Computing Semilinear Sparse Models for Approximately Eventually Periodic Signals

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Abstract: Some elements of the theory and algorithms corresponding to the computation of sparse semilinear models for discrete-time signals are presented. In this study, we will focus on approximately eventually periodic discrete-time signals, that is, signals that can exhibit an aperiodic behavior for an initial amount of time, and then behave approximately periodic afterwards. The semilinear models considered in this study are obtained by combining sparse representation methods, linear autoregressive models, and GRU neural network models, initially fitting each model independently using some reference data corresponding to some signal under consideration, and then fitting some of the resulting model parameters again using the reference data previously considered, computing sparse representations of some of the matrix parameters of the resulting model along the process. Some prototypical computational implementations are presented as well.

Keywords: Autoregressive models, neural-network models, parameter identification, least-squares approximation, time-series analysis.

1. INTRODUCTION

In this document, Some elements of the theory and algorithms corresponding to the computation of sparse semilinear models for discrete-time signals are presented. The study reported in this document is focused on approximately eventually periodic discrete-time signals, that is, signals that can exhibit an aperiodic behavior for an initial amount of time, and then behave approximately periodic afterwards.

The main contribution of the work reported in this document is the application of a *colaborative scheme* involving sparse matrix approximation methods, linear autoregressive type models and GRU neural network models, where each model is first fitted independently using some reference data corresponding to some given signal, and then the resulting model is represented as a linear combination of the previously fitted models, whose mixing coefficients are fitted using the previously considered reference data. Along the process, some of the matrices of parameters of the resulting model are fitted using sparse representation techniques. Some theoretical aspects of the aforementioned process are described in §3. As a byproduct of the work reported in this document, a toolset of Python programs for semilinear sparse signal model computation based on the ideas presented in §3 and §4 has been developed, and is available in Vides (2021a).

The applications of the sparse model identification technology developed as part of the work reported in this document, range from numerical simulation for predictive maintenance of industrial equipement and structures, to geological data analysis. Specific applications in the aforementioned fields will be the subject of future communications.

A prototypical algorithm that applies the ideas presented in §3 for the computation of sparse semilinear autoregressors is presented in §4. Some illustrative computational implementations of the prototypical algorithm presented in §4 are presented in §5.

2. PRELIMINARIES AND NOTATION

Given $\delta > 0$, let us consider the function defined by the expression

$$H_\delta(x) = \begin{cases} 1, & x > \delta \\ 0, & x \leq \delta \end{cases}.$$

Given a matrix $A \in \mathbb{C}^{m\times n}$ with singular values denoted by the expressions $s_j(A)$ for $j = 1, \ldots, \min\{m, n\}$. We will write $\mathrm{rk}_\delta(A)$ to denote the number

$$\mathrm{rk}_\delta(A) = \sum_{j=1}^{\min\{m,n\}} H_\delta(s_j(A)).$$

Given a time series $\Sigma = \{x_t\}_{t\geq 1} \subset \mathbb{C}$, a positive integer L and any $t \geq L$, we will write $\mathbf{x}_L(t)$ to denote the vector

$$\mathbf{x}_L(t) = [x_{t-L+1}\ x_{t-L+2}\ \cdots\ x_{t-1}\ x_t]^\top \in \mathbb{C}^L.$$

Given an ordered sample $\Sigma_N = \{x_t\}_{t=1}^N \subset \Sigma$ from a time series $\Sigma = \{x_t\}_{t\geq 1}$, we will write $\mathcal{H}_L(\Sigma_N)$ to denote the Hankel type trajectory matrix corresponding to Σ_N, that is determined by the following expression.

$$\mathcal{H}_L(\Sigma_N) = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{N-L+1} \\ x_2 & x_3 & x_4 & \cdots & x_{N-L+2} \\ \vdots & \vdots & \vdots & .\cdot^{\cdot} & \vdots \\ x_L & x_{L+1} & x_{L+2} & \cdots & x_N \end{bmatrix}$$

We will write I_n to denote de identity matrix in $\mathbb{C}^{n\times n}$, and we will write $\hat{e}_{j,n}$ to denote the matrices in $\mathbb{C}^{n\times 1}$ representing the canonical basis of $\mathbb{C}^n$ (each $\hat{e}_{j,n}$ is the j-column of I_n).

