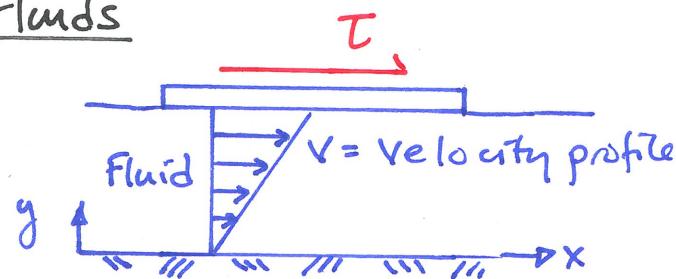


Linear viscoelasticity

(demo)
putty

- behaviour between that of a viscous fluid + elastic solid
- time, temperature dependent deformation
- polymers: engineering (e.g. nylon) + biological (e.g. collagen, cellulose).

Fluids



- fluids cannot resist shear
- flow under a shear stress

$$\tau = \mu \frac{dv_x}{dy}$$

Newton's law
"Newtonian fluid"

$$\frac{dv_x}{dy} = \text{velocity gradient}$$

= rate of shear strain = $\dot{\gamma}$

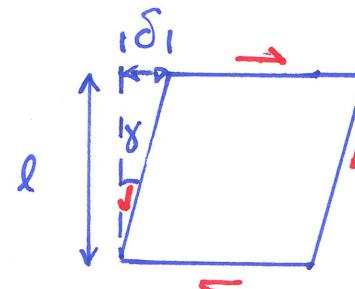
samples
of
liquids.

$$\mu = \text{viscosity [Pa s]}$$

$$= \mu_0 \exp(Q/kT)$$

temperature dependent

time dependent



$$\gamma = \delta/l$$

$$\dot{\gamma} = \dot{\delta}/l$$

Linear elastic deformation

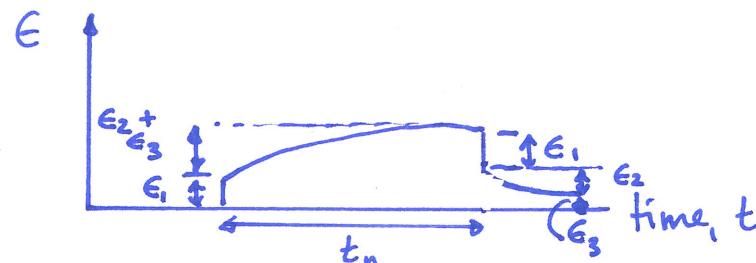
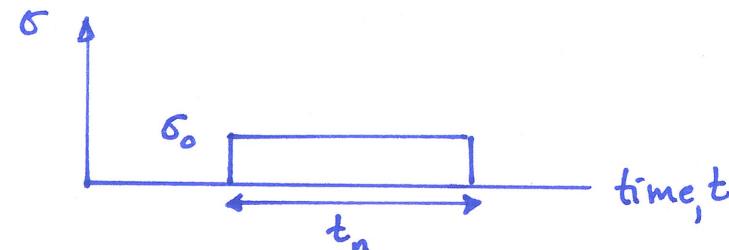
$$\sigma = E \epsilon \quad \text{Hookean solid}$$

- here, consider single mode of deformation: uniaxial or shear, small ϵ

Linear viscoelasticity

- ϵ, σ linearly related at a given time, t , & temperature, T
- e.g. if double stress, double strain (at that time, t , + temperature T)
- two common tests: creep & stress relaxation

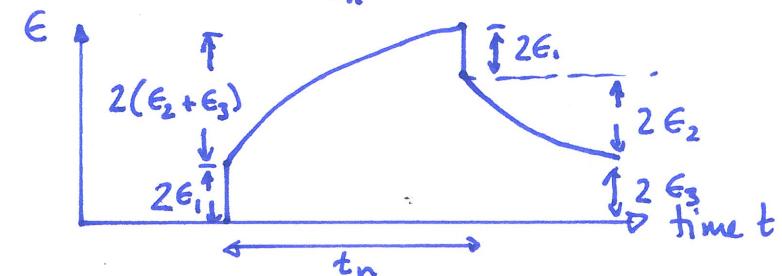
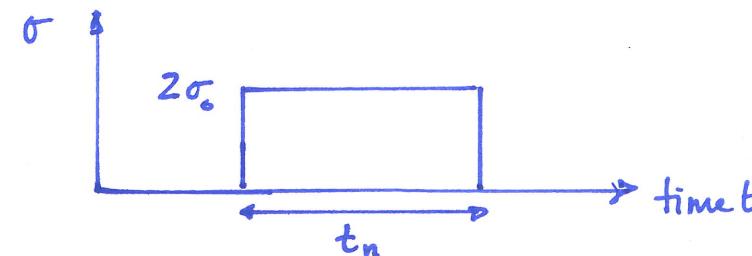
Creep test: apply constant σ , measure $\epsilon(t)$ (at const. temp T)



ϵ_1 = immediate elastic def^m

ϵ_2 = delayed elastic def^m

ϵ_3 = Newtonian flow (irrecoverable def^m)



ϵ_1 = immediate elastic def^m

ϵ_2 = delayed elastic def^m

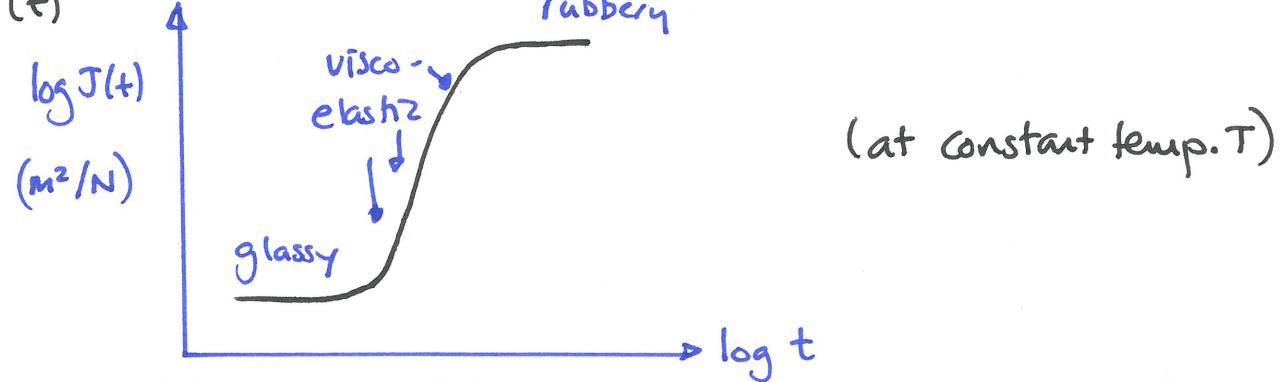
ϵ_3 = Newtonian flow (irrecoverable def^m)

