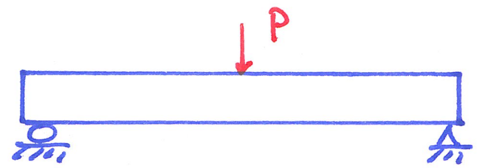
Beams



- carry transverse loads
- applications:
  - cellular solids: honeycombs, foams, trabecular bone, tissue eng. scaffolds
  - atomic force microscope - cantilever beam tip
  - thin films on substrates - thermal mismatch - bending
  - sporting equipment - skis, pole vault poles
  - structures (civil engineering).

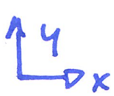
- need to find:
  - support reactions
  - internal forces + moments
  - internal stresses
  - deflections.

- notation:

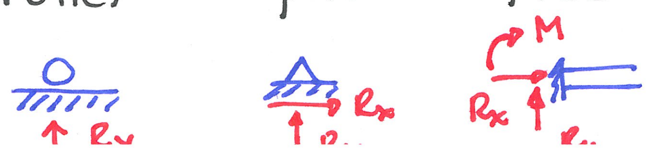
simply supported

cantilever

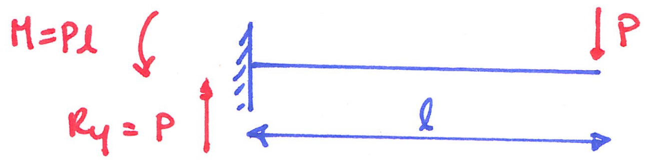
- notation:

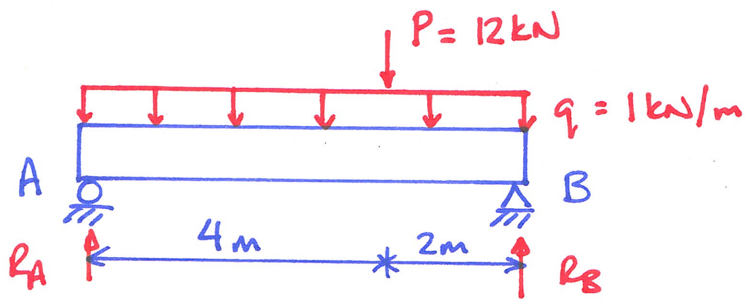


roller pin fixed end



Beams: support reactions





q: distributed load (uniform)

$\Sigma M_A = 0$

$(-1\,kN/m)(6m)(3m) - (12kN)(4m) + R_B\, 6m = 0$

$R_B = \dfrac{18+48}{6} = 11\,kN \uparrow$

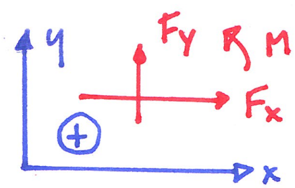
$\Sigma F_y = 0 \uparrow +$

$R_A + 11 - (1\,kN/m)(6m) - 12kN = 0$

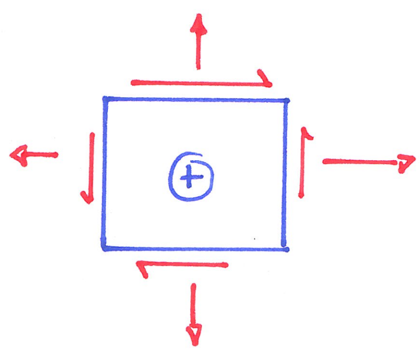
$R_A = -11 + 6 + 12 = 7\,kN \uparrow$

Sign Conventions

Static equilibrium:

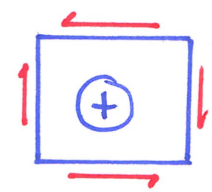


Stress:

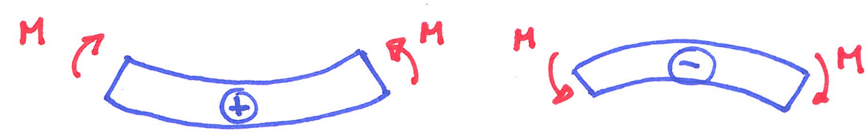


Tension positive; shear positive for positive force acting on a face with outward normal in positive direction