We will write $\mathbf{S}^1$ to denote the set $\{z \in \mathbb{C} : |z| = 1\}$. Given any matrix $X \in \mathbb{C}^{m\times n}$, we will write X^* to denote the conjugate transpose $\overline{X}^\top \in \mathbb{C}^{n\times m}$ of X. A matrix $P \in \mathbb{C}^{n\times n}$ will be called an orthogonal projector whenever $P^2 = P = P^*$. Given any matrix $A \in \mathbb{C}^{n\times n}$, we will write $\sigma(A)$ to denote the spectrum of A, that is, the set of eigenvalues of A.

3. SEMILINEAR MODELING OF APPROXIMATELY EVENTUALLY PERIODIC SIGNALS

A discrete-time signal represented by a times series $\Sigma = \{x_t\}_{t\geq 1}$ is said to be approximately eventually periodic (**AEP**) if it can be aperiodic for an initial amount of time, and then becomes approximately periodic afterwards. In other words, there are $\varepsilon > 0$ and two integers $T, S > 0$ such that $|x_{t+kT} - x_t| \leq \varepsilon$ for each $t \geq S$ and each integer $k \geq 0$. The integer T will be called the approximate period of Σ.

Remark 1. Based on the notion of approximately eventually periodic signal considered on this study, it can be seen that given an AEP signal $\Sigma = \{x_t\}_{t\geq 1}$, there is a positive integer S such that the tail $\{x_t\}_{t\geq S}$ of Σ is approximately periodic.

3.1 Sparse Semilinear Autogregressors

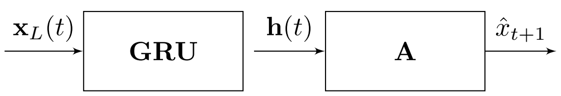
Given a time series $\Sigma = \{x_t\}_{t\geq 1} \subset \mathbb{C}$ and a lag value $L > 0$. Let us consider a semilinear signal model of the form:

$$x_{t+1} = \mathcal{L}(\boldsymbol{x}_L(t)) + \mathcal{G}(\boldsymbol{x}_L(t)) + \mathcal{E}(\boldsymbol{x}_L(t)), t \geq L. \quad (1)$$

where $\mathcal{L}$ denotes a linear operation determined by the expression

$$\mathcal{L}(\boldsymbol{x}_L(t)) = c_1 x_t + c_2 x_{t-1} + c_3 x_{t-2} + \cdots + c_L x_{t-L+1}, \quad (2)$$

the term $\mathcal{G}(\mathbf{x}_L(t))$ represents a linear combination of neural networks whose structure can be described by the block diagram



(3)

and $\mathcal{E}(\boldsymbol{x}_L(t))$ represents some suitable error term. For some given an integer $m > 0$, each GRU j-cell in the GRU block in (3) is described for each $j = 1, \ldots, m$ by the following equations:

$$\begin{aligned} r_j(t) &= \sigma\left(\hat{e}_{j,m}^\top \left(W_{ir}\mathbf{x}(t) + W_{hr}\mathbf{h}(t-1) + b_r\right)\right) \\ z_j(t) &= \sigma\left(\hat{e}_{j,m}^\top \left(W_{iz}\mathbf{x}(t) + W_{hz}\mathbf{h}(t-1) + b_z\right)\right) \\ n_j(t) &= \tanh(\hat{e}_{j,m}^\top \left(W_{in}\mathbf{x}(t) + b_n\right) \\ &\quad + r_j(t)\hat{e}_{j,m}^\top \left(W_{hn}\mathbf{h}(t-1)\right)) \\ h_j(t) &= (1 - z_j(t))n_j(t) + z_j(t)h_j(t-1) \end{aligned} \quad (4)$$

with $\mathbf{x}(t) = \mathbf{x}_L(t)$ and $\mathbf{h}(t) = [h_1(t) \cdots h_m(t)]^\top$, and where σ denotes the sigmoid function. The GRU layer configuration considered in this document has been chosen in order to prevent vanishing gradients, by taking advantage of the GRU structure presented in Cho et al. (2014).

