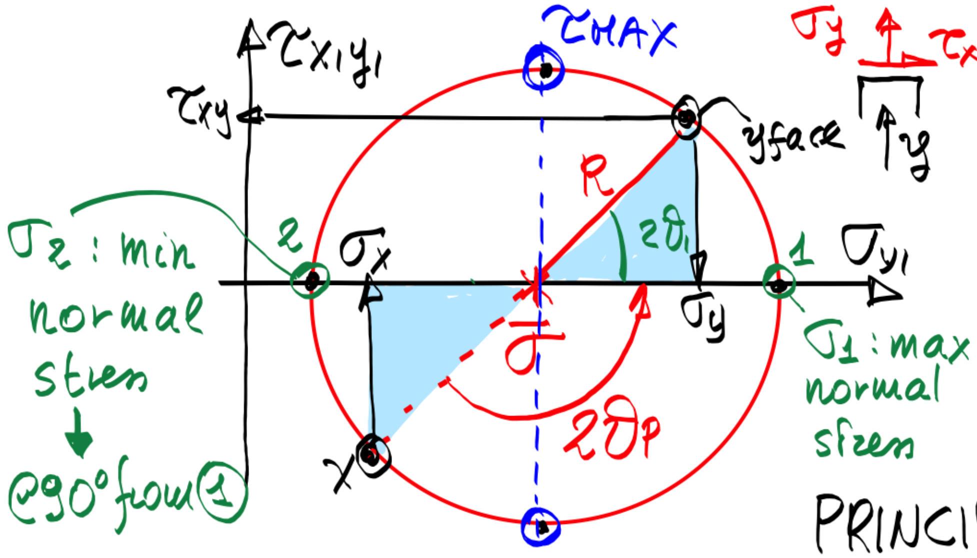
$$\underset{\sim}{\sigma}_P : \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \quad \begin{bmatrix} \sigma_{x_1} & \tau_{x_1y_1} \\ \tau_{x_1y_1} & \sigma_{y_1} \end{bmatrix} : \underset{\sim}{\sigma}_P$$

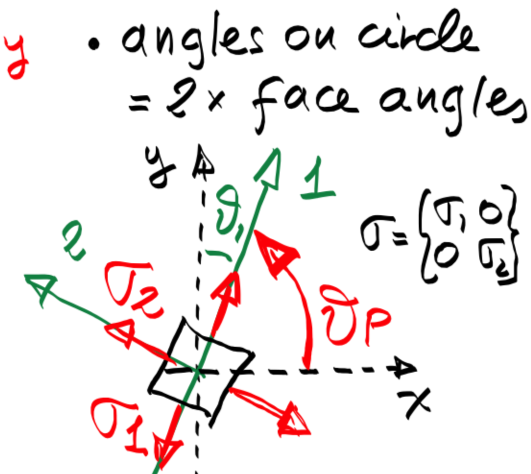
$$\sigma_{x_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\vartheta + \tau_{xy} \sin 2\vartheta$$

$$\sigma_{y_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\vartheta - \tau_{xy} \sin 2\vartheta$$

$$\tau_{x_1y_1}(\vartheta) = -\frac{\sigma_x - \sigma_y}{2} \sin 2\vartheta + \tau_{xy} \cos 2\vartheta$$

$$\bar{\sigma} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \text{INVARIANTS}$$





PRINCIPAL stresses & FRAME

- Principal stresses $\sigma_1 = \bar{\sigma} + R$, $\sigma_2 = \bar{\sigma} - R$ acting along principal axes 1,2
- Shear stress is zero in principal frame (@ 90°)
- <u>max</u> shear stress (τ_{MAX}) @ 45° from 1,2
- θ_P : angle from $\underset{x}{\overset{y}{\llcorner}}$ to $\underset{1}{\overset{2}{\llcorner}}$

$$2\theta_P = \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) \Rightarrow$$ multiple solution $2\theta_P \pm n\pi$ pick based on circle geom

given $\{\sigma_x, \tau_{xy}, \sigma_y\}$

- draw stressed elements in x,y and $x_1 y_1$ @ $\theta = 45°$
- obtain σ_1, σ_2, τ_{MAX}, θ_p