Beams.



shear positive: acts downward on right hand face



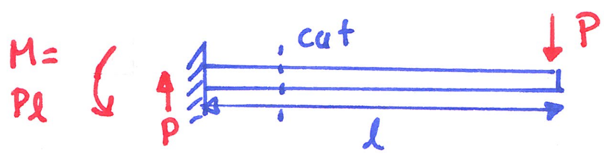
Bending moment positive if tension on bottom of beam

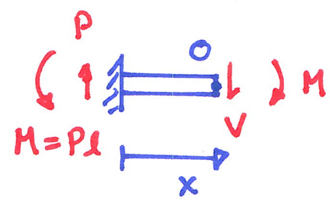
q: positive downwards

Beams - shear + bending moment diagrams

- shear force V and bending moment M can vary along length of beam
- shear stress depends on shear force
- normal stress " " bending moment
- need to know where $\tau = \tau_{max}$ (@ $V = V_{max}$) & $\sigma = \sigma_{max}$ (@ $M = M_{max}$)
- deflection also depends on bending moment

Example: Find $V(x)$, $M(x)$ for an end-loaded cantilever

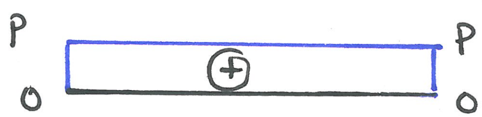




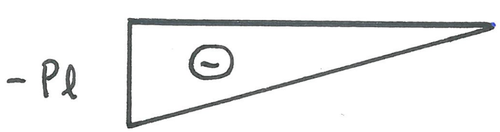
$\Sigma F_y = 0 \; +\uparrow \qquad P - V = 0 \qquad V = P$

$\Sigma M_o = 0 \; +\circlearrowleft \qquad Pl - Px - M = 0$

$M = P(l - x)$



V diagram

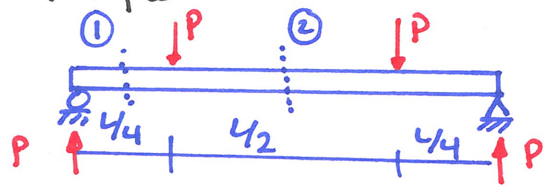


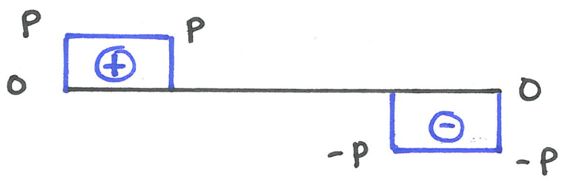
B.M. diagram

(tension on top $\Rightarrow$ negative B.M.)

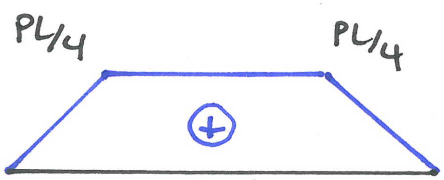
"4 POINT BENDING"

Example:





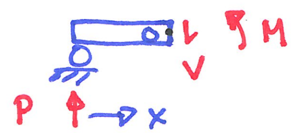
V



B.M.

constant B.M. bet'n loading points - often used in mech. testing.

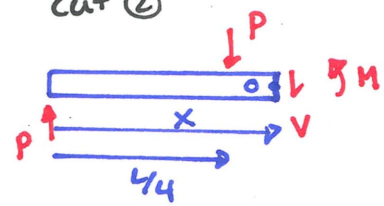
cut ①



$\Sigma F_y = 0 \uparrow +$ $\quad V = P$ (positive)

