Computing Semilinear Sparse Models for Approximately Eventually Periodic Signals

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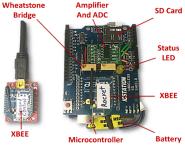
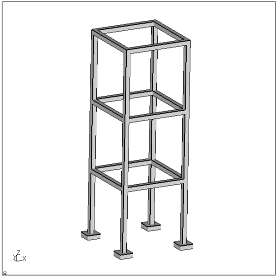
Motivation:

- Neural network model identification for approximately eventually periodic systems.
- Data driven discovery of cyber-physical systems (CPS).

Example of a periodically driven CPS

Data driven discovery of periodically driven cyber-phsycal systems





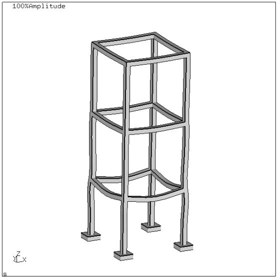


Figure: Some components of a periodically driven CPS. (Sensor courtesy of UNM CARC/SMILab)

Digital Twin Approach to Automatic Modeling of a CPS

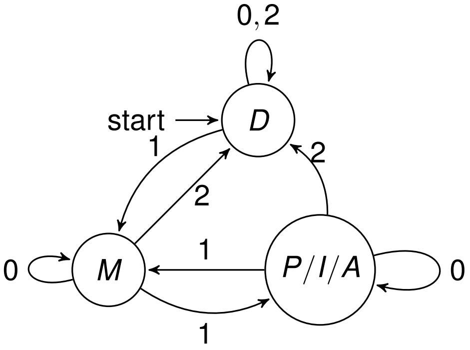


Figure: Finite automaton corresponding to a digital twin for a neural network autoregressive model.

Delay embeddings for approximately eventually periodic (AEP) signals

- A time series $\Sigma = \{x_t\}_{t \geq 1} \subset \mathbb{R}$ is approximately eventually periodic (AEP) if:
  - For each $\varepsilon > 0$, there are $S, T > 0$ such that:

    $$|x_{t+kT} - x_t| \leq \varepsilon$$

    for each $t \geq S$ and each $k > 0$.
- Given $L > 0$ and a time series $\Sigma = \{x_t\}_{t \geq 1} \subset \mathbb{R}$:

$$\mathbf{x}_L(t) = \begin{bmatrix} x_{t-L+1} & x_{t-L+2} & \cdots & x_{t-1} & x_t \end{bmatrix} \in \mathbb{R}^L$$

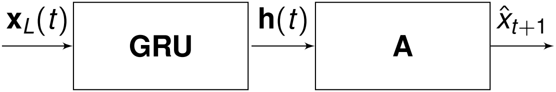
Evolution operator identification (EOI) problem:

- Given:
  - $L, M > 0$,
  - a mapping $\mathbf{T} : \mathbb{R}^L \times \mathbb{R}^{M \times R} \to \mathbb{R}^L$,
  - and a functional $\rho : \mathbb{R}^L \to \mathbb{R}$.
- Identify a matrix of parameters $\mathbf{P} \in \mathbb{R}^{M \times R}$ such that for some $N > L$:
  - $\mathbf{P} := \arg\min_{\mathbf{Q} \in \mathbb{R}^{M \times R}} \sum_{t=L}^{N} \left| x_{t+1} - \rho\left(\mathbf{T}\left(\mathbf{x}_L(t), \mathbf{Q}\right)\right)\right|^2$,
  - and $\mathbf{T}(\cdot, \mathbf{P}) \approx \mathscr{T}$.

with $\mathscr{T}(\mathbf{x}_L(t)) = \mathbf{x}_L(t+1)$ for each $L \leq t \leq N$.

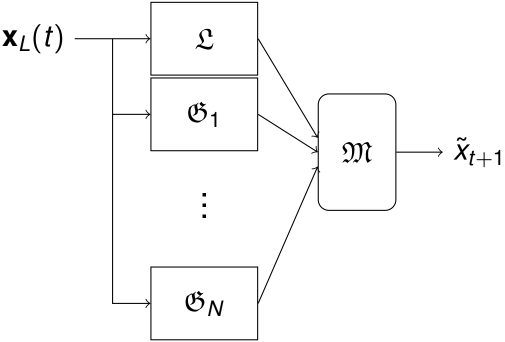
Recursive neural network representation of evolution operators

$\mathfrak{G}$ block:



(1)

Colaborative scheme for automatic modeling:



(2)

Sparse matrix representation

Sparse approximation

Given $\delta > 0$ and two matrices $X \in \mathbb{R}^{n \times m}$ and $A \in \mathbb{R}^{m \times r}$. $\hat{A}$ is an approximate sparse representation of A (with respect to X), if $\|XA - A\hat{X}\|_F \leq C\delta$ for some $C \geq 0$ that does not depend on δ, and $\hat{A}$ has fewer nonzero elements than A.

Remark

The theorem 1 in this contribution, provides a solvability criterion for sparse representation problems, in the context of Hankel matrix based system identification and model reduction.