- draw stressed elements in $(1,2)$ and on (x'', y'') @ 45° from $(1,2)$

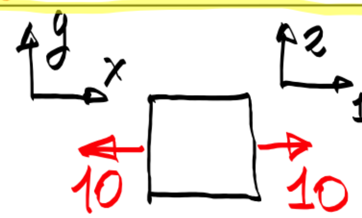
$$\begin{cases} \sigma_x = 10\ MPa \\ \sigma_y = \tau_{xy} = 0 \end{cases}$$

UNIAXIAL LOAD in X

$$\begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \xRightarrow{45°} \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix}$$

Uniaxial Tension

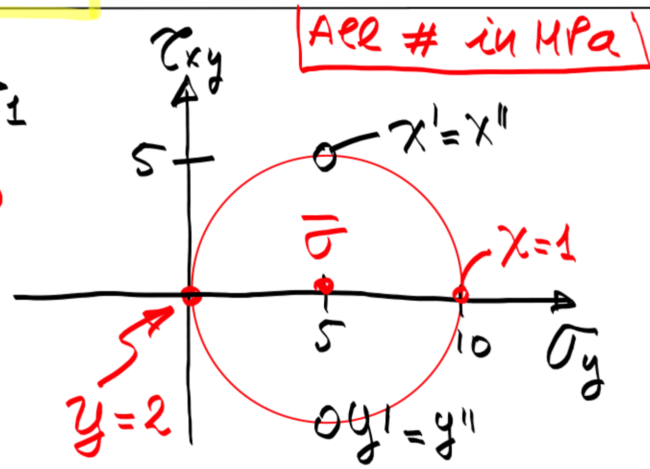
All # in MPa

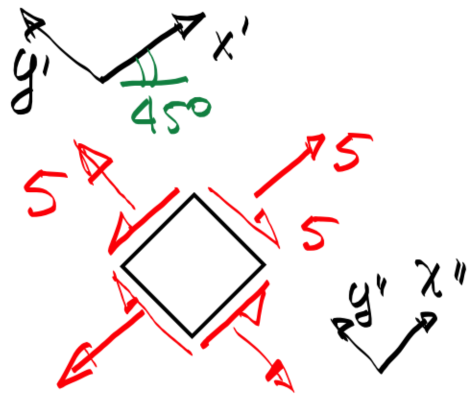


$$\bar{\sigma} = \frac{10}{2} = 5$$

$$\underbrace{R = \sqrt{(5)^2} = 5}$$

$\tau_{MAX} = 5$

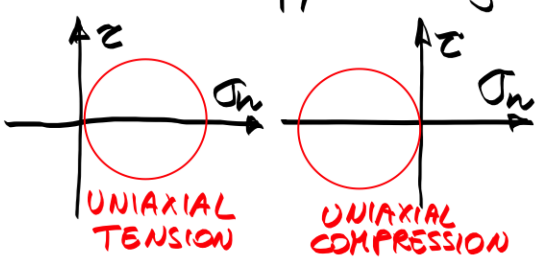




Note: x, y are principal; $x \equiv 1$, $y \equiv 2$ $\Rightarrow \theta_p = 0$

$\sigma_1 = \sigma_x = 10$; $\sigma_2 = \sigma_y = 0$

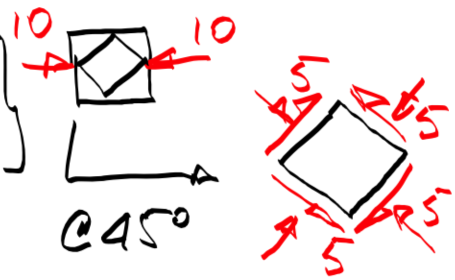
What happens if we have uniaxial COMPRESSION ?



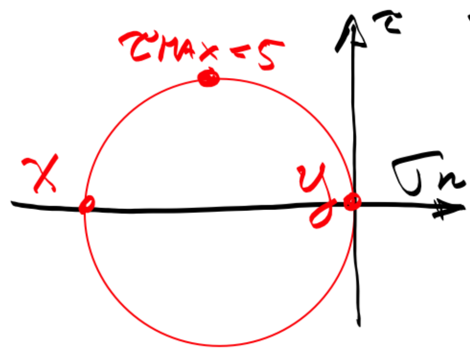
with $\sigma_x = -10$

$\sigma_y = \tau_{xy} = 0$

$$\begin{bmatrix} -10 & 0 \\ 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix}$$



$y \equiv 1$ axis

$x \equiv 2$ axis

$\theta_p = 90°$

given $\{\sigma_x, \tau_{xy}, \sigma_y\}$

- draw stressed elements in x, y and x_1, y_1 @ $\theta = 45°$
- obtain σ_1, σ_2, τ_{MAX}, θ_P

