Abstract Algebraic Foundations of Quantum Mechanics

Detailed Graduate-Level Lecture Notes

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Contents

Chapter 1

Algebraic Structures: Groups, Fields, and Modules

1.1 Abelian Groups and Fields

In abstract algebra, a *group* is a set equipped with an operation satisfying associativity, the existence of an identity element, and inverses. When the operation is also commutative, the group is said to be *abelian*.

Definition 1.1 (Abelian Group)**.** *A pair* $(G, +)$ *is an* abelian group *if:*

(i) *(Closure) For all* $a, b \in G$*, the sum* $a + b \in G$*.*

(ii) *(Associativity) For all* $a, b, c \in G$*,* $(a + b) + c = a + (b + c)$*.*

(iii) *(Identity) There exists an element* $0 \in G$ *such that for all* $a \in G$*,* $a + 0 = a$*.*

(iv) *(Inverses) For each* $a \in G$*, there exists an element* $-a \in G$ *such that* $a + (-a) = 0$*.*

(v) *(Commutativity) For all* $a, b \in G$*,* $a + b = b + a$*.*

A *field* is a set that is simultaneously an abelian group with respect to addition and a multiplicative abelian group (excluding the additive identity), where the two operations are related by distributivity.

Definition 1.2 (Field)**.** *A set* $\mathbb{F}$*, together with two operations* $+$ *and* $\cdot$*, is a* field *if:*

(i) $(\mathbb{F}, +)$ *is an abelian group with identity element* 0*.*

(ii) $(\mathbb{F} \setminus \{0\}, \cdot)$ *is an abelian group with identity element* $1 \neq 0$*.*

(iii) *(Distributivity) For all* $a, b, c \in \mathbb{F}$*,*

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

1.2 Modules and Vector Spaces

In abstract algebra, a *module* is a generalization of the notion of a vector space. When the underlying ring is a field, a module is precisely a vector space.

Definition 1.3 (Module)**.** *Let* R *be a ring and* $(M, +)$ *an abelian group. The set* M *is an* R-module *if there exists a scalar multiplication*

$$\cdot : R \times M \to M,$$

satisfying for all $r, s \in R$ *and* $x, y \in M$*:*

(a) $r \cdot (x + y) = r \cdot x + r \cdot y$,

(b) $(r + s) \cdot x = r \cdot x + s \cdot x$,

(c) $(rs) \cdot x = r \cdot (s \cdot x)$,

(d) $1_R \cdot x = x$, *where* 1_R *is the multiplicative identity in* R.

Definition 1.4 (Vector Space)**.** *Let* $\mathbb{F}$ *be a field. A* vector space *over* $\mathbb{F}$ *is an* $\mathbb{F}$*-module* $(V, +)$*; that is,* V *is an abelian group under addition together with a scalar multiplication* $\mathbb{F} \times V \to V$ *satisfying the axioms listed above.*

Example 1.1. *The set* $\mathbb{F}^n$*, consisting of* n*-tuples over* $\mathbb{F}$*, is a vector space over* $\mathbb{F}$ *under coordinatewise addition and scalar multiplication.*

Chapter 2

Inner Product Spaces and Hilbert Spaces

2.1 Inner Product as a Bilinear (or Sesquilinear) Form

In the context of vector spaces over $\mathbb{C}$, the inner product is defined as a positive-definite, sesquilinear form.

Definition 2.1 (Inner Product)**.** *Let V be a vector space over $\mathbb{C}$. An* inner product *on V is a function*

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C},$$

such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha \in \mathbb{C}$:

(i) (Conjugate Symmetry) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

(ii) (Linearity in the First Argument) $\langle \alpha\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \alpha \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

(iii) (Positive Definiteness) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$.

For real vector spaces, the inner product is bilinear.

Definition 2.2 (Hilbert Space)**.** *A* Hilbert space *$\mathcal{H}$ is an inner product space that is complete with respect to the metric induced by the norm*

$$\|\mathbf{v}\| \coloneqq \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}.$$

Remark. In quantum mechanics, the state space is modeled as a Hilbert space, where physical states are represented by unit vectors.

2.2 Orthonormal Bases and Decompositions

Definition 2.3 (Orthonormal Set)**.** *An indexed subset $\{|v_i\rangle\}_{i\in I}$ of a Hilbert space $\mathcal{H}$ is* orthonormal *if for all $i, j \in I$,*

$$\langle v_i | v_j \rangle = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta.

