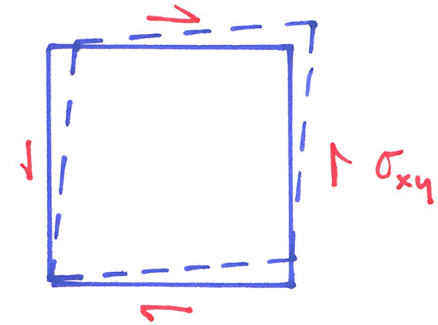
- shear stress + shear strain related in a similar way, by shear modulus, $G$

$$\sigma_{xy} = G\,\gamma_{xy} \qquad G\ [Pa]$$




- also define Poisson's ratio, $\nu = -\dfrac{\text{lateral strain}}{\text{strain in direction of applied uniaxial stress}}$

$\boxed{\text{rubber band demo}}$

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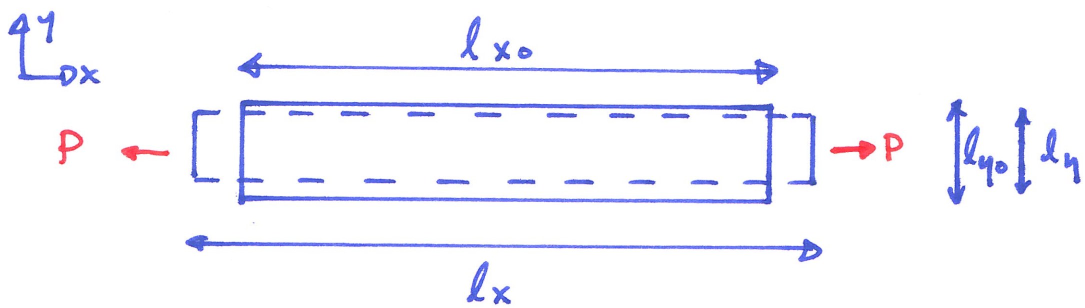



- as pull, material contracts sideways
- $\epsilon_y$ opposite sign to $\epsilon_x$

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\dfrac{l_y - l_{y0}}{l_{y0}}}{\dfrac{l_x - l_{x0}}{l_{x0}}} = -\frac{\delta_y / l_{y0}}{\delta_x / l_{x0}}$$

$\nu$ - DIMENSIONLESS [-]

- also get contraction in z direction
- if material isotropic, $\epsilon_z = \epsilon_y$

- typical values of $\nu$

| | |
|---|---|
| steel | 0.28 |
| aluminum | 0.33 |
| glass | 0.25 |
| rubber | 0.45-0.49 |

- isotropic solids: often $\nu \approx 0.3$
(except rubbers)

Demo

Cork slides

- cellular materials can have unusual values of $\nu$ (not isotropic!)
- regular hexagonal honeycomb: $\nu = 1$
- "inverted" honeycomb: $\nu < 1$
- cork: $\nu = 0$

---

- data for elastic moduli: Ashby + Jones p 39-41

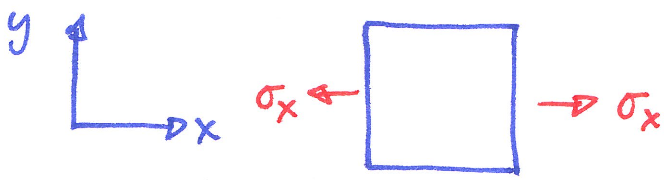
materials databases:

www.matweb.com

www.nist.gov/srd/index.cfm

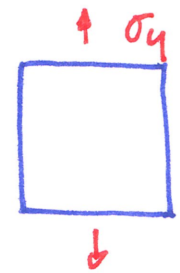
# Hooke's law: General state of stress, isotropy

- first, consider $\epsilon_x$ (one subscript ⇒ normal stress, strain)
- if apply $\sigma_x$ only




$$\epsilon_x = \frac{\sigma_x}{E}$$

- if apply $\sigma_y$ only




$$\nu = -\frac{\epsilon_x}{\epsilon_y} = -\epsilon_x \frac{E}{\sigma_y}$$

$$\therefore \ \epsilon_x = -\nu \frac{\sigma_y}{E}$$

---

- if apply $\sigma_x + \sigma_y$ simultaneously:

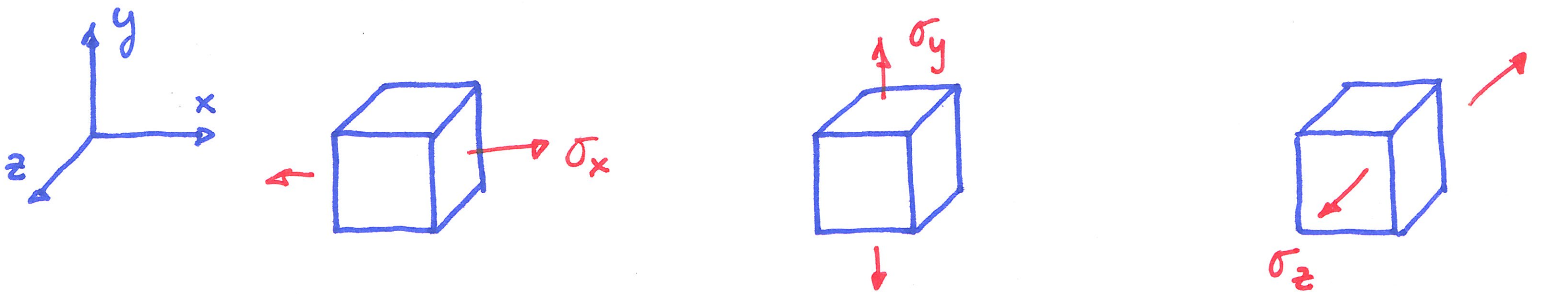
$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

- if apply $\sigma_x + \sigma_y + \sigma_z$ simultaneously:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E}$$

Poisson's ratio, isotropic material: remember:

$$\nu = -\frac{\text{lateral strain}}{\text{strain in direction of applied uniaxial stress}}$$




$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x} \qquad \nu = -\frac{\epsilon_x}{\epsilon_y} = -\frac{\epsilon_z}{\epsilon_y} \qquad \nu = -\frac{\epsilon_x}{\epsilon_z} = -\frac{\epsilon_y}{\epsilon_z}$$

- for a general state of stress (3 normal, 3 shear), can relate stress to strain by Hooke's law for an isotropic material

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \qquad \gamma_{yz} = \frac{\sigma_{yz}}{G}$$

$$\epsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \qquad \gamma_{xz} = \frac{\sigma_{xz}}{G}$$

$$\epsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \qquad \gamma_{xy} = \frac{\sigma_{xy}}{G}$$

---

- or, as matrix:

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}$$

- this set of equations completely describes the linear elastic behaviour of an isotropic material
- Young's modulus, shear modulus, Poisson's ratio are all <u>elastic moduli</u>
- for an isotropic solid, only 2 are independent

$$G = \frac{E}{2(1+\nu)} \qquad \Rightarrow \nu > -1$$

since $\nu \approx 1/3$ typically $\Rightarrow G \approx 3/8\, E$

---

- also define bulk modulus, $K$
- apply $\sigma_x = \sigma_y = \sigma_z = \sigma_h$ (hydrostatic stress, all shear stresses $=0$)

$$K = \frac{\sigma_h}{\Delta V / V_0}$$

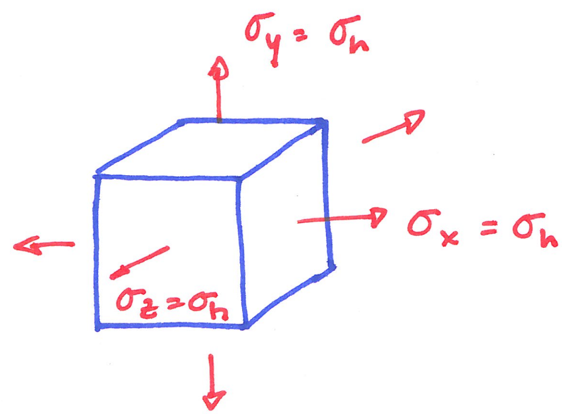
$\frac{\Delta V}{V_0}$ = VOLUMETRIC STRAIN, ALSO CALLED "DILATATION", $\Delta$.

$$K = \frac{E}{3(1-2\nu)} \qquad \Rightarrow \nu < 1/2$$

<u>bounds</u> on Poisson's ratio for isotropic solid: $-1 < \nu < 1/2$

show: $K = \dfrac{E}{3(1-2\nu)}$

$K = \dfrac{\sigma_h}{\Delta V / V_o}$




cube edge length, $l_o$

$\epsilon_x = \epsilon_y = \epsilon_z = \epsilon$

---

$\dfrac{\Delta V}{V_o} = ?$

$V_o = l_o^3$

$V = (l_o + \delta)^3 = (l_o + \epsilon l_o)^3 = l_o^3 (1+\epsilon)^3$

$\dfrac{\Delta V}{V_o} = \dfrac{l_o^3 (1+\epsilon)^3 - l_o^3}{l_o^3} = (1+\epsilon)^3 - 1 \approx 3\epsilon$