- draw stressed elements in $(1,2)$ and on (x'', y'') @ $45°$ from $(1,2)$

$$\begin{cases} \sigma_x = 10\,MPa \\ \sigma_y = \tau_{xy} = 0 \end{cases}$$

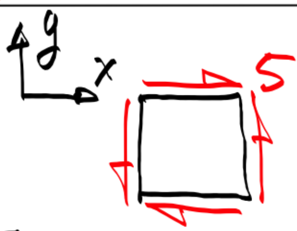
UNIAXIAL LOAD in X

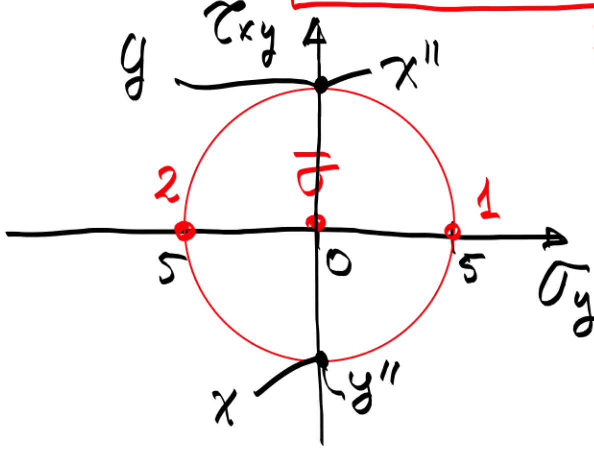
$$\boxed{\begin{array}{l} \sigma_x = \sigma_y = 0 \\ \tau_{xy} = 5\ MPa \end{array}}$$

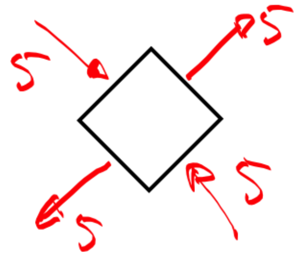
SIMPLE SHEAR

$$\begin{Bmatrix} 0 & 5 \\ 5 & 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} 5 & 0 \\ 0 & -5 \end{Bmatrix}$$

All # in MPa







$\bar{\sigma} = 0$

$R = \sqrt{5^2} = 5$

$\underbrace{\phantom{R = \sqrt{5^2} = 5}}_{\tau_{MAX} = 5}$

$\sigma_1 = \bar{\sigma} + R = 5$ $\quad \sigma_2 = \bar{\sigma} - R = -5$ $\quad \theta_P = 45°$

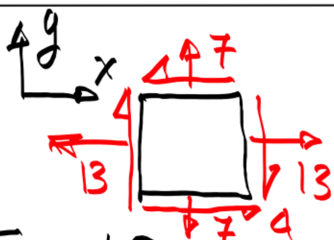
given $\{\sigma_x, \tau_{xy}, \sigma_y\}$

- draw stressed elements in x,y and $x_1 y_1$ @ $\vartheta = 45°$
- obtain $\sigma_1, \sigma_2, \tau_{MAX}, \vartheta_p$

- draw stressed elements in $(1,2)$ and on (x'', y'') @ $45°$ from $(1,2)$

$\sigma_x = 10\,MPa$, $\sigma_y = \tau_{xy} = 0$	$\sigma_x = \sigma_y = 0$, $\tau_{xy} = 5\,MPa$	$\boxed{\sigma_x = 13\,MPa,\ \sigma_y = 7\,MPa,\ \tau_{xy} = -4\,MPa}$
UNIAXIAL LOAD in X	SIMPLE SHEAR	