Definition 2.4 (Complete Orthonormal Basis)**.** *An orthonormal set $\{|v_i\rangle\}_{i\in I}$ is a* complete orthonormal basis *for $\mathcal{H}$ if every $|\psi\rangle \in \mathcal{H}$ can be uniquely expressed as*

$$|\psi\rangle = \sum_{i\in I} c_i |v_i\rangle,$$

with coefficients given by

$$c_i = \langle v_i | \psi \rangle,$$

and where the series converges in the norm induced by the inner product.

Chapter 3

Dirac's Bra–Ket Notation in an Algebraic Setting

3.1 Dual Spaces and Linear Functionals

Let V be a vector space over $\mathbb{F}$. Its dual space, denoted V^*, is the set of all linear functionals from V to $\mathbb{F}$. In quantum mechanics, every state vector $|\psi\rangle \in \mathcal{H}$ is associated with a dual vector $\langle\psi| \in \mathcal{H}^*$.

Definition 3.1 (Bra and Ket)**.** *Given a Hilbert space $\mathcal{H}$:*

(a) A ket *is an element $|\psi\rangle \in \mathcal{H}$.*

(b) Its corresponding bra *is the unique element $\langle\psi| \in \mathcal{H}^*$ defined by*

$$\langle\psi| : \mathcal{H} \to \mathbb{C}, \quad \langle\psi| \left(|\varphi\rangle\right) = \langle\psi|\varphi\rangle .$$

Remark. The notation explicitly emphasizes the duality between the abstract abelian group (the state space) and its dual space of linear functionals.

3.2 Superposition and Linear Combinations

The structure of a vector space as an abelian group under addition, coupled with scalar multiplication (the field action), naturally leads to the principle of superposition.

Definition 3.2 (Superposition Principle)**.** *Let $|\psi\rangle, |\varphi\rangle \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$. The vector*

$$|\chi\rangle = \alpha\,|\psi\rangle + \beta\,|\varphi\rangle$$

is also an element of $\mathcal{H}$. This expresses the fact that $\mathcal{H}$, as an abelian group with a field action, is closed under linear combinations.

Example 3.1 (Qubit State)**.** *In the two-dimensional Hilbert space $\mathbb{C}^2$, a* qubit *is represented by a state*

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1,$$

where

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Chapter 4

Linear Operators in the Language of Abstract Algebra

4.1 Operators as Endomorphisms

Let V be a vector space over a field $\mathbb{F}$. The set of all linear maps (endomorphisms) from V to itself,

$$\mathrm{End}(V) = \{L : V \to V \mid L \text{ is linear}\},$$

forms an $\mathbb{F}$-algebra under pointwise addition and composition.

Definition 4.1 (Linear Operator). *A function $L : V \to V$ is a* linear operator *if for all $x, y \in V$ and $a \in \mathbb{F}$,*

$$L(x + y) = L(x) + L(y) \quad \text{and} \quad L(a \cdot x) = a \cdot L(x).$$

4.2 Adjoint Operators and Hermitian Forms

When V is endowed with an inner product, one may define the *adjoint* of an operator.

Definition 4.2 (Adjoint Operator). *Let $L \in \mathrm{End}(V)$, where V is a Hilbert space. The* adjoint *of L, denoted $L^\dagger$, is defined by the relation*

$$\langle \phi | L\psi \rangle = \left\langle L^\dagger \phi \middle| \psi \right\rangle, \quad \forall \, |\phi\rangle \, , |\psi\rangle \in V.$$

Definition 4.3 (Hermitian Operator). *An operator $H \in \mathrm{End}(V)$ is* Hermitian *(or self-adjoint) if*

$$H^\dagger = H.$$

Definition 4.4 (Unitary Operator). *An operator $U \in \mathrm{End}(V)$ is* unitary *if*

$$U^\dagger U = U U^\dagger = I,$$

where I is the identity endomorphism on V.

4.3 Matrix Representations and Change of Basis

If V is finite-dimensional, fixing a complete orthonormal basis $\{v_i\}_{i=1}^n$, every operator $L \in \mathrm{End}(V)$ is represented by an $n \times n$ matrix whose entries are given by

$$L_{ij} = \langle v_i | \, L \, | v_j \rangle \, .$$

A change of basis corresponds to a similarity transformation within the algebra $\mathrm{End}(V)$.

Chapter 5

Tensor Products and Composite Systems

5.1 Tensor Product of Modules and Vector Spaces

Let V and W be vector spaces over a field $\mathbb{F}$. Their tensor product $V \otimes W$ is constructed as follows: Consider the free abelian group generated by the Cartesian product $V \times W$, and then impose the bilinearity relations to obtain an $\mathbb{F}$-module. When $\mathbb{F}$ is a field, this module is a vector space.