(neglect $\epsilon^2$, $\epsilon^3$ terms since $\epsilon$ small) use Hooke's law:

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\sigma_x = \sigma_y = \sigma_z = \sigma_h$$

$$\epsilon_x = \epsilon_y = \epsilon_z = \epsilon$$

$$\epsilon = \left(\frac{1-2\nu}{E}\right)\sigma_h$$

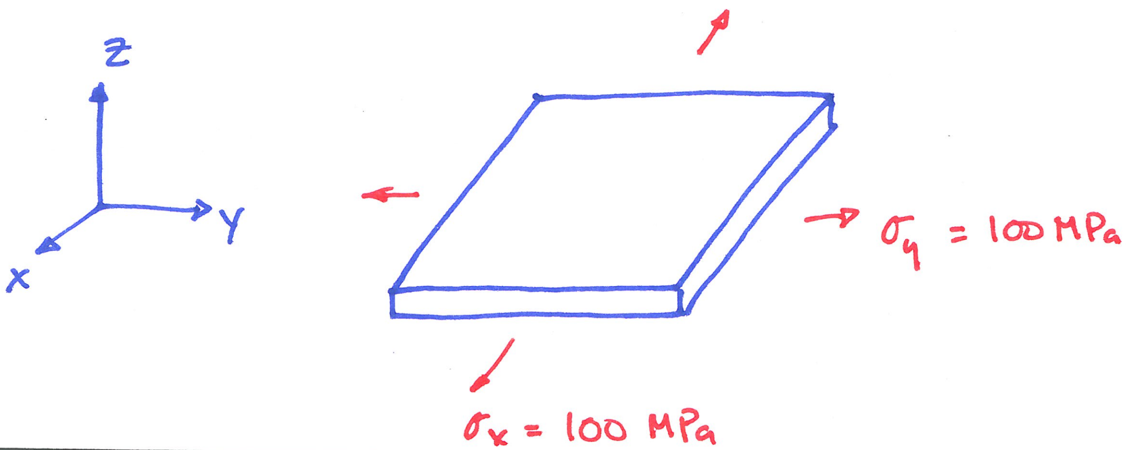
$$K = \frac{\sigma_h}{\Delta V/V_0} = \frac{\sigma_h}{3\epsilon}$$

$$= \frac{\sigma_h \; E}{3(1-2\nu)\sigma_h}$$

$$K = \frac{E}{3(1-2\nu)}$$

# Example: Hooke's law

- Al plate sees biaxial stress, $\sigma_x = \sigma_y = 100$ MPa
- What are $\epsilon_x$, $\epsilon_y$, $\epsilon_z$? Al: $E = 70$ GPa $\nu = 0.33$

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$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{(1-0.33)(100\text{ MPa})}{(70{,}000\text{ MPa})} = 9.52\times10^{-4}$$

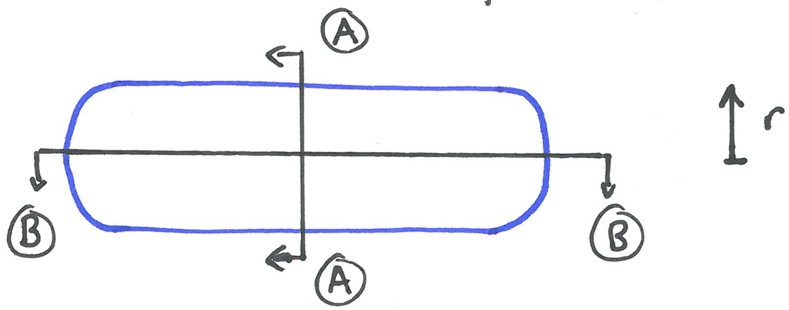
$$\epsilon_y = \epsilon_x \quad \text{(symmetry)}$$

$$\epsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{-(0.33)(2)(100\text{ MPa})}{70{,}000\text{ MPa}} = -9.52\times10^{-4}$$

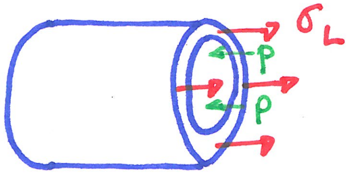
Note:
- strains small
- transverse strain can be of same order as strains in direction of applied load.

# Example: Pressure vessels (for lab 2)

Consider a thin-walled cylinder (radius $r \gg$ wall thickness, $t$) with an internal pressure, $p$. Stresses in wall = ?