$\Sigma M_o = 0 \circlearrowleft +$ $\quad M - Px = 0 \quad M = Px$

cut ②



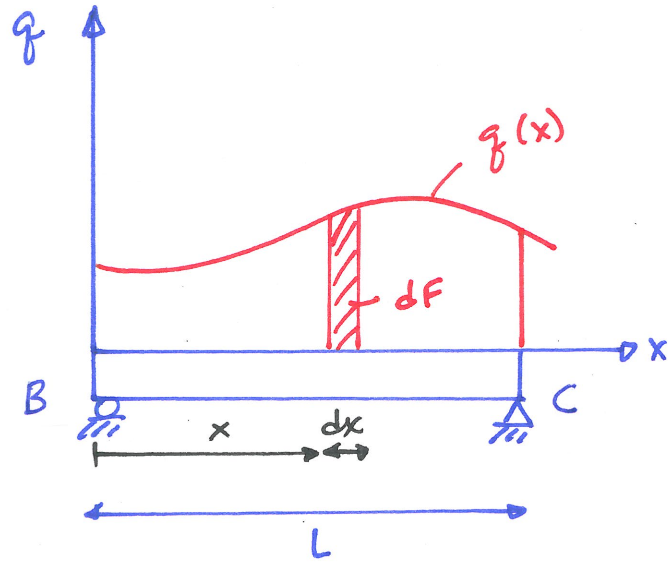
$\Sigma F_y = 0 + \uparrow$ $\quad V = 0$

$\Sigma M_o = 0 + \circlearrowleft$ $\quad -Px + P(x - L/4) + M = 0$

$$M = Px - Px + PL/4$$

$$M = PL/4$$

Distributed loads on beams

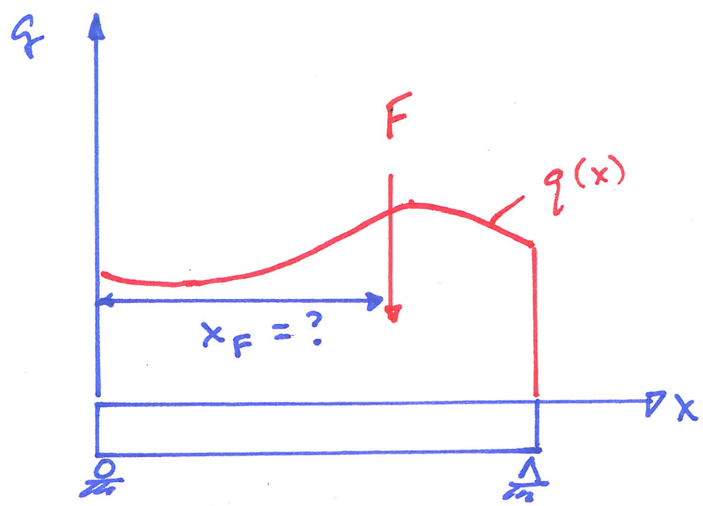


- equivalent concentrated load F

$$dF = q\,dx$$

$$F = \int_0^L q\,dx = \int dA = \text{area beneath } q \text{ diagram, } A$$

- where does this force F act to produce same moment about B as the distributed load q?



$$\int_0^L x\,dF = F\,x_F$$

$$dF = dA$$

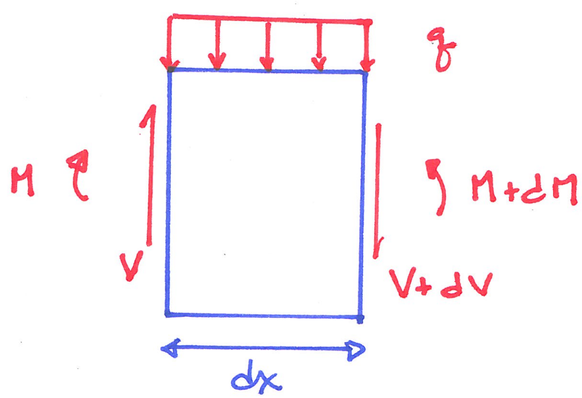
$$F = A$$

$$\int_0^L x\,dA = A\,x_F$$

$$x_F = \frac{\int_0^L x\,dA}{A} = \bar{x} \quad \text{centroid coordinate}$$

- equivalent force F acts through centroid of area of the distributed load.

