Mohamed Noordeen Alaudeen

Time Series

Typical Time Series

$$\hat{y}_{t+1} = f(y_t, y_{t-1}, y_{t-2} \dots) + f(x_1, x_2, x_3 \dots)$$

f can be linear or nonlinear

Important Concepts

Time Series Descriptive Statistics

- In descriptive statistics covered earlier (central tendencies, measures of variability, skewness, kurtosis, distributions, correlations, etc.), the order of observations in the data was of no consequence.

- In time series descriptive statistics, order of observations is of primary importance and so autocorrelations, etc. play a vital role in identifying the models and their characteristics.

Autocorrelation (ACF) and Partial ACF (PACF)

- ACF: n^{th} lag of ACF is the correlation between a day and n days before that.

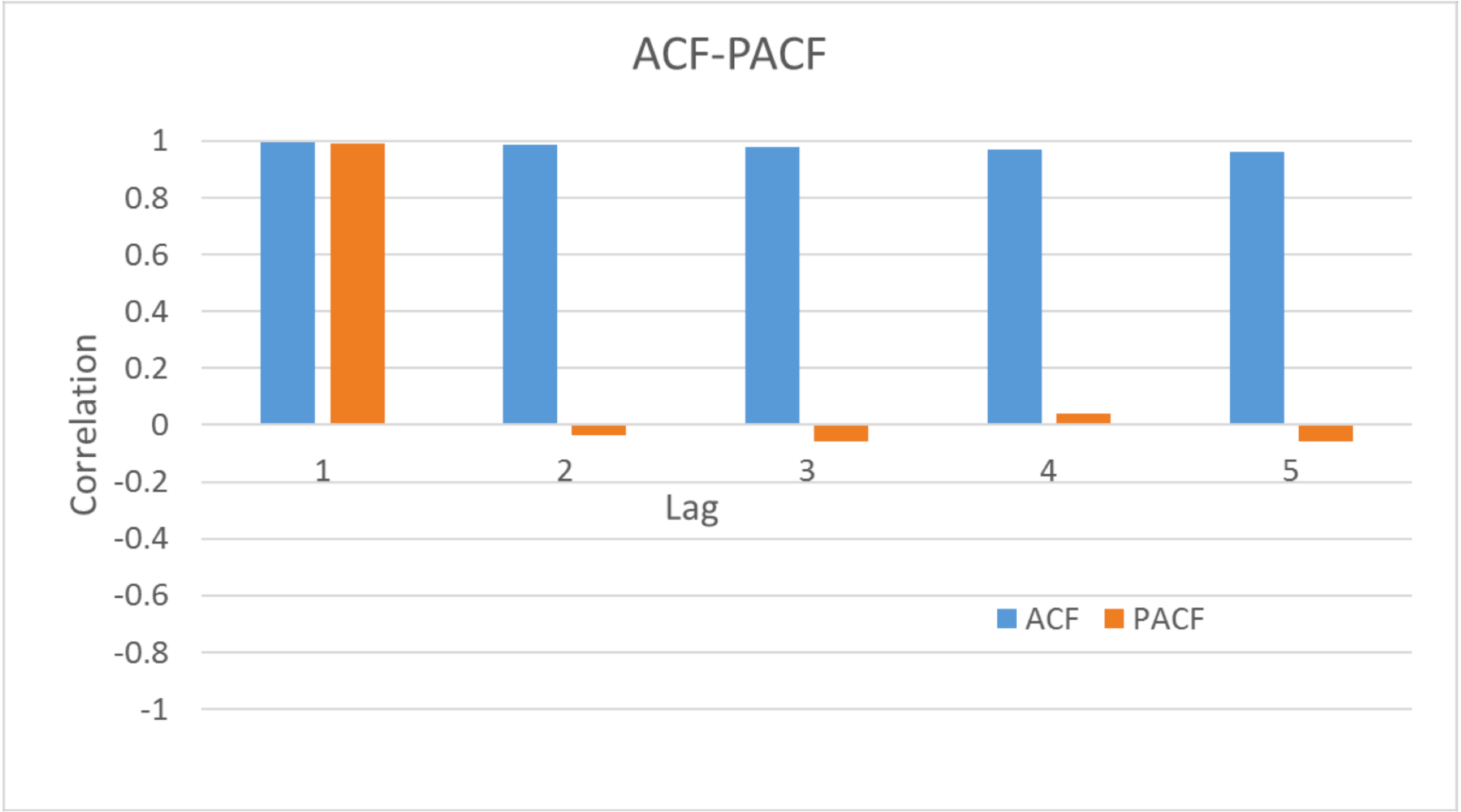
- PACF: The same as ACF with all intermediate correlations removed. It is the k_{th} coefficient of the ordinary least squares regression.

$$[y_t] = \beta_0 + \sum_{i=1}^{k} \beta_i [y_{t-i}] \text{ where}$$

$[y_t]$ is the input time series, k is the lag order and β_i is the i_{th} coefficient of the linear multiple regression.

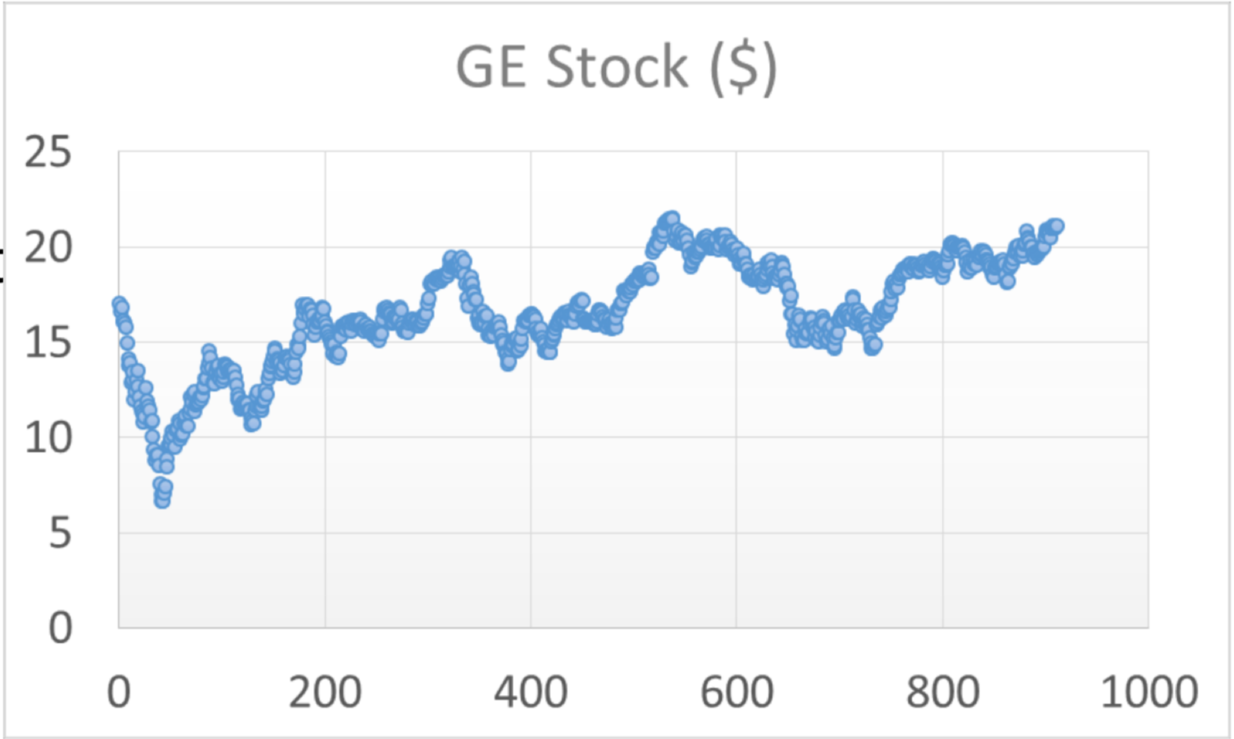
Excel Activity for ACF and PACF

Autocorrelation (ACF) and Partial ACF (PACF)

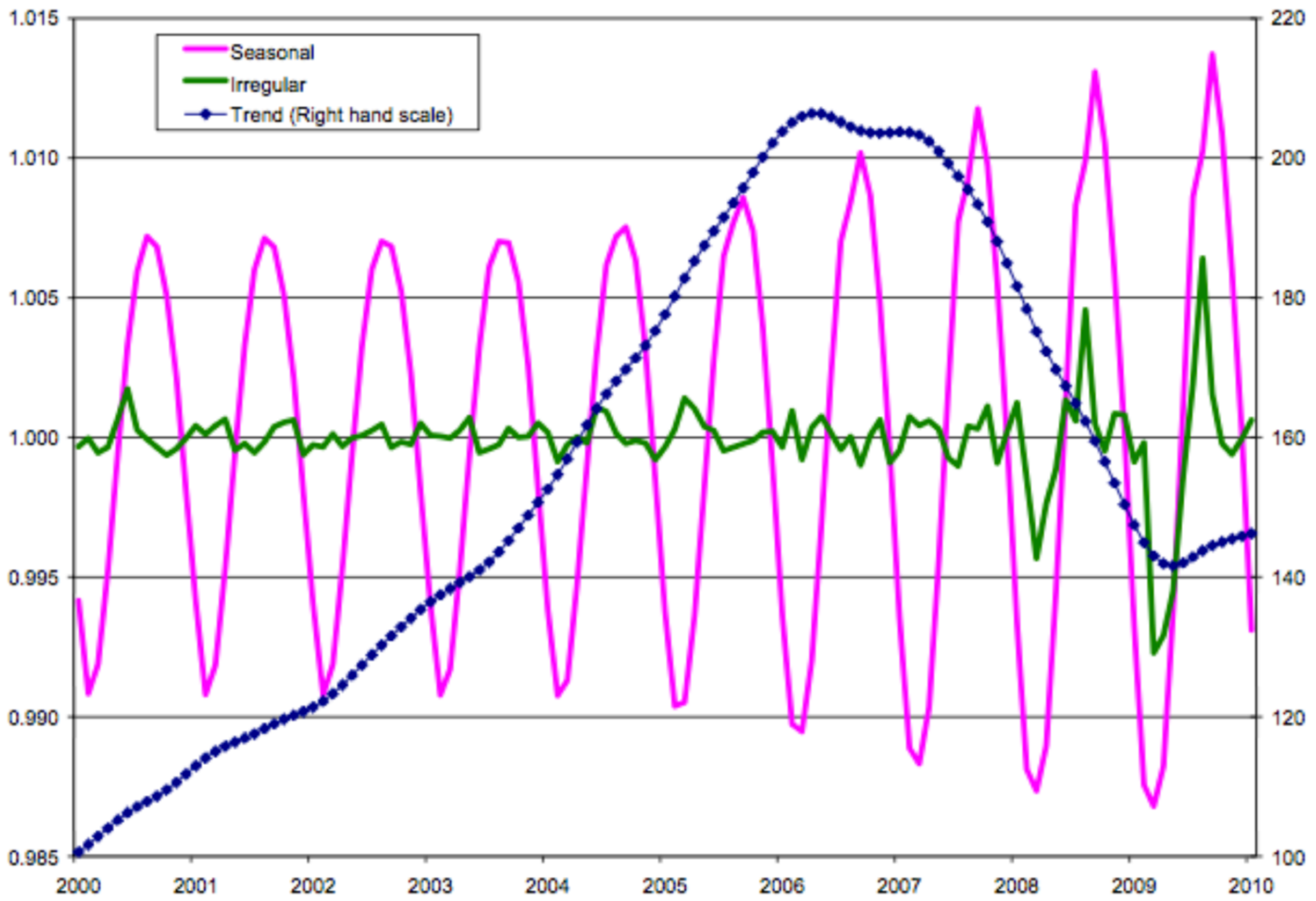


Components of Time Series

- Trend
- Seasonality
- Random component



Trend, Seasonality and Randomness

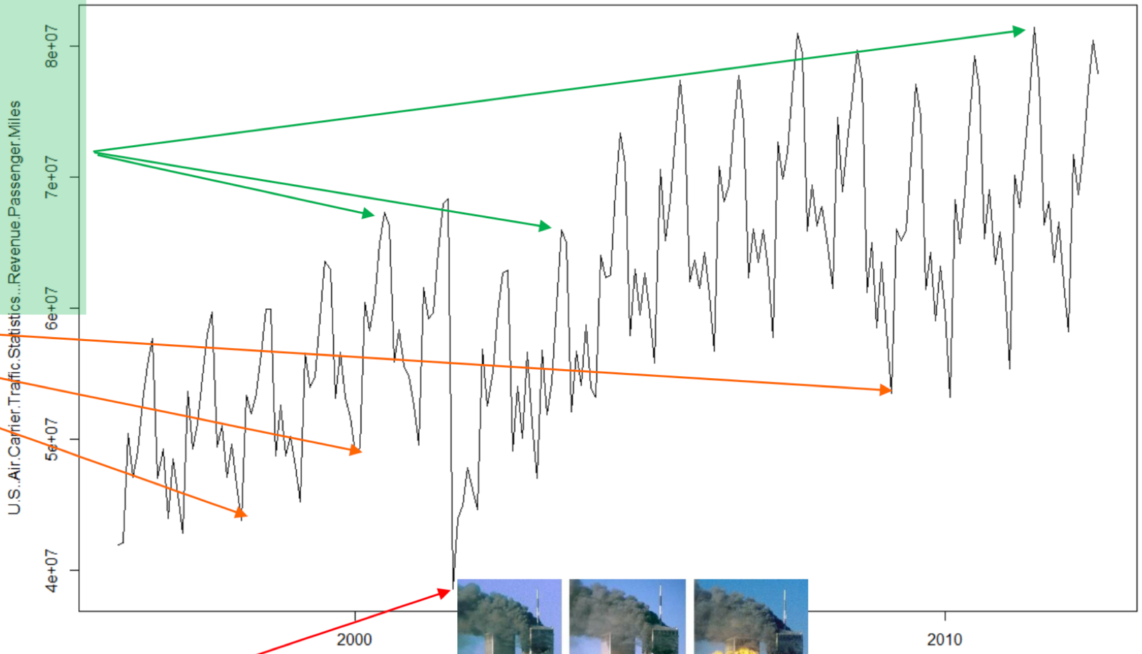


US Air Carrier Traffic – Revenue Passenger Miles (‘000) *RPM*

```
> milestimeseries <- ts(miles, frequency = 12, start = c(1996,1))
> milestimeseries
```

	Jan	Feb	Mar	Apr	May	Jun	Jul
1996	41972194	42054796	50443045	47112397	49118248	52880510	
1997	45850623	42838949	53620994	49282817	51191842	54707221	
1998	46514139	43769273	53361926	51968480	53515798	56460422	
1999	47988560	45241211	56555731	53920855	54674958	59213000	
2000	49045412	49306303	60443541	58286680	60533783	64903295	
2001	52634354	49532578	61575055	59151645	59662416	64353323	
2002	46224031	44615129	56897729	52542164	55116060	59745343	
2003	51197175	47040806	56766580	51857453	54335598	60272900	
2004	53979786	53179693	64035864	62340117	62530704	68866398	
2005	59629608	55795165	70595861	65145552	68268899	72952959	
2006	61035027	56729212	70799794	68120559	69352606	74099239	
2007	63016013	57793832	72700241	69836156	71933109	76926452	
2008	64667106	61504426	74575531	68906882	72725750	76162105	
2009	58373786	53506580	66027341	65166300	65868254	71350227	
2010	59651061	53240066	68307090	64953250	68850904	74474550	
2011	61630362	55391206	70158268	67683558	71711448	76057910	
2012	61940180	58243763	71696039	68669228	71887523	76760759	