The affine layer $\mathbf{A}$ of the neural network $\mathcal{G}$ described in (3) is determined by the expression

$$\mathbf{A}(\mathbf{h}(t)) = \mathbf{w}_A^\top \mathbf{h}(t) + b_L.$$

In order to compute models of the form (1), we can combine sparse autoregressive models of the form (2) that can be computed using the techniques presented in Vides (2021b), with GRU RNN models of the form (3) that can be computed using the computational tools provided as part of TensorFlow, Keras and PyTorch, that are described as part of Chollet et al. (2015) and Paszke et al. (2019).

An approximate representation

$$\tilde{\mathcal{L}}(\mathbf{x}_L(t)) = \tilde{c}_1 x_t + \tilde{c}_2 x_{t-1} + \tilde{c}_2 x_{t-2} + \cdots + \tilde{c}_L x_{t-L+1},$$

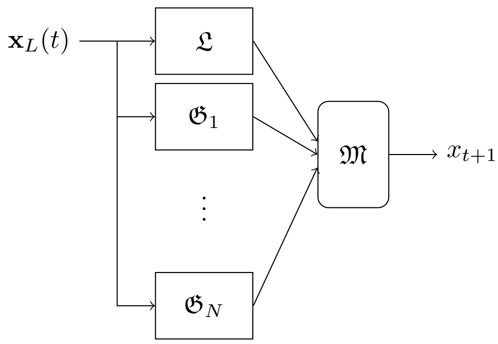
of the linear part of (1) such that

$$x_{t+1} \approx \hat{\mathcal{L}}(\mathbf{x}_L(t)), t \geq L$$

can be computed using some sample $\Sigma_N = \{x_t\}_{t=1}^N$ and a corresponding subsample $\Sigma_0 = \{x_t\}_{t=1}^{N-1} \subset \Sigma_N$ for some suitable $N > L$, by approximately solving the matrix equation

$$\mathcal{H}_L(\Sigma_0)^\top \begin{bmatrix} c_L \\ c_{L-1} \\ \vdots \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} x_{N-L+1} \\ x_{N-L+2} \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}. \quad (5)$$

Schematically the semilinear autoregressors considered in this study can be described by a block diagram of the form,



(6)

where the block $\mathfrak{L}$ is represented by (2), each block $\mathfrak{G}_j$ is represented by (3), and where the block $\mathfrak{M}$ is a *mixing* block determined by the expression

$$\mathfrak{M}(y_1(t), \ldots, y_{N+1}(t)) = \sum_{j=1}^{N+1} w_j y_j(t),$$

for some coefficients w_j to be determined and some given N, with $y_1(t) = \mathfrak{L}(\mathbf{x}_L(t))$ and $y_{k+1}(t) = \mathfrak{G}_k(\mathbf{x}_L(t))$ for each $k = 1, \ldots, N$ and each $t \geq L$.

The details of the computation of the neural network blocks of the model (1) will be omitted for brevity, for details on the theory and computation of the GRU NN models considered for this study the reader is kindly referred to Cho et al. (2014), Chollet et al. (2015), Paszke et al. (2019) and Vides (2021a).

Several interesting papers have been written on the subject of hybrid time series models that combine ARIMA and ANN, two important references on this matter are Zhang (2003) and Khandelwal et al. (2015). An important distinctive aspect of the modeling approach reported in this document, is that instead of using the GRU RNN components of (6) represented by $\mathcal{G}$ in (1) to approximate the residual $r_t = x_{t+1} - \mathcal{L}(\mathbf{x}_L(t))$. Using some suitable training subsets Σ_I, Σ_M of a given data sample Σ_N from an arbitrary AEP signal $\Sigma = \{x_t\}_{t\geq 1}$ under consideration, first the parameters of the blocks $\mathfrak{L}, \mathfrak{G}_1, \cdots, \mathfrak{G}_N$ of (6) are fitted using Σ_I, and then the coefficients of the block $\mathfrak{M}$ of (6) are fitted using Σ_M and some corresponding predicted values generated with $\mathfrak{L}$, $\mathfrak{G}_1$, $\cdots$, $\mathfrak{G}_N$. Computing sparse representations of some of the matrix parameters of the resulting model along the process.