Consider an element with a distributed load, q



$\Sigma F_y = 0 \uparrow +$

$V - q\,dx - (V+dV) = 0$

$$\boxed{q = -\frac{dV}{dx}}$$

$\Rightarrow$ if $q = 0$ $V =$ constant

(see previous examples)

Beam convention:

q positive downwards

Also, can integrate between 2 points A + B on beam

$$\int_A^B dV = -\int_A^B q\,dx$$

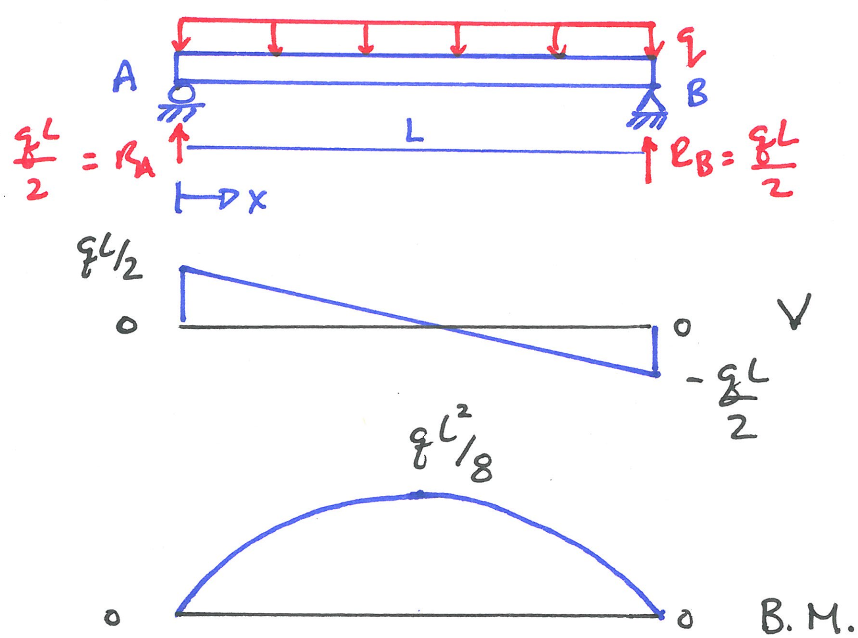
$$V_B - V_A = -\int_A^B q\,dx = -(\text{area under } q \text{ diagram between } A \,\&\, B)$$

$$\Sigma M_{LHS} = 0 \quad -M - q\,dx\frac{dx}{2} - (V+dV)\,dx + M + dM = 0 \qquad \begin{pmatrix} dx\,dx \to 0 \\ dV\,dx \to 0 \end{pmatrix}$$

$$\boxed{V = \frac{dM}{dx}} \Rightarrow \text{ if } V = 0 \quad \frac{dM}{dx} = 0 \quad \text{\& } M = M_{max} \text{ or } M_{min}$$

Also, can integrate: $\int_A^B dM = \int_A^B V\,dx$

$$M_B - M_A = \int_A^B V\,dx = (\text{area under } V \text{ diagram from } A \to B)$$



at A $\quad V_A = +qL/2$ (cut just right of A, V positive ☐)

at x along beam: $\quad V_B - V_A = -\int q\,dx$

$$V(x) = V_A - \int_0^x q\,dx$$

$$V(x) = \frac{qL}{2} - qx = q\left(\frac{L}{2} - x\right) \quad (\text{linear})$$

at $x = L$ (at B) $\quad V_B = -qL/2$

at $x = L/2 \quad V = 0$

$$M(x) = M_A + \int_0^x V\,dx$$

$$= 0 + \int q\left(\frac{L}{2} - x\right) dx$$

$$= \frac{qL}{2}x - \frac{qx^2}{2} \quad \text{(parabola)}$$

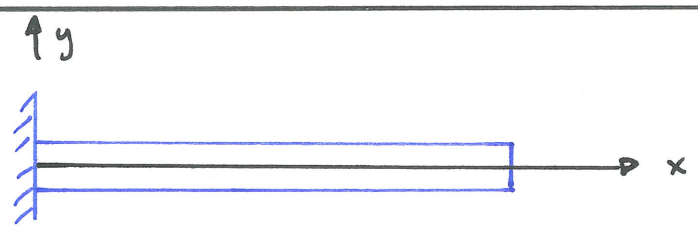
@ $x = L/2$ $V = 0$ $M = M_{max} = \dfrac{qL^2}{8}$ = area under V diagram from A → $x = L/2$