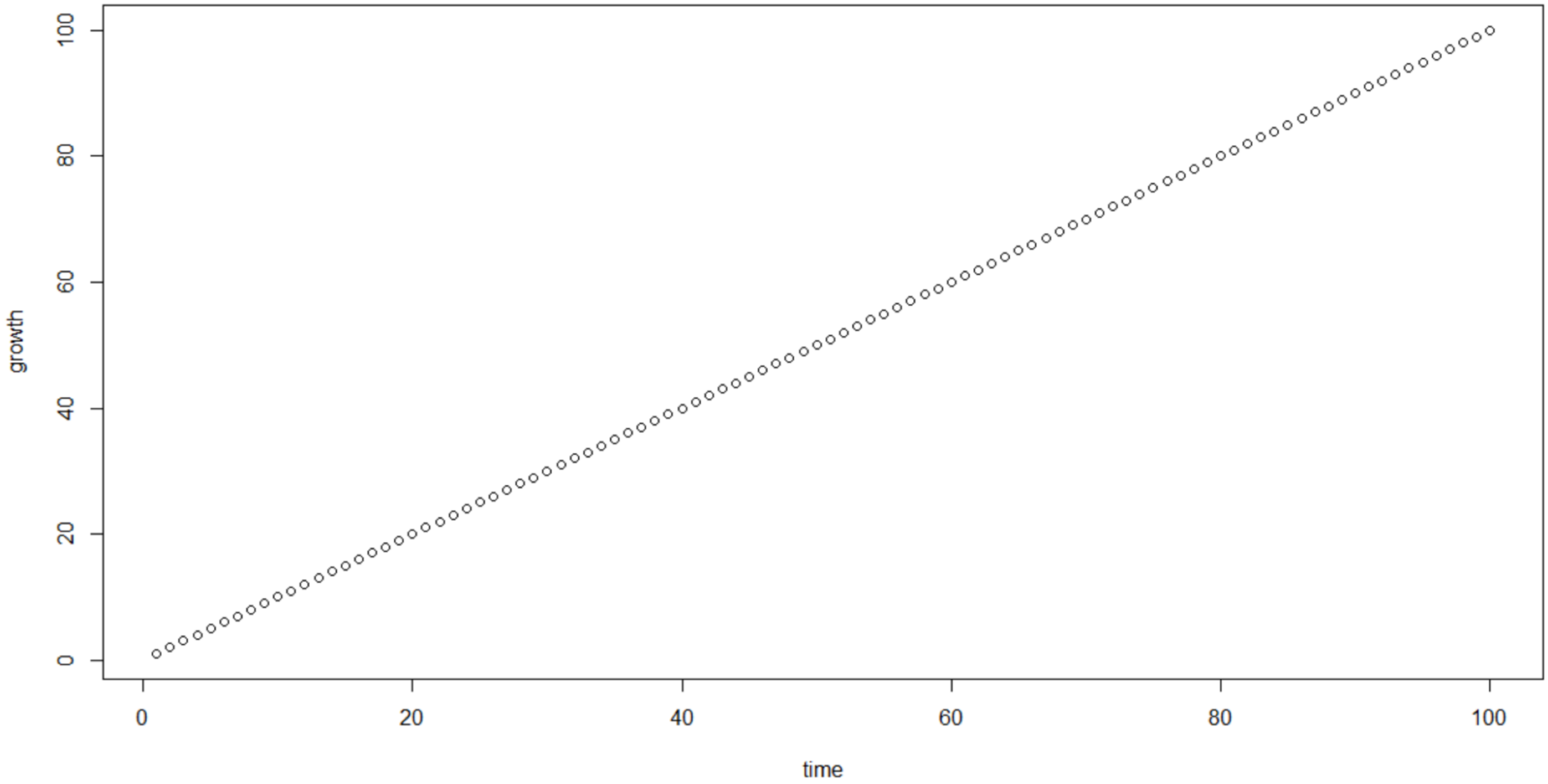
	Aug	Sep	Oct	Nov	Dec
1996	57723208	47035464	49263120	43937074	48539606
1997	59715433	49418190	51058879	47056048	49654209



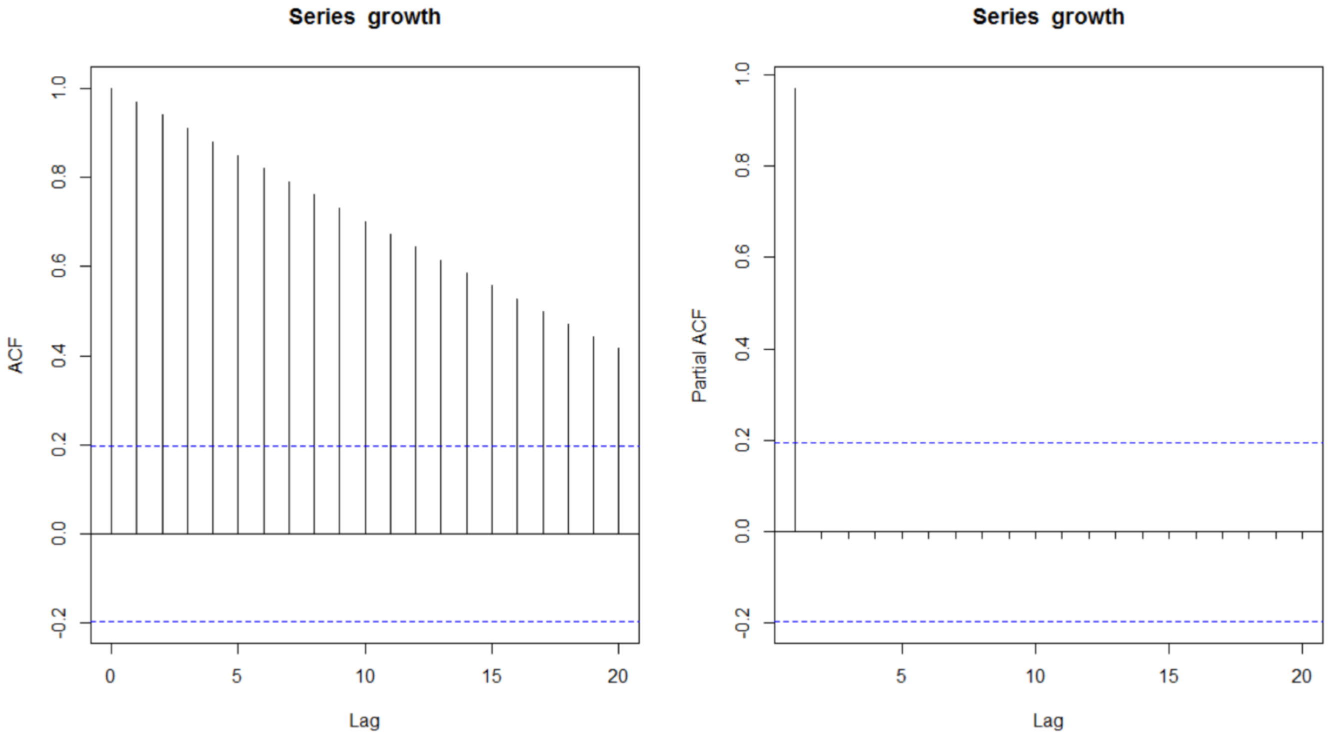
Data sources:
http://www.bts.gov/xml/air_traffic/src/index.xml
and https://datamarket.com/data/set/281x/us-air-carrier-traffic-statistics-revenue-passenger-miles
Last accessed: 31-Mar-2016

	Aug	Sep	Oct	Nov	Dec
1996	57723208	47035464	49263120	43937074	48539606
1997	59715433	49418190	51058879	47056048	49654209
1998	59927214	48751280	52578217	48734375	50208641
1999	63003663	53131972	56653901	53215500	51746821
2000	66256804	55900504	58373996	55590325	54822970
2001	68377080	38601868	43964788	44915764	47836501
2002	62944816	49096035	54019748	50106814	56656594
2003	64989766	52121480	56724551	54128776	58739845
2004	70961522	57881042	63021142	59453943	62680310

ACF and PACF – Idealized Trend



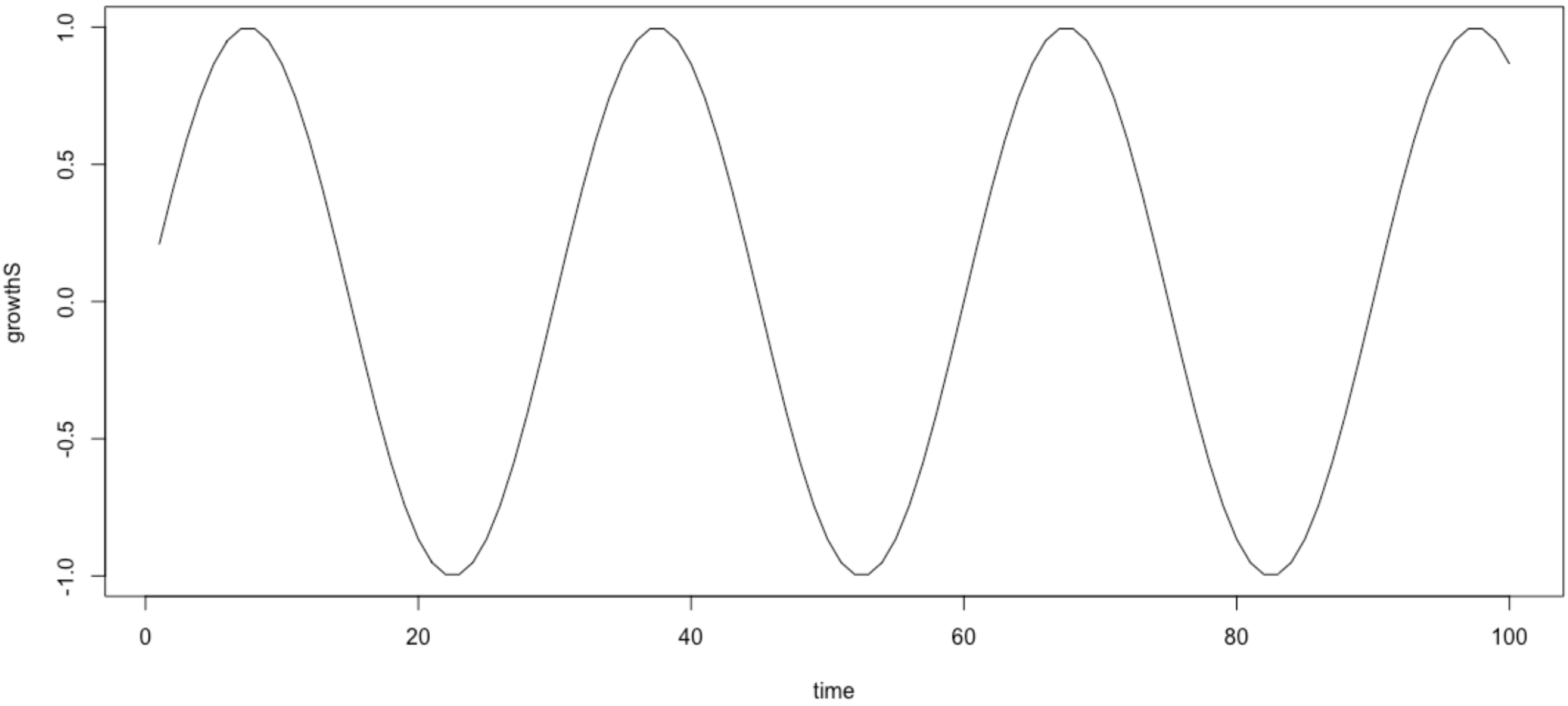
ACF and PACF – Idealized Trend



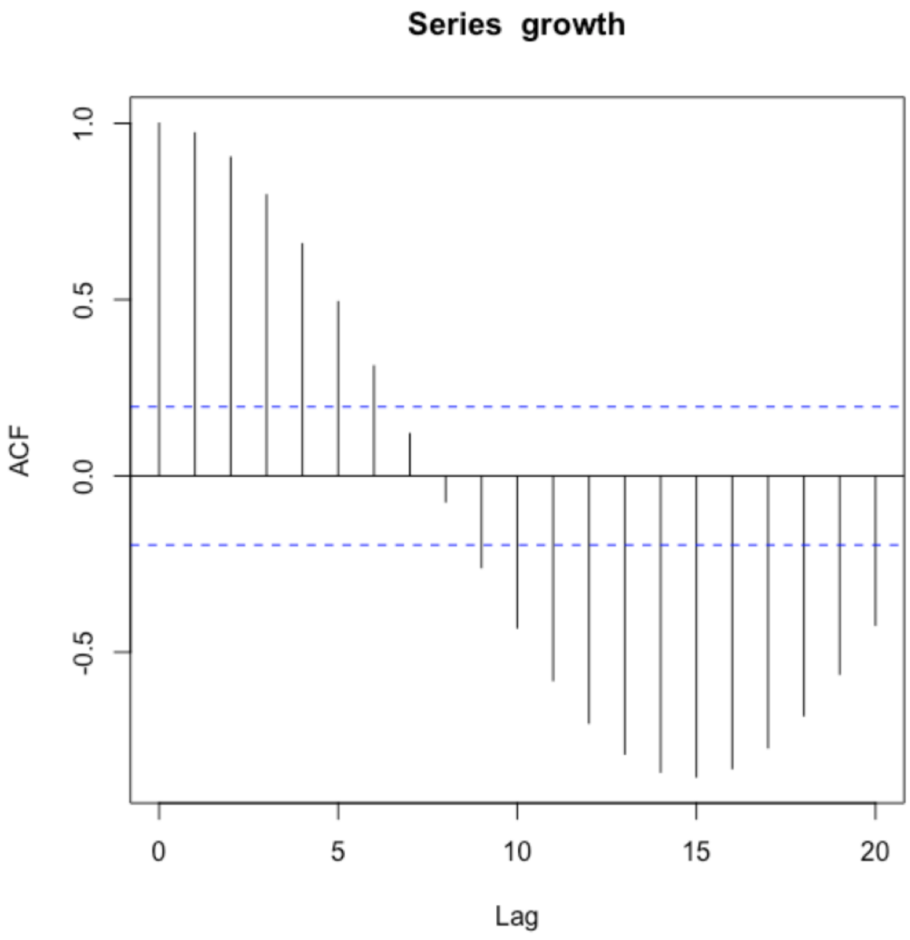
95% CI: $0 \pm \frac{1.96}{\sqrt{n}}$

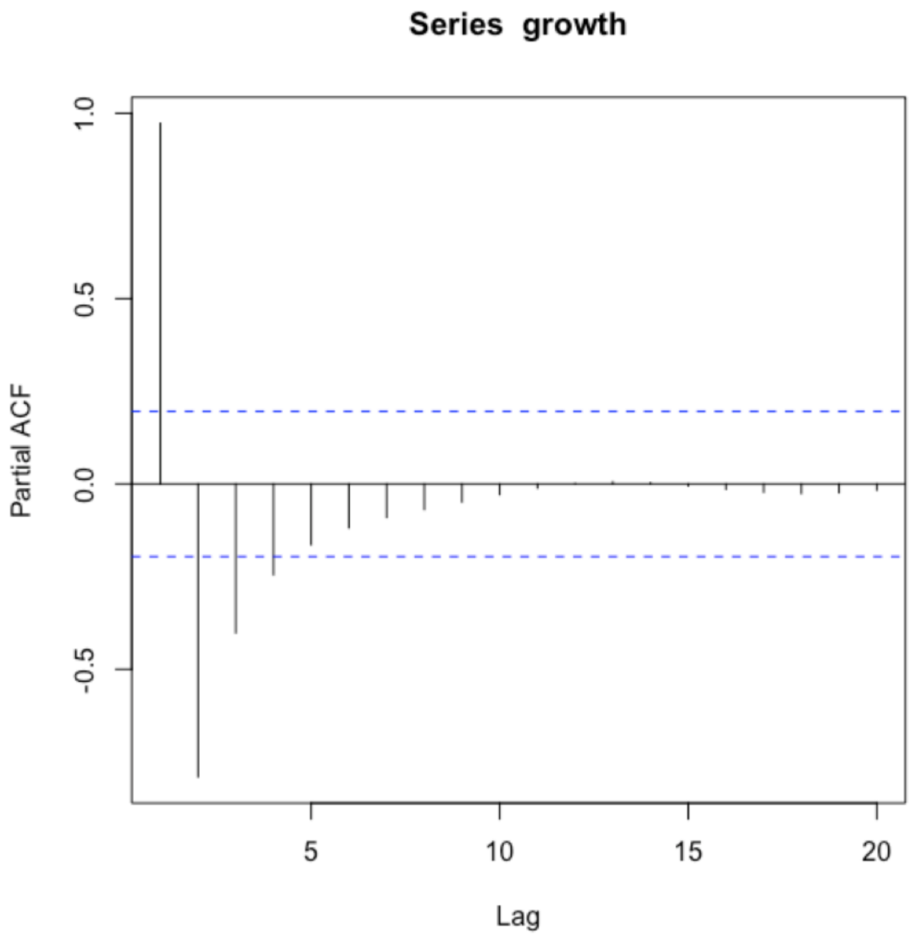
- ACF is a bar chart of correlation coefficients of the time series and its lags.
- PACF is a plot of the partial correlation coefficients of the time series and its lags.