3.2 An Operator Theoretic Approach to Semilinear Sparse Autoregressors

Given an AEP signal $\Sigma = \{x_t\}_{t\geq 1}$ whose behavior can be approximately described by a model of the form (2), that can be computed by approximately solving an equation of the form (5), and given some sample $\Sigma_0 = \{x_t\}_{t=1}^{N-1}$, if we consider any sample $\tilde{\Sigma}_0 = \{\tilde{x}_t\}_{t=1}^{N-1} \subset \Sigma$, such that for some positive integer S the states in $\tilde{\Sigma}_0$ satisfy the conditions $\tilde{x}_t = x_{t+S}$, for each $t = 1, \ldots, N-1$. We will have that the matrix

$$C_L = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 \\ c_L & c_{L-1} & \cdots & \cdots & c_2 & c_1 \end{bmatrix} \quad (7)$$

will *approximately* satisfy the condition

$$\mathcal{H}_L(\Sigma_0)^\top \left(C_L^S\right)^\top = \mathcal{H}_L(\tilde{\Sigma}_0)^\top. \quad (8)$$

Using matrices of the form (7) one can express linear models of the form (2) as follows.

$$\mathcal{L}(\mathbf{x}_L(t)) = \hat{e}_{L,L}^\top C_L \mathbf{x}_L(t) \quad (9)$$

One can observe that to each model of the form (2), there corresponds a matrix of the form (7). From here on, a matrix that satisfies the previous conditions will be called the matrix form of a linear model $\mathcal{L}$ of the form (2).

Given $\delta > 0$, and two matrices $A \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{m\times p}$, let us write $AX \approx_\delta Y$ to represent the problem of finding $X \in \mathbb{C}^{n\times p}$, $\alpha, \beta \geq 0$ and an orthogonal projector Q such that $\|AX - Y\|_F \leq \alpha\delta + \beta\|(I_m - Q)Y\|_F$. The matrix X will be called a solution to the problem $AX \approx_\delta Y$.

As a consequence of (Vides, 2021b, Theorem 3.6) we can obtain the following result.

Theorem 2. Given $\delta > 0$, two integers $L, M > 0$, a sample $\Sigma_N = \{x_t\}_{t=1}^N$ from an approximately eventually periodic signal $\Sigma = \{x_t\}_{t\geq 1}$ and a matrix $A \in \mathbb{C}^{L\times M}$. If $r = \mathrm{rk}_\delta(\mathcal{H}_L(\Sigma_N)) > 0$, then there is a sparse matrix $\hat{A} \in \mathbb{C}^{L\times M}$ with at most Mr nonzero entries such that $\mathcal{H}_L(\Sigma_N)^\top \hat{A} \approx_\delta \mathcal{H}_L(\Sigma_N)^\top A$.

Proof. Since $\mathrm{rk}_\delta(\mathcal{H}_L(\Sigma_N)^\top) = \mathrm{rk}_\delta(\mathcal{H}_L(\Sigma_N)) > 0$. This result is a consequence of the application of (Vides, 2021b, Theorem 3.6) to the problem $\mathcal{H}_L(\Sigma_N)^\top \hat{A} \approx_\delta \mathcal{H}_L(\Sigma_N)^\top A$.

Remark 3. Given some AEP signal under consideration $\Sigma = \{x_t\}_{t\geq 1}$ with approximate period T, if the corresponding residuals $r_t = |x_{t+1} - \mathcal{L}(\mathbf{x}_L(t))|$ are small, then the significative contribution of the linear component $\mathcal{L}$ of (1) to the modeling process of Σ, would be beneficial for interpretability purposes. Also, if we consider the tail $\tilde{\Sigma} = \{x_t\}_{t\geq S}$ of Σ, by applying a Krylov space approach along the lines presented in (Saad, 2011, §6.1), and as a consequence of (Vides, 2021b, Theorem 4.3.), one would expect that there are $\varepsilon > 0$ and some matrix $W_k \in \mathcal{C}^{L\times k}$ whose columns form an orthonormal basis of span $(\{x_S, C_L x_S, C_L^2 x_S, \ldots, C_L^{T-1} x_S\})$, such that each $z \in \sigma(W_k^* C_L W_k)$ satisfies the relation $|z^T - 1| \leq \varepsilon$. This interesting *approximate periodicity feature* will be further explored in future communications.

