Computing Truncated Joint Approximate Eigenbases for Model Order Reduction

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Motivation:

- Model reduction for *ND*-system realizations.
- Equivariant neural network model reduction/identification.
- Stabilization of dynamic recurrent neural network models.

Main Problem

Apoximage joint diagonalization

Given $X_1, \ldots, X_d$ in $\mathbb{R}^{n\times n}$, a d-tuple $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$. Find $W \in \mathbb{C}^{n\times r}$ with orthonormal columns such that

$$W = \arg \min_{\hat{W} \in \mathbb{C}^{n\times r}} \sum_{j=1}^{d} \left\| X_j \hat{W} - \hat{W} \Lambda_j \right\|_F^2 . \tag{1}$$

for $\Lambda_j = \in \mathbb{R}^{r\times r}$, denoting a diagonal matrix with diagonal elements realtively close to the element λ_j of λ (usually $\Lambda_j = \lambda_j I_r$).

Remark

Solutions to problem (1) can be used for model order reduction.

Quadratic localizers

Quadratic psudospectrum

Given $\varepsilon > 0$ and finitely many Hermitian matrices $X_1 \ , \ldots \ , X_d$. A d-tuple λ is an element of the quadratic ε-pseudospectrum of $(X_1 \ , \ldots \ , X_d)$ if there is unit vector $\boldsymbol{v}$ so that

$$\sqrt{\sum_{j=1}^{d} \left\| X_j \boldsymbol{v} - \lambda_j \boldsymbol{v} \right\|^2} \leq \varepsilon. \tag{2}$$

If (2) is true for $\varepsilon = 0$ then we say λ is an element of the quadratic spectrum of $(X_1 \ , \ldots \ , X_d)$.

Notation

$Q\Lambda_\varepsilon(X_1, \ldots, X_d)$ denotes the quadratic ε-pseudospectrum of $(X_1 \ , \ldots \ , X_d)$.

Autoencoders for Model Simplification

Model sparsification via local autoencoder identification

- Implicitly one computes devices of the form:

$$\xrightarrow{\mathbf{x}(t)} \boxed{\quad \mathbf{E} \quad} \xrightarrow{\mathbf{y}(t)} \boxed{\quad \mathbf{D} \quad} \xrightarrow{\hat{\mathbf{x}}(t)} \tag{3}$$

  to obtain an approximate representation $\hat{\mathbf{x}}(t)$ of the original signal $\mathbf{x}(t)$ based on coded signal $\mathbf{y}(t)$.
- The device **E** is called an *encoder*.
- The device **D** is a *decoder*.

Recursive representation of $2D$-LTI systems

$2D$-LTI system:

$$\xrightarrow{x_t^{(1)}} \boxed{\quad \mathbb{A}_1 \quad} \xrightarrow{x_{t+1}^{(1)}} \boxed{\quad \mathbb{A}_2 \quad} \xrightarrow{x_{t+1}^{(2)}} \tag{4}$$

This block diagram is equivalent to:

$$x_1(t+1) = A_1 x_1(t),\; x_2(t+1) = A_2 x_1(t+1), \tag{5}$$

Output coupling of recurrent $2D$-LTI systems

Coupled $2D$-LTI system. For the updated states generated by the recurrent system (4), we consider a system output of the form:

$$\hat{y}(t) = W \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{6}$$

QPS and model reduction

Remark

Given a discrete-time $2D$-system with states $x_1(t)$ and $x_2(t)$ in $\mathbb{R}^n$:

$$x_1(t+1) = A_1 x_1(t),\, x_2(t+1) = A_2 x_1(t+1), \qquad (7)$$
$$y(t) = W \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Let:

$$\begin{aligned} H_1 &= A_1^\top A_1, \\ H_2 &= A_2^\top A_2, \\ H_2 &= A_1^\top A_2 + A_2^\top A_1. \end{aligned}$$

QPS and model reduction

Remark

If H_1, H_2, H_3 approximately commute, we can apply the AJD algorithm presented in this contribution to find a matrix $V \in \mathbb{R}^{n\times r}$ with orthonormal columns for some suitable $0 < r \ll n$, that can be used to compute a ROM.

$$\hat{x}_1(t+1) = V^\top A_1 V \hat{x}_1(t),\ \hat{x}_2(t+1) = V^\top A_2 V \hat{x}_1(t+1),$$
$$y(t) = W \begin{bmatrix} V\hat{x}_1(t) \\ V\hat{x}_2(t) \end{bmatrix}$$

Example (LTI ROM simulation via QPS):

Let us consider the discrete-time system with states $x_1(t)$ and $x_2(t)$ in $\mathbb{R}^{400}$:

$$x_1(t+1) = A_1 x_1(t),\ x_2(t+1) = A_2 x_1(t+1), \tag{8}$$
$$y_1(t) = \hat{e}_{1,400}^{\top} x_1(t),\ y_2(t) = \hat{e}_{2,400}^{\top} x_2(t),$$

for some given matrices $A_1, A_2 \in \mathbb{R}^{400\times 400}$ such that $A_1 A_2 = A_2 A_1$, and that the corresponding hermitian matrices H_1, H_2, H_3 approximately commute.

We can compute a ROM simulation for this model obtaining the following result.

Example (LTI ROM simulation via QPS):

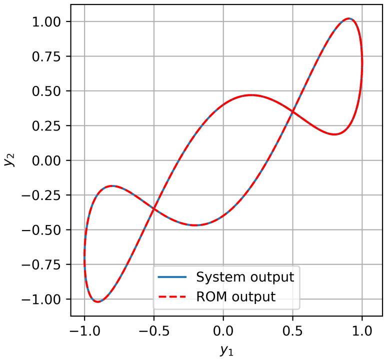

Figure: Illustration of QPS-ROM simulation for the LTI system.

Conclusion

QPS MOR techniques can be used for reduced order modeling of recurrent *ND*-system representations.

Future work

- Extend the structured matrix representation techniques presented in this contribution to other types of systems.
- Apply QPS ROM to equivariant neural network model identification/reduction.
- Combine QPS ROM Python implementations with other ROM Pyhton libraries.

References

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Questions?

Thanks!