Linear viscoelasticity

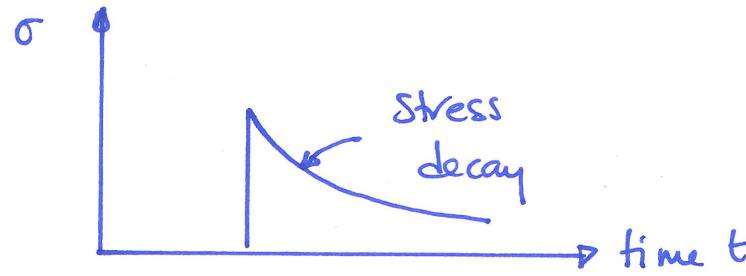
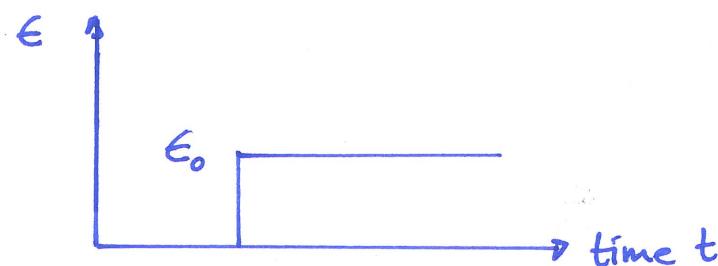
- strain at a given time \propto stress
- double stress, double strain at corresponding time

Creep Compliance $J(t)$

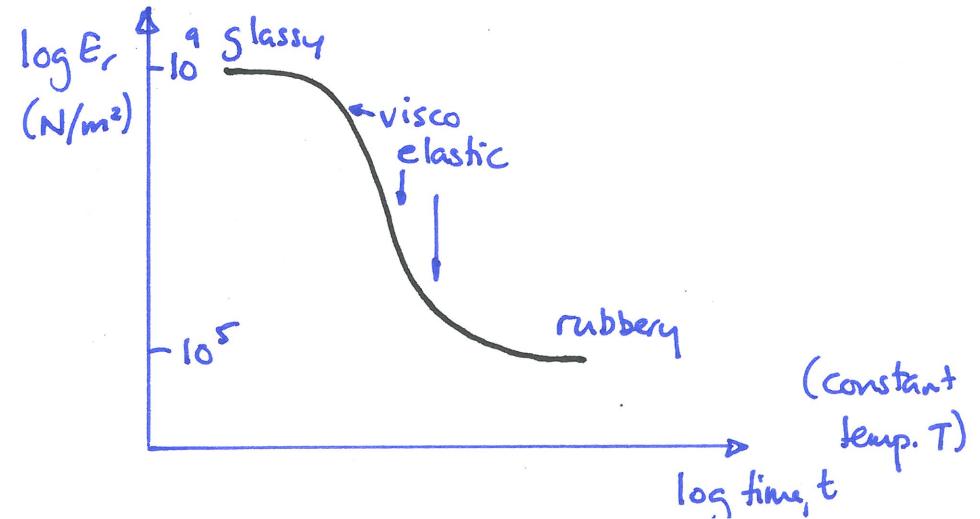
$$J(t) = \frac{E(t)}{\sigma}$$



Stress relaxation : apply constant strain, measure $\sigma(t)$

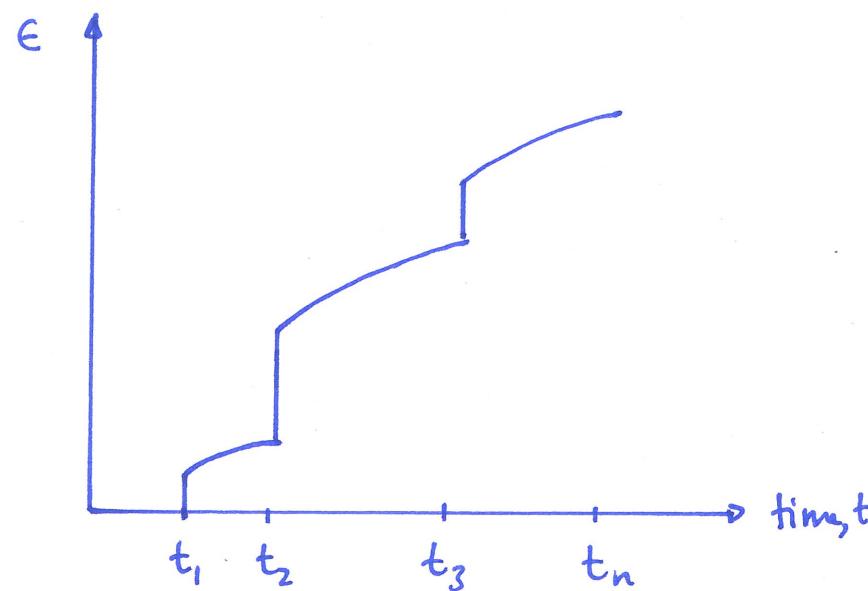
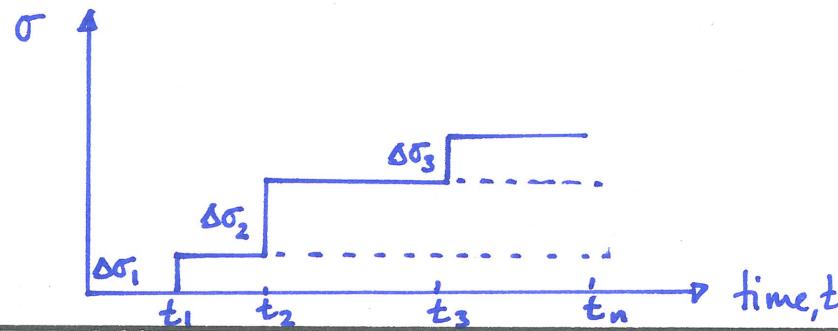


$$\text{Relaxation modulus, } E_r = \frac{\sigma(t)}{E}$$



Boltzmann Superposition principle

- Creep (or stress relaxation) is a function of entire loading history
- each loading step makes an individual contribution to the final deformation
- final deformation can be obtained by simple addition of each contribution