Definition 5.1 (Tensor Product). *The* tensor product $V \otimes W$ *is the vector space together with a bilinear map*

$$\otimes : V \times W \to V \otimes W,$$

such that for any vector space U and any bilinear map $f : V \times W \to U$, there exists a unique linear map $\tilde{f} : V \otimes W \to U$ satisfying $f(v, w) = \tilde{f}(v \otimes w)$ for all $(v, w) \in V \times W$.

Theorem 5.1 (Dimension Formula). *If $\dim V = n$ and $\dim W = m$, then*

$$\dim(V \otimes W) = nm.$$

5.2 Composite Quantum Systems

In quantum mechanics, composite systems are modeled by tensor products of Hilbert spaces. For instance, a two-qubit system is described by

$$\mathcal{H}_{\text{2-qubit}} = \mathbb{C}^2 \otimes \mathbb{C}^2,$$

with the basis elements expressed as

$$|ij\rangle \coloneqq |i\rangle \otimes |j\rangle, \quad i, j \in \{0, 1\}.$$

Example 5.1 (Two-Qubit State). *A general state of a two-qubit system is given by*

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C},$$

with the normalization condition

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1.$$

5.3 Kronecker Product as Matrix Tensor Product

When linear operators on finite-dimensional vector spaces are considered, the matrix representation of the tensor product operator is given by the Kronecker product.

Definition 5.2 (Kronecker Product)**.** *Let*

$$A \in \mathbb{F}^{n\times m} \quad \textit{and} \quad B \in \mathbb{F}^{p\times q}.$$

The Kronecker product *is defined by*

$$A \otimes B \coloneqq \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}.$$

Proposition 5.2. *For vectors $\alpha \in \mathbb{F}^n$ and $\beta \in \mathbb{F}^p$, one has*

$$(A \otimes B)(\alpha \otimes \beta) = (A\alpha) \otimes (B\beta).$$

Chapter 6

Quantum Observables and Measurement in an Algebraic Framework

6.1 Observables as Hermitian Operators

In quantum mechanics, physical observables are represented by Hermitian operators acting on a Hilbert space.

Definition 6.1 (Observable). *An* observable *is a Hermitian operator $H \in \mathrm{End}(\mathcal{H})$, i.e., $H^\dagger = H$. The eigenvalues of H represent the possible outcomes of a measurement.*

6.2 Measurement Postulate

Let $|\psi\rangle \in \mathcal{H}$ be a quantum state and let H be an observable with spectral decomposition

$$H = \sum_{\lambda} \lambda \, P_\lambda,$$

where $\{P_\lambda\}$ are orthogonal projection operators. Then the probability of obtaining the measurement outcome λ is given by

$$\Pr(\lambda) = \| P_\lambda \, |\psi\rangle \, \|^2,$$

and upon measurement, the state collapses to

$$\frac{P_\lambda \, |\psi\rangle}{\| P_\lambda \, |\psi\rangle \, \|}.$$

6.3 The Pauli Matrices

The Pauli matrices provide a concrete example of Hermitian operators on the two-dimensional Hilbert space $\mathbb{C}^2$.

Definition 6.2 (Pauli Matrices). *Define*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These operators are Hermitian and unitary, and they obey the commutation relations

$$[\sigma_i, \sigma_j] = 2i \, \epsilon_{ijk} \, \sigma_k,$$

where ϵ_{ijk} is the Levi-Civita symbol.

Example 6.1 (Eigenvalue Problem for σ_z)**.** *The eigenvalue equation for* σ_z *is*

$$\sigma_z |\psi\rangle = \lambda |\psi\rangle .$$

Direct calculation yields:

- $\lambda = 1$, *with eigenvector* $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- $\lambda = -1$, *with eigenvector* $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Thus, $\{|0\rangle, |1\rangle\}$ *is an orthonormal basis of* $\mathbb{C}^2$.

Chapter 7

Conclusion and Further Directions

In these notes we have rigorously established the algebraic foundations of quantum mechanics using the language of abstract algebra. We began by defining abelian groups, fields, and modules, thereby presenting vector spaces as abelian groups equipped with a field action. We then developed inner product spaces and Hilbert spaces, introduced Dirac's bra–ket notation, and treated linear operators (including unitary and Hermitian operators) in an abstract algebraic context. Finally, we examined tensor products, which naturally describe composite quantum systems, and discussed the representation of observables and measurement.

This comprehensive treatment lays the groundwork for advanced topics in functional analysis, representation theory, and quantum computation.

References

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