ACF and PACF – Idealized Seasonality

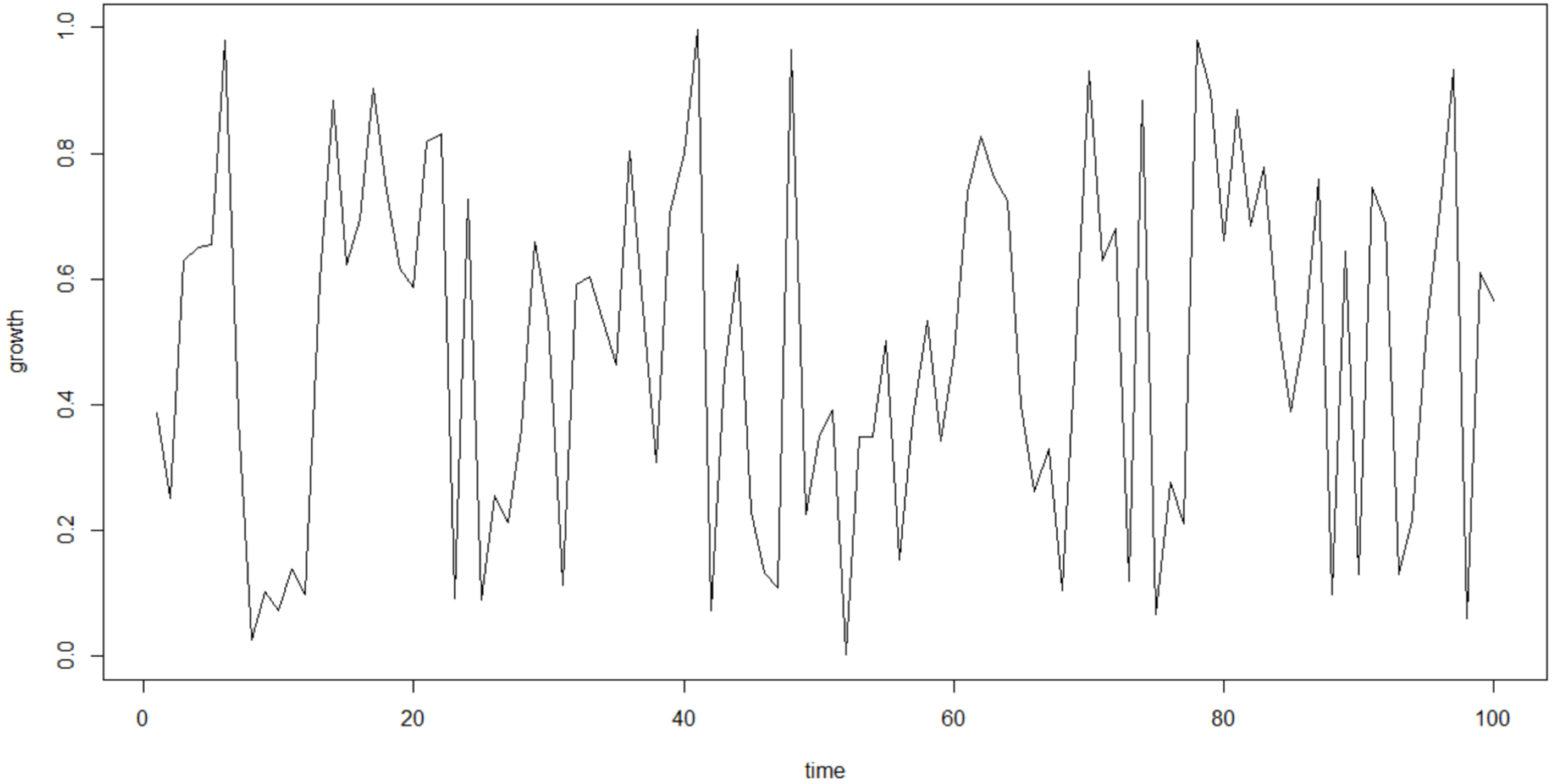


ACF and PACF – Idealized Seasonality





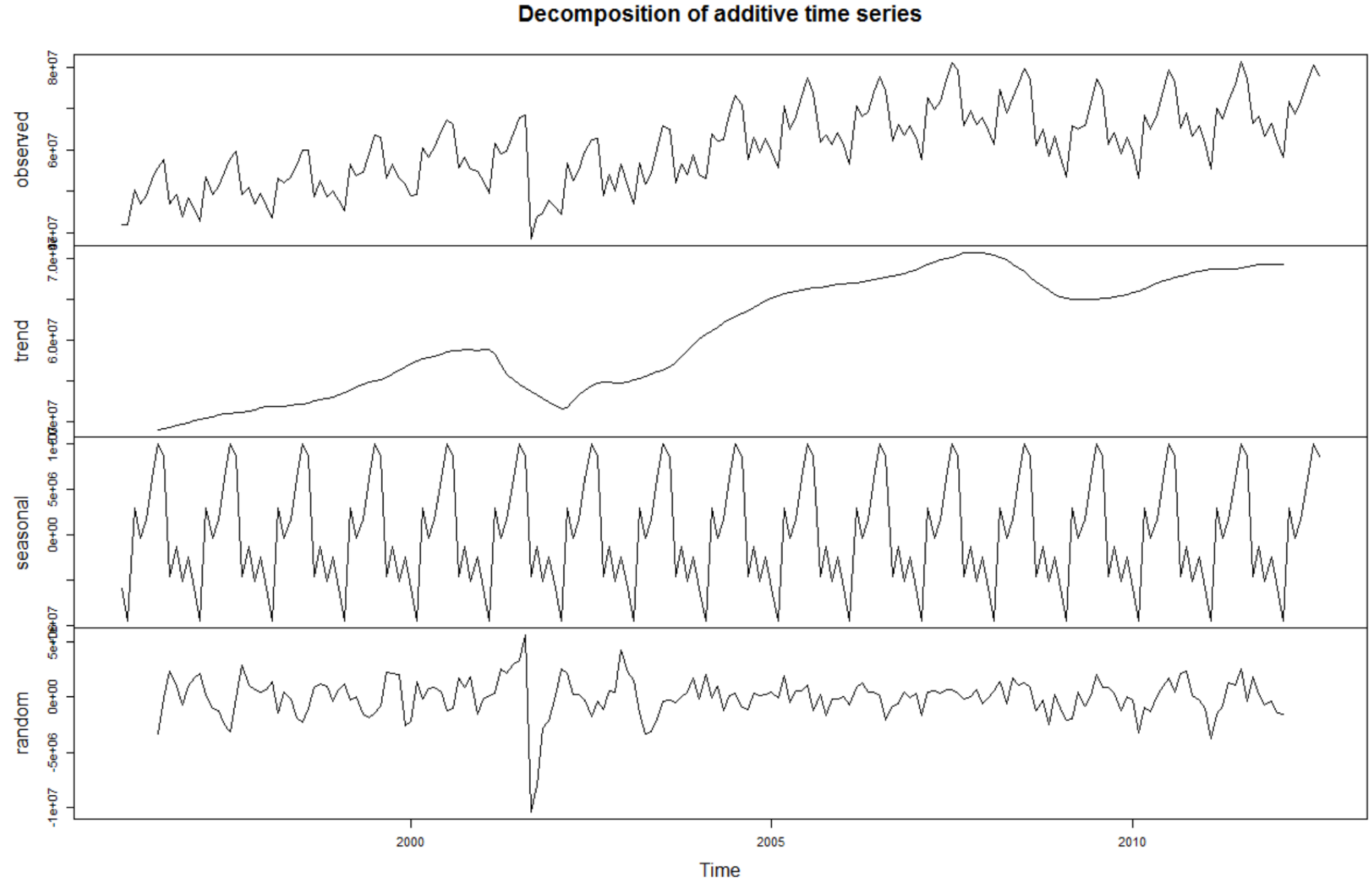
ACF and PACF – Idealized Randomness



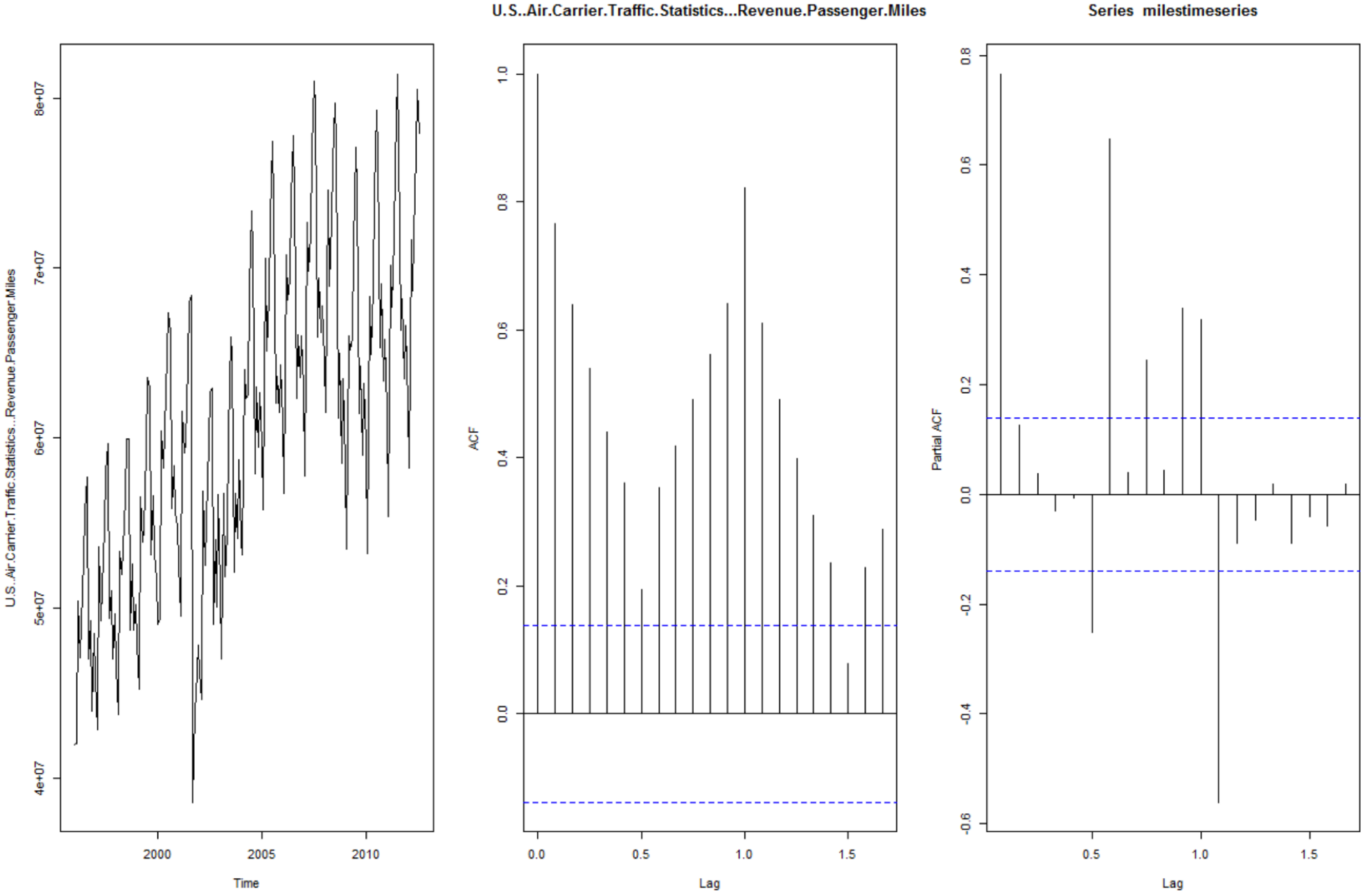
ACF and PACF – Idealized Trend, Seasonality and Randomness

- Ideal Trend: Decreasing ACF and 1 or 2 lags of PACF
- Ideal Seasonality: Cyclicality in ACF and a few lags of PACF with some positive and some negative
- Ideal Random: A spike may or may not be present; even if present, magnitude will be small

ACF and PACF (Real-world): Decomposing Time Series into the 3 Components – Revenue Passenger Miles (RPM)



ACF and PACF (Real-world): Decomposing Time Series into the 3 Components – RPM



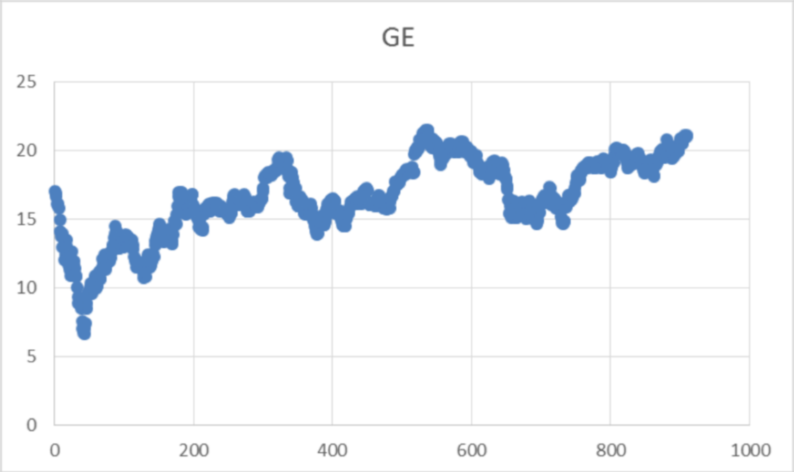
Stationary and Non-Stationary

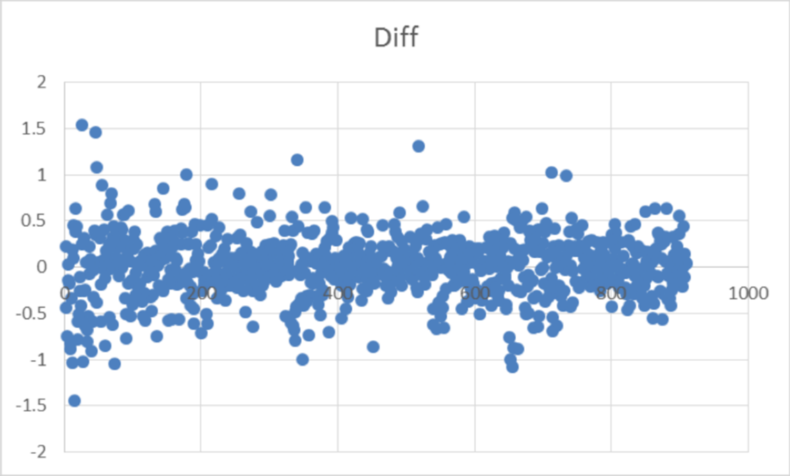
- Stationary data has constant statistical properties – mean, variance, autocorrelation, etc. – over time

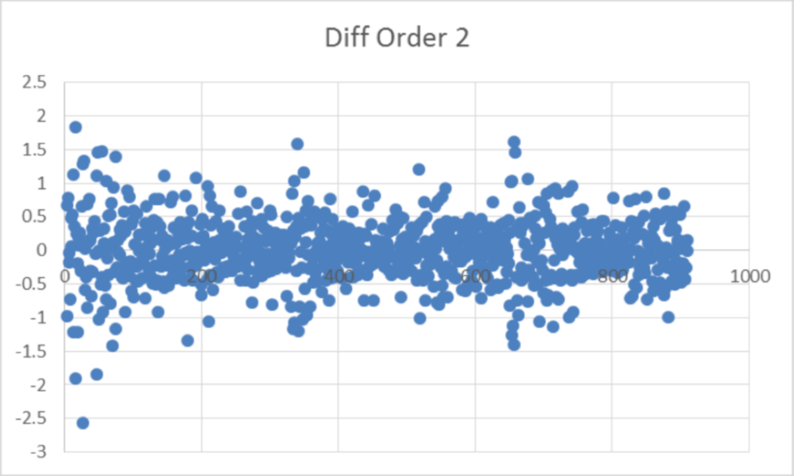
- If the data is stationary, forecasting is easier!

