Hooke's law: anisotropic materials

- new notation

$\sigma_{xx} \to \sigma_{11} \to \sigma_1$ $\quad$ $\sigma_{yz} \to \sigma_{23} \to \sigma_4$

$\sigma_{yy} \to \sigma_{22} \to \sigma_2$ $\quad$ $\sigma_{xz} \to \sigma_{13} \to \sigma_5$

$\sigma_{zz} \to \sigma_{33} \to \sigma_3$ $\quad$ $\sigma_{xy} \to \sigma_{12} \to \sigma_6$

$\epsilon_{xx} \to \epsilon_{11} \to \epsilon_1$ $\quad$ $\gamma_{yz} \to \epsilon_4$

$\epsilon_{yy} \to \epsilon_{22} \to \epsilon_2$ $\quad$ $\gamma_{xz} \to \epsilon_5$

$\epsilon_{zz} \to \epsilon_{33} \to \epsilon_3$ $\quad$ $\gamma_{xy} \to \epsilon_6$

- matrix notation: most anisotropic material possible

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \underbrace{\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ \text{symmetric} & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix}}_{\text{compliance matrix, } S} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

21 independent components

- can also write:

$$[\sigma] = [C][\epsilon]$$

↓

stiffness matrix, C

meaning of components

(a) if apply σ_1 only

$$\epsilon_1 = S_{11}\sigma_1 \Rightarrow S_{11} = 1/E_1$$

if apply σ_2 only

$$\epsilon_2 = S_{22}\sigma_2 \Rightarrow S_{22} = 1/E_2$$

if apply σ_3 only

$$\epsilon_3 = S_{33}\sigma_3 \Rightarrow S_{33} = 1/E_3$$

(b) if apply σ_4 only (shear in 2-3 plane)

$$\epsilon_4 = S_{44}\sigma_4 \Rightarrow S_{44} = 1/G_{23}$$

similarly, $S_{55} = 1/G_{13}$

$S_{66} = 1/G_{12}$

(c) if apply σ_1 only

$$\epsilon_2 = S_{21}\sigma_1$$

$$\nu_{ij} = -\frac{\epsilon_j}{\epsilon_i} \begin{array}{l}\leftarrow \text{transverse} \\ \leftarrow \text{direction of applied stress}\end{array}$$

$$\nu_{12} = -\frac{\epsilon_2}{\epsilon_1} = \frac{-S_{21}\sigma_1}{S_{11}\sigma_1} = -\frac{S_{21}}{S_{11}}$$

$$S_{21} = -\frac{\nu_{12}}{E_1}$$

similarly, $S_{31} = -\dfrac{\nu_{13}}{E_1}$

$S_{32} = -\dfrac{\nu_{23}}{E_2}$

reciprocal relation:

$$S_{21} = S_{12} \Rightarrow \frac{-\nu_{12}}{E_1} = \frac{-\nu_{21}}{E_2}$$

d) consider elements S_{14} S_{15} S_{16}

S_{14} relates ϵ_1 to σ_4

normal strain — shear stress

typically, S_{14} S_{15} S_{16}
S_{24} S_{25} S_{26} all zero
S_{34} S_{35} S_{36}

(some woven fabrics have non-zero values of these components)

e) consider elements S_{45}, S_{46}, S_{56}

S_{45} relates ϵ_4 to σ_5

shear strain in 2-3 plane — shear stress in 1-3 plane

S_{45}, S_{46}, S_{56} typically zero

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summary

$$
\begin{bmatrix}
1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & & & \\
-\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & & A & \\
-\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & & & \\
 & & & 1/G_{23} & & B \\
 & \text{symmetric} & & & 1/G_{13} & \\
 & & & & & 1/G_{12}
\end{bmatrix}
$$

A relates normal strain to shear stress

B relates shear strain in one plane to shear stress in a different plane last time:

$-\nu/E$ typically zero

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix}$$

$1/E$ typically zero

$1/G$

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anisotropic materials: symmetry

Orthotropic materials

- rotate 180° about each of 3 mutually perpendicular axes & structure remains unchanged

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & \text{sym} & & S_{44} & 0 & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix}$$

- 9 independent elastic constants

eg. wood, some fiber composites

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transversely isotropic materials

- isotropic in one plane
- eg. matrix: x_1-x_2 plane isotropic
- 5 independent elastic constants
- e.g. extruded materials