The matrix $W_k^* C_L W_k$ will be called the approximately periodic (**AP**) Σ-section of C_L and will be denoted by $C_L|_\Sigma^{AP}$.

Remark 4. When a given AEP signal $\Sigma = \{x_t\}_{t\geq 1}$ with approximate period T is *well explained* by the linear component of a semilinear model, that is, when the corresponding residual is relatively small, one would expect that the matrix $C_L|_\Sigma^{AP}$ corresponding to the model would *mimic* the approximate periodicity of the tail $\{x_t\}_{t\geq S}$ of Σ, in the sense that the number $\|(C_L|_\Sigma^{AP})^T - I_n\|$ would be relatively small for some suitable matrix norm $\|\cdot\|$ (in the sense of (Saad, 2011, §1.5)). Ideally, when plotting $\sigma((C_L|_\Sigma^{AP})^T)$ one should observe the elements of $\sigma((C_L|_\Sigma^{AP})^T)$ clustering around 1.

4. ALGORITHMS

As an applications of the results in section §3.1 we can obtain a prototypical algorithm represented by algorithm 1, that relies on (Vides, 2021b, Algorithm 1), Theorem 2 and (Vides, 2021b, Theorem 4.3.).

Algorithm 1: SLSpARModel: Semilinear Sparse Autoregressor Algorithm

Data: $\Sigma_N = \{x_t\}_{t=1}^N \subset \mathbb{C}^{n\times 1}$
Result: $\boldsymbol{c}, \mathfrak{G}_1, \ldots, \mathfrak{G}_j, \mathbf{w}_M = \mathbf{SpAutoregressor}(\Sigma_N)$

0: Estimate the lag value L using auto-correlation function based methods;
1: Approximately solve (5) for $\mathbf{c}$ using the reference data Σ_N and applying (Vides, 2021b, Algorithm 1);
2: Fit the blocks $\mathfrak{G}_j$ of (6) using the data in Σ_N.
3: For the GRU layers of each $\mathfrak{G}_j$, compute sparse representations of the corresponding input weights W_{ir}, W_{iz}, W_{in} determined by (4) when appropriate, applying Theorem 2.
4: Compute the coefficients $\mathbf{w}_M = (w_1, \ldots, w_{N+1})$ of the block $\mathfrak{M}$ of (6) using Σ_N and (Vides, 2021b, Algorithm 1);

return $\boldsymbol{c}, \mathfrak{G}_1, \ldots, \mathfrak{G}_j, \mathbf{w}_M$

We can apply algorithm 1 to compute the model parameters needed for the computation of signal models of the form (6).

5. NUMERICAL EXPERIMENTS

In this section, some computational implementations of the methods reported in this document are presented. Most of the numerical experiments documented in this section were performed on a Linux Ubuntu Server 20.04 workstation equiped with an Intel Core i7 processor with 6 cores and with 8GB RAM. Some numerical experiments where also performed on Google Colab and on IBM Quantum Lab.

The experimental results documented in this section can be replicated using the function `NumericalExperiment.py` or the Jupyter notebook `SLSpAARModelsDemo.ipynb`, that are available in Vides (2021a). The configuration required to replicate the results in this section is available as part of the aforementioned programs.

Since the models considered in this study consist of linear combinations of sparse autoregressive models with GRU RNN based models, we will refer to models of this type as **SpARGRU** models. The signal approximations computed using the SpARGRU models presented in this document are compared with the approximations obtained using standard AR models. The corresponding standard AR models are computed using the python program `Autoreg` included as part of `statsmodels` module. In this section we will write nz to denote nonzero elements.

For the experiments documented in this section two GRU RNN blocks were used, the block $\mathfrak{G}_1$ was computed using TensorFlow 2.6.0 and its input weights were replaced by their corresponding sparse representations, that were computed using (Vides, 2021b, Algorithm 1) along the lines of Theorem 2, and the block $\mathfrak{G}_2$ was computed using PyTorch 1.9.1+cpu and its input weights were left unchanged.