All # in MPa

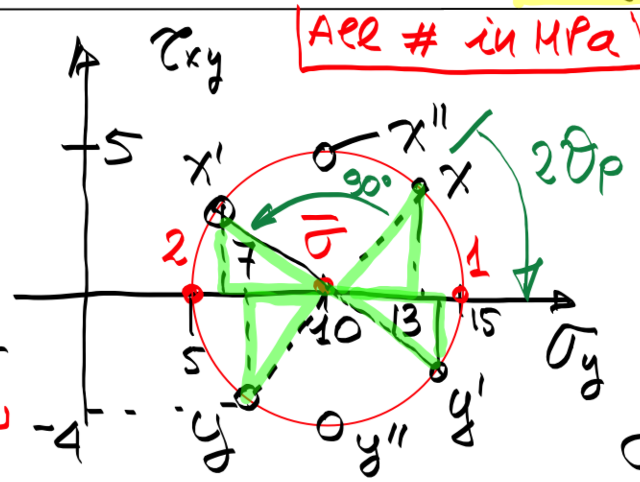


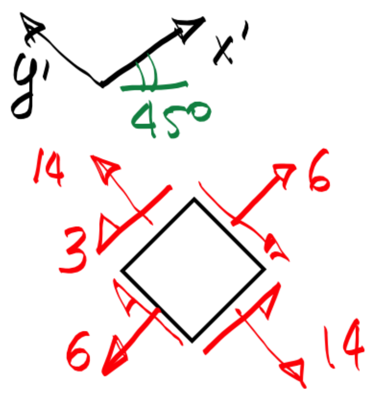
$\bar{\sigma} = 10$

$\underbrace{R = \sqrt{3^2 + 4^2} = 5}$

$\tau_{MAX} = 5$

$\sigma_1 = \bar{\sigma} + R = 15 \qquad \sigma_2 = \bar{\sigma} - R = 5$





$\sigma_x' = 10 - 4$
$\sigma_y' = 10 + 4$
$\tau_{x'y'} = -3$

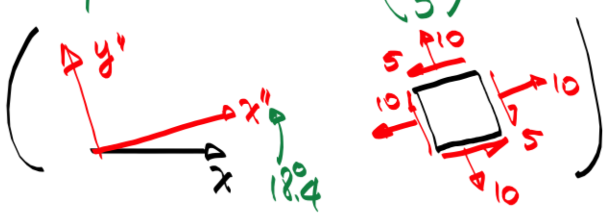
by geom or by eqns

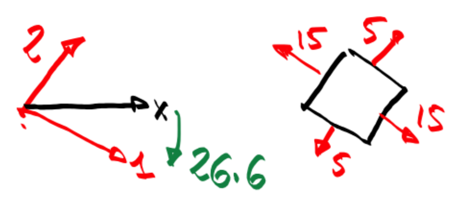
$$\sigma_{x_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2}_{\ (10)} + \frac{\sigma_x - \sigma_y}{2} \cancel{\cos 2\vartheta}^{\,0} + \tau_{xy}{}_{(-4)} \underbrace{\sin 2\vartheta}_{1}$$

$$\sigma_{y_1}(\vartheta) = \frac{\sigma_x + \sigma_y}{2}_{\ (10)} - \frac{\sigma_x - \sigma_y}{2} \cancel{\cos 2\vartheta} - \tau_{xy}{}_{(+4)} \underbrace{\sin 2\vartheta}_{1}$$

$$\tau_{x_1 y_1}(\vartheta) = \underset{(-3)}{-\frac{\sigma_x - \sigma_y}{2}} \sin 2\vartheta + \tau_{xy} \cancel{\cos 2\vartheta}$$

$$2\vartheta_p = -\tan^{-1}\left(\frac{4}{3}\right) = -0.93\ rad = 53° \implies \vartheta_p = -26.5°$$

() max τ

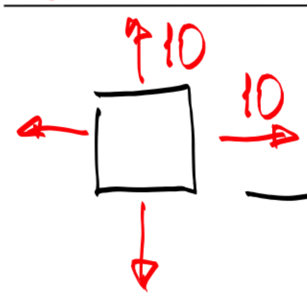


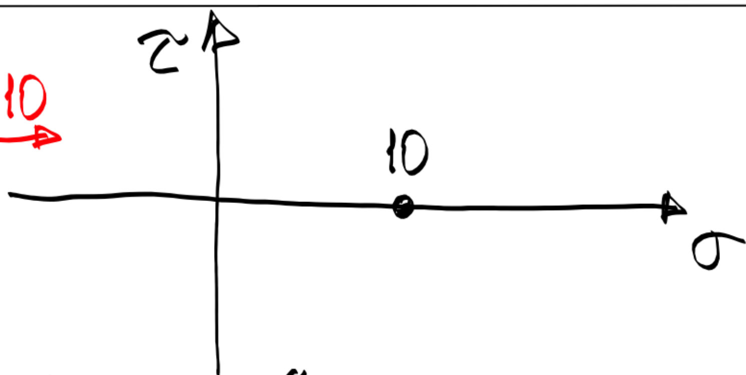
given $\{\sigma_x, \tau_{xy}, \sigma_y\}$

