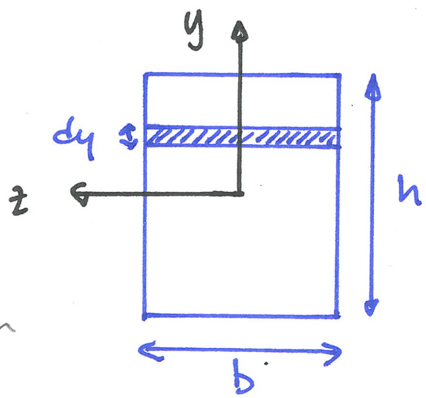
Moment of inertia

$$I_z = \int_A y^2 \, dA.$$



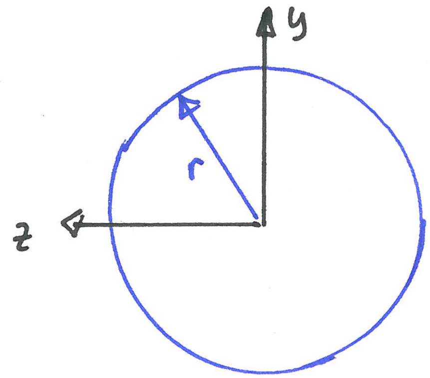
wood beam demo

$$I_z = \int_{-h/2}^{h/2} y^2 \, b \, dy$$

$$= \frac{by^3}{3}\Big|_{-h/2}^{h/2}$$

$$= \frac{b}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right)$$

$$I_z = \frac{bh^3}{12}$$



J = polar moment of inertia

$$= \int_A r^2 \, dA$$

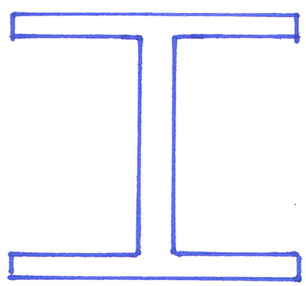
$$= \int_A (z^2 + y^2) \, dA$$

symmetry: $I_z = J/2$

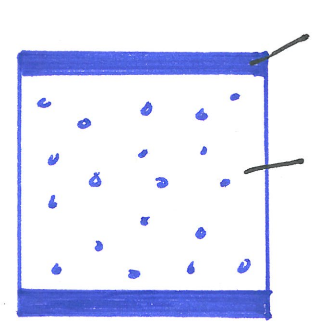
$$J = \int_0^r 2\pi r^3 dr = \frac{2\pi r^4}{4} = \frac{\pi r^4}{2}$$