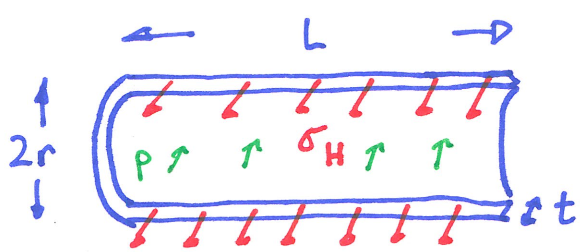

cut A-A, free body diagram




$$\sigma_L \, 2\pi r t = p \pi r^2$$

$$\sigma_L = \frac{pr}{2t}$$

cut B-B, free body diagram




$$\sigma_H \, 2Lt = p \, 2rL$$

$$\sigma_H = \frac{pr}{t}$$

and radial stress, $\sigma_R$:

at inner wall $\sigma_R = p$

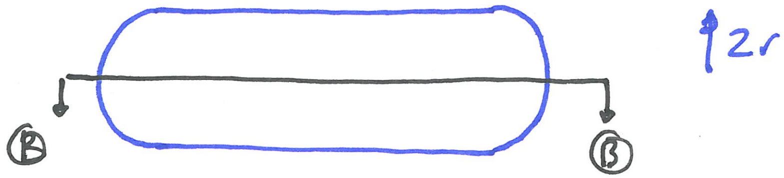
at outer wall $\sigma_R = 0$

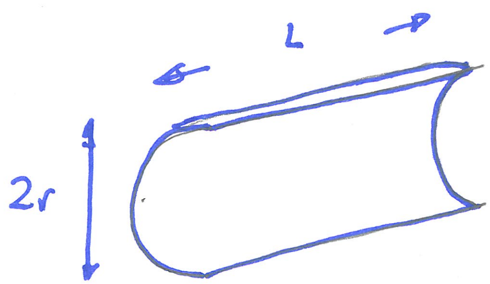
$\sigma_R \ll \sigma_H$ or $\sigma_L$

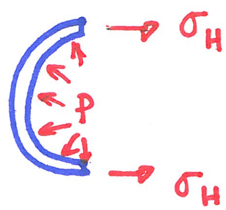
(since $r/t$ large)

$\Rightarrow$ neglect $\sigma_R$

Last time, cylindrical pressure vessel.




→ x direction.

- fluid cannot resist shear stress
  ⇒ flows under shear
- pressure acts normal to surface
- x component of force from the pressure acts on projected area with normal in x direction

---

Force balance: $\Sigma F_x = 0 \rightarrow$

$$\sigma_H \, L \, 2t = p \, L \, 2r$$

$$\sigma_H = \frac{pr}{t}.$$

## Lab 2: Pressure vessels

- measure strain on pressure vessel with strain gauge
- ratio of $\epsilon_H / \epsilon_L = ?$

$$\epsilon_H = \frac{\sigma_H}{E} - \nu \frac{\sigma_L}{E}$$

$$\epsilon_H = \frac{pr}{Et}\left(1 - \frac{\nu}{2}\right)$$

$$\epsilon_L = \frac{\sigma_L}{E} - \frac{\nu \sigma_H}{E}$$

$$= \frac{pr}{Et}\left(\frac{1}{2} - \nu\right)$$

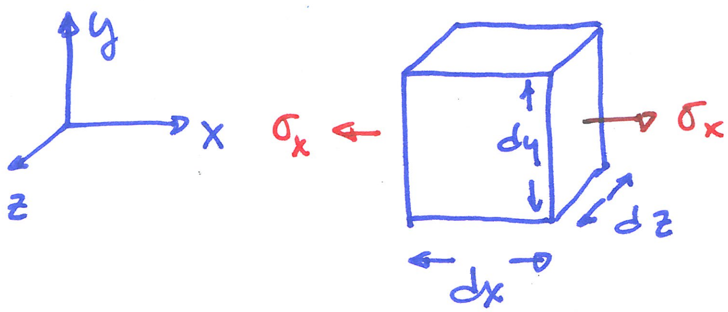
$$\frac{\epsilon_H}{\epsilon_L} = \frac{\left(1 - \nu/2\right)}{\left(\frac{1}{2} - \nu\right)}$$

e.g. $\nu = 0.3$

$\epsilon_H / \epsilon_L = 4.25$
(not 2!)