- draw stressed elements in x, y and x_1, y_1 @ $\theta = 45°$
- obtain σ_1, σ_2, τ_{MAX}, θ_P

- draw stressed elements in (1,2) and on (x'', y'') @ 45° from (1,2)

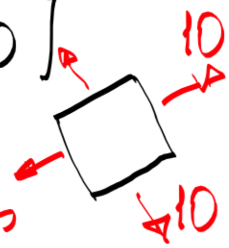
$\sigma_x = 10\ MPa$ $\sigma_y = \tau_{xy} = 0$	$\sigma_x = \sigma_y = 0$ $\tau_{xy} = 5\ MPa$	$\sigma_x = 13\ MPa$ $\sigma_y = 7\ MPa$ $\tau_{xy} = -4\ MPa$	$\sigma_x = \sigma_y = 10$ $\tau_{xy} = 0$
UNIAXIAL LOAD in X	SIMPLE SHEAR		equibiaxial





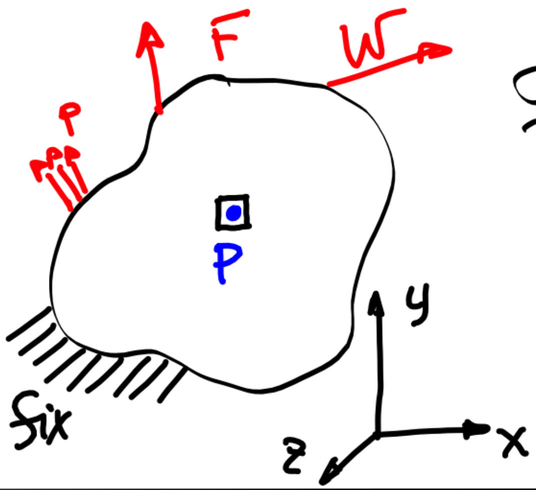
$$\sigma = \begin{Bmatrix} 10 & 0 \\ 0 & 10 \end{Bmatrix}$$

in any coordinates



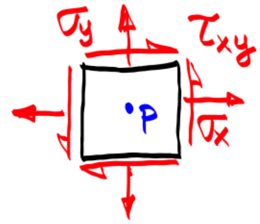
Circle "shrinks" To a point

2D Stress state: Final Considerations



$$\underset{\sim}{\sigma}(P)\;\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

zero in "PLANE STRESS"

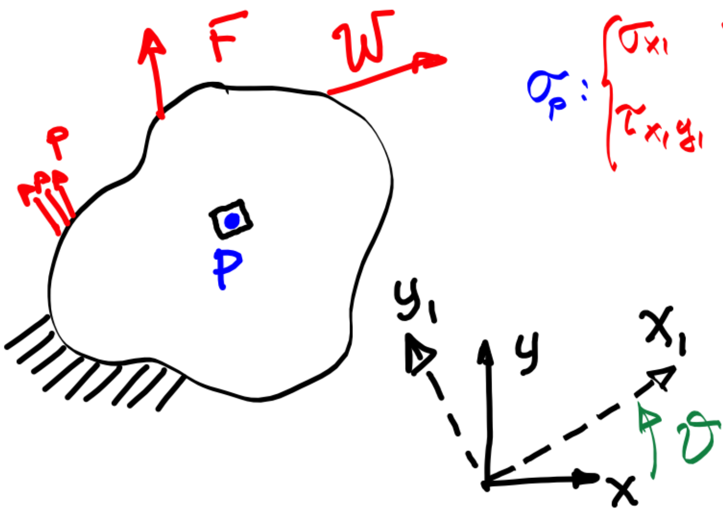


$$\frac{\sigma_x+\sigma_y}{2}=\bar{\sigma}$$

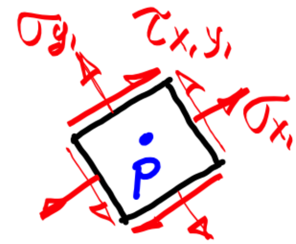
$$\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}=R$$

INVARIANTS OF $\underset{\sim}{\sigma}$

$$\sigma_P:\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$



$$\sigma_P:\begin{bmatrix} \sigma_{x_1} & \tau_{x_1y_1} \\ \tau_{x_1y_1} & \sigma_{y_1} \end{bmatrix}$$