5.1 Numerical Experiment 1

In this section, algorithm 1 is applied to compute a SpARGRU model for the signal data sample recorded in the csv file `AlmostPeriodicSignal.csv` in the DataSets folder in Vides (2021a). The graphic representations of the results produced by the command sequence

```
>>> NumericalExperiment(1)
```

are shown in figures 1 and 2, respectively.

In every figure like figure 2, the red dots represent the points in each considered spectrum, the blue lines represent $\mathbf{S}^1$, and the number T represents the estimated approximate period for each signal considered.

5.2 Numerical Experiment 2

In this section, algorithm 1 is applied to compute a SpARGRU model for the signal data sample recorded in the csv file `NLOscillatorSignal.csv` in the DataSets folder in Vides (2021a). The graphic representations of the results produced by the command sequence

```
>>> NumericalExperiment(2)
```

are shown in figures 3 and 4, respectively.

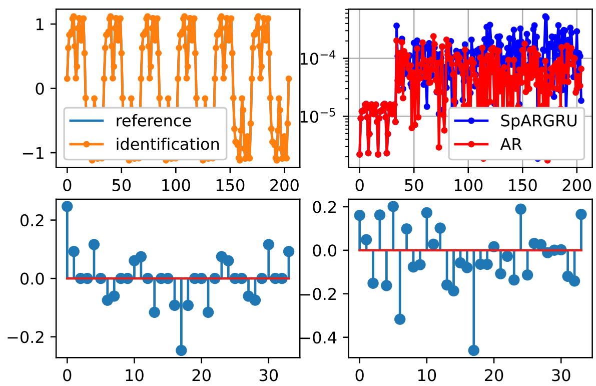


Fig. 1. Reference and identified signals (top left). Approximation errors (top right). Linear component parameters of the SpARGRU model with 18 nz (bottom left). Parameters of the linear component of standard AR model with 34 nz (bottom right).

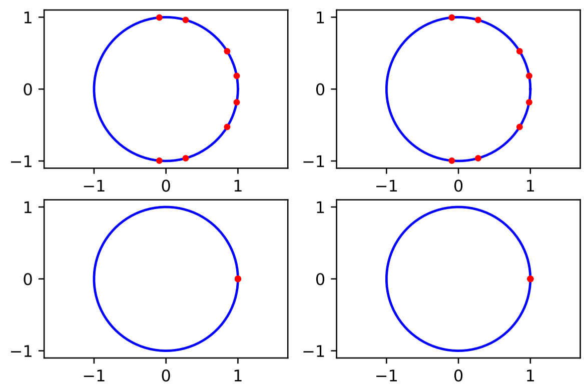


Fig. 2. $\sigma(C_L|_{\Sigma}^{AP})$ for the linear component of the SpARGRU model (top left). $\sigma(C_L|_{\Sigma}^{AP})$ for the linear component of the standard AR model (top right). $\sigma((C_L|_{\Sigma}^{AP})^T)$ for the linear component of the SpARGRU model (bottom left). $\sigma((C_L|_{\Sigma}^{AP})^T)$ for the linear component of the standard AR model (bottom right).

5.3 Numerical Experiment 3

In this section, algorithm 1 is applied to compute a SpARGRU model for the signal data sample recorded in the csv files:

- `art_daily_no_noise.csv`
- `art_daily_small_noise.csv`

that are included as part of the datasets described in Ahmad et al. (2017). The graphic representations of the results produced by the command sequence

```
>>> NumericalExperiment(3.1)
```

for the periodic signal without noise are shown in figures 5 and 6, respectively. The graphic representations of the results produced by the command sequence

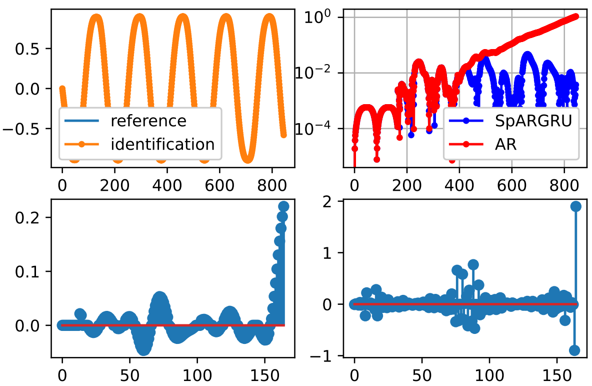


Fig. 3. Reference and identified signals (top left). Approximation errors (top right). Linear component parameters of the SpARGRU model with 118 nz (bottom left). Parameters of the linear component of standard AR model with 165 nz (bottom right).