Stresses in beams

- applied loads produce internal shear + bending moment
- gives rise to shear stresses + normal stresses
- normal stresses typically >> shear stresses ; we'll focus on normal stresses

Assumptions

- beam straight
- material linear elastic, isotropic
- x-section symmetrical
- beam is stable (no lateral buckling)



xy plane = plane of bending

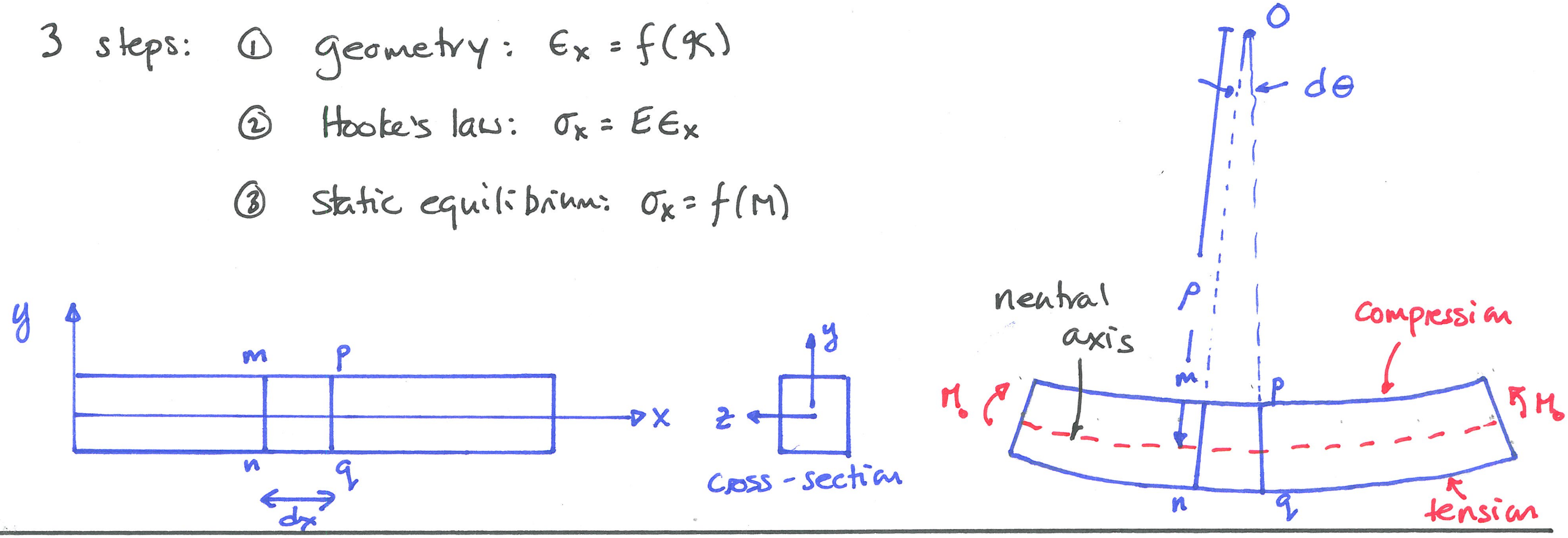
- bending: one face tension, the other compression
- plane along longitudinal axis that sees no stress "neutral axis" (N.A.)
- define pure bending: constant bending moment
  e.g. 4 point bending - central section
- if M not constant: "non-uniform bending"
- pure bending: $\boxed{\text{plane sections remain plane}}$