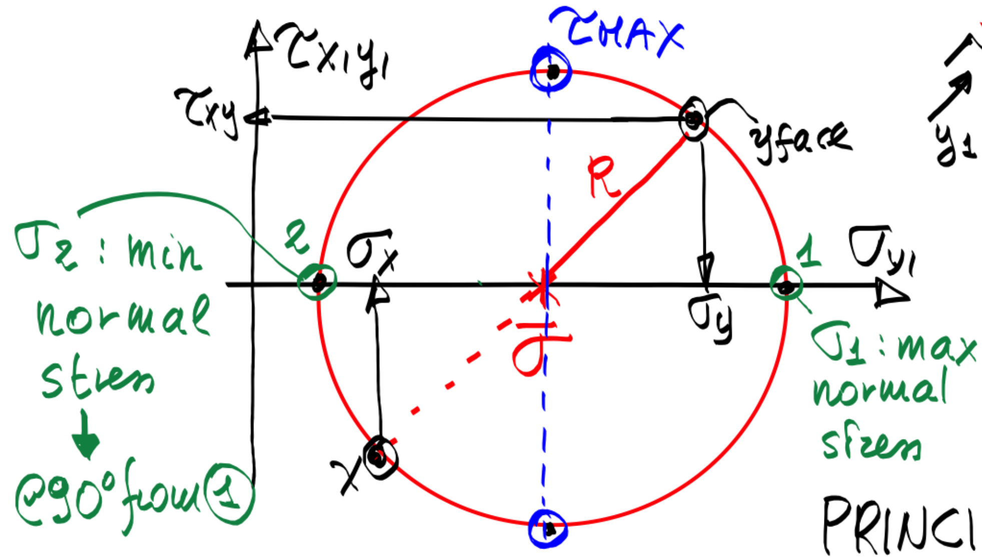


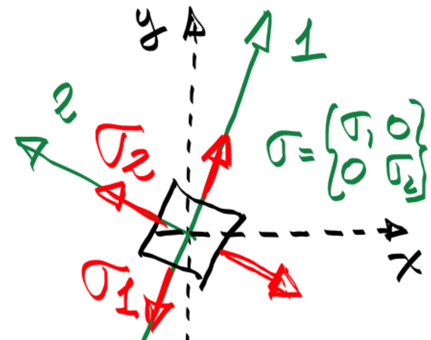
$$\frac{\sigma_{x_1}+\sigma_{y_1}}{2}=\bar{\sigma}$$

$$\sqrt{\left(\frac{\sigma_{x_1}-\sigma_{y_1}}{2}\right)^2+\tau_{x_1y_1}^2}=R$$

INVARIANTS have SAME values for any axes.

Mohr's circle: each point on circle is the stress on a face (frame orientation)





PRINCIPAL stresses & FRAME

• Principal stresses $\left.\begin{aligned}\sigma_1 &= \bar{\sigma} + R\\ \sigma_2 &= \bar{\sigma} - R\end{aligned}\right)$ acting along principal axes 1,2 (@ 90°)

INVARIANTS!

• Shear stress is zero in principal frame

• <u>max</u> shear stress (τ_{MAX}) @ 45° from 1,2

angles on circle = 2× angle between faces.

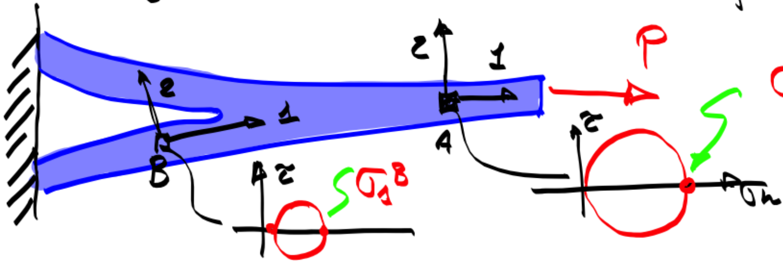
Now we have an OBJECTIVE measure of the "severity" of the stress state

→ to avoid brittle failure : limit σ_1 $\quad \underbrace{\sigma_1}_{\text{severity of stress}} \le \underbrace{\sigma_f}_{\text{material properties}} / SF$

→ to avoid ductile failure (yield) : limit τ_{MAX} $\quad \tau_{MAX} \le \tau_y / SF$

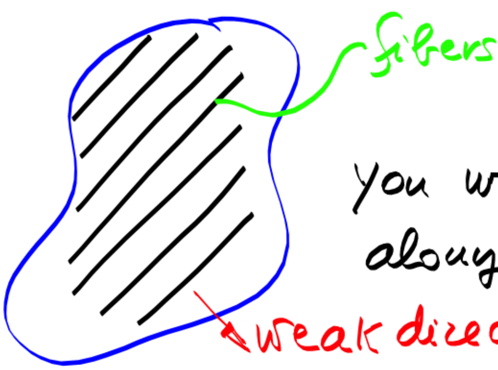
Notes

• in a loaded structure @ different points we will generally have different magnitude of stresses & different orientations of principal axes.