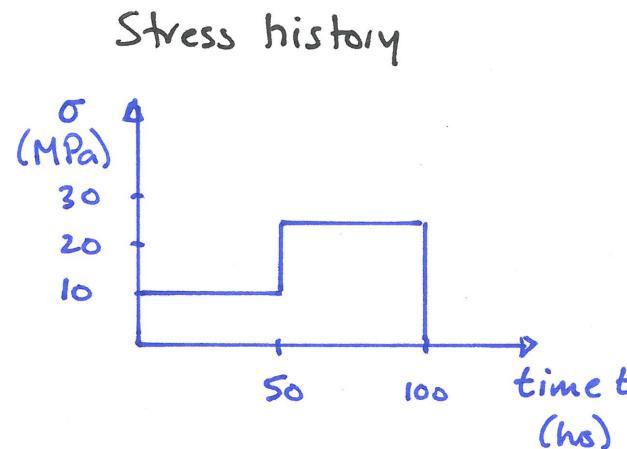
Total creep at time t_n

$$\epsilon(t=t_n) = \Delta\sigma_1 J(t_n - t_1) + \Delta\sigma_2 J(t_n - t_2) + \Delta\sigma_3 J(t_n - t_3)$$

$$J = \text{creep compliance} = \frac{\epsilon(t)}{\sigma}$$

Boltzmann Superposition: Example

t (hrs)	$J(t)$ (MPa^{-1})
0	4.66×10^{-4}
50	6.21×10^{-4}
100	6.83×10^{-4}
150	7.30×10^{-4}
200	7.76×10^{-4}



What is strain at
 $t = 0, 50, 100, 150$ hrs?
 Plot $\epsilon(t)$.

$$\Delta\sigma_1 = 10 \text{ MPa} \quad \Delta\sigma_2 = 15 \text{ MPa}$$

$$\epsilon(t=0) = \Delta\sigma_1 J(0) = 10 \text{ MPa} \times 4.66 \times 10^{-4} \text{ MPa}^{-1} = 4.66 \times 10^{-3}$$

$$\epsilon(t=50 \text{ hrs}) = \Delta\sigma_1 J(50) = 10 \text{ MPa} \times 6.21 \times 10^{-4} \text{ MPa}^{-1} = 6.21 \times 10^{-3}$$

$$\epsilon(t=50^+ \text{ hrs}) = \Delta\sigma_1 J(50) + \Delta\sigma_2 J(0) = 6.21 \times 10^{-3} + 15 \text{ MPa} (4.66 \times 10^{-4} \text{ MPa}^{-1}) = 1.32 \times 10^{-2}$$

$$\epsilon(t=100 \text{ hrs}) = \Delta\sigma_1 J(100) + \Delta\sigma_2 J(50) = 10 \times 6.83 \times 10^{-4} + 15 (6.21 \times 10^{-4}) = 1.61 \times 10^{-2}$$

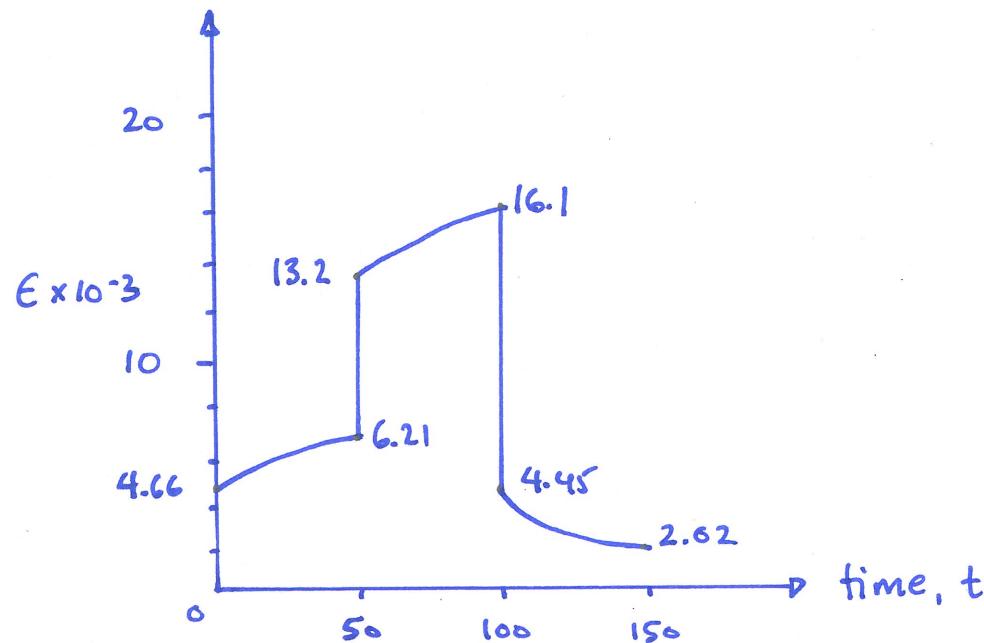
$$\begin{aligned} \epsilon(t=100^+ \text{ hrs}) &= \Delta\sigma_1 J(100) + \Delta\sigma_2 J(50) + \Delta\sigma_3 J(0) = 1.61 \times 10^{-2} - 25 \text{ MPa} (4.66 \times 10^{-4} \text{ MPa}^{-1}) \\ &= 4.45 \times 10^{-3} \end{aligned}$$

$$\epsilon(t=150 \text{ hrs}) = \Delta\sigma_1 J(150) + \Delta\sigma_2 J(100) + \Delta\sigma_3 J(50) =$$

$$= (10 \text{ MPa})(7.3 \times 10^{-4} \text{ MPa}^{-1}) + 15 \text{ MPa} \times 6.83 \times 10^{-4} \text{ MPa}^{-1} - (25 \text{ MPa})(6.21 \times 10^{-4} \text{ MPa}^{-1}) = \frac{2.02 \times 10^{-3}}{10^{-3}}$$

C

(6)



Note: for linear viscoelasticity, can measure creep response $\epsilon(t)$ at a single stress + use the resulting creep compliance $J(t)$ to get strains for any stress history

Stress relaxation

- represented in same way as creep compliance

$\Delta\epsilon_1$ applied at t_1

$\Delta\epsilon_2$ " " t_2

$\Delta\epsilon_3$ " " t_3

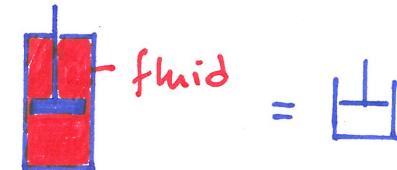
$$\sigma(t_n) = \Delta\epsilon_1 E_r(t_n - t_1) + \Delta\epsilon_2 E_r(t_n - t_2) + \Delta\epsilon_3 E_r(t_n - t_3)$$

E_r = stress relaxation modulus.