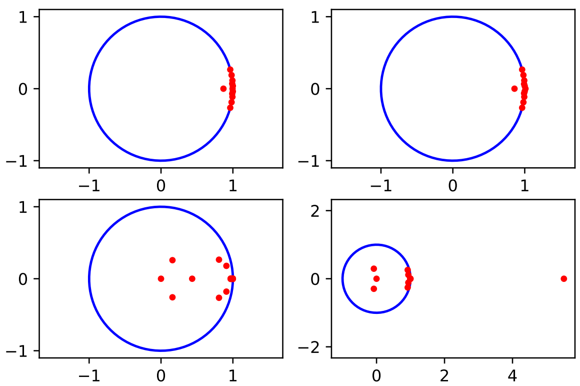


Fig. 4. $\sigma(C_L|_{\Sigma}^{AP})$ for the linear component of the SpARGRU model (top left). $\sigma(C_L|_{\Sigma}^{AP})$ for the linear component of the standard AR model (top right). $\sigma((C_L|_{\Sigma}^{AP})^T)$ for the linear component of the SpARGRU model (bottom left). $\sigma((C_L|_{\Sigma}^{AP})^T)$ for the linear component of the standard AR model (bottom right).

```
>>> NumericalExperiment(3.2)
```

for the periodic signal with noise are shown in figures 7 and 8.

5.4 Approximation Errors

The approximation root mean square errors are summarized in table 1.

Table 1. RMSE

Model	SpARGRU Model	AR Model
Experiment 1	0.0001603580	0.0001450925
Experiment 2	0.0136251438	0.3100591516
Experiment 3.1	0.0000000000	0.0000000074
Experiment 3.2	4.0464557816	4.0939437825

It is appropriate to mention that the root mean square errors can present little fluctuations as one performs several numerical simulations, due primarily to the nature of the neural-network models, as the linear components tend to present very low or no variability from simulation to simulation.

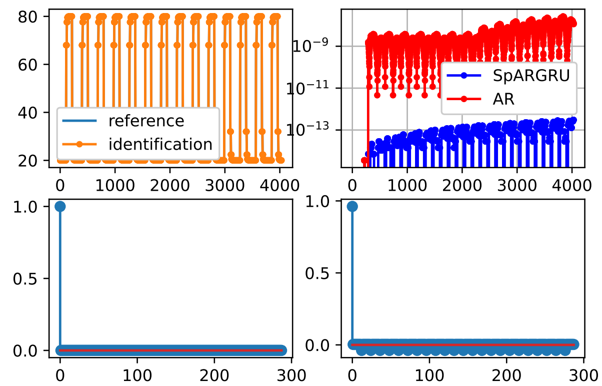


Fig. 5. Reference and identified signals (top left). Approximation errors (top right). Linear component parameters of the SpARGRU model with 8 nz (bottom left). Parameters of the linear component of standard AR model with 288 nz (bottom right).

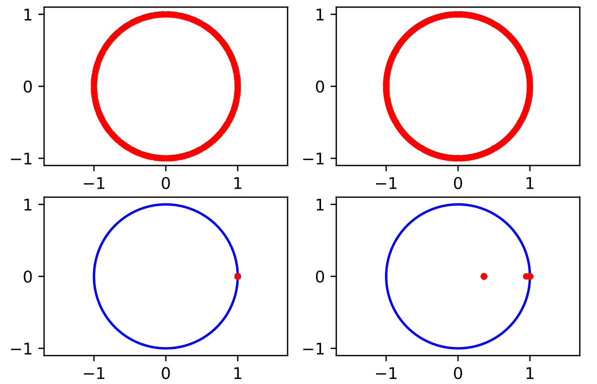


Fig. 6. $\sigma(C_L|_{\Sigma}^{AP})$ for the linear component of the SpARGRU model (top left). $\sigma(C_L|_{\Sigma}^{AP})$ for the linear component of the standard AR model (top right). $\sigma((C_L|_{\Sigma}^{AP})^T)$ for the linear component of the SpARGRU model (bottom left). $\sigma((C_L|_{\Sigma}^{AP})^T)$ for the linear component of the standard AR model (bottom right).