Semilinear sparse representation techniques

- Because of the network architecture (2) we call these models *semilinear*.
- For each block $\mathbf{B}_j$ in the input layer of (2):

$$\mathbf{B}_j \longleftrightarrow (EOI)\ \mathbf{T}(\cdot, \mathbf{P})$$

- For some EOI one can apply Theorem 1 in this contribution to find a matrix of parameters $\hat{\mathbf{P}}$ (a *sparsification* of $\mathbf{P}$) such that:
  - $\mathbf{T}(\cdot, \hat{\mathbf{P}}) \approx \mathbf{T}(\cdot, \mathbf{P})$,
  - $\hat{\mathbf{P}}$ has fewer nonzero elements than $\mathbf{P}$.
- After fitting each input block $\mathbf{B}_j$, the parameters of the mixing layer of (2) are fitted/sparsified.

Autoencoders for Model Simplification

Model sparsification via local autoencoder identification

- Implicitly one computes devices of the form:

$$\xrightarrow{\mathbf{x}(t)} \boxed{\quad \mathbf{E} \quad} \xrightarrow{\mathbf{y}(t)} \boxed{\quad \mathbf{D} \quad} \xrightarrow{\hat{\mathbf{x}}(t)} \tag{3}$$

  to obtain an approximate representation $\hat{\mathbf{x}}(t)$ of the original signal $\mathbf{x}(t)$ based on coded signal $\mathbf{y}(t)$.
- The device **E** is called an *encoder*.
- The device **D** is a *decoder*.

Topologically controlled model simplification via linear autoencoders

Model simplification (sparsification) of linear approximants of evolution operators:

- Given an integer $L > 0$, one looks for autoencoders of the form:

$$\begin{tikzcd} \mathbb{R}^L \arrow[r, dashed, "\mathbf{E}"] \arrow[rr, bend left, "id_{\mathbb{R}^L}", "\approx"'] & \mathbb{R}^n \arrow[r, dashed, "\mathbf{D}"] & \mathbb{R}^L \end{tikzcd}$$

  for some $n < L$.
- together with a matrix $\mathbb{T} \in \mathbb{R}^{n\times n}$ with spectrum $\Lambda(\mathbb{T}) \subset \mathbf{S}^1$ such that

$$\mathscr{T}(\mathbf{x}_L(t)) \approx \mathbf{T}(\mathbf{x}_L(t)), \hat{\mathbf{P}}) := \mathbf{D} \circ \mathscr{T}_{\mathbb{T}} \circ \mathbf{E}(\mathbf{x}_L(t))$$

  for each $L \leq t \leq N$ and some $N > 0$.
- with $\mathscr{T}_{\mathbb{T}}(\mathbf{x}) = \mathbb{T}\mathbf{x}$.

Van Der Pol attractor signal identification:

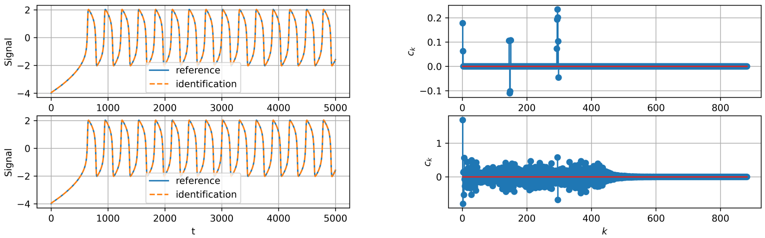


Figure: Identified signals for: SpAR model (top left), AR model (bottom left). Coefficients of linear parts of: SpAR model (top right), AR model (bottom right)

Van Der Pol attractor signal identification:

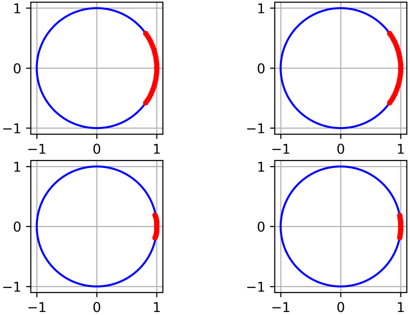


Figure: Spectra of local matrix representations $\mathbb{T}$ of linear parts of: SpAR model (top left), AR model (top right). Spectra of matrix powers $\mathbb{T}^T$ of linear parts of: SpAR model (bottom left), AR model (bottom right), for $T = 295$.

Summary for this experiment

Table: RMSE

	SpARS Model	AR Model
RMSE	0.0038019704	0.0043147522

Table: Training Data Samples Sizes

Model	Sample size
Sparse AR	899
Standard AR	1850
GRU block 1	950
GRU block 2	950
Mixing coefficients	1000

Conclusion

Semilinear sparse model representation techniques can be used for signal model identification with relatively scarce noisy data.

Future work

- Extend the structured matrix representation techniques presented in this contribution to other types of attractors.
- Consider other shallow recurrent neural network architectures.

References

1. T. Loring, F. Vides (2020). Computing Floquet Hamiltonians with Symmetries. Journal of Mathematical Physics.
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4. F. Vides. NeuralNOR GitHub web page: `https://github.com/FredyVides/SPAAR`

Questions?

Thanks!