Spring-dashpot Models of linear viscoelasticity

demo:
syringe

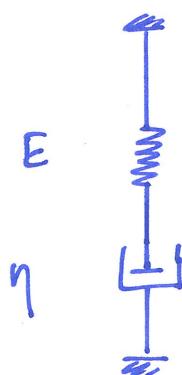
- time dependent behaviour can be represented by spring-dashpot models
- dashpot - plunger in a cylinder of fluid (at constant temp T)
- spring: $E = \sigma / \epsilon$
- dashpot: $\eta = \sigma / \dot{\epsilon}$ η = dashpot viscosity



$$\begin{aligned} \tau &= \mu \dot{\epsilon} \\ \sigma &= \eta \dot{\epsilon} \end{aligned}$$

μ, η both are viscosity.

Maxwell element



$$\sigma_s = \sigma_d = \sigma$$

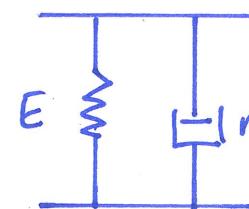
$$\epsilon = \epsilon_s + \epsilon_d$$

$$\dot{\epsilon} = \dot{\epsilon}_s + \dot{\epsilon}_d$$

$$\boxed{\frac{d\epsilon}{dt} = \frac{1}{E} \frac{d\sigma}{dt} + \frac{\sigma}{\eta}}$$

Constitutive eqn for Maxwell element

Voigt or Kelvin element



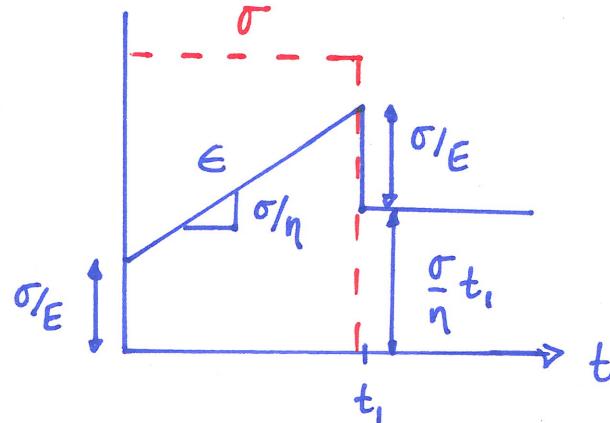
$$\sigma = \sigma_s + \sigma_d$$

$$\epsilon = \epsilon_s = \epsilon_d$$

$$\boxed{\sigma = E\epsilon + \eta \frac{d\epsilon}{dt}}$$

Constitutive eqn for
Voigt element

Maxwell element: creep



$$t < t_1, \quad \frac{d\sigma}{dt} = 0 \quad \epsilon = \frac{\sigma}{E} + \frac{\sigma}{\eta} t$$

spring ↓ dashpot ↓

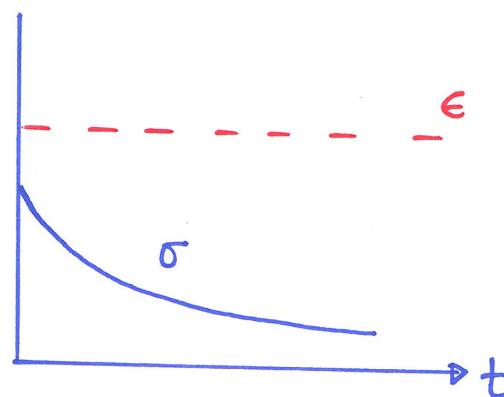
$t > t_1$

$$\epsilon = \frac{\sigma}{\eta} t_1 \quad (\text{irrecoverable flow})$$

η

(Newtonian flow)

Maxwell element: stress relaxation

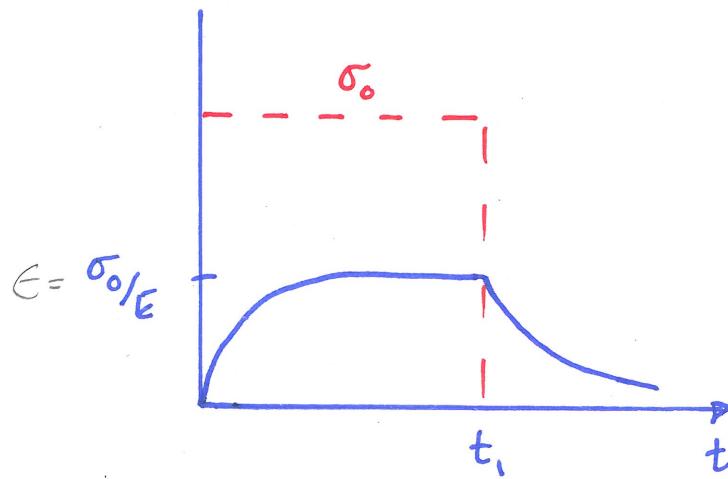


$\epsilon = \text{constant}$

$$\sigma = E \epsilon \exp\left(-\frac{E}{\eta} t\right)$$

Characteristic time $\tau = \eta/E$

Voigt element: creep

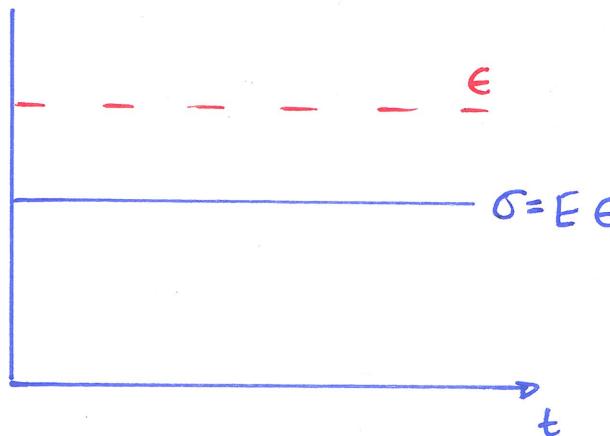


$$t < t_1, \quad \epsilon = \frac{\sigma_0}{E} \left(1 - \exp - \frac{Et}{\eta} \right)$$

$$t > t_1, \quad \epsilon = \frac{\sigma_0}{E} \left(\exp - \frac{E(t-t_1)}{\eta} \right)$$