# Elastic strain energy, $\mathcal{U}$

- work done = force × distance force moves through, in direction of force
- consider normal stress, $\sigma_x$, acting on element $dx\,dy\,dz$




Force $= \sigma_x\,dy\,dz$

If $F=0$ initially,

average force $= \dfrac{\sigma_x\,dy\,dz}{2}$ (linear elastic)

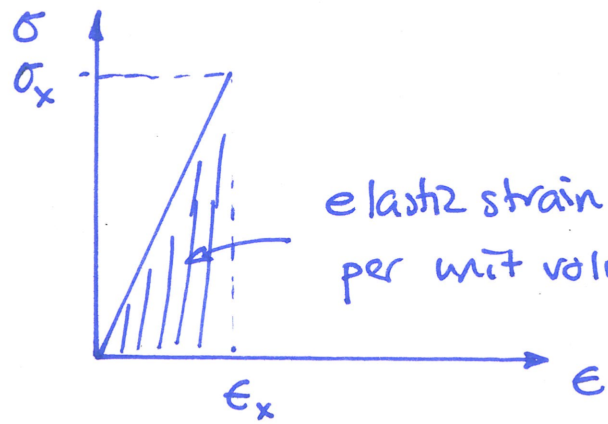
---

- deformation, $\delta = \epsilon_x\,dx$
- linear elastic, $\sigma_x \propto \epsilon_x$
- work done $= F_{avg}\,\delta = \dfrac{\sigma_x}{2}\epsilon_x\,dx\,dy\,dz$ = internal elastic strain energy, $d\mathcal{U}$.
- elastic strain energy per unit volume, $u = \dfrac{d\mathcal{U}}{dV} = \dfrac{\sigma_x\,\epsilon_x}{2}$

## Elastic strain energy, $u$

$$u = \frac{1}{2}\sigma_x \epsilon_x = \frac{\sigma_x^2}{2E} = \frac{1}{2} E \epsilon_x^2$$

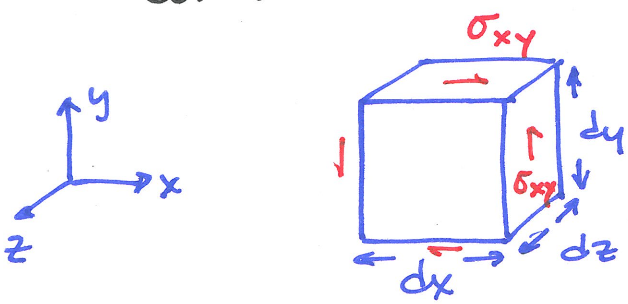
analogous expressions for $\sigma_y$, $\sigma_z$

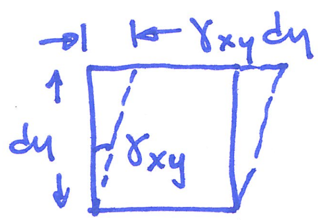



elastic strain energy per unit volume = area under $\sigma$-$\epsilon$ curve

---

- consider shear stress, $\sigma_{xy}$, applied to element



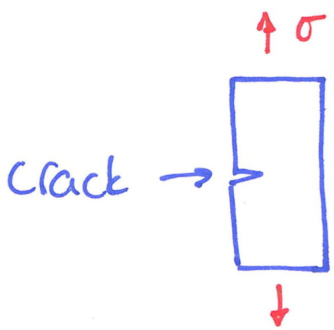

- average force on top plane $= \frac{\sigma_{xy}}{2} dx\, dz$
- distance force moves $= \gamma_{xy}\, dy$
- shear strain energy in element $= \frac{\sigma_{xy}\gamma_{xy}}{2} dx\, dy\, dz$
- shear strain energy per unit volume

$$u = \frac{\sigma_{xy}\gamma_{xy}}{2} = \frac{\sigma_{xy}^2}{2G} = \frac{\gamma_{xy}^2 G}{2}$$

## Strain energy

- in plasticity, dislocations create a stress field in the crystal lattice
- we will calculate the strain energy associated with the dislocation

- in fracture mechanics

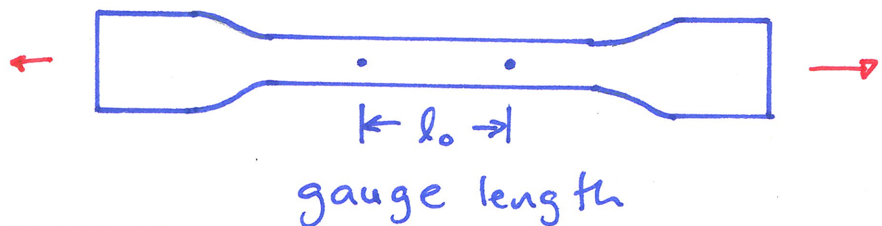



- for crack to grow, surface energy required to form new surfaces
- that energy is supplied by release of strain energy of material around the growing crack

## Stress-strain curves: test methods

- material behaviour characterized by stress-strain curves
- gives material properties independent of specimen dimensions (area, length)
- typical test: uniaxial tension on waisted specimen

e.g. "dog bone" or "waisted cylinder"