$$I_z = \frac{\pi r^4}{4}$$

- efficient cross-sections for resisting bending



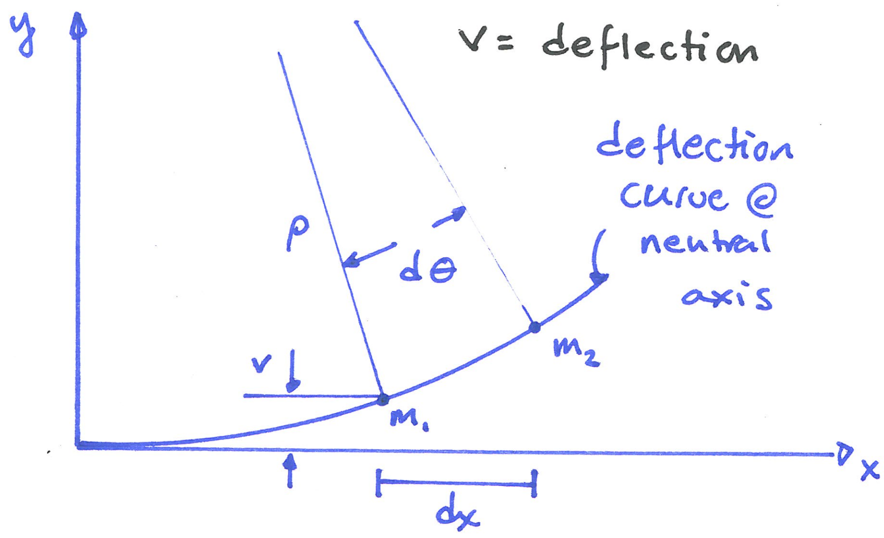
I-beam

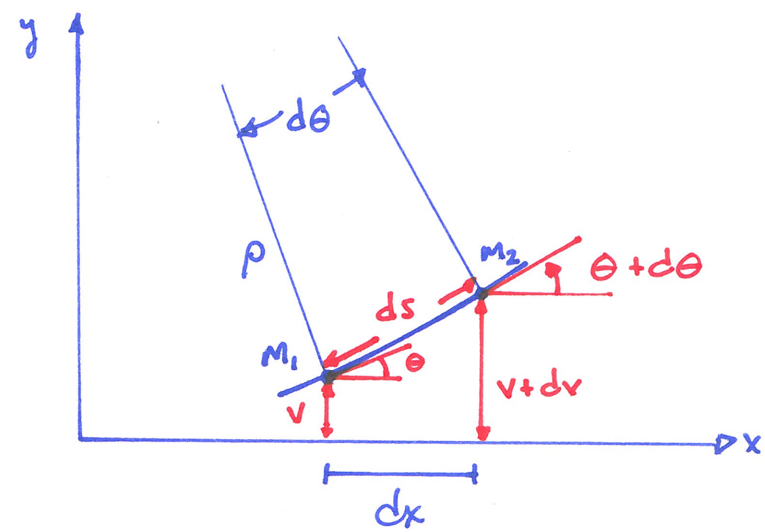


sandwich beam

slides -
natural sandwiches
palms - density gradient

Beam deflections: differential eq'n of deflection curve





- deflection v in y direction
- angle of rotation θ = angle between x axis & tangent to deflection curve
- small deflections, small angles
- deflection of m_2 = ?

deflection of m_z = ?

$\rho \, d\theta = ds \qquad \frac{1}{\rho} = \frac{d\theta}{ds} = \kappa$

approximations:

1. $\cos\theta = \frac{dx}{ds}$ small θ, $\cos\theta \approx 1$ $\quad ds \approx dx$

2. $\tan\theta = \frac{dv}{dx}$ small θ, $\tan\theta = \theta = \frac{dv}{dx}$

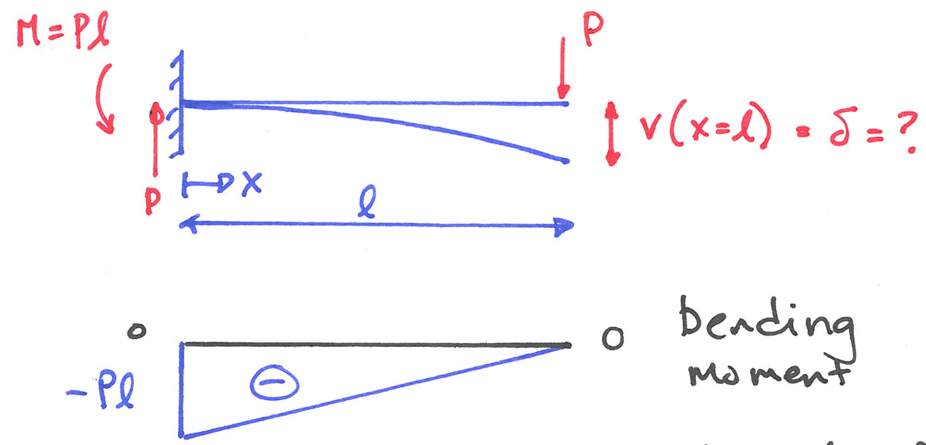
$\therefore \ \kappa = \frac{d\theta}{ds} = \frac{d\theta}{dx} = \frac{d^2v}{dx^2}$

previously had $\kappa = \frac{M}{EI}$

$$\boxed{\therefore \ \frac{d\theta}{dx} = \frac{d^2v}{dx^2} = \frac{M}{EI}}$$

- can integrate $M(x)$ twice to get deflection, v
- get constants of integration from boundary conditions
- EI: flexural rigidity, important for deflections

Beam deflection example



$$M = -P(l-x)$$

Find end deflection:

$$v(x=l) = \delta = ?$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = -\frac{P(l-x)}{EI}$$

$$\frac{dv}{dx} = -\frac{P}{EI}\left[lx - \frac{x^2}{2}\right] + C_1$$

$$v(x) = -\frac{P}{EI}\left[\frac{lx^2}{2} - \frac{x^3}{6}\right] + C_1 x + C_2$$

boundary conditions: $x=0 \quad v=0 \Rightarrow C_2 = 0$

$x=0 \quad \theta=0 \Rightarrow \frac{dv}{dx} = \theta = 0 \Rightarrow C_1 = 0$

$$\delta = v(x=\ell) = -\frac{P}{EI}\left[\frac{\ell x^2}{2} - \frac{x^3}{6}\right] = -\frac{P\ell^3}{3EI}$$