- characteristic time $\tau = \eta/E$ gives time at which 1- e^{-1} of change to steady state strain, σ_0/E , has been achieved
- $t = 3\tau \quad \epsilon = \frac{\sigma_0}{E} \left(1 - e^{-3} \right) = 0.95 \sigma_0/E$

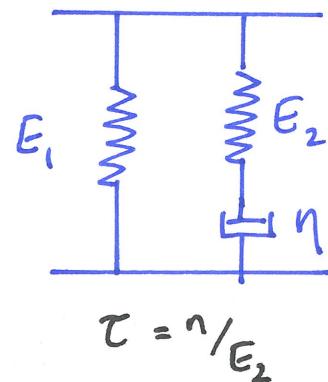
Voigt element: relaxation



$$\frac{d\epsilon}{dt} = 0 \quad \sigma = E\epsilon$$

Standard Linear Solid

- Maxwell & Voigt models not very realistic
- Better model is standard linear solid

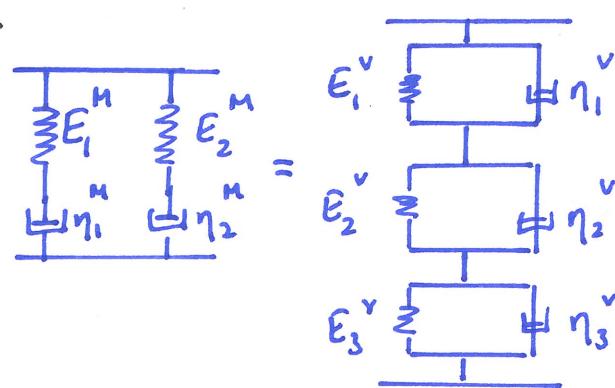


$$\text{Constitutive eqn: } \sigma + T \frac{d\sigma}{dt} = E_1 \epsilon + (E_1 + E_2) T \frac{d\epsilon}{dt}$$

- first approximation for behavior of viscoelastic materials
- gives exponential response for both creep + stress relaxation
- real materials require additional elements.

Equivalent models

e.g.



$$\text{if } \eta_1^v = 0 \text{ and } E_3^v = 0$$

- infinite number of different models that can describe viscoelastic behavior at one temperature
- models that are equivalent at one temp. are not equivalent at another
e.g. $E_3^v = f(E_1^m, E_2^m, \eta_1^m, \eta_2^m)$
 E_3^v not sensitive to temp. but η^m are sensitive to temp.
 E^v constant + $(n^m n^m)$ at different temps.

image
Tom's book - leg 

video
Hugh Herr
TED MED 2010

8'32
shows
prostheses

- spring dashpot systems also used in modelling human + animal locomotion
 - and in design of prostheses (eg. leg)
 - see Hugh Herr's website : www.media.mit.edu/people/hherr
-

Dynamic mechanical measurements

- can measure viscoelastic response from dynamic experiments
- subject specimen to sinusoidally varying stress + measure strain
apply $\sigma = \sigma_0 \sin(\omega t + \delta)$
strain $\epsilon = \epsilon_0 \sin(\omega t)$
- strain lags behind stress, by phase lag, δ .

$$\sigma = \sigma_0 \sin(\omega t + \delta)$$

$$= \sigma_0 \cos \delta \sin \omega t + \sigma_0 \sin \delta \cos \omega t$$

↓

In phase with strain

↓

90° out of phase with strain

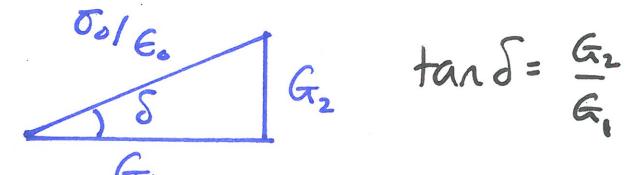
define G_1 = storage modulus (energy stored due to applied σ)

G_2 = loss modulus (out of phase with strain - dissipation in energy)

$$G_1 = \frac{\sigma_0}{\epsilon_0} \cos \delta$$

$$G_2 = \frac{\sigma_0}{\epsilon_0} \sin \delta$$

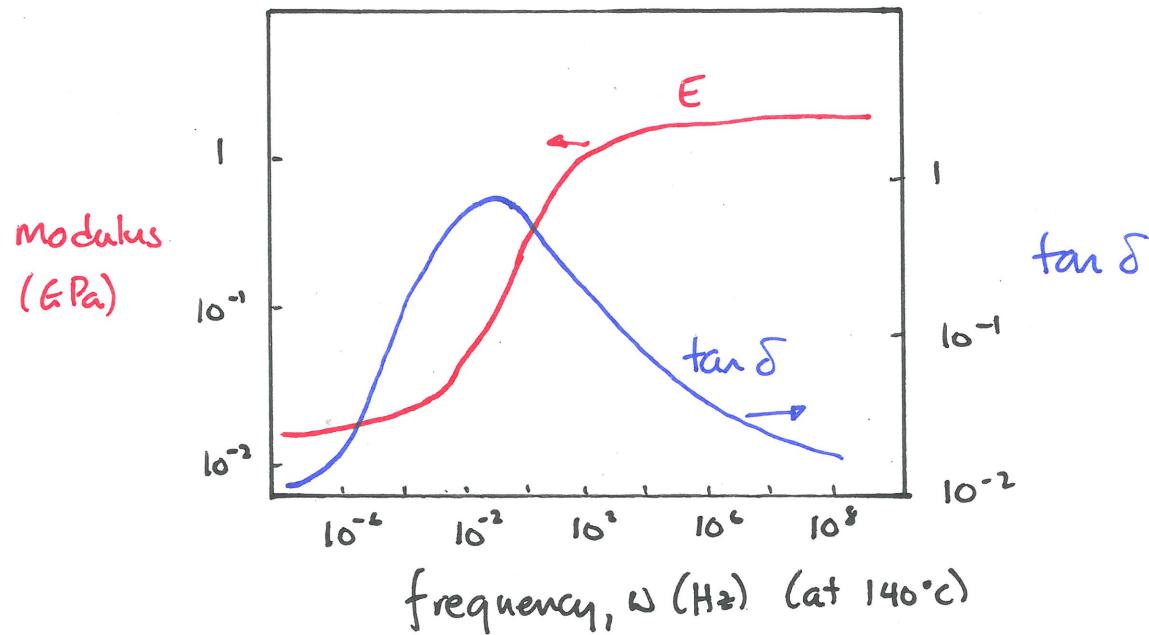
G_2 usually small compared to G_1 .