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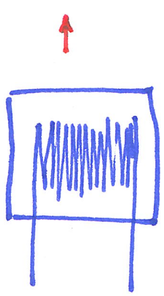
- deformation measured by extensometer, attached to specimen
- alternatively, can use strain gauge, glued onto specimen

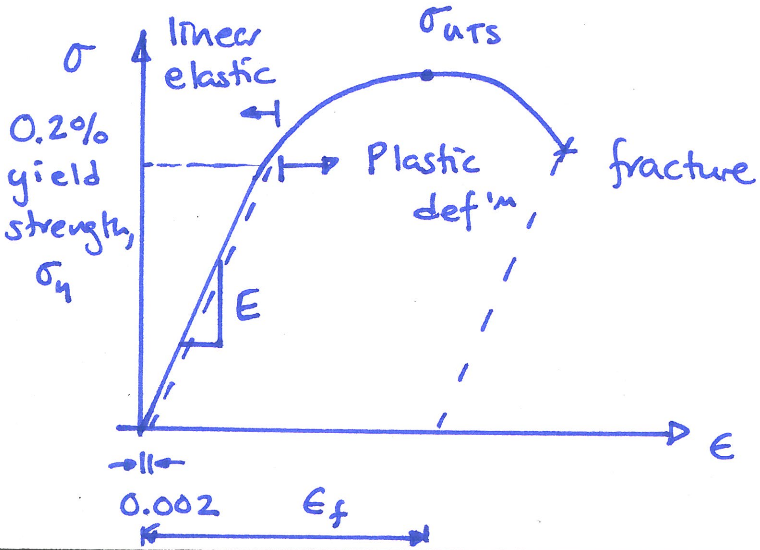


  - as pull, wire stretches, area changes
  - resistance changes → output voltage changes
  - strain gauge linear: $\Delta V \propto \delta$
  - also used in load cells

- or, crudely, can use crosshead displacement

# Typical stress-strain behaviour

## (a) Metals




Characteristic features:

- Young's modulus - initial linear elasticity
- 0.2% offset yield strength, $\sigma_y$
  - 0.2% permanent strain
  - at yield, deformation is irrecoverable "plastic"
  - ductile = material can deform to large plastic strains

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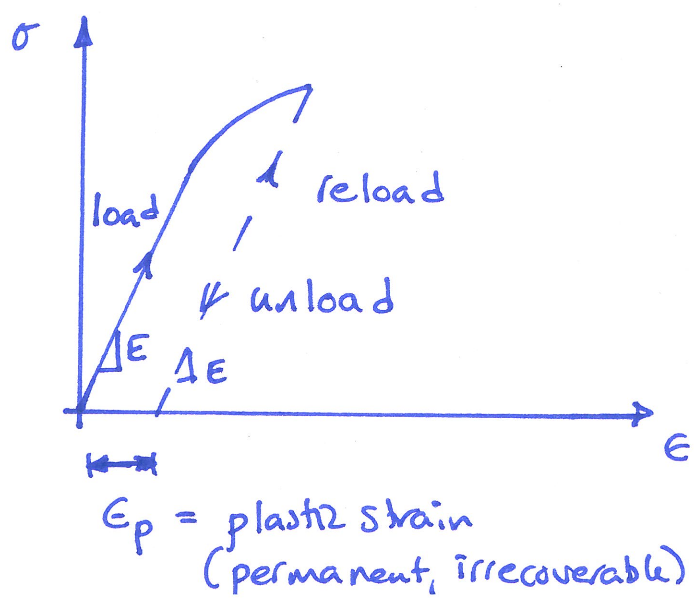
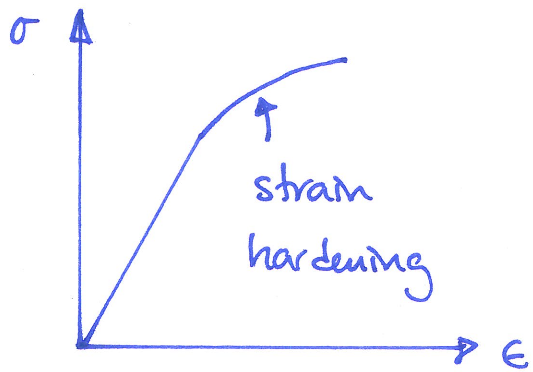
- plastic strain