- Differencing to convert non-stationary to stationary

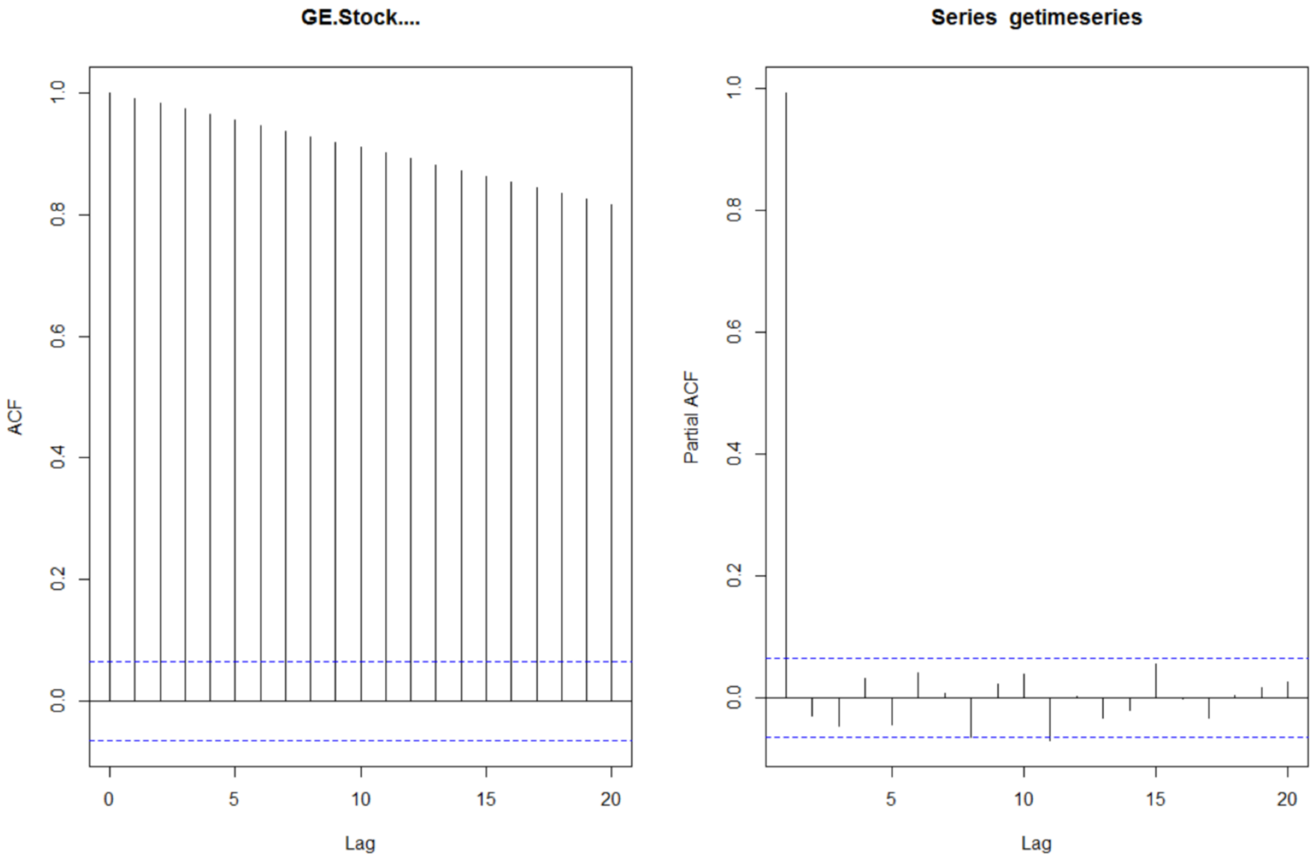
Removing Trend from Data





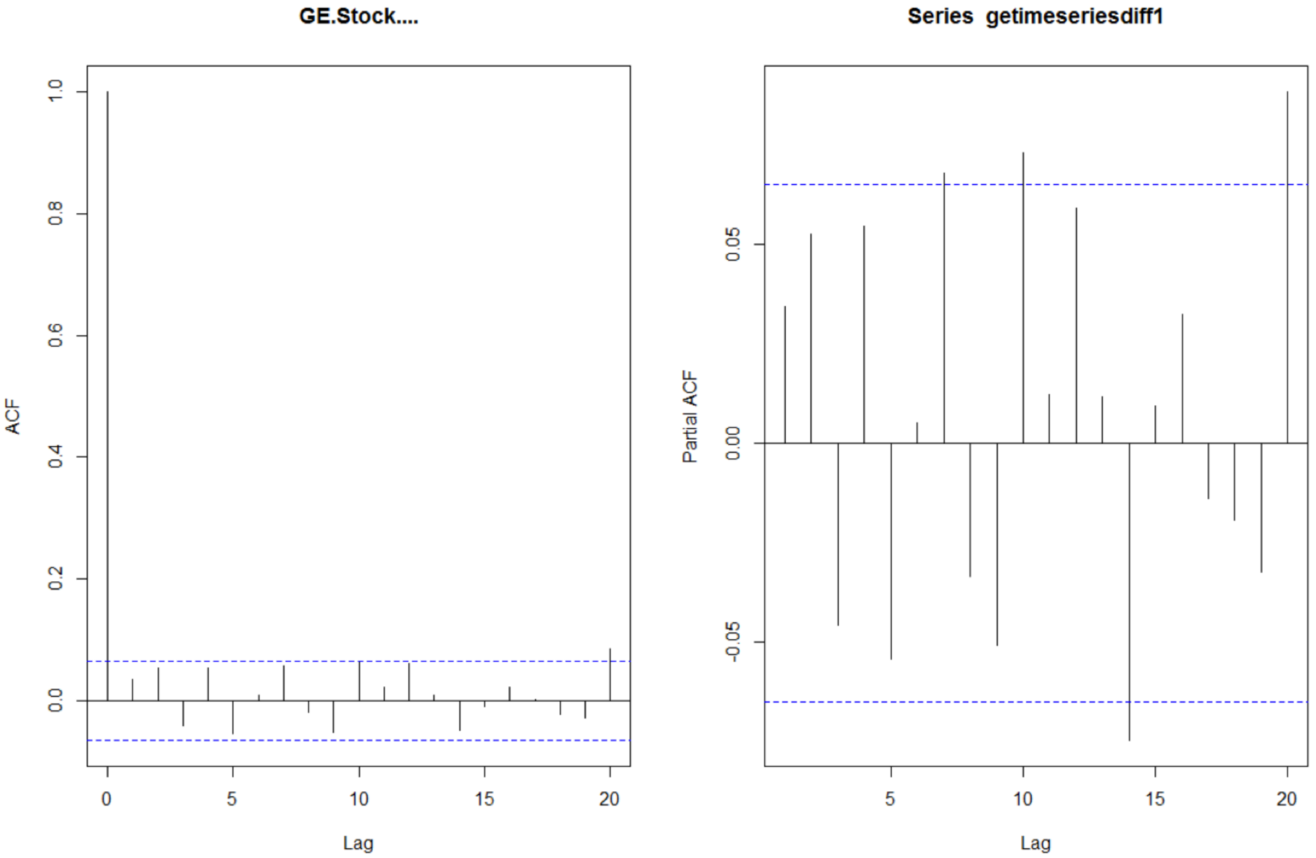


ACF and PACF of Stationary and Non-Stationary



Price of GE stock is highly correlated with the previous day's value.

ACF and PACF of Stationary and Non-Stationary



Daily changes in GE stock price are essentially random.

ACF and PACF of Stationary and Non-Stationary

- Non-stationary series have an ACF that remains significant for half a dozen or more lags, rather than quickly declining to zero.

- You must difference such a series until it is stationary before you can identify the process.

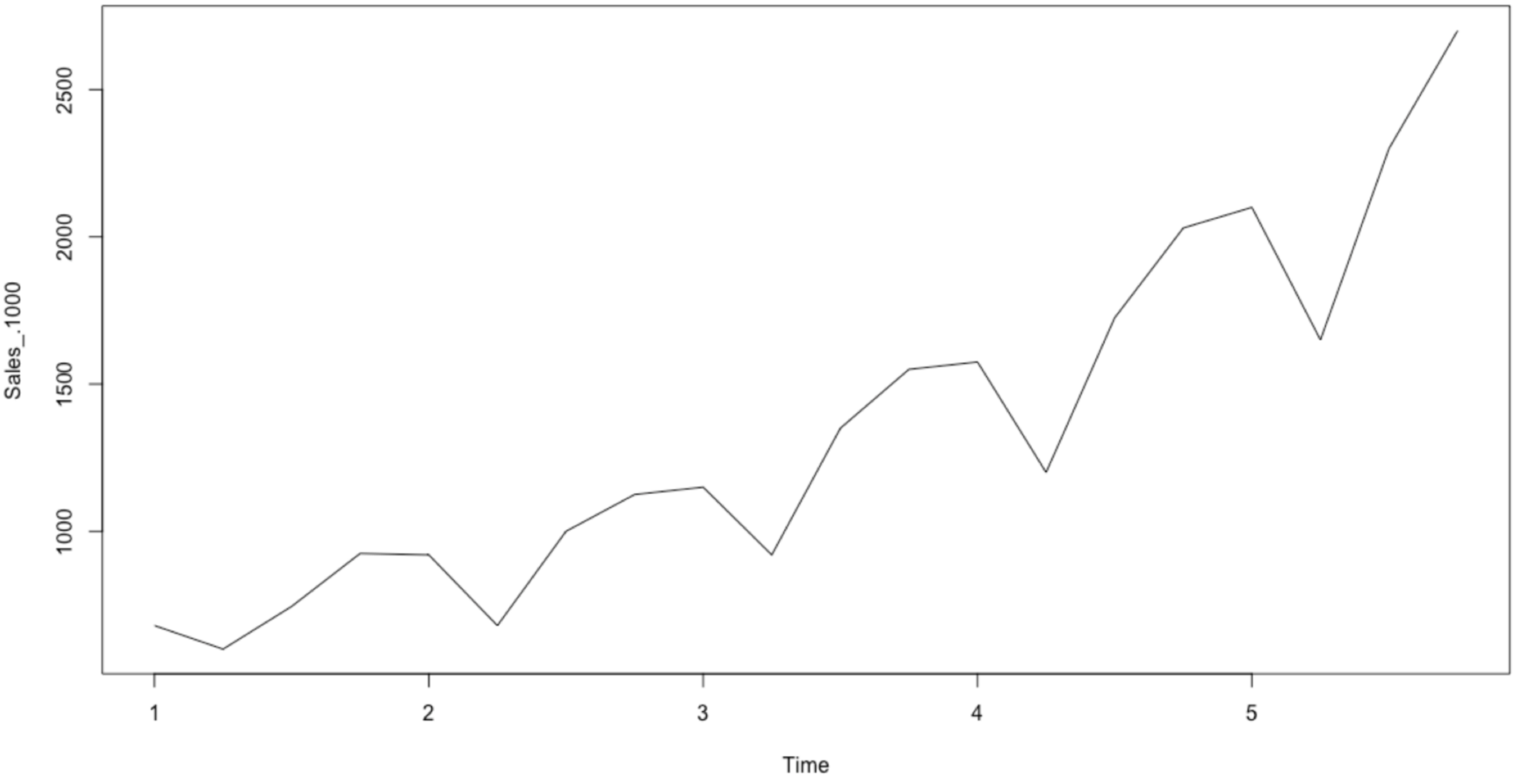
CURVE FITTING / REGRESSION ON TIME METHODS

Regression on Time

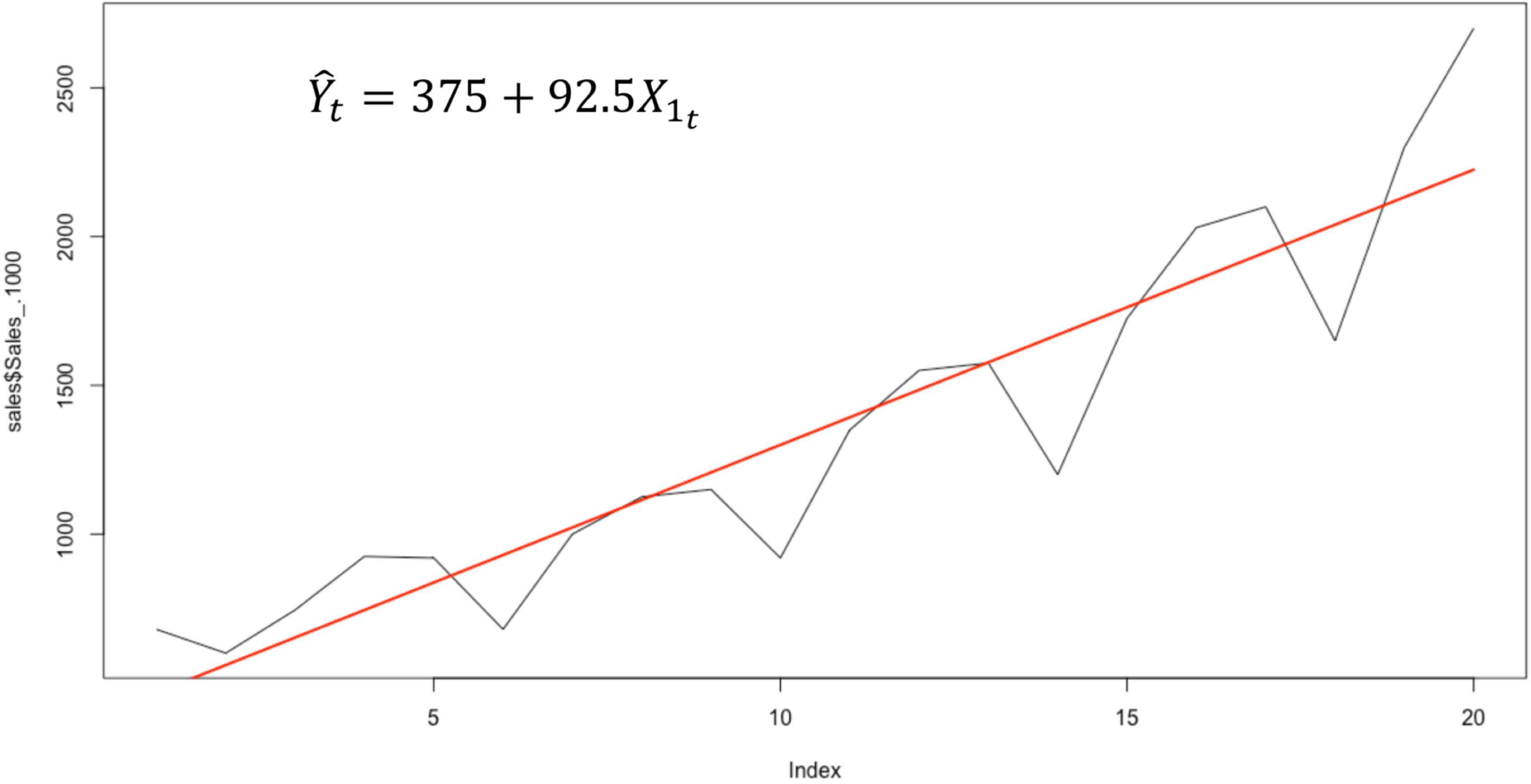
- Use when trend is the most pronounced

- ACF decays exponentially and PACF has very few spikes

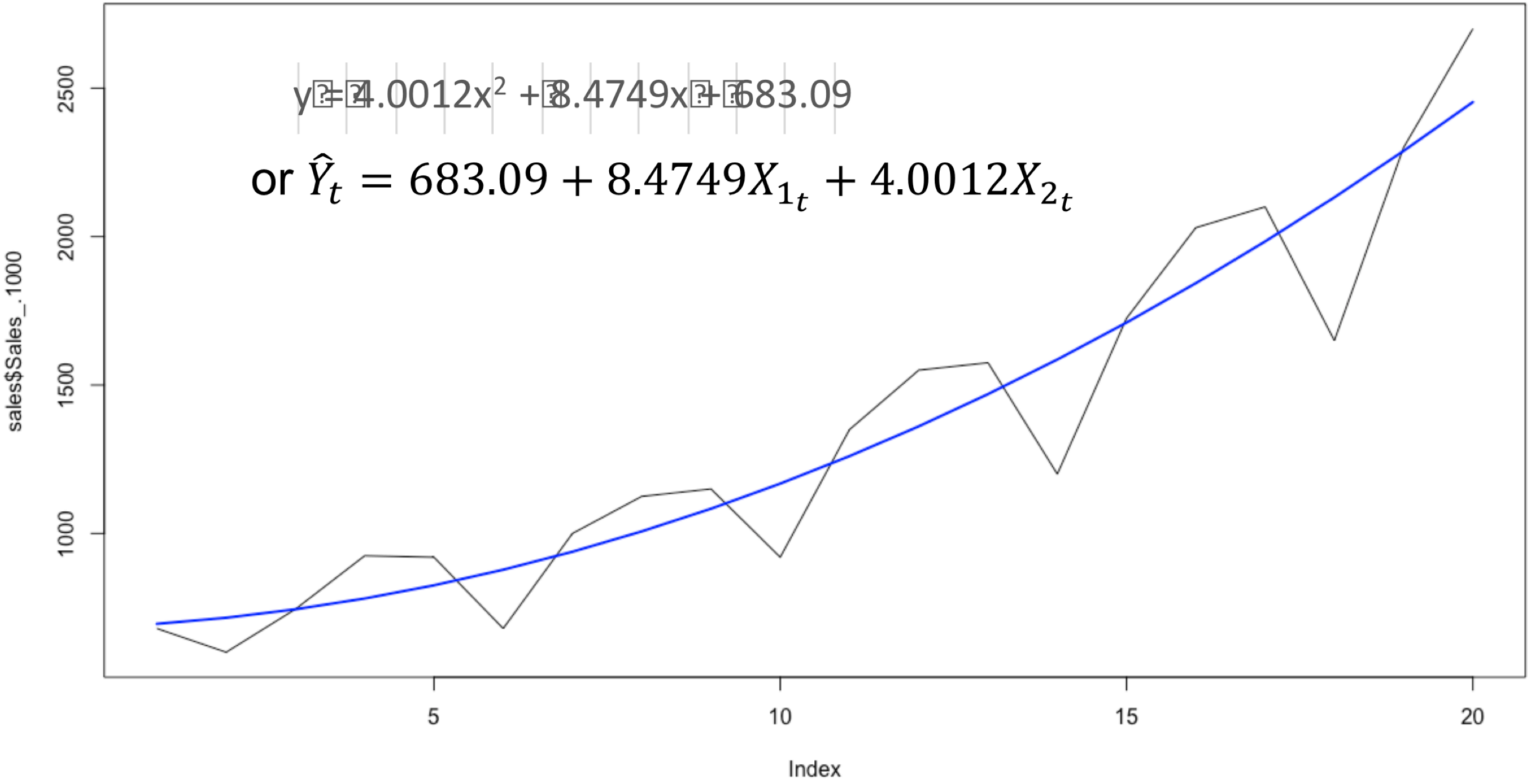
Quarter	Sales_$1000
1	680
2	600
3	745
4	925
5	920
6	680
7	1000
8	1125
9	1150
10	920
11	1350
12	1550
13	1575
14	1200
15	1725
16	2030
17	2100
18	1650