rubber ball demo

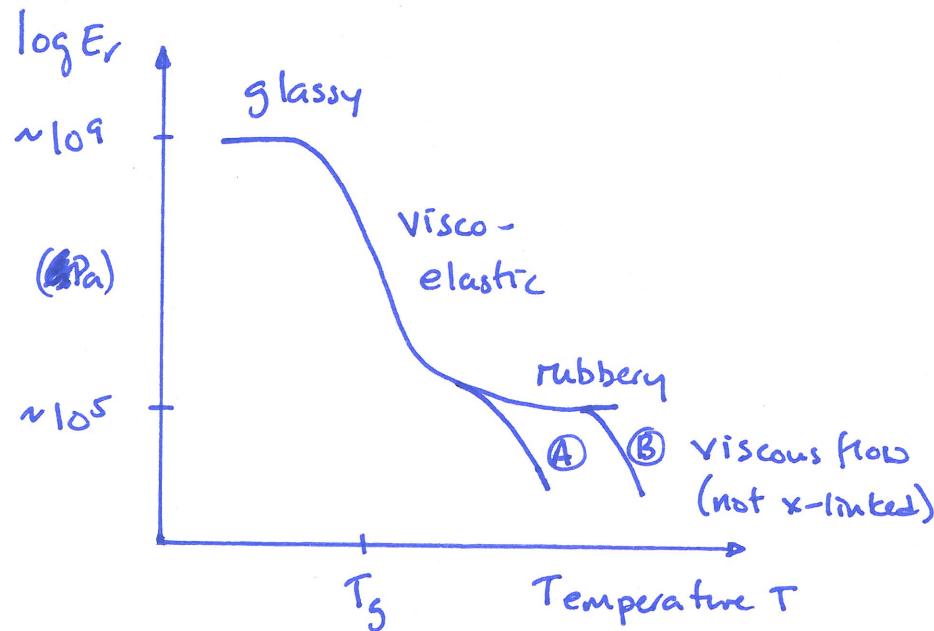
$$\sigma = \epsilon_0 G_1 \sin \omega t + \epsilon_0 G_2 \cos \omega t$$

- viscoelastic response can be described in terms of G_1 & G_2 (or G_1 & δ) & their dependence on frequency



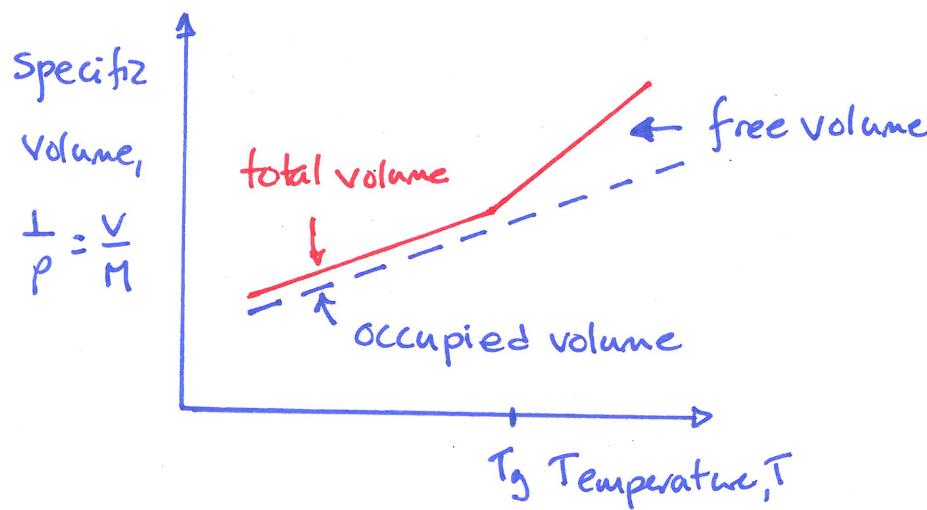
epoxy
(after MF Ashby notes)

Linear viscoelasticity in amorphous polymers



- Consider relaxation modulus, E_r , at constant time (e.g. 10 sec), as a function of temperature
- at low temp. glass-like behaviour, hard + brittle, $E \approx 1 \text{ GPa}$
- at low temp., E depends on stretching molecular bonds: $E \propto \frac{d^2U}{dE^2} \approx 10^9 \text{ Pa.}$

- as temperature increases, get viscoelastic behaviour



- at glass transition temp. T_g
- secondary bonds melt
- more open structure
- free volume increases
- sliding between short segments of polymer chains becomes possible

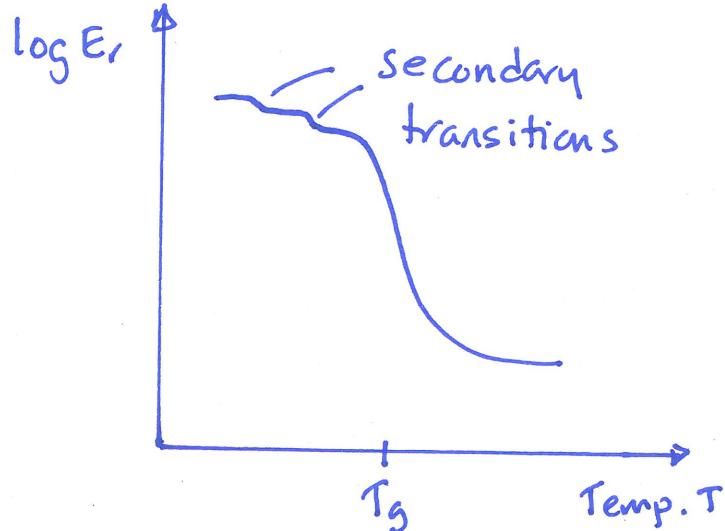
- further increase in temperature

- if have chemical cross-links - rubbery behaviour
- E depends on entropy changes associated with changes in molecular configurations $E \sim 10^5 \text{ Pa}$
- $E = -\frac{T}{2} \frac{d^2S}{dE^2} = 3n_v kT \Rightarrow E$ depends on cross-link density.
 n_v = no. chains/volume
- if no chemical cross-links, get viscous flow, as liquid
- physical entanglements can restrict molecular flow causing formation of temporary networks

- at high enough temperature, physical entanglements become labile + lead to flow
- increasing molecular weight of polymer increases physical entanglements
- on figure on last page, $M_w(B) > M_w(A)$