(down)

units: $\dfrac{[N][m^3]}{\left[\frac{N}{m^2}\right][m^4]} = [m]$

Note: for concentrated loads,

$$\delta \propto \frac{P\ell^3}{EI}$$

loading configuration determines the constant

(eg. cantilever, constant = 3)

Column buckling

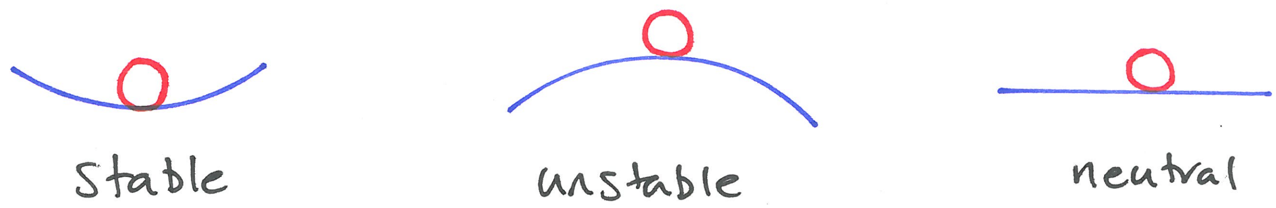
column - long slender member loaded in compression

buckling - instability phenomenon

stability - system stable if it returns to initial state after small perterbation

- unstable if small perterbation leads to larger perterbation

e.g.