Regression Analysis



Quadratic Trend

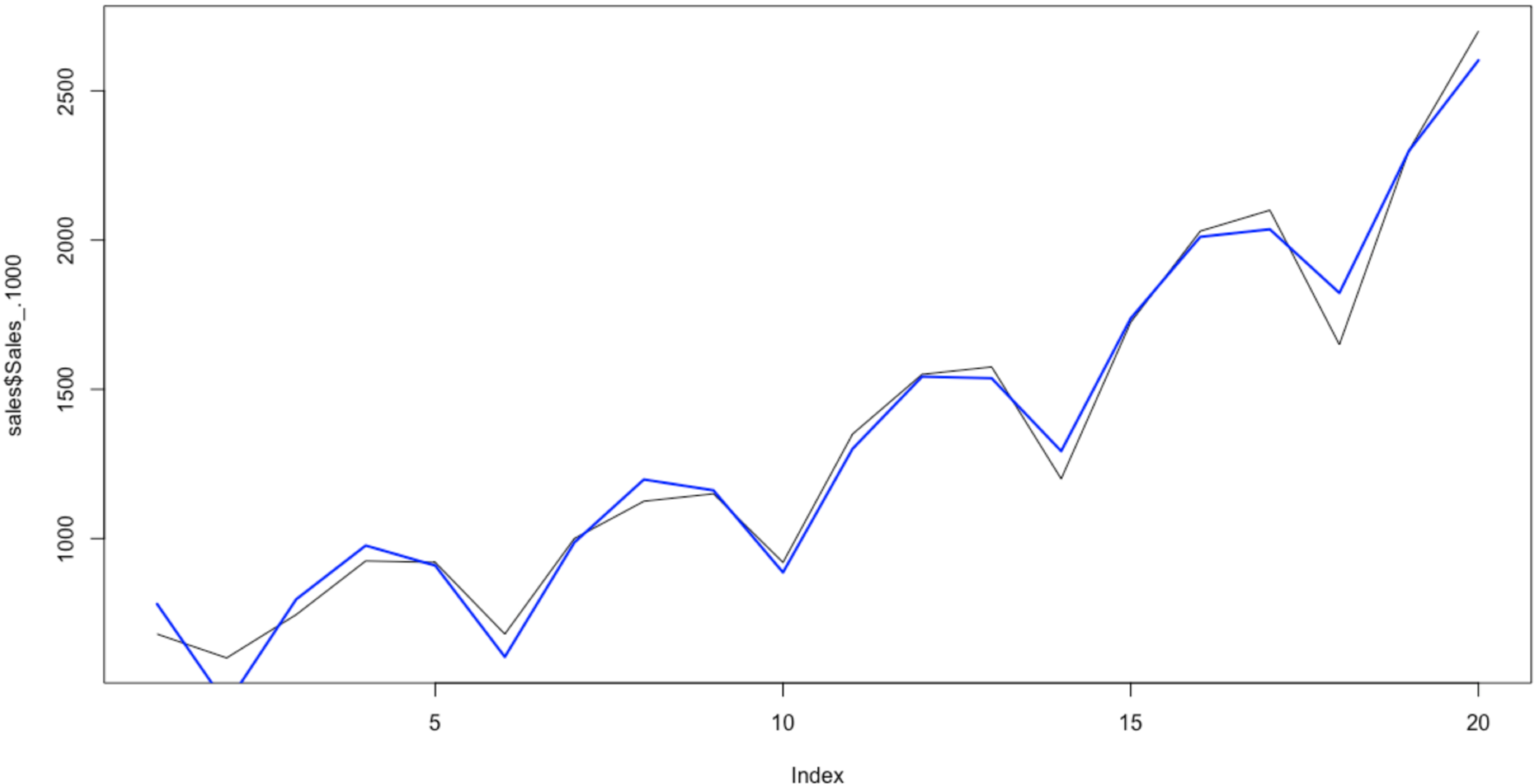


Seasonal Regression Models

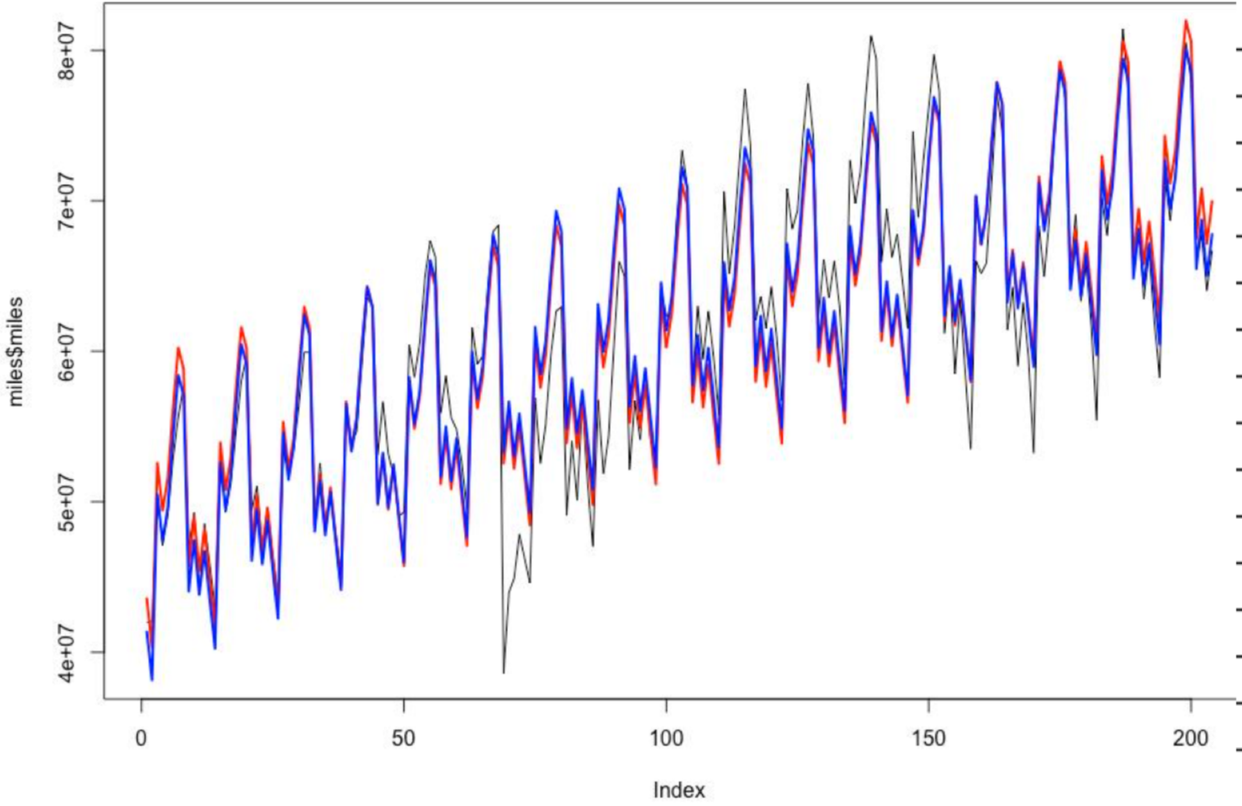
	Value of		
Quarter	X_{3_t}	X_{4_t}	X_{5_t}
1	1	0	0
2	0	1	0
3	0	0	1
4	0	0	0

$$\hat{Y}_t = \beta_0 + \beta_1 X_{1_t} + \beta_2 X_{2_t} + \beta_3 X_{3_t} + \beta_4 X_{4_t} + \beta_5 X_{5_t} + \varepsilon_t$$

Seasonal Regression Models



Seasonal Regression Models - RPM



	miles	time	seasonal
1	41972194	1	1
2	42054796	2	2
3	50443045	3	3
4	47112397	4	4
5	49118248	5	5
6	52880510	6	6
7	55664750	7	7
8	57723208	8	8
9	47035464	9	9
10	49263120	10	10
11	43937074	11	11
12	48539606	12	12
13	45850623	13	1
14	42838949	14	2
15	53620994	15	3

Another Way of Incorporating Seasonality

- Take the trend prediction and actual prediction.
- Depending on additive or multiplicative model compute the deviation and map it as seasonality effect for each prediction.
- Take averages of the seasonality value. Use this to make future predictions.

Year	Quarter	Time variable (this is created)	Revenues (in $M)
2008	I	1	10.2
	II	2	12.4
	III	3	14.8
	IV	4	15
2009	I	5	11.2
	II	6	14.3
	III	7	18.4
	IV	8	18

```
Call:
lm(formula = y ~ x)

Residuals:
    Min      1Q  Median      3Q     Max
-3.5595 -0.9384  0.4405  1.3265  1.9286

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  10.0393     1.5531   6.464  0.00065 ***
x             0.9440     0.3076   3.069  0.02196 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.993 on 6 degrees of freedom
Multiple R-squared: 0.6109,     Adjusted R-squared: 0.5461
F-statistic: 9.422 on 1 and 6 DF,  p-value: 0.02196
```

What is the Regression equation?

$$y = 10.0393 + 0.9440x$$

Seasonality: Multiplicative

Time	Observed values TSI* (assuming no impact of cyclicality)	Predicted values (per the regression) T*	SI* = TSI/T
1	10.2	10.983	0.929
2	12.4	11.927	1.040
3	14.8	12.871	1.150
4	15.0	13.815	1.086
5	11.2	14.759	0.759
6	14.3	15.703	0.911
7	18.4	16.647	1.105
8	18.0	17.591	1.023

* T: Trend; S: Seasonal; I: Irregular

Quarterly Seasonality

Time	Average seasonality factor
Q1	$0.844 \left(= \frac{0.929+0.759}{2}\right)$
Q2	0.975
Q3	1.127
Q4	1.054

Time	Observed values TSI* (assuming no impact of cyclicality)	Predicted values (per the regression) T*	SI* = TSI/T
1	10.2	10.983	0.929
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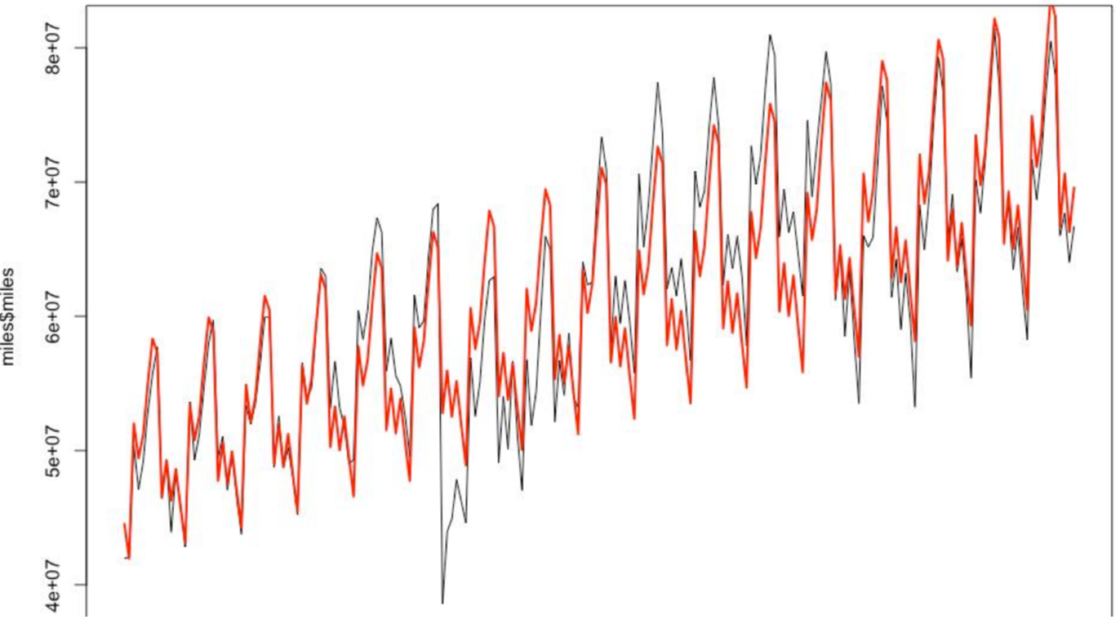
7	18.4	16.647	1.105
8	18.0	17.591	1.023

Computations

- Trend $Y_9 = 10.039 + 0.944(9) = 18.535$

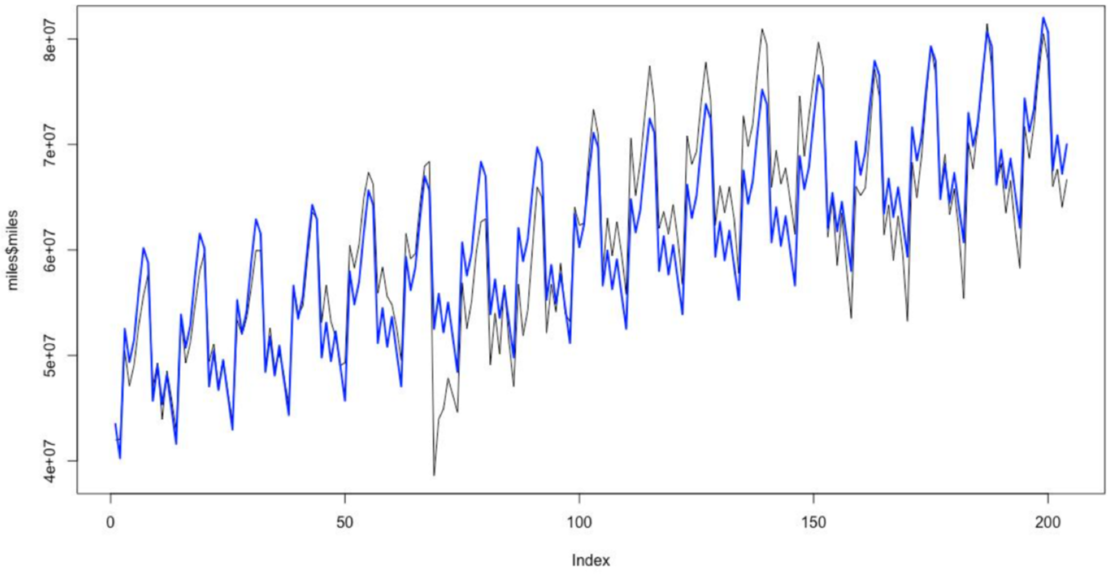
- Corrected for seasonality and randomness: $18.535 * 0.844 = 15.643$

Seasonality: Multiplicative



	miles	time	seasonal	mae
1	41972194	1	1	0.849386
2	42054796	2	2	0.8491019
3	50443045	3	3	1.016129
4	47112397	4	4	0.946865
5	49118248	5	5	0.9849257
6	52880510	6	6	1.057953
7	55664750	7	7	1.111125
8	57723208	8	8	1.149602
9	47035464	9	9	0.9346292
10	49263120	10	10	0.9766855
11	43937074	11	11	0.8691307
12	48539606	12	12	0.9580177
13	45850623	13	1	0.9029174
14	42838949	14	2	0.8417232
15	53620994	15	3	1.051224

Seasonality: Additive



	miles	time	seasonal	mae
1	41972194	1	1	-7442550
2	42054796	2	2	-7473763
3	50443045	3	3	800670.5
4	47112397	4	4	-2643793
5	49118248	5	5	-751757.1
6	52880510	6	6	2896690
7	55664750	7	7	5567114
8	57723208	8	8	7511757
9	47035464	9	9	-3289802
10	49263120	10	10	-1175962
11	43937074	11	11	-6615823
12	48539606	12	12	-2127106
13	45850623	13	1	-4929905
14	42838949	14	2	-8055394
15	53620994	15	3	2612836

Issues with Regressing on Time

- If there is no trend or if seasonality and fluctuations are more important than trend, then the coefficients behave weirdly

Goodness of Fit

- MAE (Mean absolute error) $\frac{\sum|y_i - \hat{y}_i|}{n}$
- MSE (Mean square error) $\frac{\sum(y_i - \hat{y}_i)^2}{n}$
- RMSE (Root mean square error) $\sqrt{\frac{\sum(y_i - \hat{y}_i)^2}{n}}$
- MAPE (Mean absolute percent error) $\frac{1}{n}\left(\frac{\sum|y_i - \hat{y}_i|}{y_i}\right) * 100$

SIMPLE FORECASTING / BENCHMARKING METHODS

Average Method

$$\hat{y}_{t+h} = \bar{y} = \frac{y_1 + \cdots + y_t}{t}$$

- Forecasts of all future values are equal to the **mean of the historical data**

Naïve Method

$$\hat{y}_{t+h} = y_t$$

- Forecasts of all future values are equal to the last **observed** value
- A very useful method in many economic and financial time series

Seasonal Naïve Method

$$\hat{y}_{t+h} = y_{t+h-km}$$

Integer part

where m is the seasonal period and $k = \left\lfloor \frac{h-1}{m} \right\rfloor + 1$

- Forecasts of all future values for a particular period (e.g., July) are equal to the last **observed** value for the same period (e.g., last July)

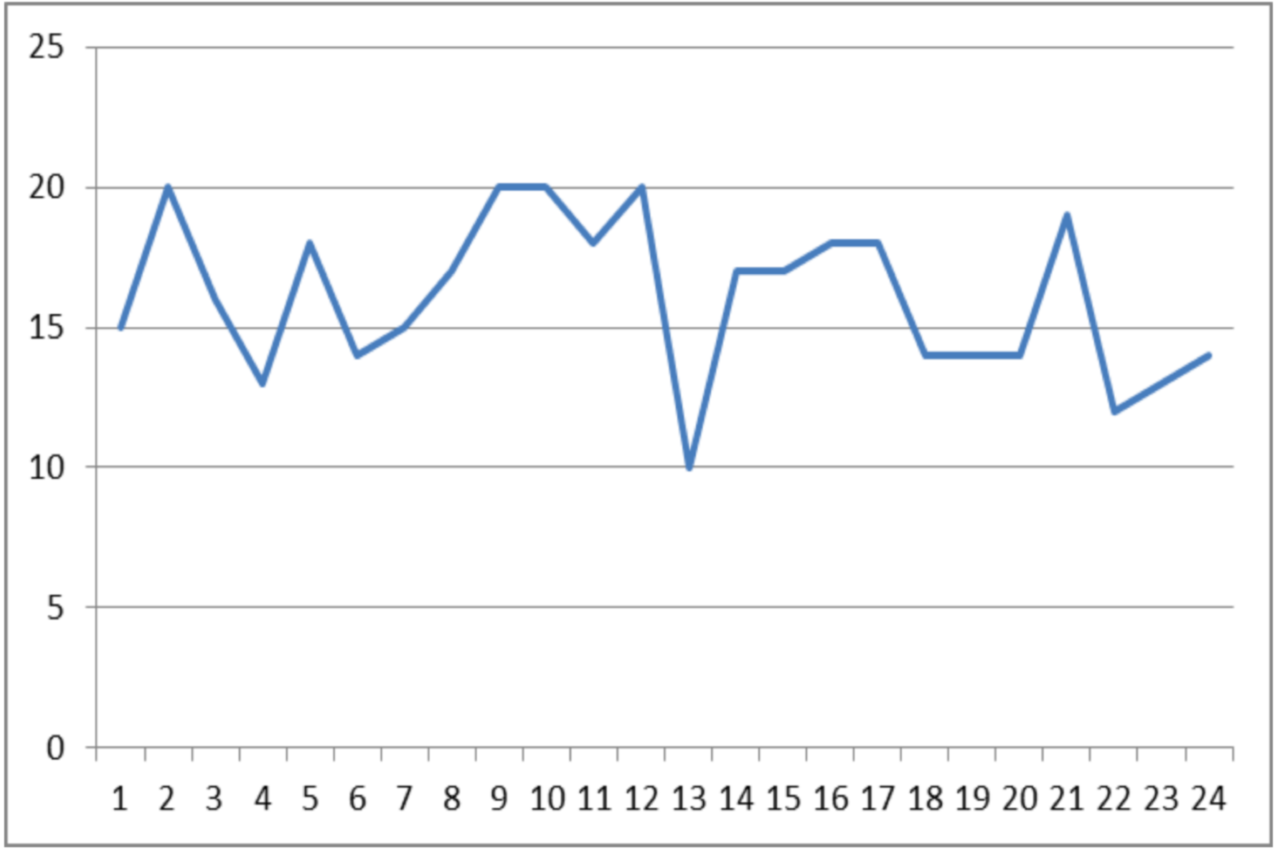
Drift Method

$$\hat{y}_{t+h} = y_t + \frac{h}{t-1}\sum_{t=2}^{t}(y_t - y_{t-1}) = y_t + h\left(\frac{y_t - y_1}{t-1}\right)$$