- We have assumed only 1 transition

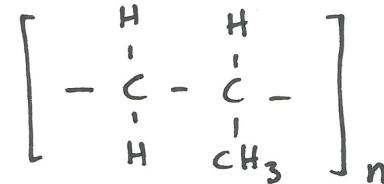
- In practice, there are several



- glass transition associated with melting of secondary bonds

- secondary transitions associated with motion of side groups

e.g. methyl $-CH_3$ in polypropylene



Linear viscoelasticity in crystalline polymers

- transitions less clearly defined, more gradual
- Modulus drop smaller $10^9 \rightarrow 10^7$ Pa [vs. 10^9 to 10^5 Pa in amorphous polymers].
- at high temperatures, molecular mobility in crystalline regions is severely curtailed + not rubber-like

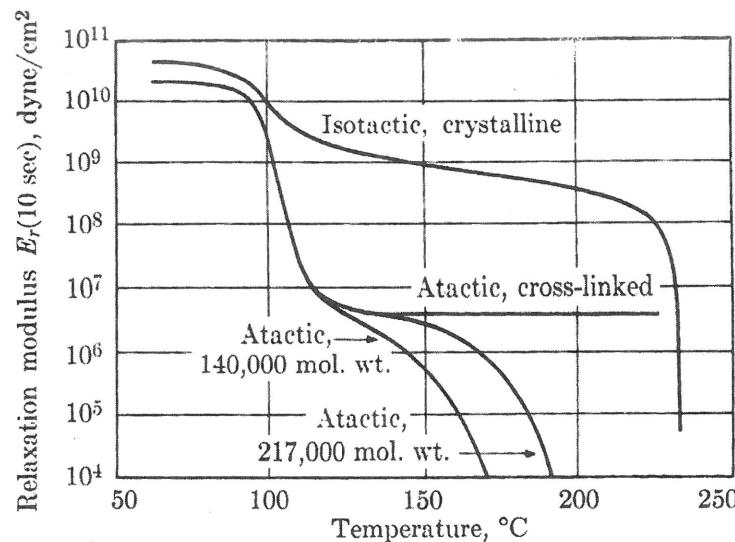


FIG. 6.8. Ten-second relaxation modulus for various forms of polystyrene. Cross-linking by 0.25 mole % divinyl benzene. (Tobolsky, 1960, 1963. Courtesy of Wiley.)

isotactic - all side groups
on same side

atactic - side groups random

(from McClintock + Argon 1966)

Viscoelasticity in biomaterials

- structural materials in nature often viscoelastic
- collagen - basic structural fiber of animal kingdom
- cellulose - basic structural fiber of plant kingdom
- both are non-linear viscoelastic

Collagen

- often most abundant protein in animal
- one of main constituents of bone, cartilage, tendon, skin
- structure: 3 non-coaxial helical poly peptides stabilized by interchain H-bonds
- forms fibrils, fibers
- data for E_r : rat tail tendon (70% collagen)
 - note E_r depends on $\epsilon \Rightarrow$ Non-linear viscoelastic.

Rat Tail Tendon: 70% collagen

A: 3.5% strain

B: 7.5% strain

Non-linear: different responses at different strains

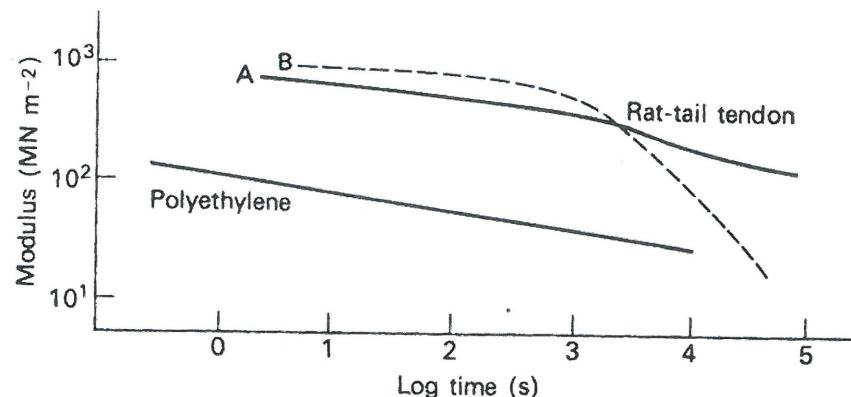


Fig. 3.20 Stress-relaxation curves for collagen fibres. Solid and dashed curves are for rat-tail tendon strained 3.5% and 7.5% respectively (RIGBY *et al.*, 1959); with permission of the *J. Gen. Physiol.* The stress-relaxation curve of bulk crystallized polyethylene is included for comparison (after BECKER, 1961).

From Wainwright SA, Biggs WD, Currey JD and Gosline JM (1978)
Mechanical Design in Organisms. Princeton University Press.

Bone

- collagen + hydroxyapatite + water
- bone creeps due to collagen content
- creep data for compact bone, trabecular bone + demineralized bone
 - all have $\dot{\epsilon} \propto \sigma^{15}$ \Rightarrow non-linear viscoelastic

Uterine cervix

- has collagen fibrils
- see plot for rat cervix

- rat pregnancy lasts 21 days
- before pregnancy + up to 21 days pregnant - creep curve A
- at term (21 days), creep behaviour of cervix changes dramatically
 - much higher - creep curve B
 - due to changes in cross-linking of collagen network
 - as cross-links break down, cervix more compliant
 - due to hormonal changes
- cross links reform post-partum - creep resistance increases
- "insufficient cervix" - condition where cervix begins to open prematurely during pregnancy.

Demineralized cortical bone

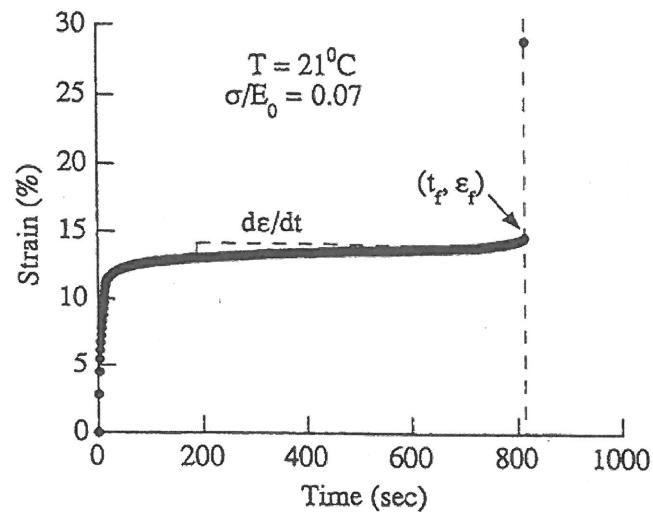


Fig. 1 Typical strain versus time curve for a waisted specimen of demineralized bovine cortical bone subjected to tensile creep loading, illustrating the methods used to determine time-to-failure t_f , steady-state creep rate $d\epsilon/dt$, and creep failure strain ϵ_f (Bowman et al., 1994). Note also the three classical regimes of creep, indicated by periods of decreasing (primary), constant (secondary), and increasing (tertiary) creep rates. These three regimes were observed in all specimens, independent of applied normalized stress σ/E_0 and temperature T . The lone data point at ~28 percent strain is shown only to confirm that complete specimen fracture has occurred.