Normal stresses in beams

3 steps: ① geometry: $\epsilon_x = f(\kappa)$

② Hooke's law: $\sigma_x = E\epsilon_x$

③ Static equilibrium: $\sigma_x = f(M)$



- sections mn & pq rotate with respect to each other, about z axis, by dθ
- extrapolations of mn & pq intersect at O ⇒ center of curvature
- distance O → neutral axis = ρ, radius of curvature
- distance between mn & pq at neutral axis is constant = dx

$$\rho \, d\theta = dx$$

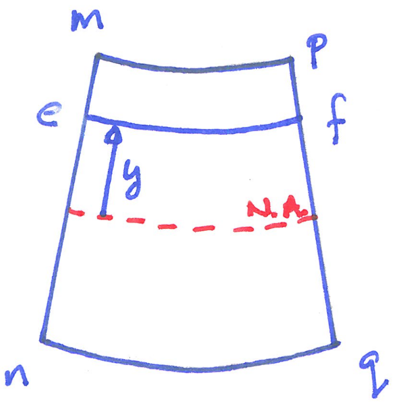
$$\frac{1}{\rho} = \frac{d\theta}{dx} = \kappa, \text{ curvature } [m^{-1}]$$

"pure bending": ρ = constant along length of beam

"non-uniform bending": $\rho = \rho(x)$

signs:   (same as B.M).

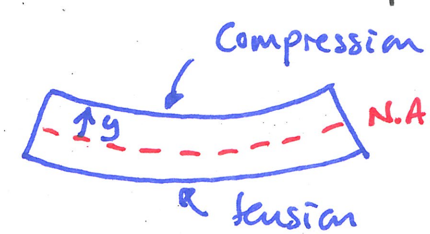
- all longitudinal planes, other than N. A., change length, producing longitudinal strain, ϵ_x



- Consider longitudinal fiber, ef, a distance y from N. A.
- deformed length $= L_1 = (\rho - y)\,d\theta$
- undeformed length $= L_0 = dx = \rho\, d\theta$
- normal strain $\epsilon_x = \dfrac{L_1 - L_0}{L_0} = \dfrac{(\rho - y)\,d\theta - \rho\, d\theta}{\rho\, d\theta}$

$$\epsilon_x = -y/\rho \qquad \boxed{\epsilon_x = -\kappa y} \qquad (1)\ \epsilon_x = f(\kappa).$$

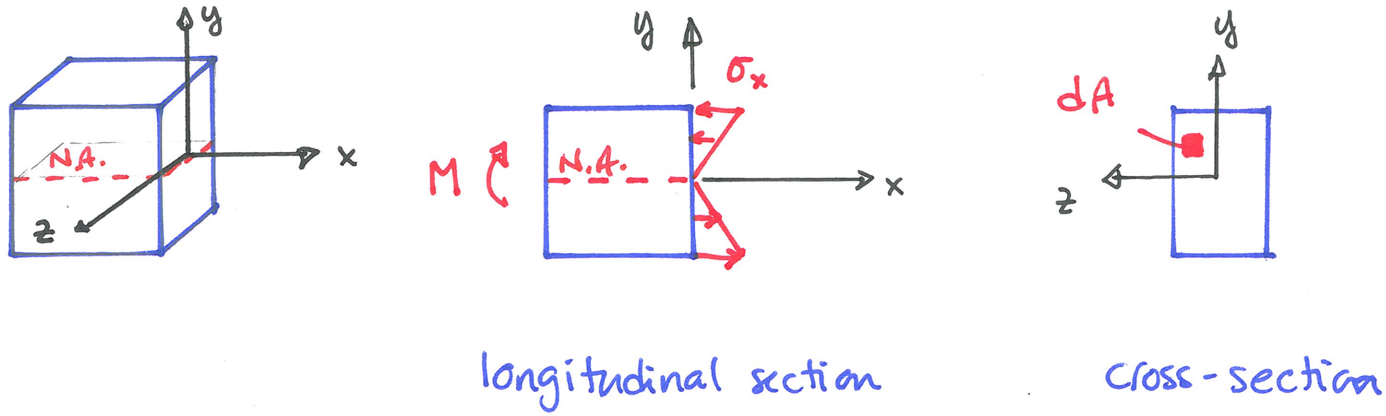
- ϵ_x varies linearly with y, distance from N. A. "plane sections remain plane"
- positive κ, positive y $\Rightarrow$ $\epsilon_x < 0$ $\Rightarrow$ compression
- positive κ, negative y $\Rightarrow$ $\epsilon_x > 0$ $\Rightarrow$ tension



