# Deutsch's Algorithm:
# An Algebraic and Quantum Computational Approach

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# Chapter 1

# Preliminaries

## 1.1 Abstract Algebraic Foundations

In our treatment, a *vector space* is regarded as an abelian group equipped with a compatible field action. In particular, let $\mathbb{F}$ be a field (usually $\mathbb{C}$) and let $(V, +)$ be an abelian group. A scalar multiplication

$$\cdot : \mathbb{F} \times V \to V$$

endows $V$ with the structure of an $\mathbb{F}$-module if for all $\alpha, \beta \in \mathbb{F}$ and all $x, y \in V$, we have:

(i) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$,

(ii) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$,

(iii) $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$,

(iv) $1_{\mathbb{F}} \cdot x = x$.

When $\mathbb{F}$ is a field, an $\mathbb{F}$-module is precisely a vector space.

**Definition 1.1** (Hilbert Space)**.** *A* Hilbert space $\mathcal{H}$ *is a vector space over* $\mathbb{C}$ *(viewed as an abelian group with scalar action) equipped with an inner product*

$$\langle \cdot \mid \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C},$$

*which is positive-definite and sesquilinear, and is complete with respect to the norm*

$$\|\psi\| = \sqrt{\langle \psi \mid \psi \rangle}.$$

## 1.2 Dirac's Bra–Ket Notation

To facilitate discussion in quantum mechanics, we employ Dirac's bra–ket notation.

**Definition 1.2** (Bra and Ket)**.** *Let* $\mathcal{H}$ *be a Hilbert space. An element* $|\psi\rangle \in \mathcal{H}$ *is called a* ket*, while the corresponding* bra *is the linear functional* $\langle\psi| \in \mathcal{H}^*$ *defined by*

$$\langle\psi| \left(|\varphi\rangle\right) = \langle \psi \mid \varphi \rangle, \quad \forall\ |\varphi\rangle \in \mathcal{H}.$$

The superposition principle follows from the abelian group structure:

$$\text{If } |\psi\rangle, |\varphi\rangle \in \mathcal{H} \text{ and } \alpha, \beta \in \mathbb{C}, \quad \alpha|\psi\rangle + \beta|\varphi\rangle \in \mathcal{H}.$$

# Chapter 2

# Deutsch's Algorithm: Problem Statement and Classical Preliminaries

## 2.1 Problem Statement

Consider a function

$$f : \{0, 1\} \to \{0, 1\},$$

with the promise that $f$ is either *constant* or *balanced*. In this context, since the domain has only two elements, $f$ is:

- *Constant* if $f(0) = f(1)$,
- *Balanced* if $f(0) \neq f(1)$.

**Task:** Given oracle (black box) access to $f$, determine whether $f$ is constant or balanced.

## 2.2 Classical Approach

In classical computation, one must evaluate $f(0)$ and $f(1)$ independently, requiring *two* evaluations to decide the problem.

## 2.3 The Oracle and the Unitary Operator

In quantum computation, the function $f$ is embedded into a unitary operator (oracle) $U_f$, defined by:

$$U_f \left| x, y \right\rangle = \left| x,\, y \oplus f(x) \right\rangle,$$

where $x \in \{0, 1\}$, $y \in \{0, 1\}$, and $\oplus$ denotes addition modulo 2. This construction ensures that $U_f$ is reversible, as required for quantum evolution.

# Chapter 3

# Deutsch's Algorithm: Quantum Circuit and Detailed Analysis

## 3.1 Circuit Description

The Deutsch algorithm is implemented by the following quantum circuit:

**Step 1: Initialization:** Prepare the state

$$|\psi_0\rangle = |0\rangle \otimes |1\rangle .$$

**Step 2: Hadamard Transform:** Apply the Hadamard gate $H$ to each qubit. Recall that

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Thus, the state becomes

$$|\psi_1\rangle = (H \otimes H)|0,1\rangle = \frac{1}{2}\Big[(|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle)\Big].$$

**Step 3: Oracle Query:** Apply the oracle $U_f$ to $|\psi_1\rangle$. Because

$$U_f|x,y\rangle = |x, y \oplus f(x)\rangle ,$$

the resulting state is

$$|\psi_2\rangle = \frac{1}{2}\Big[|0,\, 0 \oplus f(0)\rangle - |0,\, 1 \oplus f(0)\rangle + |1,\, 0 \oplus f(1)\rangle - |1,\, 1 \oplus f(1)\rangle\Big].$$

**Step 4: Interference via Hadamard:** Apply the Hadamard transform $H$ to the first qubit:

$$|\psi_3\rangle = (H \otimes I)|\psi_2\rangle .$$

A detailed calculation shows that the amplitude on the first qubit encodes the quantity

$$\frac{1}{2}\Big[(-1)^{f(0)} + (-1)^{f(1)}\Big].$$

**Step 5: Measurement:** Measure the first qubit in the computational basis. One obtains

$$\begin{cases} |0\rangle , & \text{if } (-1)^{f(0)} + (-1)^{f(1)} \neq 0, \quad (\text{i.e., } f(0) = f(1)); \\ |1\rangle , & \text{if } (-1)^{f(0)} + (-1)^{f(1)} = 0, \quad (\text{i.e., } f(0) \neq f(1)). \end{cases}$$

## 3.2 Detailed State Transformations

For completeness, we detail the state evolution through the algorithm.