$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(P)$

• the criteria above are very simplified and only appropriate for <u>isotropic</u> materials.



For materials with oriented structure (like wood or fiber-reinforced composite) you will have different failure criteria along the different material directions.

weak direction: danger if $\left(\begin{smallmatrix}\text{1-axis}\\ \text{of stress}\end{smallmatrix}\right)$ along weak direction!

Mohr's circle for <u>STRAIN</u> → Same derivations apply

$\underset{\sim}{\sigma}$ & $\underset{\sim}{\varepsilon}$ are both symm. 2nd order Tensors.

2D:
PLANE strain

zero

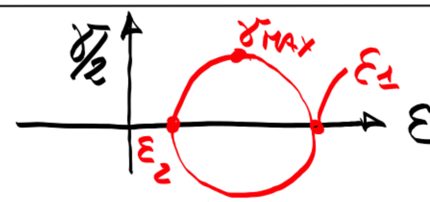
$$\underset{\sim}{\varepsilon}(P) = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

symmetric
2nd order
(3D) Tensor

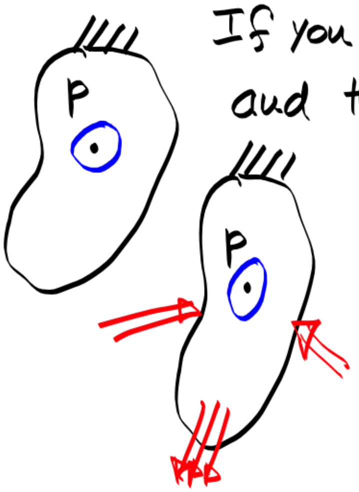
(2D)
$$\varepsilon : \begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 \\ \gamma_{xy}/2 & \varepsilon_y \end{bmatrix}$$

in general in x-y frame there will be axial & shear

<u>But</u> (just as for stress):



- @ every point, P, as the axes rotate the cartesian normal & shear will be on a circle
- there will be a special axes orientation $(1_\varepsilon, 2_\varepsilon)$ along which there is NO shear, are the max/min axial strains ε_1 & ε_2

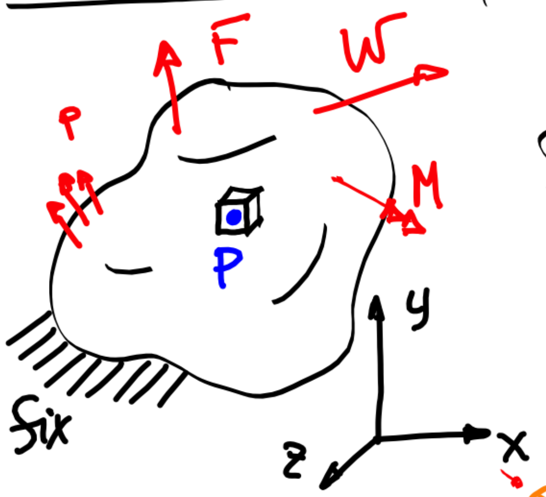


If you draw a circle @ P on the UNLOADED BODY and then load the body, so it deforms, the circle will become an ellipse. the axes of the ellipse are the 1 & 2 principal axes of strain @ P

- the principal axes of stress are always the <u>same</u> as the principal axes of strain if the material is <u>isotropic</u>

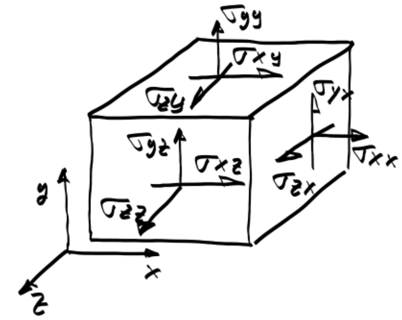
(if material is anisotropic, it depends on the relative orientation of principal axes & material symmetry axes)