5.5 Data Availability

The Python programs that support the findings of this study are openly available in the SPAAR repository, reference number Vides (2021a). The time series data used for the experiments 1 and 2 documented in §5 are available as part of Vides (2021a), and the time series data used for experiment 3 in §5 are available as part of the Numenta Anomaly Benchmark (NAB) described in Ahmad et al. (2017).

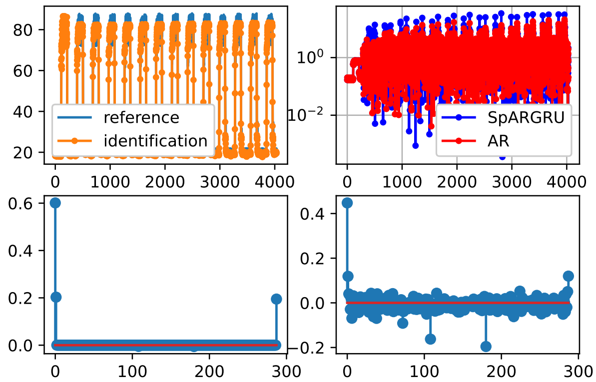


Fig. 7. Reference and identified signals (top left). Approximation errors (top right). Linear component parameters of the SpARGRU model with 5 nz (bottom left). Parameters of the linear component of standard AR model with 288 nz (bottom right).

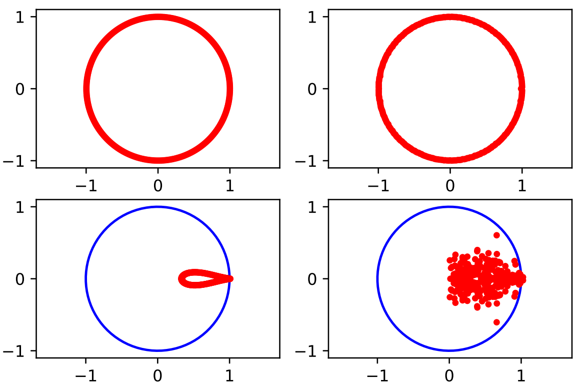


Fig. 8. $\sigma(C_L|_{\Sigma}^{AP})$ for the linear component of the SpARGRU model (top left). $\sigma(C_L|_{\Sigma}^{AP})$ for the linear component of the standard AR model (top right). $\sigma((C_L|_{\Sigma}^{AP})^T)$ for the linear component of the SpARGRU model (bottom left). $\sigma((C_L|_{\Sigma}^{AP})^T)$ for the linear component of the standard AR model (bottom right).

6. CONCLUSION

The results observed in §5 are consistent with the theoretical elements presented in section §3. In particular, although in some experiments in §5 the corresponding root mean square errors corresponding to the AR and SpARGRU models are similar, the AP Σ-sections corresponding to the SpARGRU models exhibit a better *mimetic approximately periodic behavior* in the sense of remarks 3 and 4, as it can be visualized in figures 2, 4, 6 and 8. This mimetic behavior is interesting not just from a theoretical point of view, as it provides a criteria for how well the linear component of a given model *mimics or captures* the eventual approximate periodic beavior of the signal under study, but also for practical computational reasons, as long term predictions or simulations can be affected when the eigenvalues of the AP Σ-section of the matrix form corresponding to the linear component of a signal model, lie outside the set $\mathbf{D}^1 = \{z \in \mathbb{C} : |z| \leq 1\}$, as one can observe in figures 3 and 4. Another advantage of the sparsity of the linear component of the SpARGRU models is the reduction of the computational complexity of the corresponding linear component, when compared to a nonsparse linear model based on the same reference data.

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The structure preserving computations corresponding to the numerical experiments documented in this paper were performed with computational resources from the Scientific Computing Innovation Center of UNAH, as part of the research project PI-063-DICIHT. Some experiments were performed on Google Colab and IBM Quantum Lab as well.

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