### After Hadamard Transforms

Starting with

$$|\psi_0\rangle = |0\rangle \otimes |1\rangle,$$

after applying $H \otimes H$, we obtain

$$\begin{aligned} |\psi_1\rangle &= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) \\ &= \frac{1}{2}\Big[|0,0\rangle - |0,1\rangle + |1,0\rangle - |1,1\rangle\Big]. \end{aligned}$$

### After Oracle Application

The oracle acts as

$$U_f |x,y\rangle = |x, y \oplus f(x)\rangle,$$

so

$$|\psi_2\rangle = \frac{1}{2}\Big[|0,\, 0 \oplus f(0)\rangle - |0,\, 1 \oplus f(0)\rangle + |1,\, 0 \oplus f(1)\rangle - |1,\, 1 \oplus f(1)\rangle\Big].$$

Noting that addition modulo 2 satisfies

$$0 \oplus f(x) = f(x) \quad \text{and} \quad 1 \oplus f(x) = 1 - f(x),$$

we can rewrite the state as

$$|\psi_2\rangle = \frac{1}{2}\Big[|0, f(0)\rangle - |0, 1 - f(0)\rangle + |1, f(1)\rangle - |1, 1 - f(1)\rangle\Big].$$

### After Final Hadamard on the First Qubit

Apply $H$ to the first qubit:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Thus,

$$|\psi_3\rangle = (H \otimes I)|\psi_2\rangle = \frac{1}{2\sqrt{2}} \sum_{x\in\{0,1\}} \Big[(-1)^{x\cdot 0}|0\rangle + (-1)^{x\cdot 1}|1\rangle\Big] \otimes \Big[\cdots\Big],$$

which, after grouping terms, yields an amplitude on $|0\rangle$ proportional to

$$(-1)^{f(0)} + (-1)^{f(1)}.$$

### Measurement Outcome

A measurement in the computational basis on the first qubit then distinguishes:

$$|0\rangle \quad \text{if } (-1)^{f(0)} + (-1)^{f(1)} \neq 0 \quad \Longleftrightarrow \quad f(0) = f(1),$$

and

$$|1\rangle \quad \text{if } (-1)^{f(0)} + (-1)^{f(1)} = 0 \quad \Longleftrightarrow \quad f(0) \neq f(1).$$

Thus, Deutsch's algorithm determines whether $f$ is constant or balanced with a single query (oracle evaluation) on a quantum computer.

## 3.3 Phase Kickback and Oracle Implementation

A key concept in Deutsch's algorithm is *phase kickback*. In the implementation of $U_f$, the target qubit is prepared in a state $|-\rangle$, where

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

The action of $U_f$ then imprints a phase $(-1)^{f(x)}$ on the control qubit:

$$U_f |x\rangle |-\rangle = |x\rangle |- \oplus f(x)\rangle = (-1)^{f(x)} |x\rangle |-\rangle .$$

This phase encoding is then revealed by subsequent interference (Hadamard transform) on the control qubit.

## 3.4 Summary of Algorithmic Steps (Slides Overview)

For completeness, we list the conceptual steps as they might appear in a lecture slide sequence:

**Slide 1: Deutsch Algorithm Overview:** Problem statement and promise on $f$.

**Slide 2: Function Definition:** $f : \{0, 1\} \to \{0, 1\}$, constant versus balanced.

**Slide 3: Classical Evaluation:** Necessity of two evaluations.

**Slide 4: Oracle Construction:** Definition of $U_f |x, y\rangle = |x, y \oplus f(x)\rangle$.

**Slide 5: Quantum Circuit Initialization:** Prepare $|0\rangle \otimes |1\rangle$.

**Slide 6: Hadamard Transformation:** Transition to superposition.

**Slide 7: Oracle Application:** State transformation under $U_f$.

**Slide 8: Phase Kickback:** Imprinting phase $(-1)^{f(x)}$.

**Slide 9: Interference via Hadamard:** Extraction of global phase information.

**Slide 10: Measurement:** Readout distinguishing constant vs balanced.

**Slide 11: Conclusion:** Deutsch's algorithm solves the problem with one oracle query.

# Chapter 4

# Exercises and Further Directions

## Assignment

**Problem 1:** **State Decomposition:** Express the two-qubit state at the red-dashed point (as indicated in the lecture circuit diagram) as a linear combination of the computational basis vectors $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

**Problem 2:** **Measurement Probabilities:** Suppose that $f(0) = f(1) = 1$ (i.e., $f$ is constant with value 1). Compute the probabilities of obtaining outcomes 0 and 1 when measuring the first qubit.

## Further Reading

Readers are encouraged to consult the following references for additional background on quantum algorithms and the algebraic foundations of quantum mechanics:

- M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information.*
- P. Shor, *Introduction to Quantum Algorithms.*
- J. Preskill, *Lecture Notes for Physics 229: Quantum Information and Computation.*

# References

[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information.*

[2] D. Deutsch, "Quantum theory, the Church–Turing principle and the universal quantum computer," *Proc. R. Soc. Lond. A* **400**, 97–117 (1985).

[3] S. Lang, *Algebra.*

[4] J.J. Sakurai, *Modern Quantum Mechanics.*