3D Stress state



$$\underset{\sim}{\sigma}(P): \begin{Bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{Bmatrix}$$

NORMAL SHEAR SYM



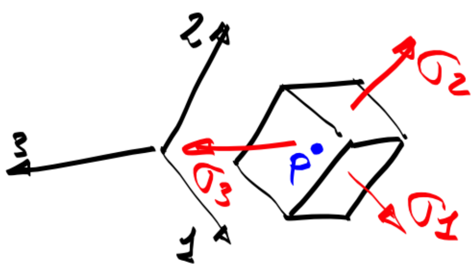
1st INVARIANT: $\sigma_{xx} + \sigma_{yy} + \sigma_{zz}$ (trace σ)

note: $\sigma_h = \dfrac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$ = hydrostatic stress

$p = -\sigma_h$

Also in 3D @ every point P

- there are 3 orthogonal principal directions $\{1,2,3\}$: Principal Frame (right handed)
- on faces $\perp$ to $\{1,2,3\}$ there are ONLY <u>normal</u> stresses : Principal Stresses $\begin{Bmatrix} \sigma_1 & \text{MAX} \\ \sigma_2 & \text{MED} \\ \sigma_3 & \text{MIN} \end{Bmatrix}$



in principal frame $\quad \underset{\sim}{\sigma_P} : \begin{Bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{Bmatrix}$

(note: $\sigma_h = (\sigma_1 + \sigma_2 + \sigma_3)/3$)

<u>Note</u> axes 1, 2, 3 are the EIGENVECTORS of $\underset{\sim}{\sigma_P}$

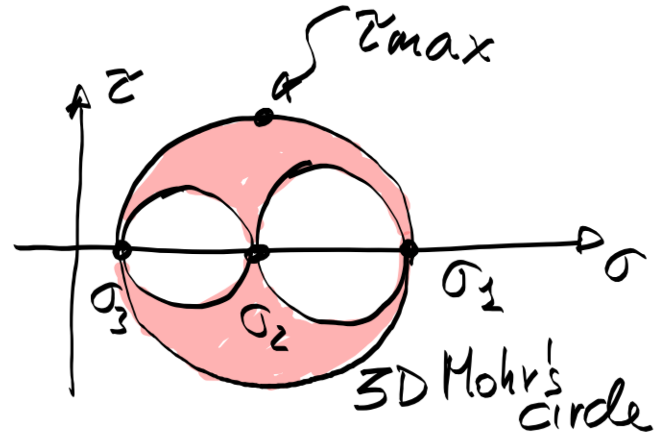
$\sigma_1 \; \sigma_2 \; \sigma_3$ are the EIGENVALUES of $\underset{\sim}{\sigma_P}$

(GIVEN a matrix of Cartesian components of $\underset{\sim}{\sigma_P}$ in <u>any</u> reference frame (x,y,z) MATLAB can calculate eigenvalues $\sigma_1, \sigma_2, \sigma_3$ & eigenvectors 1, 2, 3)

$\sigma_1 \; \sigma_2 \; \sigma_3$ Are <u>INVARIANTS</u> (get same value in any x,y,z)

3D Mohr's circle for stress

Can draw Mohr's circle on each principal plane



<u>all</u> stresses on <u>all</u> faces are in shaded area!

Note: $\tau_{MAX} = \sigma_1 - \sigma_3$ (also invariant)

Failure criteria in 3D –

for brittle <u>materials</u> still worry about σ_1

$\sigma_1 \leq \sigma_f / SF$ (also some effects of σ_2)

for ductile materials worry about shear yield

• Tresca criterion: $\tau_{MAX} \leq \underbrace{\tau_y}_{\text{material shear strength}} / SF$

---

for many materials
get better predictions for yield conditions using:

• von Mises criterion: $\bar{\sigma} \leq \underbrace{\sigma_y}_{\text{yield strength of the material (from tests in uniaxial loading)}} / SF$

$\bar{\sigma}$: von Mises stress

$$\bar{\sigma} = \sqrt{\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_1-\sigma_3)^2}{2}} = \sqrt{\frac{(\sigma_x-\sigma_y)^2+(\sigma_x-\sigma_z)^2+(\sigma_y-\sigma_z)^2}{2}+3(\sigma_{xy}^2+\sigma_{xz}^2+\sigma_{yz}^2)}$$

(invariant) —— can obtain from xyz components as