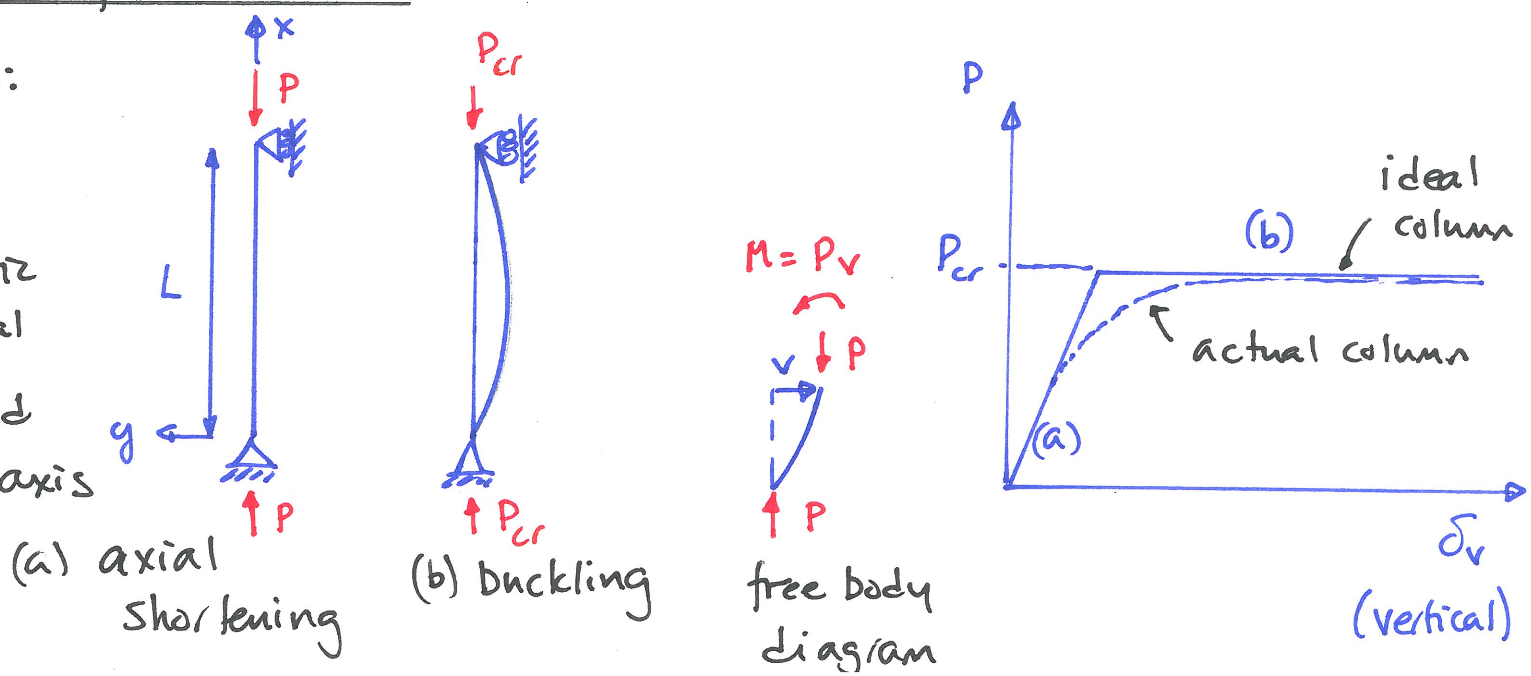


Column with pinned ends

ideal column:

- straight
- linear elastic material
- load aligned with column axis

(demo)



- Small P - column is stable, compresses linearly elastically + axially shortens
- increase P - at some critical load, becomes unstable → buckles

$P_{cr} = ?$

$M = Pv$

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = \frac{Pv}{EI}$$ differential eq'n of buckled state

solution of form: $$v = C_1 \sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_2 \cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

---

boundary conditions

$x = 0 \quad v = 0 \Rightarrow C_2 = 0$

$x = L \quad v = 0 \Rightarrow C_1 \sin\left(\sqrt{\frac{P}{EI}}\,L\right) = 0$

$C_1 = 0 \Rightarrow$ column remains straight

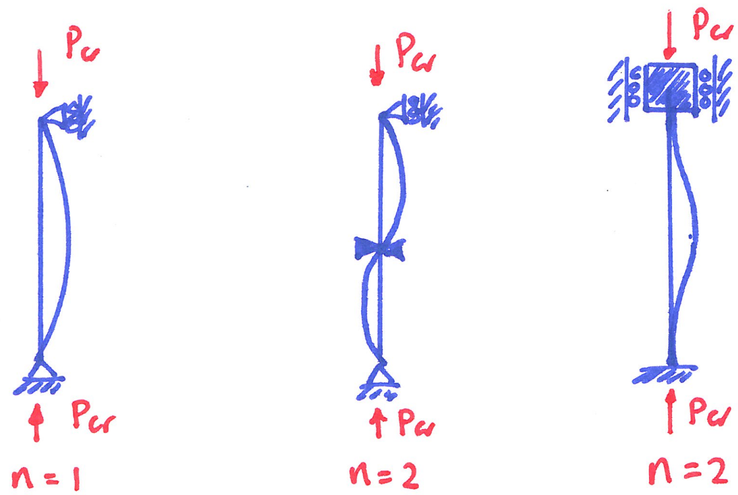
or $\sin\sqrt{\frac{P}{EI}}\,L = 0 \Rightarrow \sqrt{\frac{P}{EI}}\,L = n\pi \quad n = 1, 2, 3 \ldots n$

$$\boxed{P_{cr} = \frac{n^2 \pi^2 EI}{L^2}}$$ $n = 1, 2, 3$

"Euler load" "Euler buckling"

n = number of half-wavelengths along column length, L (end constraint factor)

$n = 1 \Rightarrow$ corresponds to pin-pin ends of column



$$P_{cr} = \frac{n^2 \pi^2 E I}{L^2}$$

- critical buckling load increases with:
  - material modulus
  - moment of inertia
  - decreased length

(demo)

- consider moment of inertia
- want constant I about all cross-sectional axes to have same buckling load about all axes
- circular cross-section attractive
- compare solid circular section + thin-walled hollow cylinder

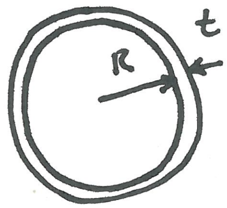
Solid

$$I_s = \frac{\pi r^4}{4}$$

$$m_s = \rho \pi r^2$$

- for same mass, (solid + tube) $\quad \rho \pi r^2 = \rho 2\pi R t$
$$r = (2Rt)^{1/2}$$

Tube (thin-walled)



$$I_t = \pi R^3 t$$

$$m_t = \rho 2\pi R t$$

- for same mass, $\dfrac{I_t}{I_s} = \dfrac{\pi R^3 t}{\frac{\pi}{4}(2Rt)^2} = \dfrac{R}{t}$

- for same mass, tube has larger I than solid circular section, by a factor of R/t

---

- how large can R/t be? $\Rightarrow$ limited by local buckling (kinking)

(demo)
$$\sigma_{\substack{local\\buckling}} = \frac{Et}{R\sqrt{3(1-\nu^2)}}$$
(axial compression of cylindrical tube)

- can increase resistance to local buckling by incorporating longitudinal + circumferential stiffeners (eg. space shuttle fuel tanks, offshore oil platform legs)

- or by using honeycomb or foam core

eg. plant stems, animal quills
Inside Plants article.