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{11} & S_{13} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & \text{sym} & & S_{44} & 0 & 0 \\ & & & & S_{44} & 0 \\ & & & & & 2(S_{11}-S_{12}) \end{bmatrix}$$

Note: $2(S_{11}-S_{12}) = 2\left(\frac{1}{E_1} + \frac{\nu_{12}}{E_1}\right)$

$$= \frac{2(1+\nu_{12})}{E_1} = \frac{1}{G_{12}}$$

cubic materials

- $E_1 = E_2 = E_3 = 1/S_{11}$
- $G_{23} = G_{13} = G_{12} = 1/S_{44}$
- all ν equal $= -S_{12}/S_{11}$
- 3 independent elastic constants

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{11} & S_{12} & 0 & 0 & 0 \\ & & S_{11} & 0 & 0 & 0 \\ & \text{symmetric} & & S_{44} & 0 & 0 \\ & & & & S_{44} & 0 \\ & & & & & S_{44} \end{bmatrix}$$

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isotropic materials

- $E_1 = E_2 = E_3 = 1/S_{11}$
- all $G = \dfrac{E}{2(1+\nu)} = \dfrac{1}{2(S_{11} - S_{12})}$
- all ν equal.

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{11} & S_{12} & 0 & 0 & 0 \\ & & S_{11} & 0 & 0 & 0 \\ & & & 2(S_{11}-S_{12}) & 0 & 0 \\ & & & & 2(S_{11}-S_{12}) & 0 \\ & & & & & 2(S_{11}-S_{12}) \end{bmatrix}$$

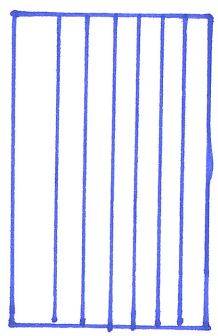
SCALING + STRESS:

WOODPECKER TALK

Composite materials

- combination of 2 or more materials

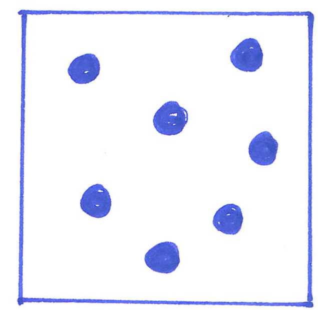
e.g. fibre reinforced materials



e.g. glass fibers / poly ester resin

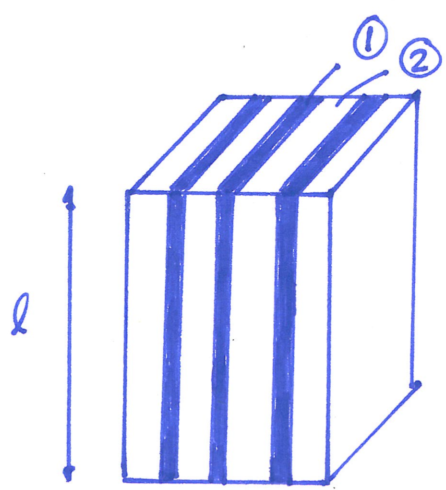
metal matrix composites

particulate composites



eg. rubber toughened resins.

Young's modulus



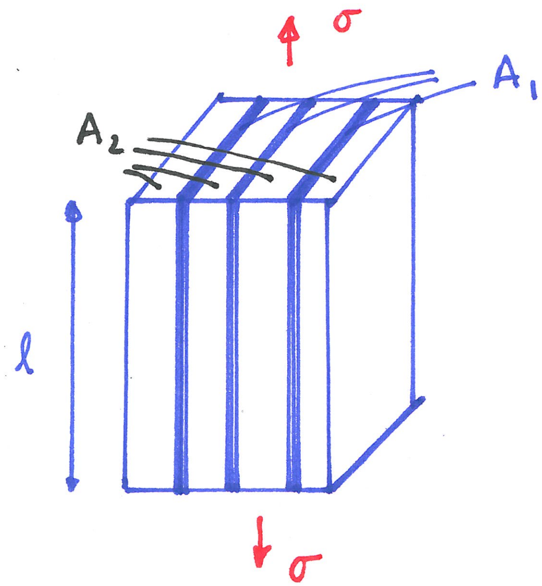
phases 1 & 2 : E_1 E_2

composite : no subscript (σ, ϵ, E)

$\updownarrow$ load $\parallel$ layers : deformation same in all layers

$\leftrightarrow$ load $\perp$ layers : load same in all layers.