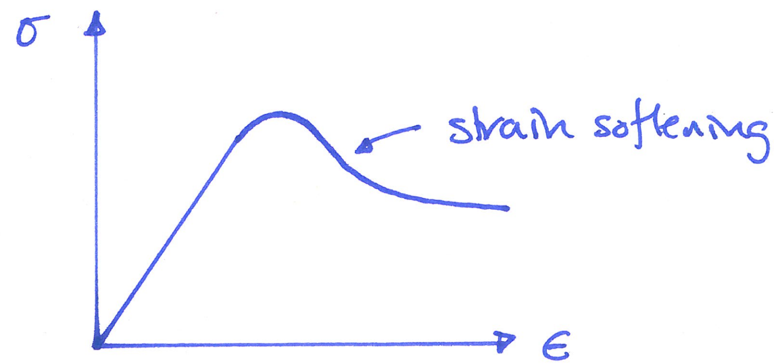

- on unloading, do not recover plastic strain
- recover elastic strain (with same slope as E)

Strain hardening




- as strain increases in plastic regime, stress increases

Strain softening




- as strain increases in plastic regime, stress decreases

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Ultimate tensile strength, $\sigma_{UTS}$

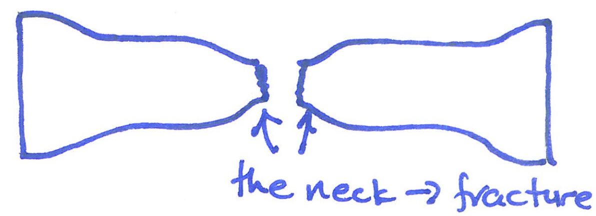
- maximum stress reached in a uniaxial test

Ductility, $\epsilon_f$

- plastic strain at fracture

## Necking

- as pull on some materials (eg. ductile metals), plastic deformation becomes localized
- one section weaker, or has smaller area
  - stress increases, strain increases, area further reduced
  - stress increases again ⇒ instability
  - eventually, fracture
- called "necking"

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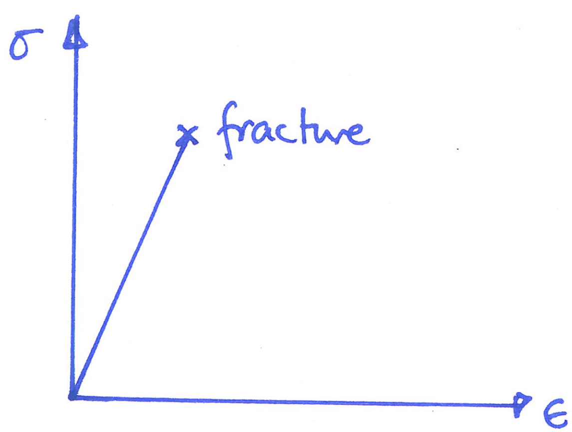
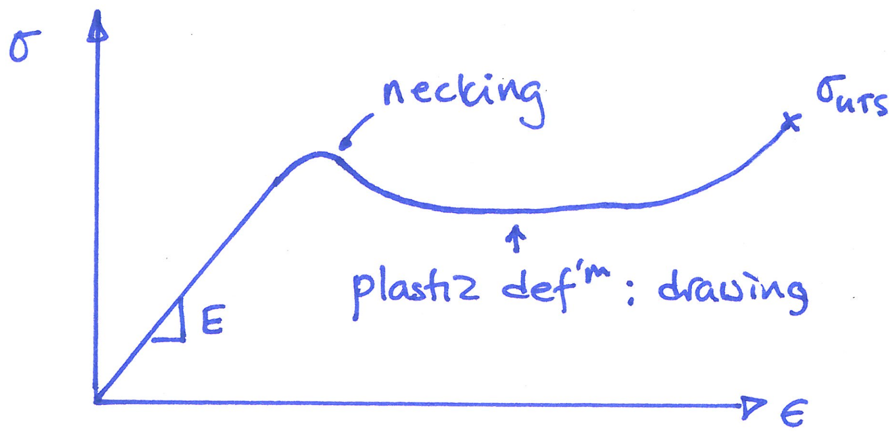
## Ceramics




- E generally higher than metals
- brittle - fail suddenly at small strain (e.g. 0.001)

## Polymers

- some polymers can be drawn




e.g. nylon, thermoplastics

- at local max. stress, neck forms
- but, as continue def'm, polymer chains become aligned, which strengthens the necked region
- if strengthening > area reduction, the neck propagates
- "drawing"