- Forecasts increase or decrease over time
- Amount of change over time (drift) is the average historical change *(In Naïve method, this change is zero)*

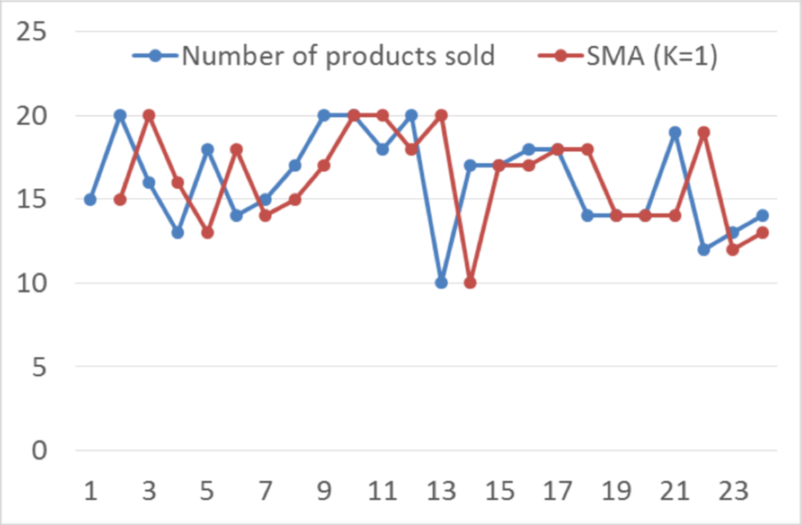
SMOOTHING METHODS

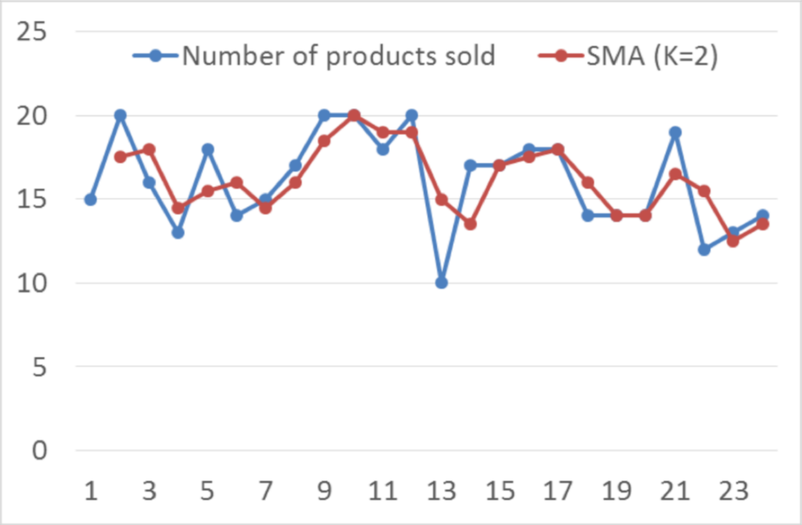
Stationary Model: Moving Averages

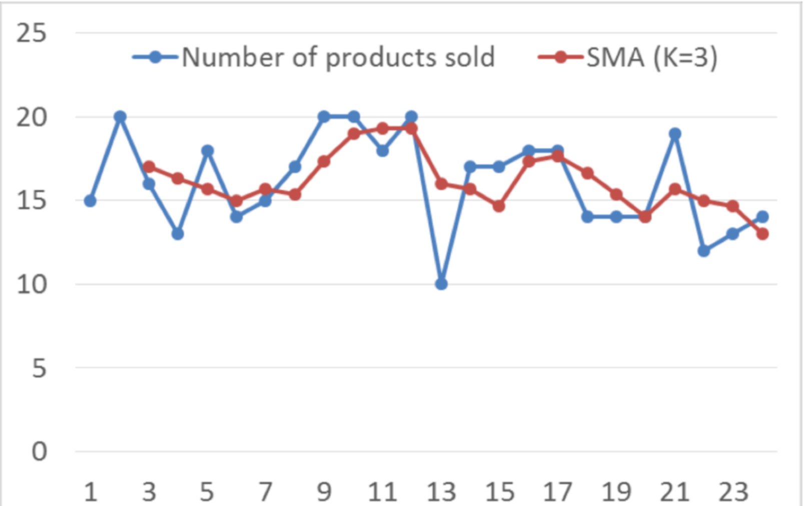


Stationary Model: Case 1 – Simple Moving Averages

	Number of products sold	SMA (K=1)	Error	SMA (K=2)	Error	SMA (K=3)	Error
2	15						
3	20	15	5	17.5	2.5		
4	16	20	4	18	2	17	1
5	13	16	3	14.5	1.5	16.333333	3.33333
6	18	13	5	15.5	2.5	15.666667	2.33333
7	14	18	4	16	2	15	1
8	15	14	1	14.5	0.5	15.666667	0.66667
9	17	15	2	16	1	15.333333	1.66667
10	20	17	3	18.5	1.5	17.333333	2.66667
11	20	20	0	20	0	19	1
12	18	20	2	19	1	19.333333	1.33333
13	20	18	2	19	1	19.333333	0.66667
14	10	20	10	15	5	16	6
15	17	10	7	13.5	3.5	15.666667	1.33333
16	17	17	0	17	0	14.666667	2.33333
17	18	17	1	17.5	0.5	17.333333	0.66667
18	18	18	0	18	0	17.666667	0.33333
19	14	18	4	16	2	16.666667	2.66667
20	14	14	0	14	0	15.333333	1.33333
21	14	14	0	14	0	14	0
22	19	14	5	16.5	2.5	15.666667	3.33333
23	12	19	7	15.5	3.5	15	3
24	13	12	1	12.5	0.5	14.666667	1.66667
25	14	13	1	13.5	0.5	13	1
26			**2.91304**		**1.45652**		**1.78788**







Only decision point is K

Stationary Model: Case 2 – Weighted Moving Averages

$$\hat{Y}_{t+1} = w_1 Y_t + w_2 Y_{t-1} + \cdots + w_k Y_{t-k+1}$$

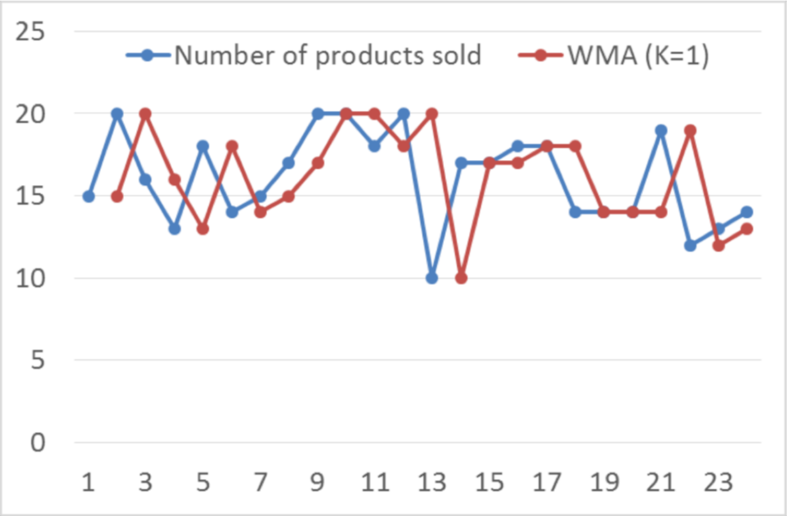
- Typically we choose a time period of moving average and weights are chosen such that the error is minimized

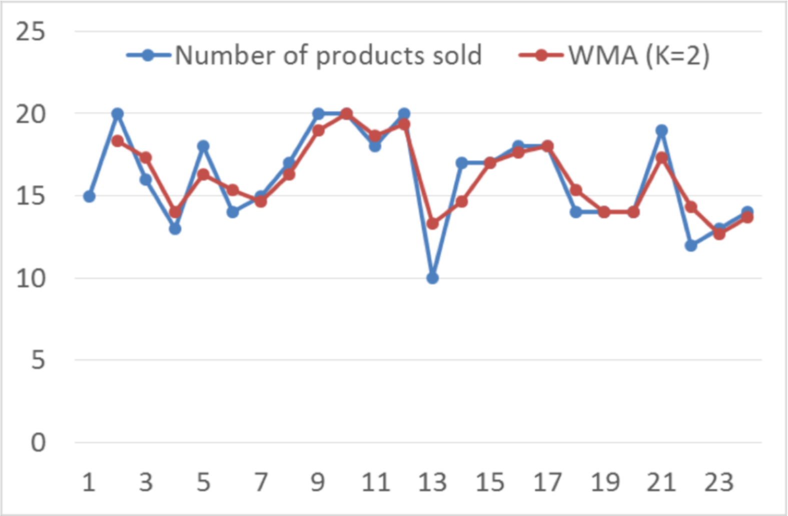
Stationary Model: Case 2 – Weighted Moving Averages

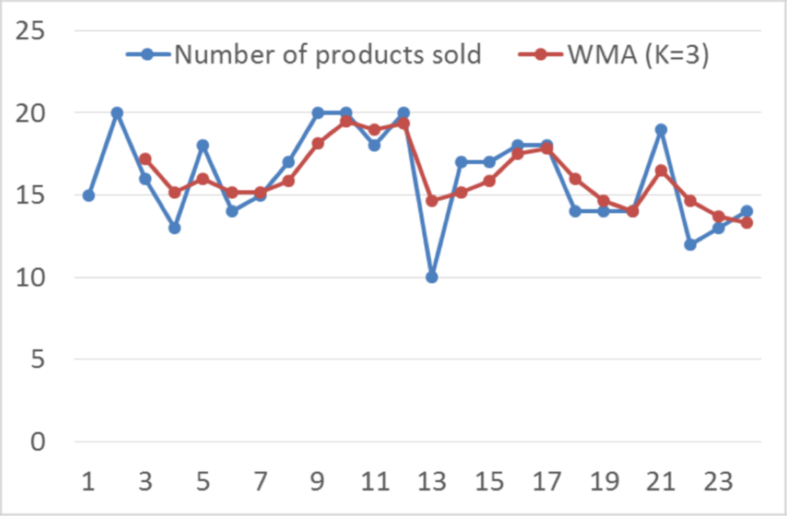
	Number of products sold	WMA (K=1)	Error	WMA (K=2)	Error	WMA (K=3)	Error
1							
2	15						
3	20	=A2*1	=ABS(B3-A3)	=(A3*2+A2*1)/3	=ABS(A3-D3)		
4	16	=A3*1	=ABS(B4-A4)	=(A4*2+A3*1)/3	=ABS(A4-D4)	=(A4*3+A3*2+A2*1)/6	=ABS(A4-F4)
5	13	=A4*1	=ABS(B5-A5)	=(A5*2+A4*1)/3	=ABS(A5-D5)	=(A5*3+A4*2+A3*1)/6	=ABS(A5-F5)
6	18	=A5*1	=ABS(B6-A6)	=(A6*2+A5*1)/3	=ABS(A6-D6)	=(A6*3+A5*2+A4*1)/6	=ABS(A6-F6)
7	14	=A6*1	=ABS(B7-A7)	=(A7*2+A6*1)/3	=ABS(A7-D7)	=(A7*3+A6*2+A5*1)/6	=ABS(A7-F7)
8	15	=A7*1	=ABS(B8-A8)	=(A8*2+A7*1)/3	=ABS(A8-D8)	=(A8*3+A7*2+A6*1)/6	=ABS(A8-F8)
9	17	=A8*1	=ABS(B9-A9)	=(A9*2+A8*1)/3	=ABS(A9-D9)	=(A9*3+A8*2+A7*1)/6	=ABS(A9-F9)
10	20	=A9*1	=ABS(B10-A10)	=(A10*2+A9*1)/3	=ABS(A10-D10)	=(A10*3+A9*2+A8*1)/6	=ABS(A10-F10)
11	20	=A10*1	=ABS(B11-A11)	=(A11*2+A10*1)/3	=ABS(A11-D11)	=(A11*3+A10*2+A9*1)/6	=ABS(A11-F11)
12	18	=A11*1	=ABS(B12-A12)	=(A12*2+A11*1)/3	=ABS(A12-D12)	=(A12*3+A11*2+A10*1)/6	=ABS(A12-F12)
13	20	=A12*1	=ABS(B13-A13)	=(A13*2+A12*1)/3	=ABS(A13-D13)	=(A13*3+A12*2+A11*1)/6	=ABS(A13-F13)
14	10	=A13*1	=ABS(B14-A14)	=(A14*2+A13*1)/3	=ABS(A14-D14)	=(A14*3+A13*2+A12*1)/6	=ABS(A14-F14)
15	17	=A14*1	=ABS(B15-A15)	=(A15*2+A14*1)/3	=ABS(A15-D15)	=(A15*3+A14*2+A13*1)/6	=ABS(A15-F15)
16	17	=A15*1	=ABS(B16-A16)	=(A16*2+A15*1)/3	=ABS(A16-D16)	=(A16*3+A15*2+A14*1)/6	=ABS(A16-F16)
17	18	=A16*1	=ABS(B17-A17)	=(A17*2+A16*1)/3	=ABS(A17-D17)	=(A17*3+A16*2+A15*1)/6	=ABS(A17-F17)
18	18	=A17*1	=ABS(B18-A18)	=(A18*2+A17*1)/3	=ABS(A18-D18)	=(A18*3+A17*2+A16*1)/6	=ABS(A18-F18)
19	14	=A18*1	=ABS(B19-A19)	=(A19*2+A18*1)/3	=ABS(A19-D19)	=(A19*3+A18*2+A17*1)/6	=ABS(A19-F19)
20	14	=A19*1	=ABS(B20-A20)	=(A20*2+A19*1)/3	=ABS(A20-D20)	=(A20*3+A19*2+A18*1)/6	=ABS(A20-F20)
21	14	=A20*1	=ABS(B21-A21)	=(A21*2+A20*1)/3	=ABS(A21-D21)	=(A21*3+A20*2+A19*1)/6	=ABS(A21-F21)
22	19	=A21*1	=ABS(B22-A22)	=(A22*2+A21*1)/3	=ABS(A22-D22)	=(A22*3+A21*2+A20*1)/6	=ABS(A22-F22)
23	12	=A22*1	=ABS(B23-A23)	=(A23*2+A22*1)/3	=ABS(A23-D23)	=(A23*3+A22*2+A21*1)/6	=ABS(A23-F23)
24	13	=A23*1	=ABS(B24-A24)	=(A24*2+A23*1)/3	=ABS(A24-D24)	=(A24*3+A23*2+A22*1)/6	=ABS(A24-F24)
25	14	=A24*1	=ABS(B25-A25)	=(A25*2+A24*1)/3	=ABS(A25-D25)	=(A25*3+A24*2+A23*1)/6	=ABS(A25-F25)
26			**=AVERAGE(C3:C25)**		**=AVERAGE(E3:E25)**		**=AVERAGE(G3:G25)**