Young's modulus: parallel to layers. ($E_{\parallel}$)



$$\delta = \delta_1 = \delta_2$$

$$l_1 = l_2 = l$$

$$\epsilon_1 = \epsilon_2 = \epsilon$$

$$P = P_1 + P_2$$

$$= E_1 \epsilon A_1 + E_2 \epsilon A_2$$

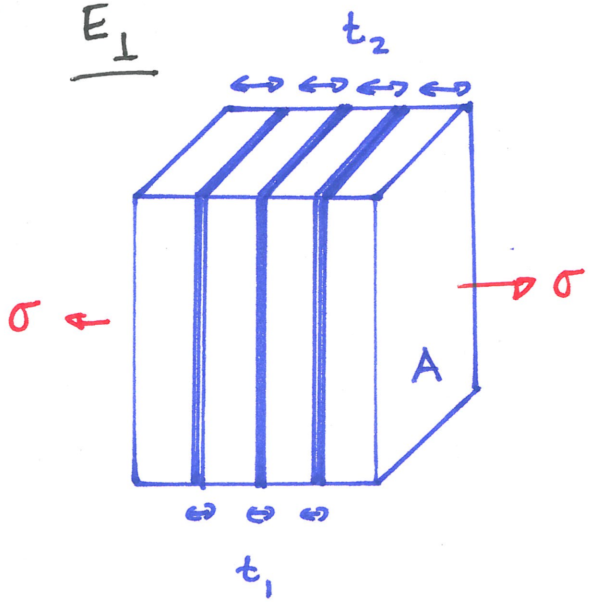
$$\sigma = \left(\frac{E_1 A_1 + E_2 A_2}{A} \right) \epsilon$$

$$\boxed{E_{\parallel} = E_1 V_1 + E_2 V_2}$$

rule of mixtures upper bound

V_1, V_2 = volume fraction of 1, 2

$E_{\perp}$



$$P_1 = P_2 = P$$

$$A_1 = A_2 = A$$

$$\sigma_1 = \sigma_2 = \sigma$$

$$\delta = \delta_1 + \delta_2$$

$$= \frac{P t_1}{A E_1} + \frac{P t_2}{A E_2}$$

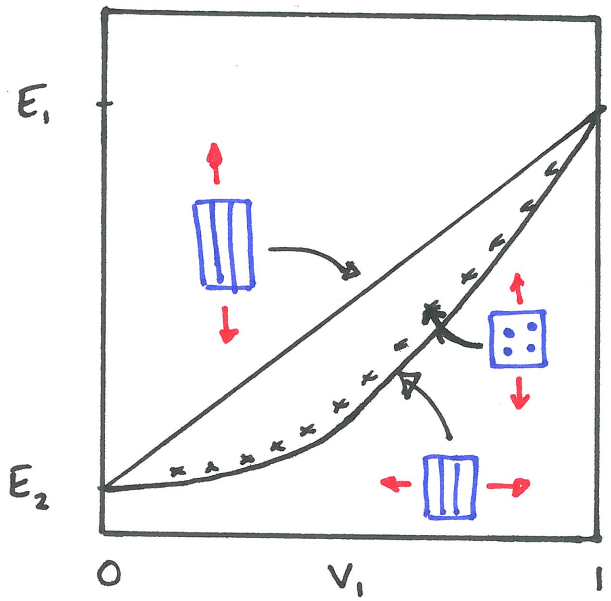
$$= \sigma \left(\frac{t_1}{E_1} + \frac{t_2}{E_2} \right)$$

$$\epsilon = \frac{\delta}{t} = \sigma \left(\frac{t_1}{t E_1} + \frac{t_2}{t E_2} \right) = \sigma \left(\frac{V_1}{E_1} + \frac{V_2}{E_2} \right)$$

$$\boxed{E_{\perp} = \frac{E_1 E_2}{E_1 V_2 + E_2 V_1}}$$

lower bound

Composites



unidirectional fiber composites

- loaded parallel to fibers - close to upper bound
- loaded $\perp$ to fibers - close to lower bound

particulate composites - close to lower bound