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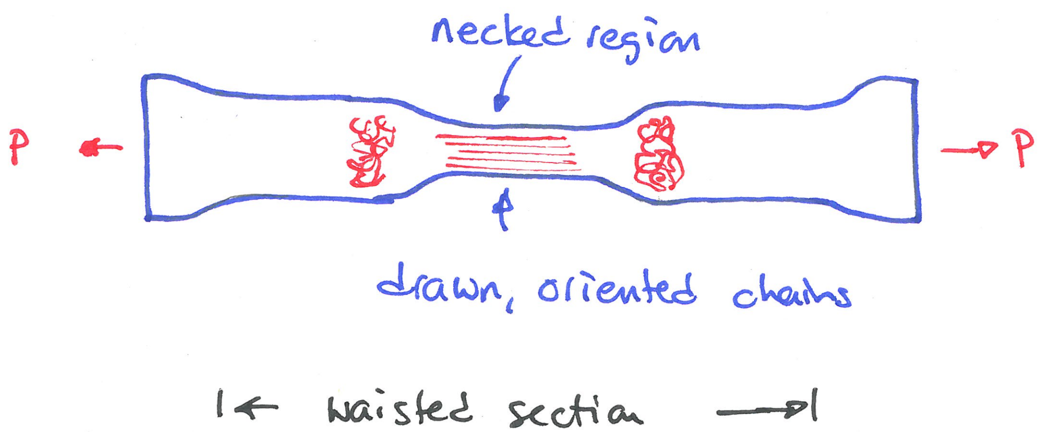
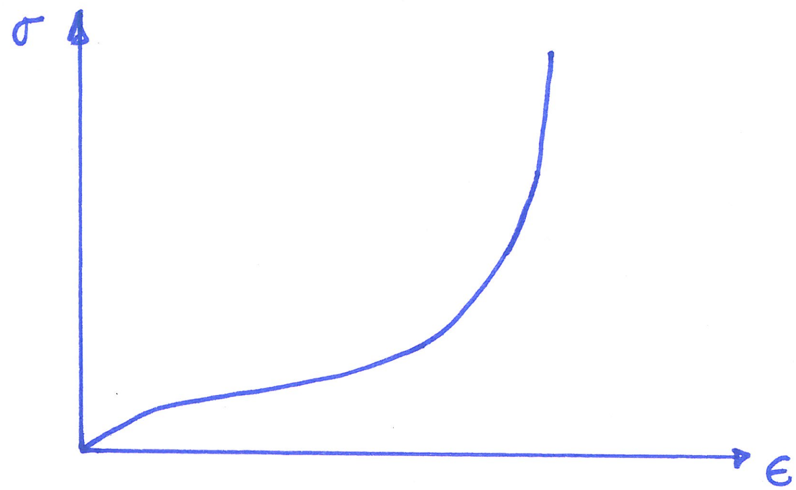
- drawing can extend over almost entire length of specimen
- ε of several hundred percent possible.



## Rubbers






rubbers - polymer chains, occasional secondary cross-links

- at low $\epsilon$, chains slide over each other $\Rightarrow$ low $E$
- as deform to large $\epsilon$, chains straighten
- as chains straighten, rubber stiffer; local slope increases

---

Above curves, for metals, ceramics, polymers, rubbers

all used: engineering stress, $\sigma_{eng} = \frac{P}{A_o}$

& engineering strain, $\epsilon_{eng} = \frac{\delta}{l_o}$

Also have true stress + true strain

- useful for materials at large deformations
- used in plasticity.

## True stress + strain

$$\sigma_{true} = \frac{P}{A} \leftarrow \text{instantaneous area (current)}$$

relate $\sigma_{true}$ to $\sigma_{eng}$ & $\epsilon_{eng}$

plastic deformation: volume = constant

$$A_o l_o = A l$$

initial values ($A_o l_o$) instantaneous values ($A l$)

$$\epsilon_{eng} = \frac{l - l_o}{l_o} = \frac{A_o}{A} - 1 \; ; \; \frac{A_o}{A} = 1 + \epsilon_{eng}$$

$$\frac{\sigma_{true}}{\sigma_{eng}} = \frac{P}{A}\frac{A_o}{P} = \epsilon_{eng} + 1$$

$$\therefore \sigma_{true} = \sigma_{eng}(1 + \epsilon_{eng})$$

for plasticity.

---

true strain

$$d\epsilon_{true} = \frac{dl}{l} \leftarrow \text{instantaneous (current) length}$$

$$\epsilon_{true} = \int_{l_o}^{l} \frac{dl}{l} = \ln\frac{l}{l_o} = \ln\frac{l_f}{l_o} \quad \text{(logarithmic strain)}$$

$$\frac{l}{l_o} = \exp(\epsilon_{true}) = \frac{A_o}{A} = 1 + \epsilon_{eng}$$

$$\epsilon_{true} = \ln(1 + \epsilon_{eng}) \quad \text{for plasticity.}$$