Stationary Model: Case 2 – Weighted Moving Averages

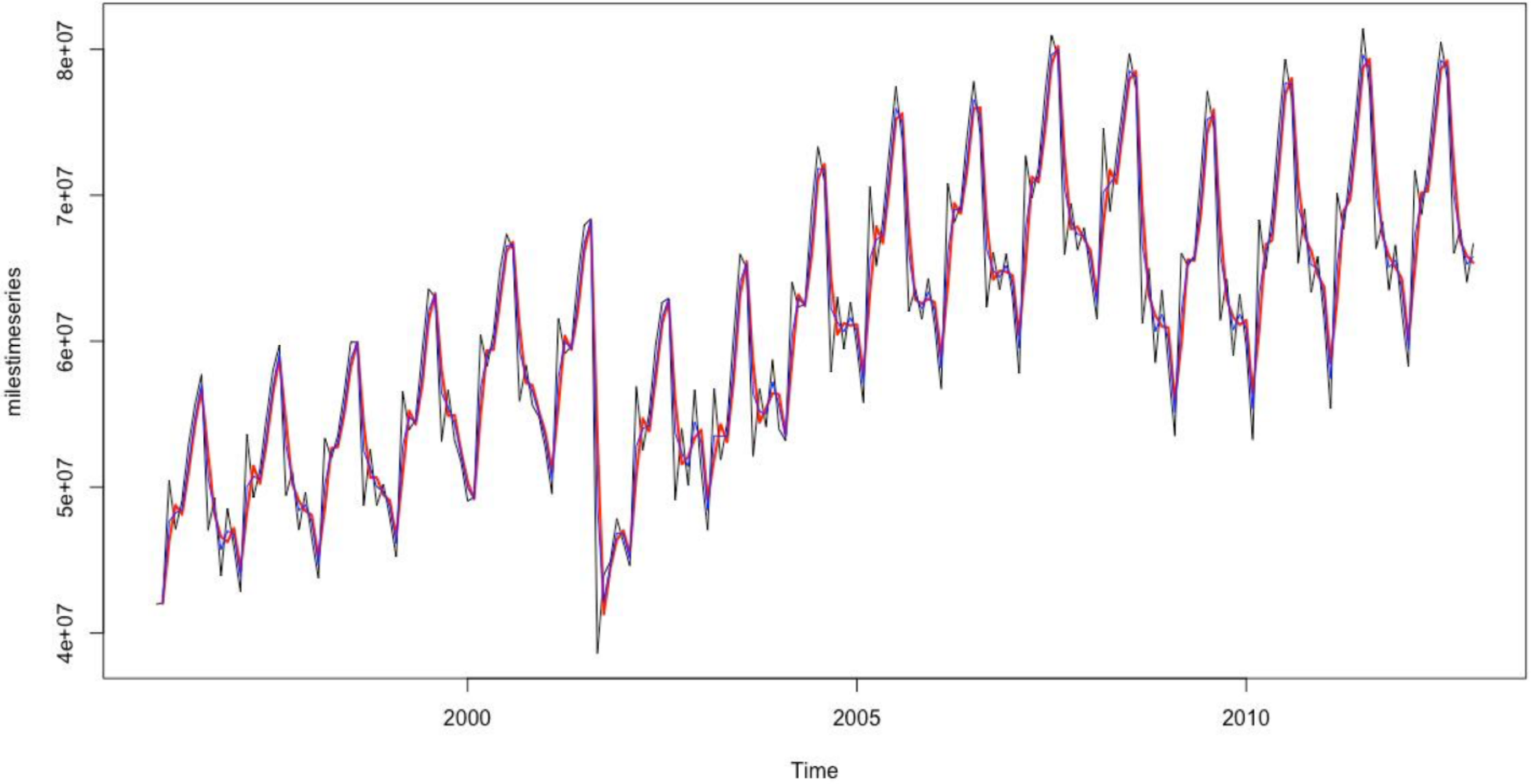
1	Number of products sold	WMA (K=1)	Error	WMA (K=2)	Error	WMA (K=3)	Error
2	15						
3	20	15	5	18.3333333	1.66666667		
4	16	20	4	17.3333333	1.33333333	17.1666667	1.16666667
5	13	16	3	14	1	15.1666667	2.16666667
6	18	13	5	16.3333333	1.66666667	16	2
7	14	18	4	15.3333333	1.33333333	15.1666667	1.16666667
8	15	14	1	14.6666667	0.33333333	15.1666667	0.16666667
9	17	15	2	16.3333333	0.66666667	15.8333333	1.16666667
10	20	17	3	19	1	18.1666667	1.83333333
11	20	20	0	20	0	19.5	0.5
12	18	20	2	18.6666667	0.66666667	19	1
13	20	18	2	19.3333333	0.66666667	19.3333333	0.66666667
14	10	20	10	13.3333333	3.33333333	14.6666667	4.66666667
15	17	10	7	14.6666667	2.33333333	15.1666667	1.83333333
16	17	17	0	17	0	15.8333333	1.16666667
17	18	17	1	17.6666667	0.33333333	17.5	0.5
18	18	18	0	18	0	17.8333333	0.16666667
19	14	18	4	15.3333333	1.33333333	16	2
20	14	14	0	14	0	14.6666667	0.66666667
21	14	14	0	14	0	14	0
22	19	14	5	17.3333333	1.66666667	16.5	2.5
23	12	19	7	14.3333333	2.33333333	14.6666667	2.66666667
24	13	12	1	12.6666667	0.33333333	13.6666667	0.66666667
25	14	13	1	13.6666667	0.33333333	13.3333333	0.66666667
26			**2.91304348**		**0.97101449**		**1.33333333**


Number of products sold
WMA (K=1)


Number of products sold
WMA (K=2)


Number of products sold
WMA (K=3)

SMA and WMA – Revenue Passenger Miles




```
> MAPE-SMA 4.093731 > MAPE-WMA 2.729154
```

Stationary Model: Case 3 – Exponential ~~Weighted Moving Averages~~ or Exponential Smoothing

Averaging over long periods dampens fluctuations, removing not only the noise but also trend and seasonality.

Moving averages over short recent periods maintains trend and seasonality but determining an optimum number for periods is tricky, even when using metrics like MAE. If averaged over too few periods, irregularities continue to remain and if averaged over long periods, dampening again becomes a problem.

Exponential smoothing **retains all older periods** while giving a greater weight to more recent periods (hence not a MOVING average).

Caution: It doesn't make any one method superior for all situations.

Stationary Model: Case 3 – Exponential ~~Weighted Moving Averages~~ or Exponential Smoothing

$$\hat{Y}_{t+1} = \hat{Y}_t + \alpha\left(Y_t - \hat{Y}_t\right)$$

Above equation indicates that the predicted value for time period t+1 ($\hat{Y}_{t+1}$) is equal to the predicted value for the previous period ($\hat{Y}_t$) plus an adjustment for the error made in predicting the previous period's value ($\alpha\left(Y_t - \hat{Y}_t\right)$).

The parameter α can assume any value between 0 and 1 ($0 \le \alpha \le 1$).

Various Ways of Understanding Exponential Smoothing

- Forecast
  - Interpolation between previous *forecast* and previous *observation*

$$= \alpha Y_t + (1-\alpha)\hat{Y}_t$$

  - Previous *forecast* plus fraction of previous error

$$= \hat{Y}_t + \alpha(Y_t - \hat{Y}_t)$$

  - Previous *observation* minus fraction 1- of previous error

$$= Y_t - (1-\alpha)(Y_t - \hat{Y}_t)$$

  - *Exponentially weighted (i.e., discounted) moving average*

$$= \alpha Y_t + \alpha(1-\alpha)Y_{t-1} + \alpha(1-\alpha)^2 Y_{t-2} + \cdots + \alpha(1-\alpha)^n Y_{t-n} + \cdots$$

Exponential Smoothing

- Y at t+1 $\hat{Y}_{t+1} = \hat{Y}_t + \alpha(Y_t - \hat{Y}_t)$
- Y at t+2
- All future predictions are same! This is in accordance with **stationary** assumption.

Exponential Smoothing

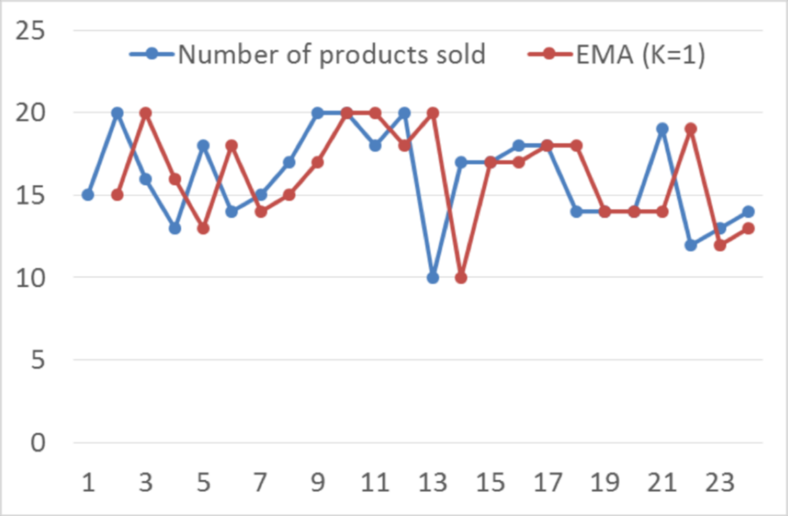
	A	B	C	D	E	F	G	H	I
1	Number of products sold	EMA (K=1)	Error	EMA (K=2)	Error	EMA (K=3)	Error	EMA (K=4)	Error
2	15								
3	20	=A2*1	=ABS(B3-A3)	=AVERAGE(A2:A3)					
4	16	=A3*K2+B3*L2	=ABS(B4-A4)	=A4*K3+D3*L3	=ABS(A4-D4)	=AVERAGE(A2:A4)	=ABS(A4-F4)		
5	13	=A4*K2+B4*L2	=ABS(B5-A5)	=A5*K3+D4*L3	=ABS(A5-D5)	=A5*K4+F4*L4	=ABS(A5-F5)	=AVERAGE(A2:A5)	=ABS(H5-A5)
6	18	=A5*K2+B5*L2	=ABS(B6-A6)	=A6*K3+D5*L3	=ABS(A6-D6)	=A6*K4+F5*L4	=ABS(A6-F6)	=A6*K5+H5*L5	=ABS(H6-A6)
7	14	=A6*K2+B6*L2	=ABS(B7-A7)	=A7*K3+D6*L3	=ABS(A7-D7)	=A7*K4+F6*L4	=ABS(A7-F7)	=A7*K5+H6*L5	=ABS(H7-A7)
8	15	=A7*K2+B7*L2	=ABS(B8-A8)	=A8*K3+D7*L3	=ABS(A8-D8)	=A8*K4+F7*L4	=ABS(A8-F8)	=A8*K5+H7*L5	=ABS(H8-A8)
9	17	=A8*K2+B8*L2	=ABS(B9-A9)	=A9*K3+D8*L3	=ABS(A9-D9)	=A9*K4+F8*L4	=ABS(A9-F9)	=A9*K5+H8*L5	=ABS(H9-A9)
10	20	=A9*K2+B9*L2	=ABS(B10-A10)	=A10*K3+D9*L3	=ABS(A10-D10)	=A10*K4+F9*L4	=ABS(A10-F10)	=A10*K5+H9*L5	=ABS(H10-A10)
11	20	=A10*K2+B10*L2	=ABS(B11-A11)	=A11*K3+D10*L3	=ABS(A11-D11)	=A11*K4+F10*L4	=ABS(A11-F11)	=A11*K5+H10*L5	=ABS(H11-A11)
12	18	=A11*K2+B11*L2	=ABS(B12-A12)	=A12*K3+D11*L3	=ABS(A12-D12)	=A12*K4+F11*L4	=ABS(A12-F12)	=A12*K5+H11*L5	=ABS(H12-A12)
13	20	=A12*K2+B12*L2	=ABS(B13-A13)	=A13*K3+D12*L3	=ABS(A13-D13)	=A13*K4+F12*L4	=ABS(A13-F13)	=A13*K5+H12*L5	=ABS(H13-A13)
14	10	=A13*K2+B13*L2	=ABS(B14-A14)	=A14*K3+D13*L3	=ABS(A14-D14)	=A14*K4+F13*L4	=ABS(A14-F14)	=A14*K5+H13*L5	=ABS(H14-A14)
15	17	=A14*K2+B14*L2	=ABS(B15-A15)	=A15*K3+D14*L3	=ABS(A15-D15)	=A15*K4+F14*L4	=ABS(A15-F15)	=A15*K5+H14*L5	=ABS(H15-A15)
16	17	=A15*K2+B15*L2	=ABS(B16-A16)	=A16*K3+D15*L3	=ABS(A16-D16)	=A16*K4+F15*L4	=ABS(A16-F16)	=A16*K5+H15*L5	=ABS(H16-A16)
17	18	=A16*K2+B16*L2	=ABS(B17-A17)	=A17*K3+D16*L3	=ABS(A17-D17)	=A17*K4+F16*L4	=ABS(A17-F17)	=A17*K5+H16*L5	=ABS(H17-A17)
18	18	=A17*K2+B17*L2	=ABS(B18-A18)	=A18*K3+D17*L3	=ABS(A18-D18)	=A18*K4+F17*L4	=ABS(A18-F18)	=A18*K5+H17*L5	=ABS(H18-A18)
19	14	=A18*K2+B18*L2	=ABS(B19-A19)	=A19*K3+D18*L3	=ABS(A19-D19)	=A19*K4+F18*L4	=ABS(A19-F19)	=A19*K5+H18*L5	=ABS(H19-A19)
20	14	=A19*K2+B19*L2	=ABS(B20-A20)	=A20*K3+D19*L3	=ABS(A20-D20)	=A20*K4+F19*L4	=ABS(A20-F20)	=A20*K5+H19*L5	=ABS(H20-A20)
21	14	=A20*K2+B20*L2	=ABS(B21-A21)	=A21*K3+D20*L3	=ABS(A21-D21)	=A21*K4+F20*L4	=ABS(A21-F21)	=A21*K5+H20*L5	=ABS(H21-A21)
22	19	=A21*K2+B21*L2	=ABS(B22-A22)	=A22*K3+D21*L3	=ABS(A22-D22)	=A22*K4+F21*L4	=ABS(A22-F22)	=A22*K5+H21*L5	=ABS(H22-A22)
23	12	=A22*K2+B22*L2	=ABS(B23-A23)	=A23*K3+D22*L3	=ABS(A23-D23)	=A23*K4+F22*L4	=ABS(A23-F23)	=A23*K5+H22*L5	=ABS(H23-A23)
24	13	=A23*K2+B23*L2	=ABS(B24-A24)	=A24*K3+D23*L3	=ABS(A24-D24)	=A24*K4+F23*L4	=ABS(A24-F24)	=A24*K5+H23*L5	=ABS(H24-A24)
25	14	=A24*K2+B24*L2	=ABS(B25-A25)	=A25*K3+D24*L3	=ABS(A25-D25)	=A25*K4+F24*L4	=ABS(A25-F25)	=A25*K5+H24*L5	=ABS(H25-A25)
26			**=AVERAGE(C3:C25)**		**=AVERAGE(E3:E25)**		**=AVERAGE(G3:G25)**		**=AVERAGE(I3:I25)**

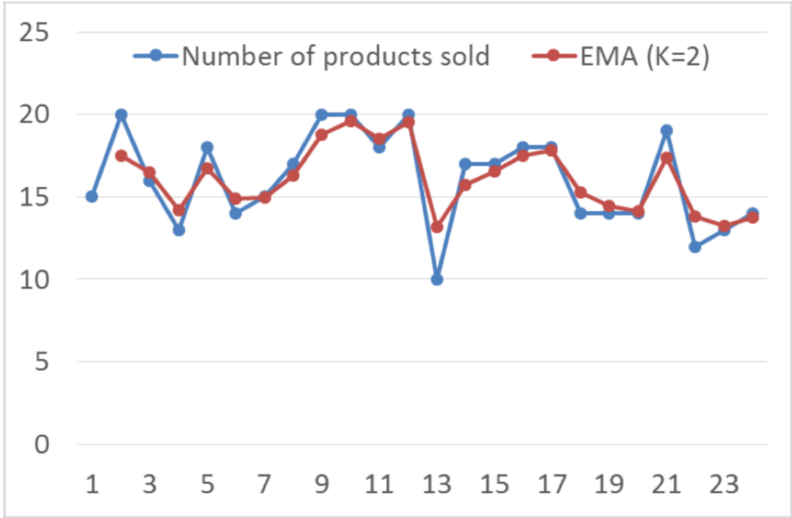
J	K	L
K	2/(K+1)	1-[2/(K+1)]
1	1	=1-K2
2	=2/(J3+1)	=1-K3
3	=2/(J4+1)	=1-K4
4	=2/(J5+1)	=1-K5

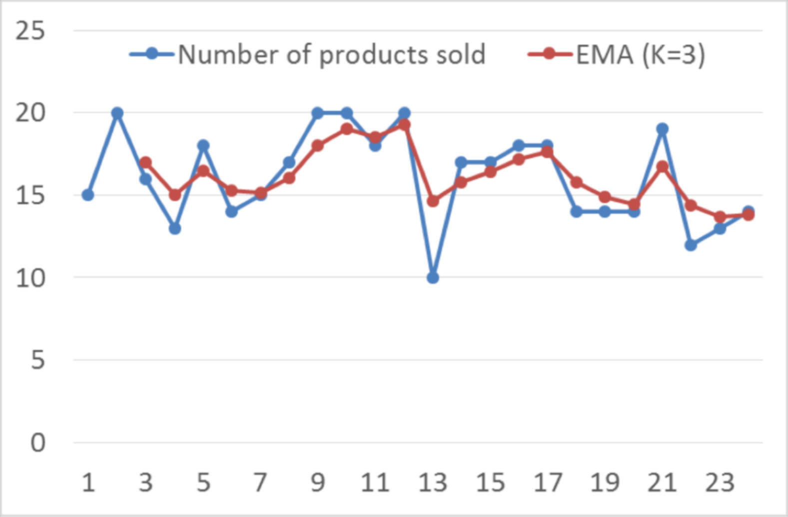
$$\hat{Y}_{t+1} = \alpha Y_t + (1-\alpha)\hat{Y}_t$$