Hooke's law $\quad \boxed{\sigma_x = E\,\epsilon_x = -E\,\kappa y} \quad (2)$

- normal stress varies linearly with y

- next, relate σ to moment M using static equilibrium:



- Consider element dA

$$\Sigma F_x = 0 \xrightarrow{+} \qquad \Sigma F_x = \int_A \sigma_x \, dA = -\int_A E \kappa y \, dA = 0$$

$$E, \kappa \neq 0 \Rightarrow \int_A y \, dA = 0 \qquad \text{centroid } \bar{y} = \frac{\int y \, dA}{\int dA} = 0$$

- z axis passes through centroid of cross-section
- z axis also the neutral axis (N. A.)
- neutral axis passes through centroid of cross-section

$\Sigma M_o = 0$ $+\circlearrowleft$ $\qquad -M - \int_A \sigma_x y \, dA = 0$ (if y positive σ_x compression, negative)

$$M = -\int_A \sigma_x y \, dA = E\kappa \int_A y^2 \, dA$$

Moment of inertia, $I = \int y^2 dA$ $[m^4]$

$$\therefore \quad M = \kappa E I$$

$$\kappa = \frac{M}{EI}$$

$EI \rightarrow$ flexural rigidity

E - material property

I - geometry

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finally, $\sigma_x = -E\kappa y$

$$= -E \frac{M}{EI} y$$

3 $\quad \boxed{\sigma_x = -\frac{My}{I}}$

M positive, y positive $\sigma < 0$ compression on top

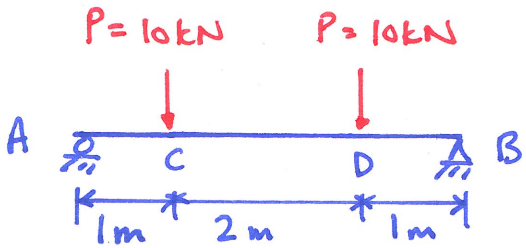
M positive, y negative $\sigma > 0$ tension on bottom ✓

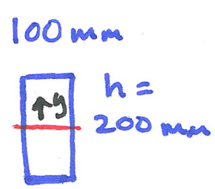
$\sigma_x \propto M$, moment

$\propto y$, distance from neutral axis

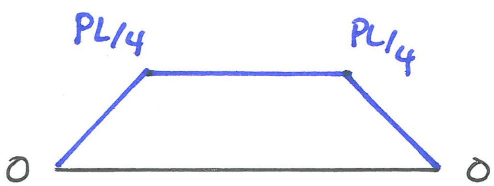
$\propto 1/I$, moment of inertia

Example : 4 point bending : $\sigma_{max} = ?$





cross-section



M diag (previous example)

$$\sigma = \frac{-My}{I}$$

$$\sigma_{max} = \frac{-M_{max}\, y_{max}}{I}$$

σ_{max} between C & D

max compression at top of beam

max tension at bottom of beam

$$M_{max} = \frac{PL}{4} = \frac{(10\,000\text{ N})(4\text{ m})}{4} = 10{,}000 \text{ N m}$$

$$y_{max} = \pm\, 0.1 \text{ m}$$

$$I = \frac{bh^3}{12} = \frac{(0.1\text{ m})(0.2\text{ m})^3}{12} = 6.67 \times 10^{-5} \text{ m}^4$$

$$\sigma_{max} = \frac{-M_{max}\, y_{max}}{I} = \frac{-(10{,}000 \text{ N m})(\pm 0.1 \text{ m})}{6.67 \times 10^{-5} \text{ m}^4}$$

$\sigma_{max} = 15 \times 10^6 \text{ N/m}^2 = 15$ MPa. (compression on top, tension on bottom).