Column buckling example

A pin-ended Al column, with a solid circular cross-section, has a radius $r = 5mm$ & length $l = 0.1\,m$. $E_{Al} = 69\,GPa$, $\nu = 0.33$

(a) what is its critical Euler buckling load?

(b) what is the maximum critical Euler buckling load of a hollow Al cylindrical tube of the same mass & length? What are the tube's dimensions?

(a) $P_{cr} = \dfrac{n^2 \pi^2 E I}{l^2}$ $n = 1$; $I = \pi r^4/4 = \pi(.005)^4/4 = 4.91\times10^{-10}\,m^4$; $E = 69GPa$

$$P_{cr} = \frac{\pi^2 (69\times10^9)(4.91\times10^{-10})}{(0.1)^2} \frac{N}{m^2}\frac{m^4}{m^2} = 33.4\text{ kN}$$

(b) solid cylinder radius r

hollow cylinder radius, R, wall thickness, t

- same mass: $\pi r^2 = 2\pi R t$

$$Rt = r^2/2 = 0.005^2/2 = 1.25\times10^{-5}\,m^2$$

- many combinations of R & t possible - want to maximize buckling load
- R/t limited by local buckling : equate $\sigma_{\text{local buckling}} = \sigma_{\text{Euler buckling}}$.

(i.e. make R/t as large as possible & still avoid local buckling)

$$(\sigma_{Euler}) \quad \frac{n^2 \pi^2 \cancel{E} I}{l^2 (2\pi R t)} = \frac{\cancel{E} t}{R\sqrt{3(1-\nu^2)}} \quad (\sigma_{local})$$

$n = 1$

$\nu = 0.33$

$\sqrt{3(1-\nu^2)} = 1.64$

$$\frac{\pi^2 \; \pi R^3 t}{l^2 (2\pi R t)} = \frac{t}{1.64 R}$$

$$\frac{\pi^2 R^2}{2 l^2} = \frac{t}{1.64 R}\frac{R}{R} = \frac{1.25 \times 10^{-5} \; m^2}{1.64 \; R^2}$$

$$R = \left(\frac{1.25 \times 10^{-5} \; m^2}{1.64} \; \frac{2 \, (0.1 \; m)^2}{\pi^2} \right)^{1/4}$$

$$R = 0.0111 \; m$$

$$t = \frac{1.25 \times 10^{-5} \; m^2}{0.0111} = 0.00112 \; m$$

Note: $\frac{R}{t} = \frac{.0111}{.00112} \sim 10$

& P_{cr} hollow tube $\sim 10 \, P_{cr}$ solid cylinder

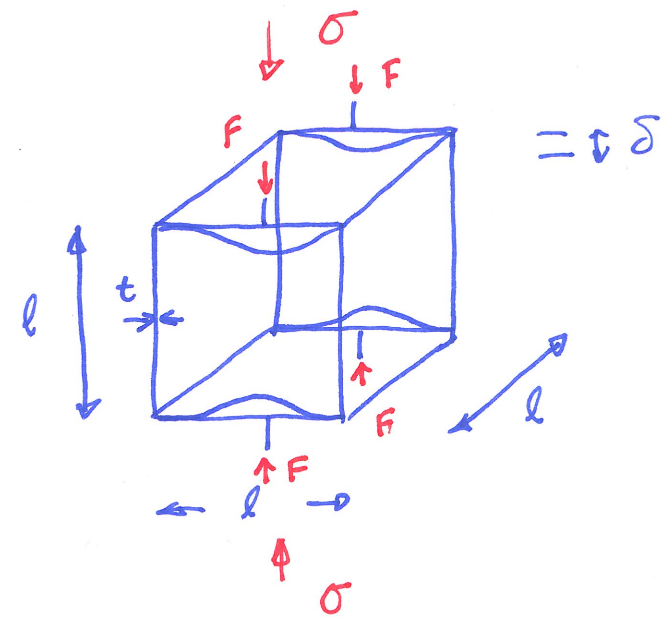
$$P_{cr} = \frac{\pi^2 E I}{l^2} = \frac{\pi^2 E \; \pi R^3 t}{l^2}$$