Exponential Smoothing

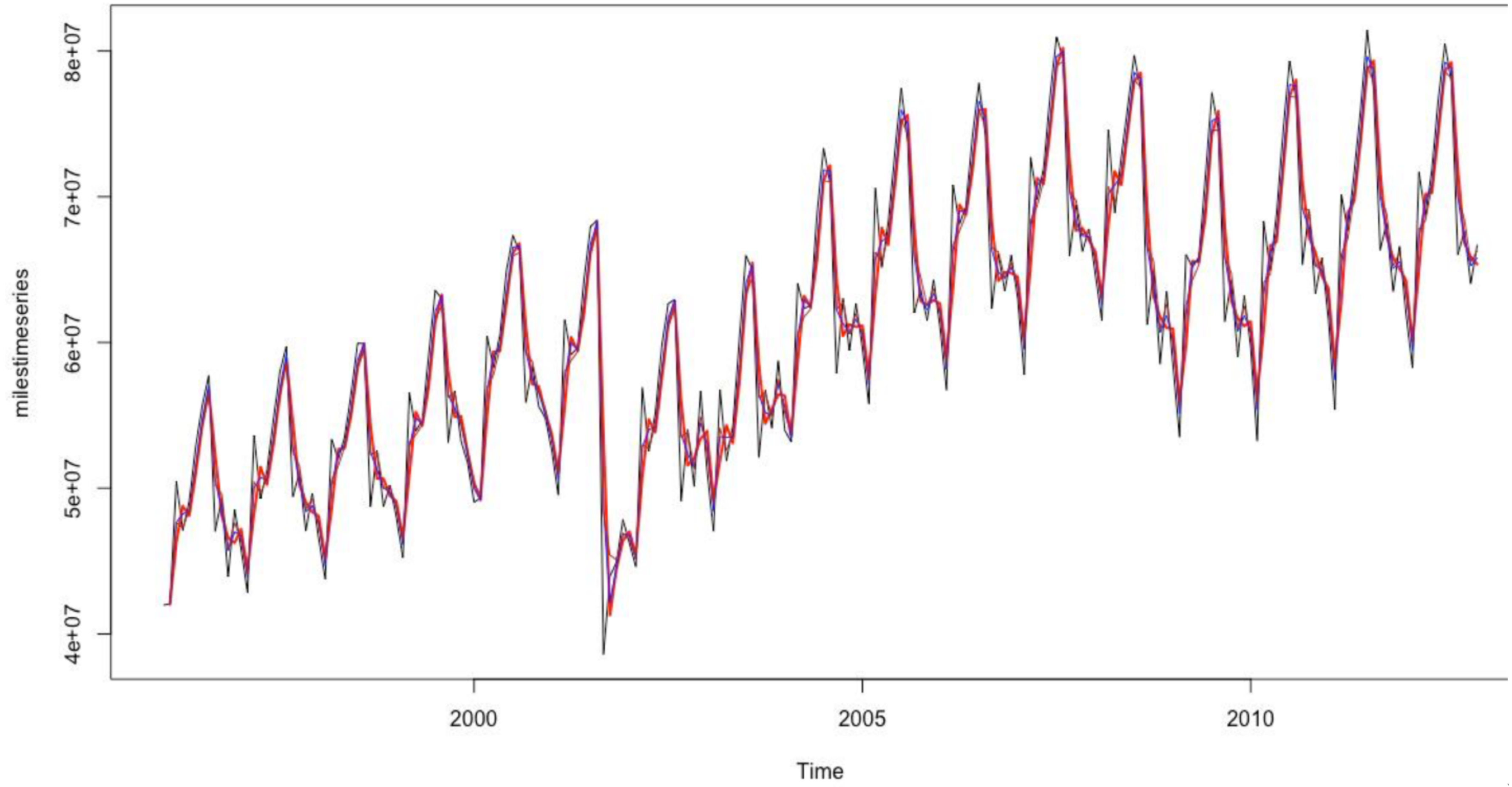
1	Number of products sold	EMA (K=1)	Error	EMA (K=2)	Error	EMA (K=3)	Error	EMA (K=4)	Error
2	15								
3	20	15	5	17.5					
4	16	20	4	16.5	0.5	17	1		
5	13	16	3	14.1666667	1.16666667	15	2	16	3
6	18	13	5	16.7222222	1.27777778	16.5	1.5	16.8	1.2
7	14	18	4	14.9074074	0.90740741	15.25	1.25	15.68	1.68
8	15	14	1	14.9691358	0.0308642	15.125	0.125	15.408	0.408
9	17	15	2	16.3230453	0.67695473	16.0625	0.9375	16.0448	0.9552
10	20	17	3	18.7743484	1.22565158	18.03125	1.96875	17.62688	2.37312
11	20	20	0	19.5914495	0.40855053	19.015625	0.984375	18.576128	1.423872
12	18	20	2	18.5304832	0.53048316	18.5078125	0.5078125	18.3456768	0.3456768
13	20	18	2	19.5101611	0.48983895	19.2539063	0.74609375	19.0074061	0.99259392
14	10	20	10	13.1700537	3.17005368	14.6269531	4.62695313	15.4044436	5.40444365
15	17	10	7	15.7233512	1.27664877	15.8134766	1.18652344	16.0426662	0.95733381
16	17	17	0	16.5744504	0.42554959	16.4067383	0.59326172	16.4255997	0.57440029
17	18	17	1	17.5248168	0.4751832	17.2033691	0.79663086	17.0553598	0.94464017
18	18	18	0	17.8416056	0.1583944	17.6016846	0.39831543	17.4332159	0.5667841
19	14	18	4	15.2805352	1.2805352	15.8008423	1.80084229	16.0599295	2.05992954
20	14	14	0	14.4268451	0.42684507	14.9004211	0.90042114	15.2359577	1.23595772
21	14	14	0	14.1422817	0.14228169	14.4502106	0.45021057	14.7415746	0.74157463
22	19	14	5	17.3807606	1.61923944	16.7251053	2.27489471	16.4449448	2.55505522
23	12	19	7	13.7935869	1.79358685	14.3625526	2.36255264	14.6669669	2.66696687
24	13	12	1	13.264529	0.26452895	13.6812763	0.68127632	14.0001801	1.00018012
25	14	13	1	13.754843	0.24515702	13.8406382	0.15936184	14.0001081	0.00010807
26			**2.91304348**		**0.84055449**		**1.23867161**		**1.48027795**


25
20
15
10
5
0
Number of products sold
EMA (K=1)
1 3 5 7 9 11 13 15 17 19 21 23


25
20
15
10
5
0
Number of products sold
EMA (K=2)
1 3 5 7 9 11 13 15 17 19 21 23

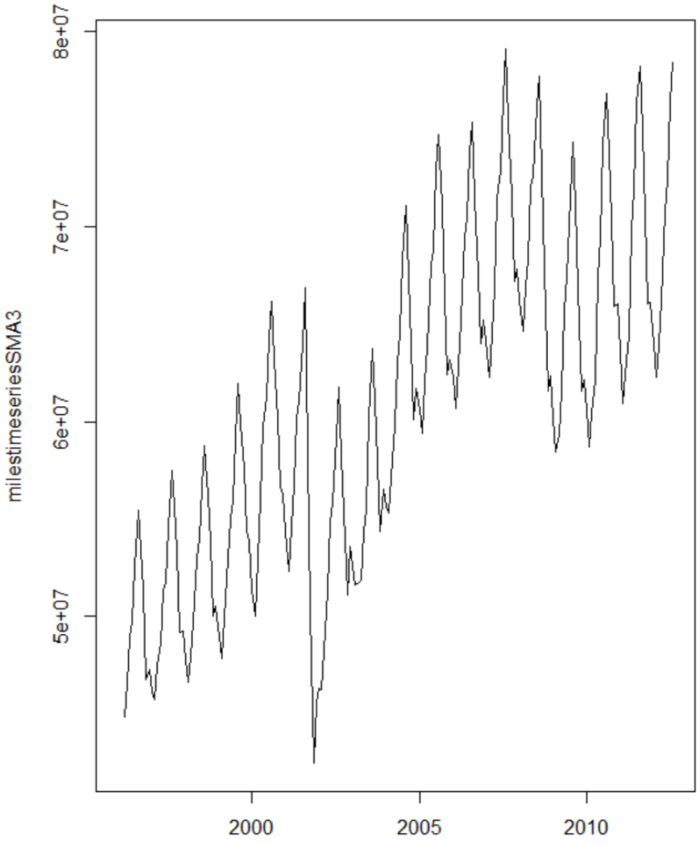

25
20
15
10
5
0
Number of products sold
EMA (K=3)
1 3 5 7 9 11 13 15 17 19 21 23

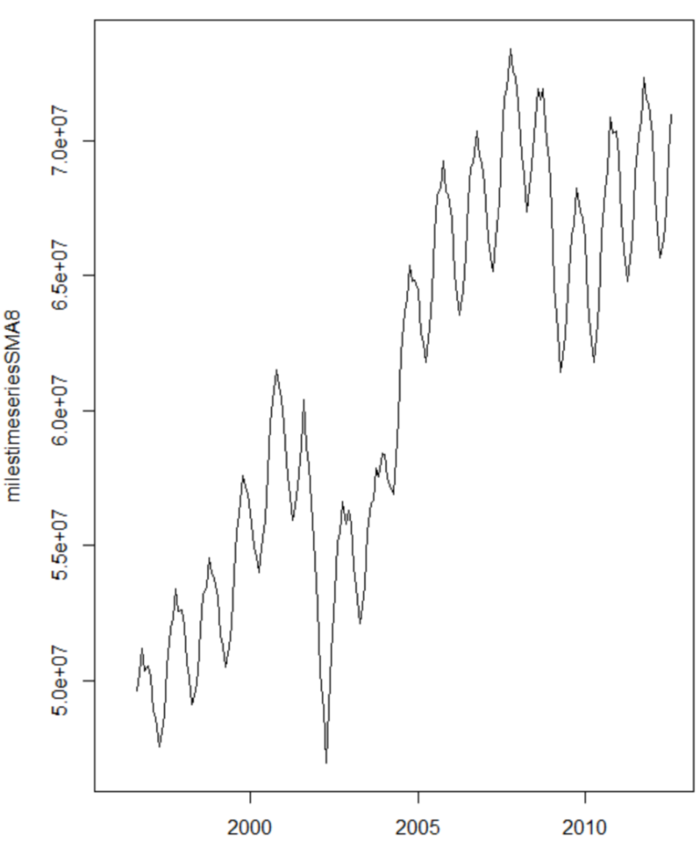
SMA, WMA and Exponential Smoothing – RPM




```
> MAPE-SMA 4.087519 > MAPE-WMA 2.725013 > MAPE-EMA 2.542173
```

Effect of *k* – Revenue Passenger Miles





SMA

K=3 vs K=8

Advanced Time Series Methods
ARIMA

AutoRegressive (AR) Models

- $\hat{y}_t$ depends only on its own past values y_{t-1}, y_{t-2}, y_{t-3}, etc. Thus,

  $\hat{y}_t$=f(y_{t-1}, y_{t-2}, y_{t-3}, ….., ε_t)

- Autoregressive model where it depends on **p** of its **past values** *(lagged values having significant relationship with the most recent value)* is an **AR(p)** model and is represented as:

$$\hat{y}_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + e_t$$

Moving Average (MA) Models

- A moving average model is one where $\hat{y}_t$ **depends** only on the **random error term**, i.e.,

  $\hat{y}_t$ = f(ε_t, $\varepsilon_{t\text{-}1}$, $\varepsilon_{t\text{-}2}$, $\varepsilon_{t\text{-}3}$, …..)

- Moving average model where it depends on **q** of its **past values** *(residuals or lagged errors from earlier estimates)* is an **MA(q)** model and is represented as

  $$\hat{y}_t = c + e_t + \theta_1 e_{t-1} + \vartheta_2 e_{t-2} + \ldots + \vartheta_q e_{t-q}$$

AutoRegressive Moving Average (ARMA) Models

- ARMA(p,q)
  - Mix of both AR and MA
- General form of such a time-series model

$$\hat{y}_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} +$$

$$e_t + \theta_1 e_{t-1} + \vartheta_2 e_{t-2} + \ldots + \vartheta_q e_{t-q}$$

Integration of order d

- **Non-stationary** series can be made **stationary** series after **differencing**.
- **Integration** of order **d**
  - A series that is **differentiated d times**
  - Denoted as **I(d)**
    - If the series is differentiated **once** then it is **I(1)**.
    - **I(0)** means series is **stationary** without **differencing**

Box–Jenkins Methodology

- Model identification and model selection
  - Make sure variables are stationary. Difference as necessary to get a constant mean and transformations to get constant variance.
  - Check for seasonality: Decays and spikes at regular intervals in ACF plots.
- Parameter estimation
  - Compute coefficients that best fit the selected model.
- Model checking
  - Check if residuals are independent of each other and constant in mean and variance over time (white noise).

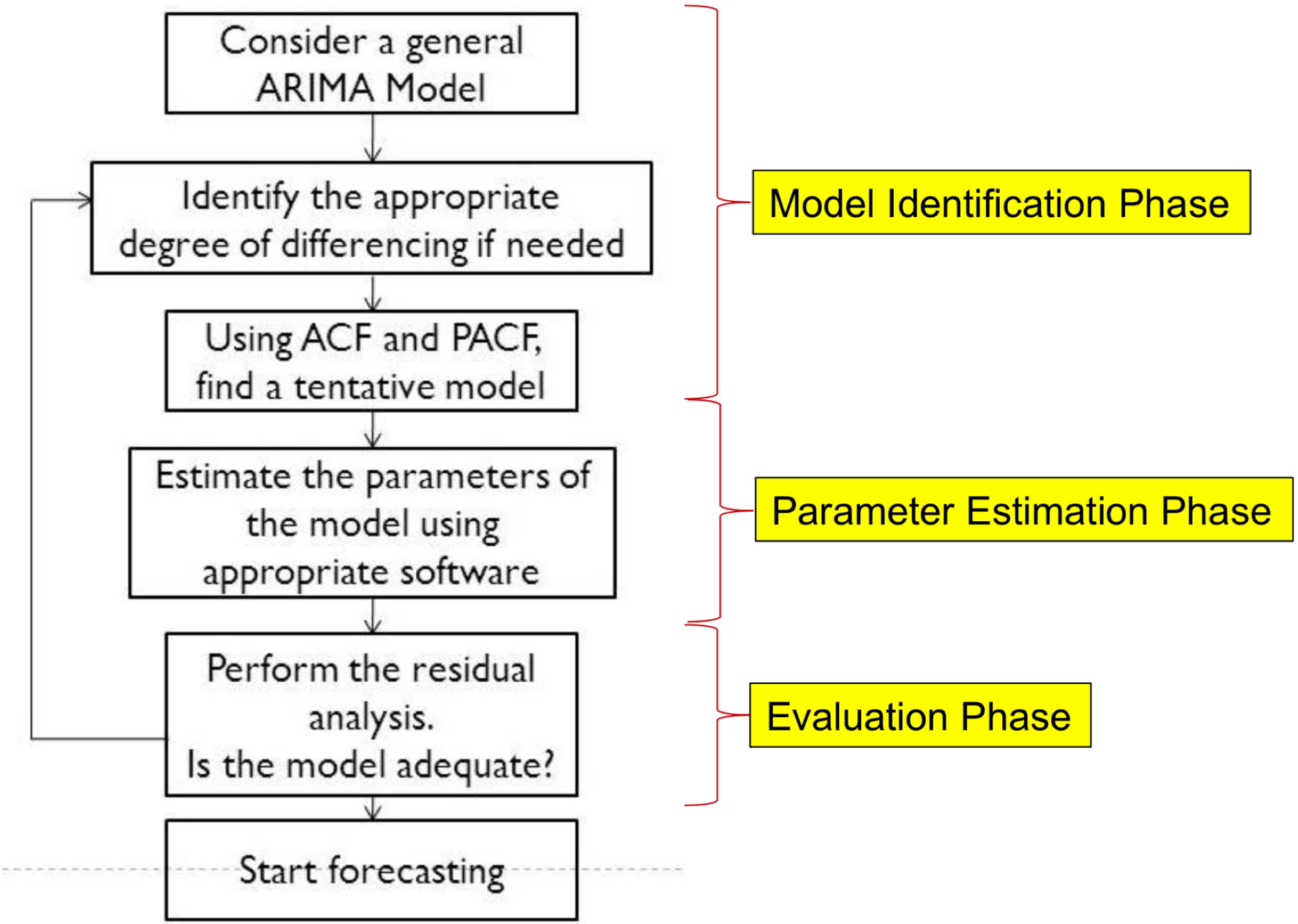
https://www.ncss.com/wp-content/themes/ncss/pdf/Procedures/NCSS/The_Box-Jenkins_Method.pdf

ARIMA(p,d,q) Model

- *p* is the number of autoregressive [AR(p)] terms (a linear regression of the current value of the series against one or more prior values of the series)
  - Maximum lag beyond which PACF is 0
- *d* is the number of non-seasonal differences (order of the differencing) used to make the time series stationary [I(d)]
- *q* is the order of the moving average [MA(q)] model
  - Maximum lag beyond which the ACF is 0

- Non-seasonal ARIMA models are denoted ARIMA(p,d,q)
- Seasonal ARIMA (SARIMA) models are denoted ARIMA(p,d,q)$(P,D,Q)_m$, where m refers to the number of periods in each season and (P,D,Q) refer to the autoregressive, differencing and moving average terms of the seasonal part of the ARIMA model.

Time Series Model Building Using ARIMA



THANK YOU