From Bowman SM, Gibson LJ, Hayes WC and McMahon TA (1999) J. Biomech. Eng. **121**, 253-258.

Bone

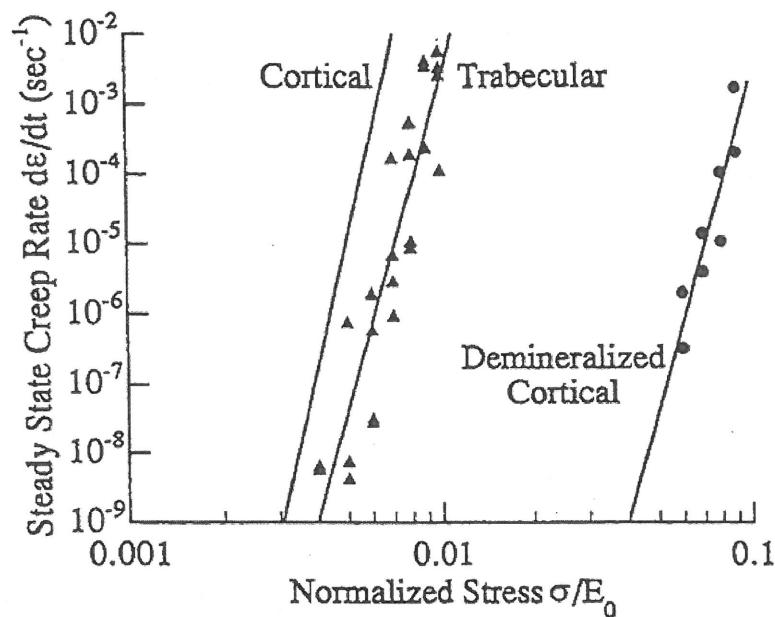


Fig. 2 Log-log plot of steady-state creep rate $d\varepsilon/dt$ versus applied normalized stress σ/E_0 for bovine trabecular ($n = 24$, compressive) (Bowman et al., 1996a, 1998) and demineralized cortical ($n = 8$, tensile) bone creep tests conducted at room temperature, and the best-fit linear regressions of the data (Table 1). Note also the regression (data not available) obtained from room temperature tensile creep tests on nine bovine cortical bone specimens (Fondrk et al., 1988). All three tissues (cortical, trabecular, and demineralized bone) have similar creep exponents (18.9, 15.6, and 15.5, respectively), suggesting that collagen (i.e., demineralized bone) is the phase responsible for creep of both cortical and trabecular bone. See Table 1 for actual regression equations.

From Bowman SM, Gibson LJ, Hayes WC and McMahon TA (1999) J. Biomech. Eng. 121, 253-258.

Rat Uterine Cervix

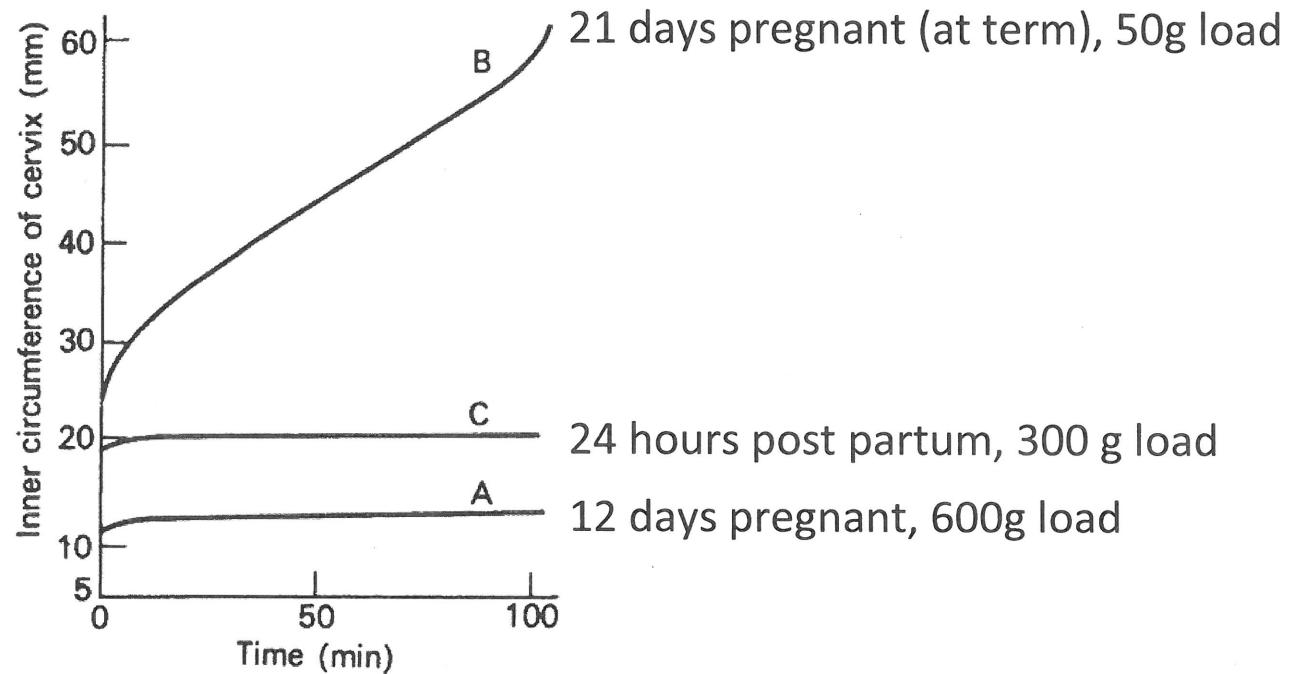


Fig. 4.11 Curves for creep under constant load of whole cervix from rats at (A) 12 days pregnant (load = 600 g), (B) 21 days pregnant (load = 50 g), (C) 24 h post partum (load = 300 g) (after HARKNESS and HARKNESS, 1959).

From Wainwright SA, Biggs WD, Currey JD and Gosline JM (1978)
Mechanical Design in Organisms. Princeton University Press.