$$= \frac{\pi^3 (69 \times 10^9)(.0111)^3 (.00112)}{(0.1)^2} \; \frac{N}{m^2} \; \frac{m^3 \, m}{m^2}$$

$$P_{cr} = 328 \; kN$$

Open cell foams - Young's modulus

- foams - complex geometry polyhedral cells
- can do first order model using dimensional arguments
- assume cell geometry of foams of different densities is similar.



Solid modulus, E_s

$\sigma \propto \dfrac{F}{l^2}$ $\quad \epsilon \propto \dfrac{\delta}{l}$ $\quad \delta \propto \dfrac{Fl^3}{E_s I}$ $\quad I \propto t^4$

$$E^* = \frac{\sigma}{\epsilon} \propto \frac{F}{l^2}\frac{l}{\delta} \propto \frac{F}{l}\frac{E_s t^4}{F l^3} \propto E_s\left(\frac{t}{l}\right)^4$$

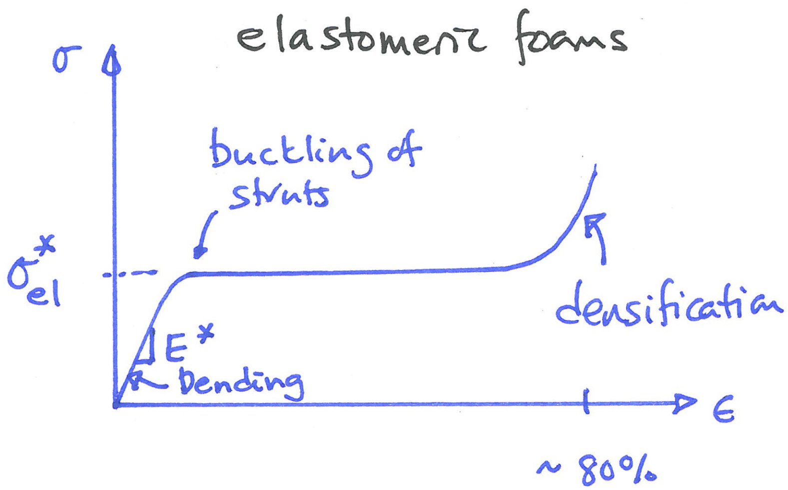
$*$ = for foam

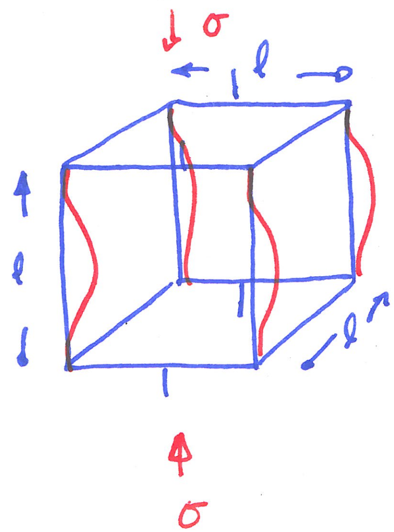
also: $\dfrac{\rho^*}{\rho_s} \propto \dfrac{M_s}{V_{total}}\dfrac{V_s}{M_s} \propto \dfrac{t^2 l}{l^3} \propto \left(\dfrac{t}{l}\right)^2$

$$\therefore \quad E^* = C_1 \, E_s \, (\rho^*/\rho_s)^2$$

C_1: cell geometry; E_s: solid property; (ρ^*/ρ_s): relative density (volume fraction of solids)

Open cell foams - elastic buckling compressive strength





$$\sigma^*_{el} \propto \frac{P_{cr}}{l^2} \propto \frac{n^2 \pi^2 E_s I}{l^2\, l^2} \propto E_s \left(\frac{t}{l}\right)^4$$

$$\sigma^*_{el} = C_2 \, E_s \, (\rho^*/\rho_s)^2$$

cell geometry | solid | relative density

- bands of cells progressively buckle until all have collapsed at high strains (~0.8)
- then opposing cell walls touch - become more difficult to strain further, stress rises sharply (densification)
- long stress plateau - high energy absorption @ low stress
  - foams excellent at energy absorption